Weight drop in ϕ^4 transcendentals

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Recall

The **Kirchoff polynomial** of a graph G is

$$\Psi_G = \sum_{\substack{T \text{ spanning} \\ \text{tree of } G}} \prod_{e \notin T} a_e$$

A **Feynman graph in** ϕ^4 is a graph with vertices of valence at most 4 (external edges make up the missing valence). It is **1PI** if it is 2-edge-connected.

The associated **Feynman integral**, after using Schwinger parameters, is

$$\int_{e_i \geq 0} \frac{\prod de_i}{\Psi_G^2}$$

Divergence

In interesting cases these integrals diverge.

Graphs are said to be **divergent** if they lead to divergent integrals.
Graphs are said to be **primitive** if they have no divergent subgraphs.

Renormalization is a procedure to convert formal integrals to convergent integrals. In general this is complicated and there are choices to be made.

Today

Today we are interested in primitive 4-point graphs in ϕ^4. In this case

- There are no divergence problems for any proper subsets of the integration variables.
- The overall integral is log divergent.

In this case any sane choice will give the same transcendental. We can set the external momenta to 0 and take one linear relation among the variables, say

$$\sum_e a_e = 1$$

(projectivising)

That is...

Given a 2-edge-connected graph with

- vertex degree at most 4,
- n independent cycles, $2n$ edges, and
- no proper subgraph with cycles to edges in this ratio.

We are interested in

$$\int_{e_i \geq 0} \frac{\Omega}{\Psi^2}$$

where $\Omega = \sum_i (-1)^i e_i de_1 \wedge \cdots \wedge \widehat{de_i} \wedge \cdots \wedge de_{2n}$ and Ψ is the Kirchhoff polynomial of the graph

$$\Psi = \sum_{\substack{T \text{spanning} \\ \text{tree}}} \prod_{e \notin T} e$$

Multiple zeta values

$$\zeta(s_1, \ldots, s_n) = \sum_{a_1 > \cdots > a_n \geq 1} \frac{1}{a_1^{s_1} \ldots a_n^{s_n}}$$

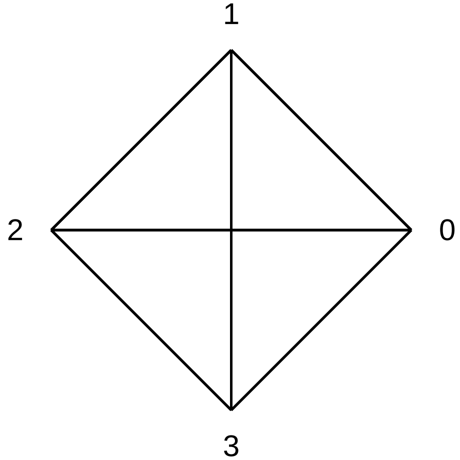
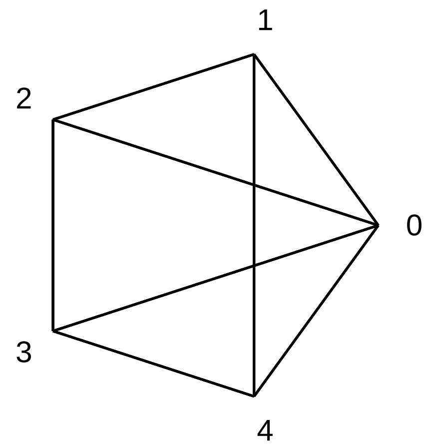
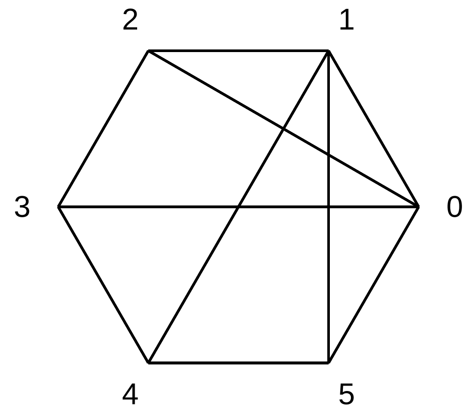
The **weight** of $\zeta(s_1, \ldots, s_n)$ is $s_1 + \cdots + s_n$.

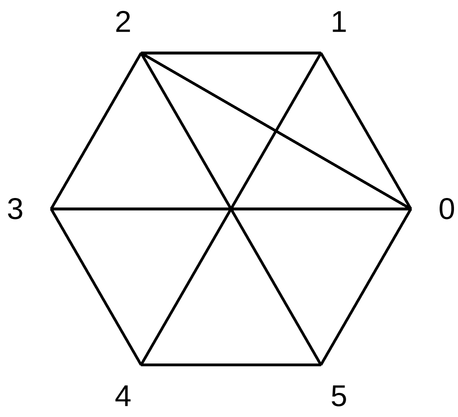
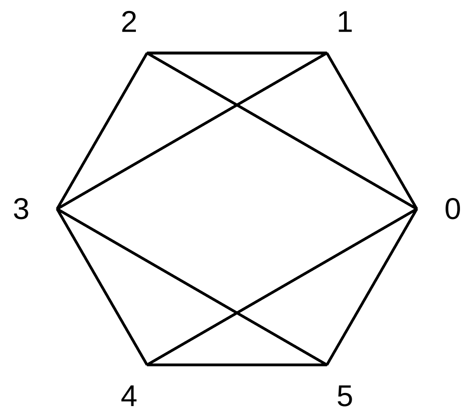
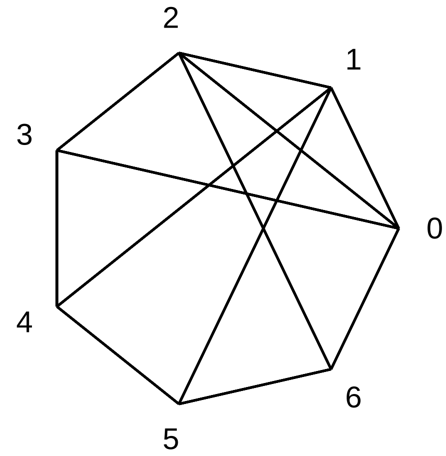
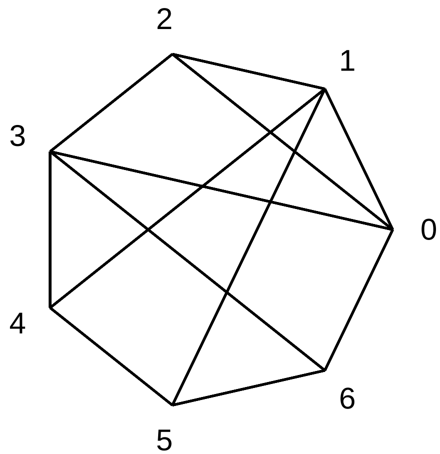
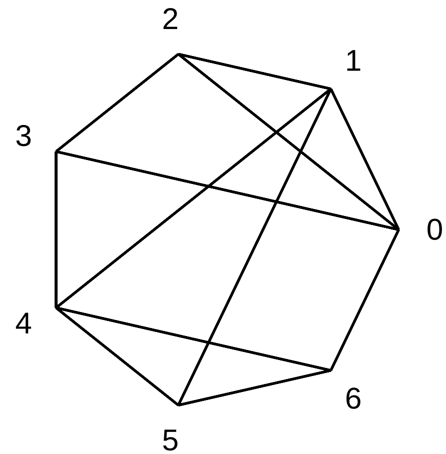
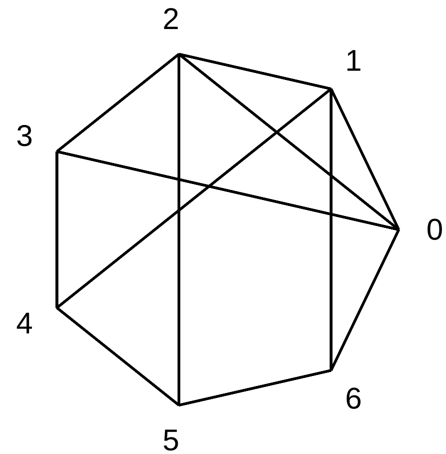
Multiple zeta values

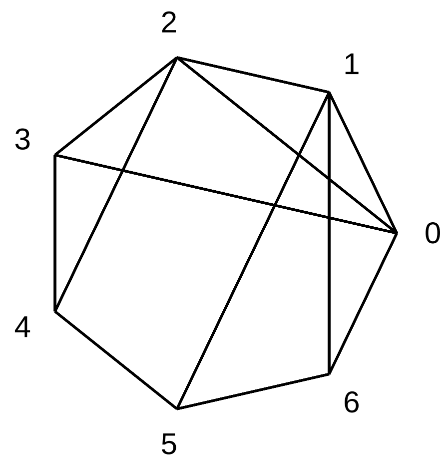
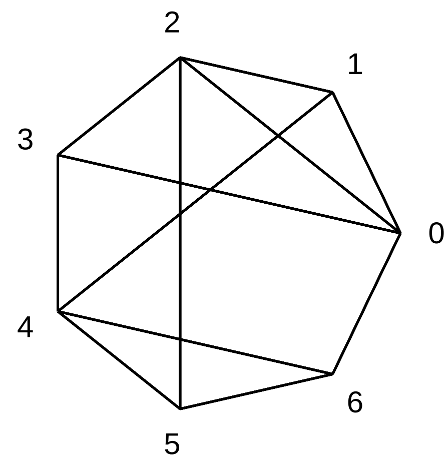
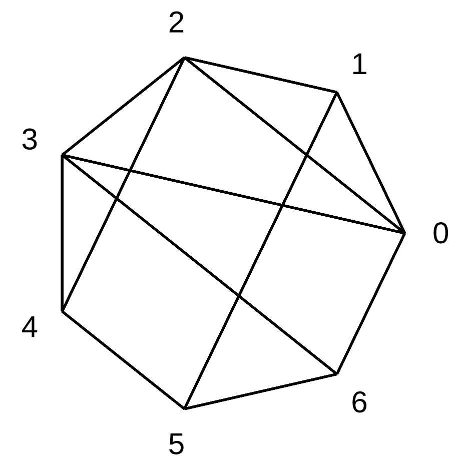
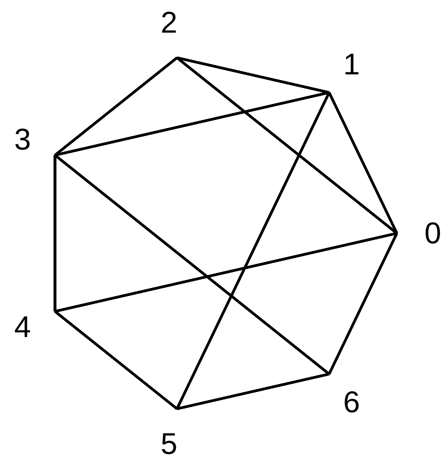
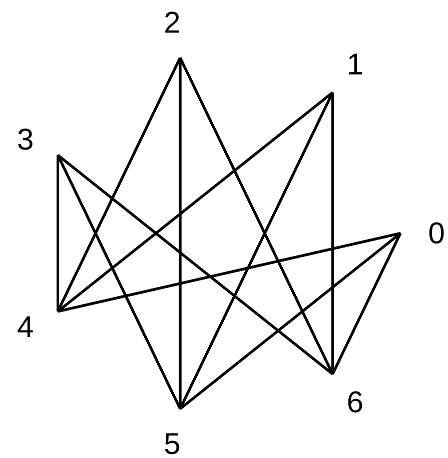
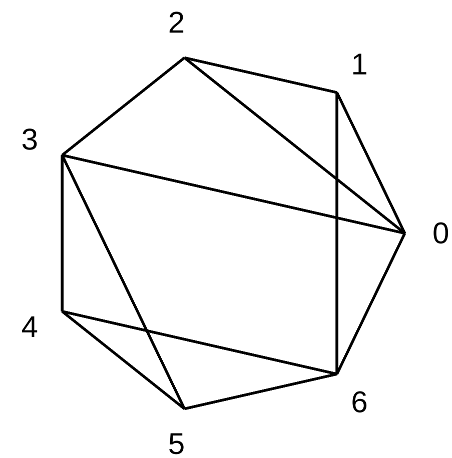
- generalize special values of the Riemann zeta function
- have an interesting algebra structure and relations
- count things (ask David Broadhurst)
- are the periods of moduli spaces
- ...

Feynman periods

In known examples these are multiple zeta values. Computations due to David Broadhurst, Oliver Schnetz, ...

		
$6\zeta(3)$	$20\zeta(5)$	$\frac{441}{8}\zeta(7)$

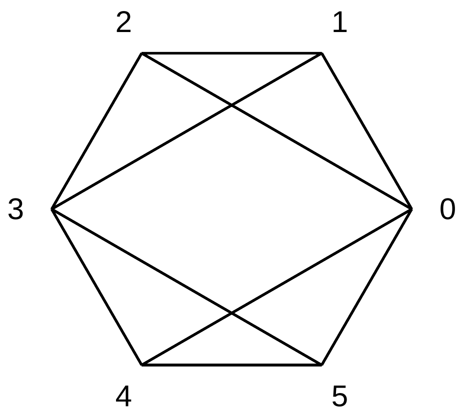
		
$36\zeta(3)^2$	$36\zeta(3)^2$	$120\zeta(3)\zeta(5)$
		
$\frac{1063}{9}\zeta(9)+8\zeta(3)^3$	$\frac{1063}{9}\zeta(9)+8\zeta(3)^3$	$\frac{1063}{9}\zeta(9)+8\zeta(3)^3$

		
$168\zeta(9)$	$36\zeta(5,3) - 87\zeta(8) + 105\zeta(3)\zeta(5)$	$120\zeta(3)\zeta(5)$
		
$36\zeta(5,3) - 87\zeta(8) + 105\zeta(3)\zeta(5)$	$1392\zeta(8) - 576\zeta(5,3) - 240\zeta(3)\zeta(5)$	$120\zeta(3)\zeta(5)$

and many more

Useful facts – 2VR graphs

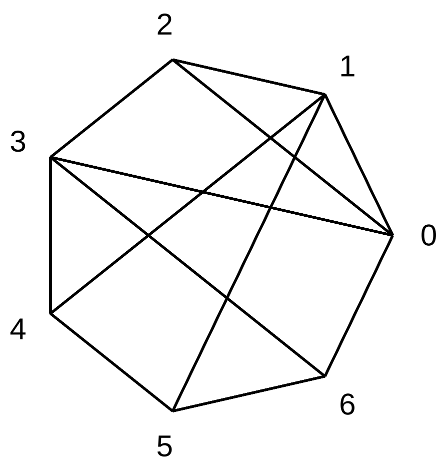
Suppose a graph has vertex connectivity 2 (i.e. we can remove 2 vertices in such a way as to disconnect the graph; *2 vertex reducible* in physics jargon). Then the period of the graph is the product of the two pieces in the following way.

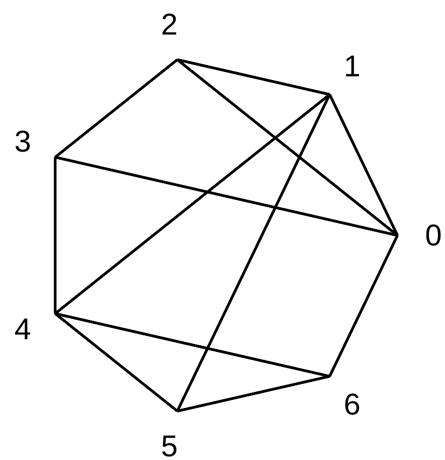


Useful facts – tying together the external legs

Add a new vertex to a graph, and connect it to existing vertices so that all vertices are 4-valent.

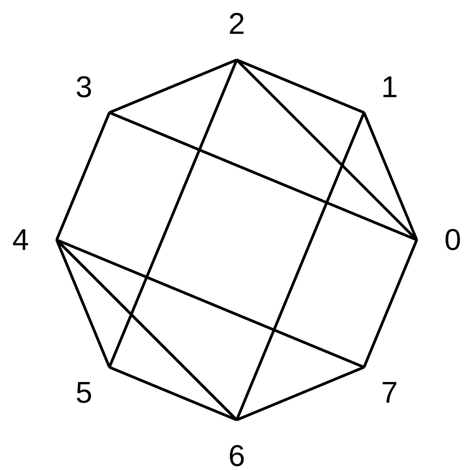
Remove any vertex to obtain another graph of the sort we are interested in. Both graphs have the same period.

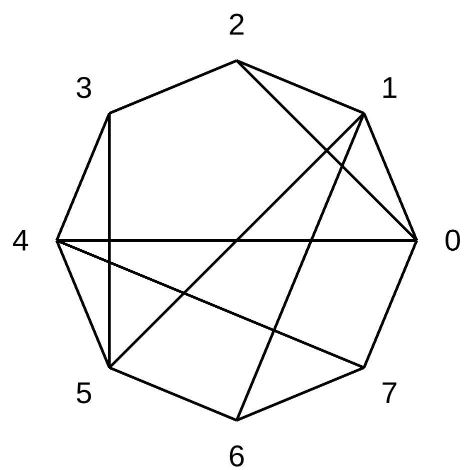




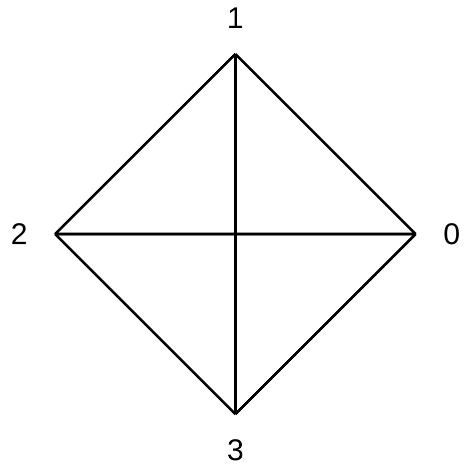
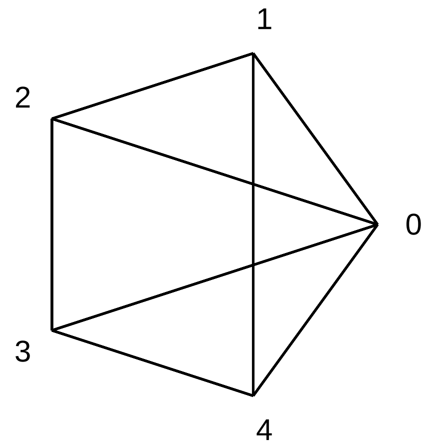
Useful facts – dual graphs

Should a graph happen to be planar and its dual be a graph of the sort we are interested in, then both graphs have the same period.





Consider just the weight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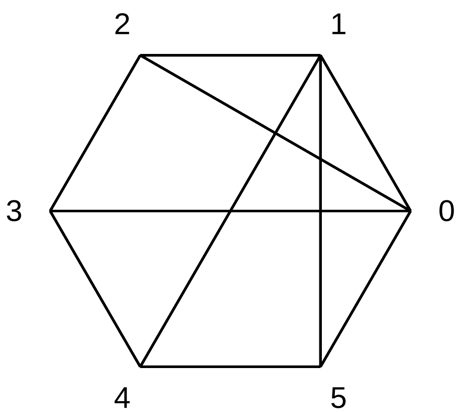
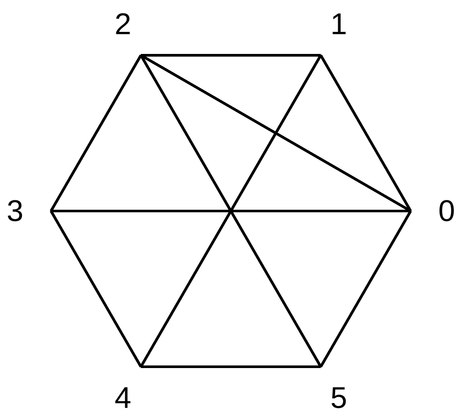
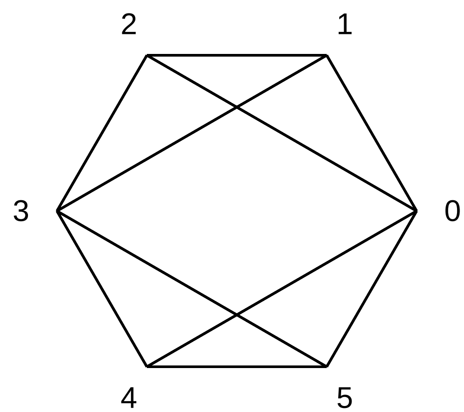
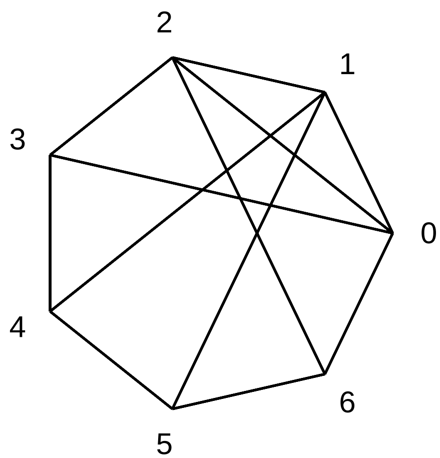
		
7	6	6



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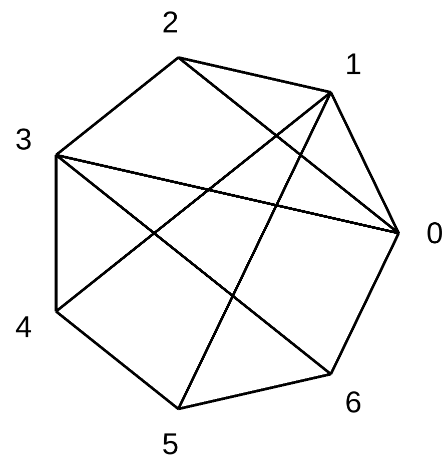

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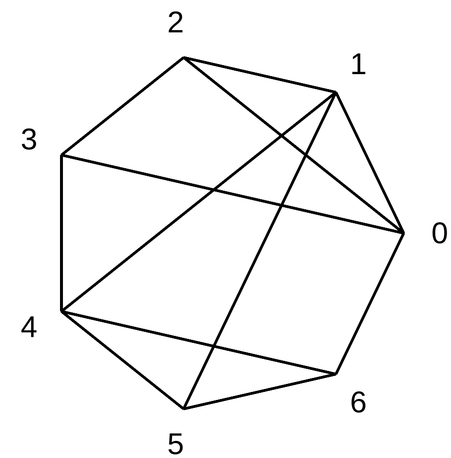

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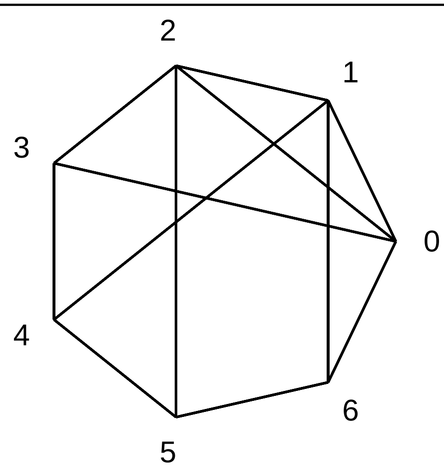

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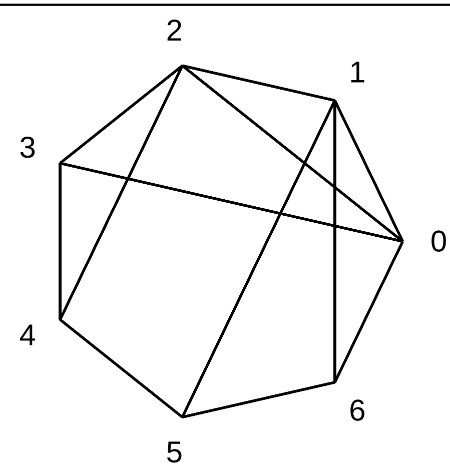

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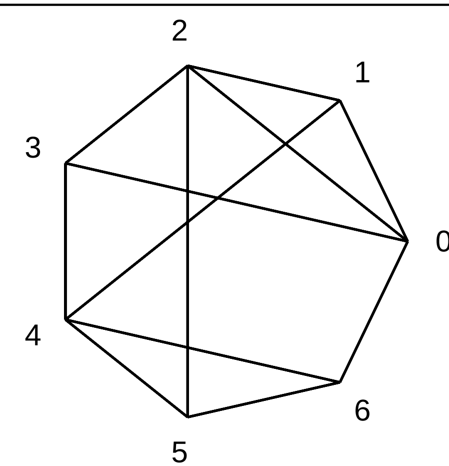

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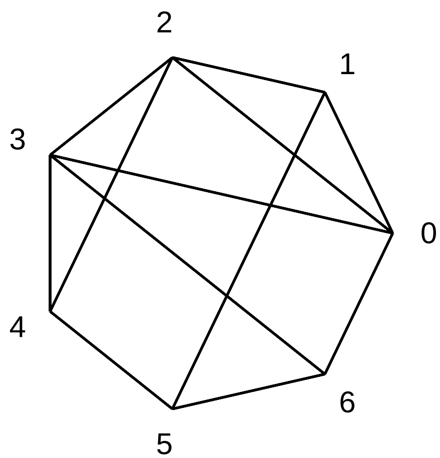

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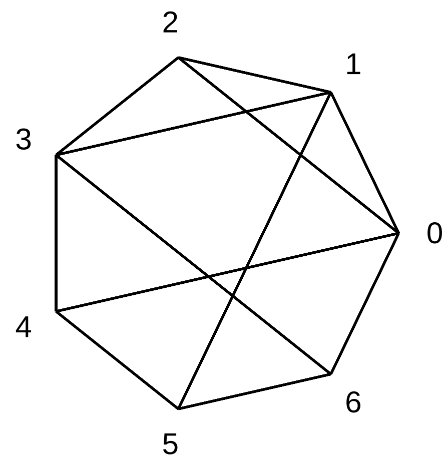

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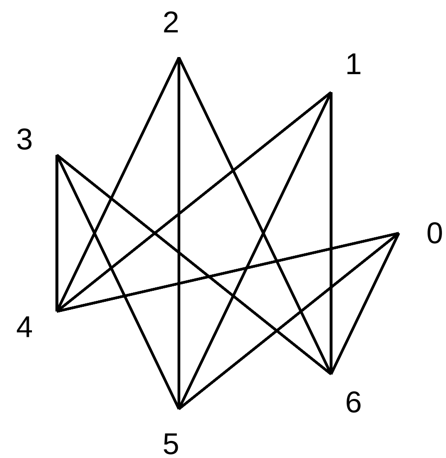

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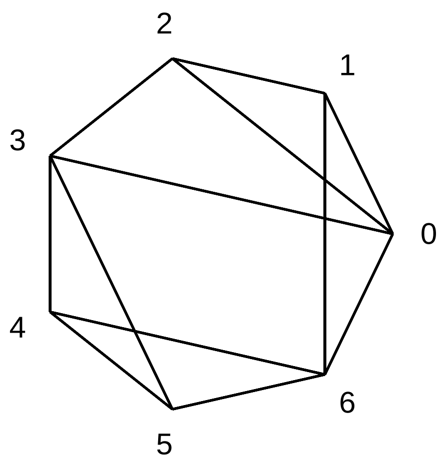

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Note

The maximum weight is

$$2\ell - 3$$

where ℓ is the first Betti number of the graph. When is the weight less.

A graph which is 2VR has one cycle broken when it is split in half and 2 created when the halves are put back together

$$(2(k+1)-3)+(2(\ell-k-1+1)-3)=2\ell-4$$

Weight drop!

But can we explain more weight drops? Can we find an infinite family of graphs with weight drop which are not generated by this construction either directly or through the other useful facts?

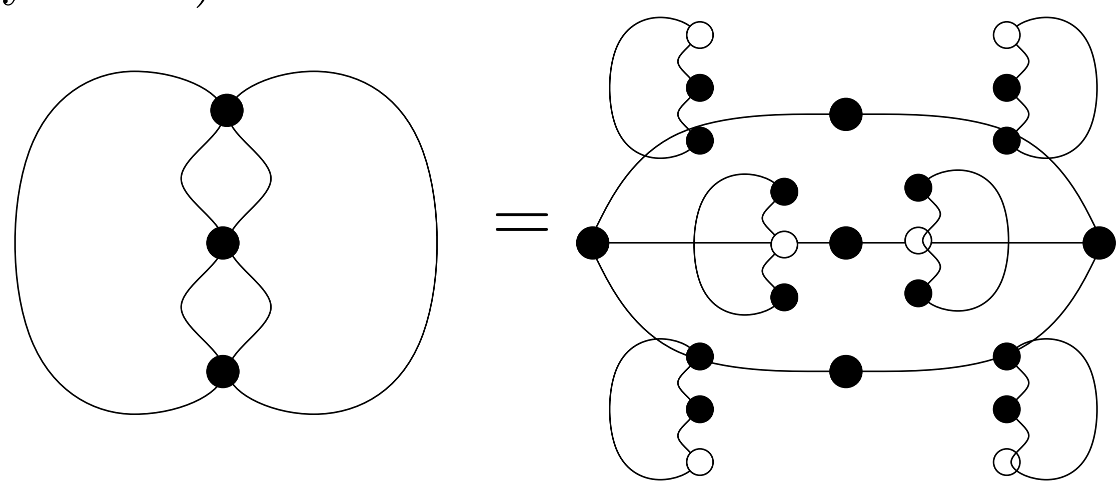
Recall

Joint with Francis Brown.

For a graph G with a colouring C of some of its vertices, consider spanning forests with the properties that

- there are exactly as many trees as colours and
- vertices which are the same colour are in the same tree.

Then (abusing notation by letting a graph stand for its spanning forest polynomial)



One edge at a time

Integrate the Feynman integral one edge variable at a time.

So long as there is always a variable e so that the numerator is a product of two linear polynomials in e,

$$(Ae + B)(Ce + D),$$

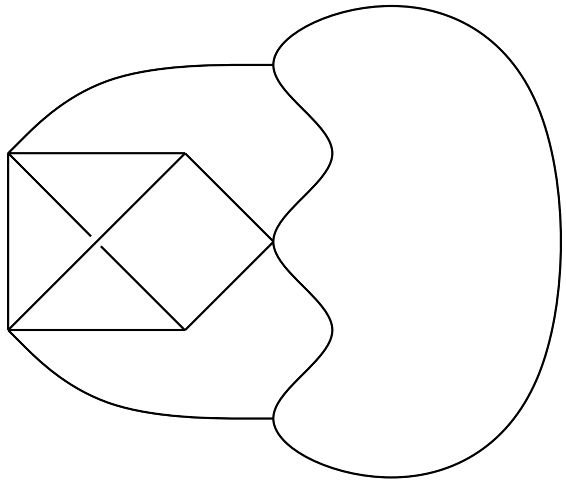
then we can do the integral, getting increasingly complex polylogarithms in the numerator and

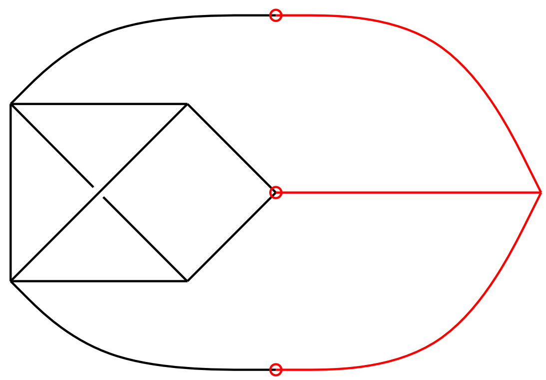
$$AD - BC$$

in the denominator.

We will get a weight drop when the denominator has a factor which is a square or one of the edge variables is missing entirely. There could also be more subtle weight drops where full weight terms in the numerator cancel. This hasn't been seen yet.

A weight drop family

Graphs of the form  have weight drop.

Apply the 3-join to get  .

This is a graph known to have weight drop, with two edges subdivided. We're fine as long as the weight drop occurs after integrating only the black edges. Which it does.