An Analysis of Combinatorial Dyson Schwinger Equations for Massless ϕ^4 Theory

by

William Gregory Dallaway

Submitted in Partial Fulfillment of the

Requirements for PHYS 437A

Supervised by: Karen Yeats and Raffi Budakian

Department of Physics and Astronomy

University of Waterloo

Waterloo, Ontario

2022

Acknowledgments

I would like to thank both my supervisor Karen Yeats and my co-supervisor Raffi Budakian both of whom without this project would not have been possible. I would also like to thank Guillaume van Baalen and Dirk Kreimer both of whom together with Karen Yeats did a large amount of work on a project similar to this and without their insights this project would have been considerably more difficult. Lastly I would like to thank the members of Karen Yeats' research group for the extra information and insights shared at group meetings.

Abstract

In this project we investigate the ϕ^4 beta function through a coupled set of ordinary differential equations which are obtained through combinatoric techniques. Specifically we use the results of Professor Yeats' thesis combining the renormalization group equation and combinatorial Dyson-Schwinger equations to reduce the equations to a set of coupled ordinary differential equations in terms of the anomalous dimensions and auxiliary functions. We use standard methods of analysing ordinary differential equations to produce three implicit solutions each which gives us some information on the conditions on the auxiliary functions and initial conditions for which global solutions to the beta function exist. We also present a numerical analysis of several different solutions for different functional forms of the auxiliary functions and show how these examples support our conditions on the auxiliary functions and initial conditions. We also present a preliminary analysis of the three dimensional phase space plots of these solutions and separatrix surfaces separating the solutions with Landau poles with those without. It is our intention to derive more information on these surfaces as well as consider other models in the continuation of this project next term (PHYS 437B).

Contents

List of Tables

List of Figures

1. Introduction

Quantum field theory is an important part of modern physics with many amazing successes ranging from the prediction of the Higgs boson [7] and the prediction of the anomalous magnetic moment of the electron [13]. These and many other predictions have made this a very highly regarded physical theory. A central and important aspect of almost all quantum field theories is renormalization [11][16] since naive evaluaton of Feynamn diagrams leds to infinite physical quantities.

There has been a lot of work on understanding renormalization in physics and this is generally done pert-erbatively, order by order see the examples in [11][16]. This is however, not the only way to understand this problem. Another approach is to use combinatorial methods specifically, hopf algebras in graph theory to analyse this problem. Combinatorial methods have been applied successfully to obtain interesting results see for example [3][10][5][4][9] and the references therein. Our goal is to use these techniques and specifically the equations derived in [19], to investigate the renormalization of ϕ^4 theory. This is an interesting testing ground for these methods for a few reasons. First the rrenormalization of ϕ^4 theory is well understood and so we already know what to expect and secondly as pointed out in [18] this is the simplest field theory that leads to a genuine system of equations to analyse.

Here we numerically and analytically investigate solutions of the differential equations which govern the evolution of the anomalous dimensions and the beta function in ϕ^4 theory. We will provide a sufficient condition for global solutions of $\beta(\lambda)$ to exist and present results relating to the existence of the anomalous dimensions as well. I also present numerical examples of the solutions as well as the slope fields and numerical investigations into where the beta function has zeros which under certain conditions lead to necessarily non-physical solutions. I will also lay out the different work which is planned for the future relating to this work. This purpose of this work is to understand when global solutions to the beta function of massless ϕ^4 theory exist and in this work I present an example of a global solution obtained numerically. The rest of the work is structured as follows, chapter two gives the necessary physical and combinatorial background. Chapter three presents a derivation of all of the necessary equations which will be used in the rest of the work as well as some preliminary analysis and the numerical methods used to do all of the numerical analysis. Finally chapter four gives the main results as well as the conclusions of the paper.

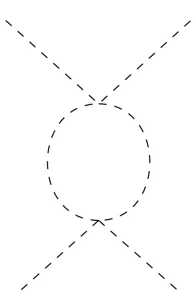
2. Background

2.1 Physics of the Beta Function

In this part I will describe the physics of the beta function and give a brief overview of quantum field theory and in particular re-normalization. First however I will lay out notation which will be used throughout this section. The notation $:\phi(\mathbf{x})\phi(\mathbf{y}):$ denotes the normal ordering of two fields, the notation $\phi(\mathbf{x})\phi(\mathbf{y})$ is the contraction of two fields, similarly with similar notation for contractions with states. The n-point correlator hereafter referred to as the n-point Green's function is $G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \langle\Omega|\mathcal{T}(\phi(\mathbf{x}_1), \phi(\mathbf{x}_2), \ldots, \phi(\mathbf{x}_n))|\Omega\rangle$, where $\mathcal{T}(\cdot)$ is the time ordering operator and $|\Omega\rangle$ is the ground state of the full interacting theory. Lastly we will use λ do denote the running coupling and L will as usual denote $\log(q^2/\mu^2)$.

2.1.1 Why We Need Renormalization

In this part I will give a brief example showing why we need renormalizaiton through an example in ϕ^4 theory. Most of this section is loosely based on the discussions in chapter 12 of [11] and in chapter 10 of [16]. Consider the second order diagram for $\phi\phi$ scattering in ϕ^4 theory



According to the momentum space Feynman rules for this theory we have a 1 for each external edge and a factor of $\frac{i}{p^2-m^2+i\epsilon}$ for each of the two propagators, we also get two factors of $-i\lambda$ for each of the vertices, we then enforce momentum conservation for each momentum and integrate over the undetermined momentum. We thus see this digram is equivalent to the integral.

$$\frac{(-i\lambda)^2}{(2\pi)^4}\int d^4p\,\frac{i}{p^2-m^2+i\epsilon}\frac{i}{(p-k)^2-m^2+i\epsilon}$$

Where $k := p_1 + p_2$ is the sum of the initial momenta. We can simplify these integral using a four dimensional analogue of spherical coordinates in which $d^4p = p^3\sin^2(\theta)\sin(\varphi)\,dpd\theta d\varphi d\psi$ and we thus have evaluating the angular integrals which evaluate to order unity constants we get that the amplitude is proportional to the integral

$$V\left(k^2\right) = \int_{p=0}^{p=\infty} dp\,\frac{p^3}{(p^2-m^2+i\epsilon)((p-k)^2-m^2+i\epsilon)}$$

We can evaluate the integral explicitly using Feynman parameters but, to see the problem it is enough to consider the large p limit since we will need to integrate to infinite momentum we see that

$$V\left(k^2\right) \longrightarrow_{\text{large } p} \int_{p_0}^{\infty} dp\,\frac{1}{p} \propto \lim_{p\to\infty}\log(p/p_0) = \infty$$

So we can see that this integral will diverge in the infinite momentum limit. This is clearly a problem for evaluating these second order corrections and so there is something we need to change. The answer is that we have set up our theory's lagrangian in terms of the bare mass and coupling constants which are to us unobservable. In the next section I will show how to rewrite the theory's lagrangian in order to fix this problem.

2.1.2 A Brief Introduction to Renormalization

In this section I will give a brief introduction to renormalization from the perspective of physics. For this section I will consider as an example ϕ^4 theory since this will be the primary theory for the focus of my project. The Lagrangian for ϕ^4 theory is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m_0^2(\phi')^2 - \frac{\lambda_0}{4!}(\phi')^4 \tag{2.1}$$

I have used m_0 and λ_0 for the bare mass and coupling constant respectively. We consider the field strength rescaling $\phi' = Z^{\frac{1}{2}}\phi$ then the lagrangian in (2.1) becomes

$$\mathcal{L} = \frac{Z}{2}\partial_\mu\phi\partial^\mu\phi - \frac{Z}{2}m_0^2\phi^2 - \frac{Z^2\lambda_0}{4!}\phi^4 \tag{2.2}$$

Now we can't observe the bare mass and coupling constant but we consider the quantities $D_Z = Z-1$, $D_m = Zm_0^2 - m^2$ and $D_\lambda = Z^2\lambda_0 - \lambda$ where the quantities m and λ are the physical mass and coupling constant. Using these quantities we can rewrite the lagrangian for the theory in terms of these quantities as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{D_Z}{2}\partial_\mu\phi\partial^\mu\phi - \frac{D_m}{2}\phi^2 - \frac{D_\lambda}{4!}\phi^4 \tag{2.3}$$

In terms of Feynman diagrams the "additional"[1] terms in the Lagrangian will lead to counter terms which will cause integrals such as the ones considered in the previous subsection to converge to unambiguous values. A very similar procedure to this works in other physical theories and is a very important part of quantum field theory and this is the heart of renormalization. In the interest of brevity I will not go into depth on how the physical mass m and physical coupling constant λ are defined[2] but it can be done and can also be done in other theory such as quantum electrodynamics.

2.1.3 The Renormalizaiton Group Equation

In this final part of the physics introduction I will present a derivation of the renormalization group equation[3] this equation will be central to our later discussions so this section more than any of the other ones so far is very important. There is a problem with the renormalization scheme I presented in the previous section. The problem comes in the case where our field is massless since some of the previous renormalization conditions will still lead to divergences[4] so the question becomes how can we deal with massless theories? The answer is we choose a specific momentum scale μ and we impose the same renormalization conditions as in [11],

We would like physical quantities like the n-point Green's function to be independent of our choice of renormalization scale. By enforcing this we can get an equation which must be satisfied by the n-point Green's function. Specifically suppose we make the infinitesimal shift $\mu \to \mu + d\mu$ which can induce a change in coupling constant $\lambda \to \lambda + d\lambda$ and change in the field strength $\phi \to \phi + d\varphi$ which for we can always parametrize using $d\phi = d\eta\phi$ i.e η can be defined through $\frac{d\eta}{d\phi} = \frac{1}{\phi}$. With this parametrization we can write $\phi \to \phi + d\eta\phi$ we get that the change in the n-point Green's function is simply the change induced by the field strength so that the total change is given by $G \longrightarrow (1 + nd\eta)G$ so we get that

$$dG = nd\eta G \tag{2.4}$$

We can use the chain rule to write

$$dG = \frac{\partial G}{\partial \mu}d\mu + \frac{\partial G}{\partial \lambda}d\lambda \tag{2.5}$$

[1] These terms aren't really additional, they of course were always there since we have only rewritten the Lagrangian. I will however continue to use the abuse of terminology since it makes talking about this significantly easier.

[2] The mass as well as the field strength renormalization can be defined as the pole of the two-point Green's function of the full interacting theory and the residue respectively and the coupling constant as the scattering amplitude at zero momentum.

[3] Also called the Callan-Symanzik eqauation

[4] This is why I mostly skipped over these conditions.

And by combining these we find

$$\left(\frac{\partial}{\partial\mu}d\mu+\frac{\partial}{\partial\lambda}d\lambda-nd\eta\right)G=0 \tag{2.6}$$

It is conventional to express the last two terms in terms of different quantities by defining $\beta=\mu\frac{d\lambda}{d\mu}$ and $\gamma=\mu\frac{d\eta}{d\mu}$ using these we find that (2.6) becomes

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}-n\gamma(\lambda)\right)G(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)=0 \tag{2.7}$$

The quantity $\beta(\lambda)=\mu\frac{d\lambda}{d\mu}$ is called the beta function and as we can see it tells us how our coupling constant changes with renormalziaion scale. We call the function $\gamma(\lambda)$ the anomalous dimension for the theory and this quantity is something we will be coming back to in subsequent sections. We can write the beta function in terms of a different quantity $L=\log\left(q^2/\mu^2\right)$ so that $dL=\frac{1}{\mu}d\mu$ and so and then we can see from the definition of the beta function that

$$\beta(\lambda(L))=\frac{d\lambda}{dL} \tag{2.8}$$

And this is another equation we will be coming back to later when we analyse these theories in a different way. In the next section I will present a more mathematical and combinatorial approach to these same ideas and this will allow us to get a different version of these equations which we will be analysing for the main results of this work.

2.2 Combinatorics of the Beta Function

In this section I will present a general overview of a combinatorial perspective on the same ideas of the beta function we described through the physics of the renormalization in the previous section. Almost all of the content in this section is based primarily on [19] as well as [17] and many of the topics are expanded upon in both of these works.

2.2.1 Feynman Diagrams and Combinatorics

Here I will present some of the ideas which will allow us to talk about Feynman diagrams as combinatorial objects called graphs. We will need to modify some of the standard ideas of graph theory in order to do this; so for those familiar with more standard graph theory do note that we are modifying the standard notions of graph theory for this and all subsequent sections. With that being said here I will mention some notions in graph theory which will not change for us. A graph is connected if there is a path via the edges of G between any two vertices of G.[5] A graph is called 2-edge connected if it remains connected whenever a single edge is removed the graph remains connected[6]. With these definitions being borrowed from our standard graph theory we are able to define our

[5]When a graph is not connected it can be decomposed into more than one connected component.
[6]This is often called being one particle irreducible in physics

notions of graphs called *Feynman graphs*. A Feynman graph is a combinatorial object comprised of a set of half edges E, each element of E can be put into a many to one correspondence with the set $\{1, 2, \ldots, n\}$ for some $n > 1$. Each integer in the set $\{1, 2, \ldots, n\}$ can be associated with a label called a half edge type and so the total number of half edge types in the theory is n. Each Feynman graph also has an associated set V called the vertex set, each vertex is a set of size $N \geq 3$ half edge types, each set defines a specific vertex type and the allowed vertex types we take as prescribed by the physical theory we are considering.

For example in quantum electrodynamics there are three half edge types (half photon edge, front half fermion edge and a back half fermion edge) and a single vertex type consisting of one of each of the half edge types. For ϕ^4 theory we have a single half edge type consisting of the half scalar edge and a single vertex type consisting of 4 of the scalar half edges.

We now have to consider the operations on these graphs, which will be needed for when we define the Hopf algebra structure for these graphs. For those familiar with graph theory this first definition will correspond to our version of the standard graph theory operation contraction. Before doing this however we need to make a definition which will come in handy. In each Feynman graph the half edges which are adjacent to at most one vertex or other half edge is called an external edge. The set of external edges in a Feynman graph corresponds to at most one edge or one vertex type and this edge or vertex type is called the external leg structure of the Feynman graph. With this definition we are ready to define some operations on Feynman graphs.

Let Γ be a Feynman graph and γ be a subgraph of Γ and suppose γ has external leg structure of vertex type v then we define the Feynman graph Γ/γ to be the graph with all of the vertices of γ removed from Γ and all the internal edges of γ removed from Γ and with the adjacencies induced from Γ along with the adjacencies of the external edges of γ with the new vertex.

We can do this in a similar way if the external leg structure of Γ is an edge type we define the graph Γ/γ to be the graph obtained by Γ by removing all of the half edges of γ with the adjacencies induced by Γ and with the external edges of γ retained with the induced adjacencies in Γ[7].

The operation I described in the preceding two paragraphs is called contraction and we will be using it when we describe the the Hopf algebra structure of these types of graphs. We would also like to give some notion of insertion for these graphs. We again in this case need to be careful because we can't insert arbitrary subgraphs into arbitrary graphs since we only have a given set of possible vertex types we also need to worry about having two half edges adjacent which are not the same type which is also not allowed. We therefore need to define an insertion operation which will take these different possible problems into account. The way this is done will be the subject of the next paragraph.

Let Γ and γ be a Feynman graphs and suppose γ is a connected graph with an external leg structure of vertex type v. Let $\tilde{v}$ be a vertex of type v in Γ. Let f be a bijection from the external edges of γ to the adjacent edges

[7] In the case that γ is not connected we define this operation to be the previous two paragraphs applied to each connected component.

in Γ which preserve half edge type. Then $G \circ_{\tilde{v},f} \gamma$ is the graph consisting of the vertices of Γ except for $\tilde{v}$ in disjoint union with the vertices of γ and the half edges of Γ and those of γ with identifications given by f and the adjacencies induced by both Γ and γ.

Let Γ and γ be Feynman graphs and suppose γ is a connected graph with an external leg structure of edge type e and let $\tilde{e}$ be an edge in Γ of type e. Let f be a bijection from the external edges of γ to the half edges composing $\tilde{e}$ such that if e' is an external edge of γ then $(e', f(e'))$ is a permissible half edge half edge adjacency. The $\Gamma \circ_{\tilde{e},f} \gamma$ is the graph consisting of, the vertices of Γ in disjoint union with the vertices of γ. The half edges of Γ in disjoint union with those of γ with the adjacencies e' and $f(e')$ for each external edge e' along with the induced adjacencies from Γ and γ. For both of these operations it is possible to show that inserting a 2-edge connected Feynman graph into a 2-edge connected Feynman results in a 2-edge connected Feynman graph and contracting any subgraph of a 2-edge connected Feynman graph also results in a 2-edge connected Feynman graph, the details of both of these are found in [19].

These two definition give us two operations on graphs which we will be using frequently in the next part describing the Hopf algebra on these graphs.[8] The goal of the next part of this section will be to describe this.

2.2.2 Hopf Algebra of Feynman Graphs

The goal of this part of the section is to describe a Hopf algebra structure on a subset of these Feynman graphs. First however we need to discuss divergence in graphs. The reason for the term divergence is that when we consider these graphs in a physical theory they will be associated to integrals[9]. To this end for a Feynman graph we define the superficial degree of divergence to be

$$D\ell - \sum_e w(e) - \sum_v w(v) \tag{2.9}$$

Where D is the dimension of spacetime and $w(\cdot)$ denotes the weight of an edge or vertex and ℓ is the number of independent loops in the graph. This essentially measures how divergent these integrals which are obtained from the Feynman rules are. A Feynman graph with a non-negative degree of divergence is called a divergent graph. The motivation for this definition physically is that the reason we needed to talk about renormalization in the previous section was for the diagrams which are divergent. For instance in our example in the previous section with ϕ^4 theory, the number of loops is 1 and the dimension of spacetime is 4 and the weight of each of the two internal edges is 2. Therefore the superficial degree of divergence is 0 and so we can expect the logarithmic UV divergence which is what we computed analytically. Given the idea of a superficial degree of divergence and a divergent graph we are able to define the Hopf algebra of Feynman graphs.

[8] An introduction to Hopf algebras is given in the appendix.

[9] If we only care about the combinatorics and not for example calculating lifetimes or cross sections for scattering then we can consider these integrals to be purely formal.

Let $\mathcal{H}$ be the vector space formed by the $\mathbb{Q}$-span of disjoint unions of 2-edge connected divergent Feynman graphs along with the empty graph denoted by $\mathbb{1}$[10]. It is very easy to see that $\mathcal{H}$ also has the structure of an algebra with the multiplication given by disjoint union and the identity given by the empty graph. Once this algebra structure has been noted it is easy to see that this algebra can be made into a Hopf algebra. To show this we need to define the unit map $\iota : \mathbb{Q} \to \mathcal{H}$ by $\iota(q) = q\mathbb{1}$. We can also define the coproduct by $\Delta(\Gamma) = \sum_{\gamma \subseteq \Gamma} \gamma \otimes \Gamma/\gamma$ where each γ in the sum is 2-edge connected and divergent including Γ and $\mathbb{1}$. We also extend this sum to all of $\mathcal{H}$ by algebra homomorphism. We can also define the algebra homomorphism $\eta : \mathcal{H} \to \mathbb{Q}$ which satisfies $\eta(\mathbb{1}) = 1$ and $\eta(G) = 0$ for any connected Feynman graph G. These two maps give a coalgebra structure on $\mathcal{H}$ with coproduct Δ and with counit η given in the previous sentence. We also define recursively the antipode $S : \mathcal{H} \to \mathcal{H}$ by $S(\mathbb{1}) = \mathbb{1}$ and $S(\Gamma) = -\Gamma - \sum_{\gamma \subset \Gamma} S(\gamma)\Gamma/\gamma$ where again the sum is over products of 2-edge connected divergent Feynman graphs but this time we do not include Γ and $\mathbb{1}$ in the sum. These maps give $\mathcal{H}$ a Hopf algebra structure and we will refer to this as the standard Hopf algebra of Feynman graphs. All of the components of the proof that this indeed has a Hopf algebra structure can be found collectively in [1] and [14]. Given this Hopf algebra structure we can adopt some of the standard terminology from this subject, for example a Feynman graph Γ is primitive if $\Delta(\Gamma) = \mathbb{1} \otimes \Gamma + \Gamma \otimes \mathbb{1}$. Note that for a 2-edge connected Feynman graph to be primitive it has to have no divergent subgraphs. These previous two subsections have given the necessary combinatorial and algebraic background of Feynman graphs. In the next subsection I will present the main ideas which lead to the equations which will be the basis of my work. Specifically I will introduce combinatoric and analytic Dyson-Swinger equations.

2.3 Combinatorial and Analytic Dyson-Schwinger Equations

In this section I introduce on of the main tools for deriving the equations we will be analysing. Using the graph theoretic and Hopf algebraic structure of the Feynman graphs presented in the previous two sections I will present a summarized version of combinatoric and analytic Dyson-Schwinger equations and how they lead to the main equations we analyse namely equations (3.1) and (3.2). All of this is essentially a summary of chapter 3-7 of [19] and more details can be found there, however the details presented here should be sufficient for this analysis.

In order to do this however we need to talk about one more operation on Feynman graphs as well as some of the subtleties associated with this operation. In order to understand this definition for Feynman graphs however we first will follow the approach of [19] and first mention a similar definition which is an operation on rooted trees. First recall a rooted tree is a tree which has a special designated vertex called the root. These trees also admit a Hopf algebra structure similar to the structure we defined on Feynman graphs which is called the Connes-Kreimer Hopf algebra on rooted trees. Now the operation denoted B^+ takes two rooted trees and forms a single rooted tree in the following way, insert a new root vertex and connected it by a single edge to the roots of input trees.

[10]This is needed so that $\mathcal{H}$ has an identity element

This operation has a very useful and important property called the Hochschild 1-cocycle property meaning that $\Delta B^{+} = \left(\mathrm{id} \otimes B^{+}\right)\Delta + B^{+} \otimes \mathbb{1}$, see [1]. This property is required for our Feynman graphs for many of the proofs to be valid and so this is a desirable property to have in our analogous definition of B^{+} for Feynman graphs. We would have this property immediately if all of the insertion places for our graphs would form a tree structure, however this fails for potentially two cases. First there could be overlapping insertion points meaning we don't have an insertion tree structure. The other problem is that in gauge theories there may be different ways to get the same graph by inserting two graphs with different external leg structure. The solution to the second problem found in [19] which cites [9] is to define

$$B_{+}^{\gamma}(X) = \sum_{\Gamma \subseteq \mathcal{H}_{\mathrm{lin}}} \frac{\mathrm{bij}(\gamma, X, \Gamma)}{|X|} \frac{1}{\mathrm{maxf}(\Gamma)(\Gamma|X)} \Gamma \tag{2.10}$$

With the definitions of these quantities being found in [19] this definition fixes the first problem since the combinatoric factors take care of the issue of overlapping divergences. The second problem is solved by considering $\sum B_{+}^{\gamma}$ where the sum is over γ with a given loop number and external leg structure, the Ward identities in QED and the Slavnov-Taylor identities in QCD ensures that these sums have the Hochschild 1-cocycle property.

Now I will introduce the idea of a combinatorial Dyson-Schwinger equation, this is a recursive equation or system of equations for the edges or vertices of a theory written in terms of the B_{+}^{γ} operator. These equations can be derived in a similar way to standard recursive combinatorial objects such as rooted trees and lattice paths, using combinatorial classes as well as the function B_{+}^{γ}. For an example of this see [4] where they study a special case of Yukawa, where the combinatorial Dyson Schwinger equation can be found exactly and it is

$$X = \mathbb{1} - B_{+}\left(\frac{1}{X}\right) \tag{2.11}$$

A derivation and further explanation is found in both [19] and [17], but to briefly summarise the derivation of this is similar to that of the derivation for the generating function for plane rooted trees. The reason for this similarity is that the the insertion tree for this structure is a plane rooted tree. We can then derive analytic Dyson-Swinger equations by applying the Feynman rules to the combinatoric Dyson-Schwinger equations. There are some subtleties associated with this that are once again discussed in [19]. Now I will give a summary of the main results of [19]. First (systems of) combinatorial Dyson-Swinger equation(s) of the form[11]

$$X^{r}(x) = \mathbb{1} - \mathrm{sgn}\left(s_{r}\right) \sum_{k \geq 1} \sum_{i=0}^{t_{k}^{r}} x^{k} B_{+}^{k,i;r}\left(XQ^{k}\right) \tag{2.12}$$

Where $s_{r} \in \mathbb{Z}$, k is an index for the loop number and r is an index running over the external leg structures and

[11] Here x is the coupling constant I write it as x to be consistent with [19] but after this chapter it will be renamed to λ.

i indexes the primitives with given loop number and external leg structure r and $Q = \prod_{r \in \mathcal{R}} X^r(x)^{-s_r}$ is the combinatorial invariant charge[12]. The Dyson-Schwinger equations associated to equation (2.12) is given defined in [19] to be

$$G^r(x, L_1, L_2, \ldots, L_j) = 1 - \operatorname{sgn}(s_r) \sum_{k \geq 1} \sum_{i=0}^{t_k^r} x^k G^r\left(x, \partial_{-\rho_1^r}\right)^{-\operatorname{sgn}(s_r)} \cdots G^r\left(x, \partial_{-\rho^r_{\operatorname{sgn}(s_r)\operatorname{sgn}(s_r k - 1)}}\right)^{-\operatorname{sgn}(s_r)}$$
$$\prod_{t \in \mathcal{R} \setminus \{r\}} G^t\left(x, \partial_{-\rho_1^t}\right)^{-\operatorname{sgn}(s_t)} \cdots G^t\left(x, \partial_{-\rho^t_{\operatorname{sgn}(s_t)(s_t k)}}\right)^{-\operatorname{sgn}(s_t)}$$
$$\left(e^{-L\left(\rho_1 + \rho_2 + \cdots + \rho_{n_{k,i;r}}\right)}\right) F^{k,i;r}\left(\rho_1, \cdots, \rho_{n_{k,i;r}}\right)\Big|_{\rho_1 = \rho_2 = \cdots = \rho_{n_{k,i;r}} = 0} \tag{2.13}$$

Where $F^{k,i;r}$ is the Mellin transform of the primitive $B_+^{k,i;r}$. This is then combined with the expansion of the Green's function in terms of the anomalous dimension as

$$G^r(x, L) = 1 - \operatorname{sgn}(s_r) \sum_{k \geq 1} \gamma_k^r(x) L^k \tag{2.14}$$

Having these two equations I will go over very briefly the ideas of reducing these equations into a more useful form for us the steps are

- Finding a first recursion.
- Reducing to a single insertion place.
- Reducing to a geometric series.
- Finding a second recursion.

In order to derive the first recursion, the expansion for the Green's function in terms of the anomalous dimension (equation (2.13)) is inserted into the renormalization group equation[13] after this one obtains [19]

$$\gamma_k^r = \frac{1}{k}\left(-\operatorname{sgn}(s_r)\gamma_1^r(x) + \sum_{j \in \mathcal{R}} |s_j| \gamma_1^j(x) x \frac{\partial}{\partial x}\right) \gamma_{k-1}^r(x) \tag{2.15}$$

Once this equation is derived the next step is to reduce to the case of a single insertion plane so that the Mellin transform becomes a function of a single variable. This is done using the ideas of coloured insertion trees. The end result of this is [19] chapter 5 theorem 5.13

$$X^r = 1 - \operatorname{sgn}(s_r) \sum_{k \geq 1} x^k R_+^{q_k^r}\left(X^r Q^k\right) \tag{2.16}$$

The next step is to reduce to the case of a geometric series, again the end result of this is [19] theorem 6.3 which

[12]See [19] for the exact relationship between this quantity and the standard invariant charge used in physics.
[13]This can also be done in a different way see [19] section 4.2

says there are unique $r_k^j, r_{k,i}^j$ such that

$$\sum_{k\geq 1}\sum_{i=0}^{t_k} x^k \left(1-\operatorname{sgn}(s_j)\gamma^j(-\partial_\rho)\right)^{1-s_r k} \prod_{j\in\mathcal{R}\setminus\{r\}} \left(1-\operatorname{sgn}(s_j)\gamma^j(-\partial_\rho)\right)^{-s_j k}\left(e^{-L\rho}-1\right) F^{k,i}(\rho)\Big|_{\rho=0} =$$

$$\sum_{k\geq 1}\sum_{i=0}^{t_k} x^k \left(1-\operatorname{sgn}(s_r)\gamma^r(-\partial_\rho)\right)^{1-s_r k} \prod_{j\in\mathcal{R}\setminus\{r\}} \left(1-\operatorname{sgn}(s_j)\gamma^j(-\partial_\rho)\right)^{-s_j k}\left(e^{-L\rho}-1\right)\left(\frac{r_{k,i}^r}{\rho(1-\rho)}+\sum_{1\leq i<k}\frac{r_{k,i}^r L^i}{\rho}\right)\Bigg|_{\rho=0}$$

The final step is deriving the second recursion, this involves subbing in the previous results and extracting the first two coefficients. This will present a remarkable simplification but it will come at the cost of introducing new functions which are called $p^r(k) = r_k^r - 2r_{k,1}^r$. The end result is [19] theorem 7.2,

$$\gamma_1^r = -\sum_{k\geq 1} p^r(k)x^k + \operatorname{sgn}(s_r)\left(\gamma_1^r(x)\right)^2 - \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)x\frac{\partial\gamma_1^r(x)}{\partial x} \tag{2.17}$$

By defining $P^r(x) := -\sum_{k\geq 1} p^r(k)x^k$ we finally obtain the two equation which will be the main focus of this analysis.

$$\gamma_1^r = -P^r(x) + \operatorname{sgn}(s_r)\left(\gamma_1^r(x)\right)^2 - \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)x\frac{\partial\gamma_1^r(x)}{\partial x}$$

and,

$$\beta(x) = x\sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)$$

There two equations for a given combinatorial physical theory will generate a system of differential equations which we can solve for the anomalous dimension and the beta function.

3. Deriving the Equations and Preliminary Analysis

In this chapter I will derive the equations which will be the basis of my analysis for this project, present some preliminary analyses of these equations and give a comparison to the same type of equations derived and analysed in both [2] and [19]. Lastly I will discuss the particulars regarding the numerical analysis of the equations we performed.

3.1 Deriving the Equations

Recall from the previous chapter the main result of the work in [19][1] is the following pair of equations.

$$\gamma_1^r(\lambda) = P^r(\lambda) - \operatorname{sgn}(s_r)\,\gamma_1^r(\lambda)^2 + \left(\sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(\lambda)\right)\lambda\frac{\partial\gamma_1^r}{\partial\lambda} \tag{3.1}$$

$$\beta(\lambda) = \lambda\left(\sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(\lambda)\right) \tag{3.2}$$

Now we only need to specialize this to ϕ^4. As mentioned in the example in section (2.2.1) there are two external leg structures one is the the single vertex type in ϕ^4 theory and the other is the single edge type. This corresponds to $\mathcal{R}$ being the finite set of size 2 which we immediately identify with $\{+,-\}$ and we will keep with this convention for the remainder of this work. The only thing remaining therefore is to determine s_+ and s_-. This is also an immediate corollary of the combining equations (3.5) and (3.6) in [19]. Specifically equation (3.6) with a single edge with $m=4$ and $\mathrm{val}(v)=4$ as appropriate for the single vertex in ϕ^4 theory gives

$$Q_{\phi^4} = \frac{X^v}{(X^e)^2} \tag{3.3}$$

[1] I am writing λ instead of x, which is used in Yeats 2011 for the value of the coupling constant. Since we are going to specialize to ϕ^4 theory and this notation should be more familiar to readers in physics.

Comparing this to equation (3.5) in [19] which says $Q = \prod_{r\in\mathcal{R}}(X^r)^{-s_r}$ we see that for this theory we must have, $s_v = s_+ = -1$ and $s_e = s_- = 2$. With these substitutions and isolating the derivative we get the following set of equations[2]

$$\frac{d\gamma^-}{d\lambda} = \frac{\gamma^- + \left(\gamma^-\right)^2 - P^-(\lambda)}{\lambda(\gamma^+ + 2\gamma^-)} \tag{3.4}$$

$$\frac{d\gamma^+}{d\lambda} = \frac{\gamma^+ - \left(\gamma^+\right)^2 - P^+(\lambda)}{\lambda(\gamma^+ + 2\gamma^-)} \tag{3.5}$$

$$\beta(\lambda) = \lambda(\gamma^+ + 2\gamma^-) \tag{3.6}$$

It is these equations which will be the main focus of our analysis. These are very similar equations to the equations analysed in [2] and so I will be using the results of this paper as an analogy quite frequently from now on in this work. With that said there is an important difference between this and the case of massless QED. This is that the QED case can be reduced to a single equation by combining the Ward identities with [8] however in this case we are not able to do that so we must analyse this as a system. This can lead to a lot of interesting behaviour which we will see further on in the work. For now I will present an equivalent set of equations to (3.4)-(3.6) which for some of the analysis will be better then the system of three equations we currently have. Recall from the physics introduction we have the relation

$$\frac{d\lambda}{dL} = \beta\left(\lambda(L)\right) \tag{3.7}$$

Where $L = \log\left(q^2/\mu^2\right)$ is the physical scale. Using this relation and the chain rule in the form $\frac{d}{dL} = \frac{d\lambda}{dL}\frac{d}{d\lambda}$ we can write the equations in terms of the physical scale parameter by multiplying both sides of (3.4) and (3.5) by $\beta(\lambda)$ and using (3.6) to get the equivalent set of equations.

$$\frac{d\gamma^-}{dL} = \gamma^- + \left(\gamma^-\right)^2 - P^-(L) \tag{3.8}$$

$$\frac{d\gamma^+}{dL} = \gamma^+ - \left(\gamma^+\right)^2 - P^+(L) \tag{3.9}$$

And still supplementing with equation (3.6). Some discussion of equations (3.8) and (3.9) is warranted. Although it may appear we have decoupled the system this apparent simplicity is slightly deceiving. The reason for this is that the evolution variable L is now related to the coupling in the following way

$$L(\lambda) = L_0 + \int_{\lambda(L_0)}^{\lambda} \frac{1}{\beta(\lambda')}\, d\lambda' \tag{3.10}$$

[2]We have suppressed the λ dependence of γ on λ and converted the partials to ordinary derivatives since there is only a single variable of interest.

This is undesirable since we are interested in solutions for $\beta(\lambda)$ so having the evolution variable coupled to the desired solution in this way is not ideal. Specifically this is a problem since given a functional form for $P^{\pm}(\lambda)$ there is no simple way to convert this analytically to $P^{\pm}(L)$ since finding L is difficult. Numerically, however this form of the equation is very advantageous for solutions where $\beta(\lambda) \approx 0$ since all of the problems go into the evolution variable but we can still evolve γ^+, γ^- and λ and get sensible numerical results. I will discuss this more further on but I mention it now for motivating this rearrangement. Another useful rearrangement will be to write a single differential equation for the beta function this can be done through combining (3.4) and (3.5) and using the product rule on equation (3.6) the end result of this can take many forms, here I will give only two which I will use later, and the derivations of both of these equations can be found in appendix B.

$$\frac{d\beta}{d\lambda} = 1 + \frac{2\beta(\lambda)}{\lambda} - \lambda \frac{2(\gamma^+ + \gamma^-)^2 + Q(\lambda)}{\beta(\lambda)} \tag{3.11}$$

Another useful form is

$$\frac{d\left(\gamma^+ + 2\gamma^-\right)}{d\lambda} = \frac{1}{\lambda} + \frac{\gamma^+ + 2\gamma^-}{\lambda} - \frac{2\left(\gamma^+ + \gamma^-\right)^2 + Q(\lambda)}{\lambda(\gamma^+ + 2\gamma^-)} \tag{3.12}$$

Where $Q(\lambda) := P^+(\lambda) + 2P^-(\lambda)$ is a simple linear combinations of the two functions. These rearrangements are useful because they are the closest we can get to decoupling equations (3.4) and (3.5) and will allow us derive results analogous to some results in [2]. This is the last rearrangement we will need to do everything in the analysis section.

Finally it will be useful it will be useful for us to have equations (3.8) and (3.9) in a single matrix-vector equation this can be written as

$$\frac{d\boldsymbol{\gamma}}{dL} = \boldsymbol{A}\boldsymbol{\gamma} - \boldsymbol{P} \tag{3.13}$$

Where $\boldsymbol{\gamma} = \begin{pmatrix}\gamma^+ & \gamma^- & \lambda\end{pmatrix}^T$, $\boldsymbol{P} = \begin{pmatrix}P^+(L) & P^-(L) & 0\end{pmatrix}^T$ and $\boldsymbol{A} = \begin{pmatrix} 1-\gamma^+ & 0 & 0 \\ 0 & 1+\gamma^- & 0 \\ \lambda & 2\lambda & 0 \end{pmatrix}$

Writing the equation in this form will allow us to solve the equations numerically more easily and it will also allow us to perform a stability analysis on the system when viewed as a function of L. Of course if we are not content with specifying $P^{\pm}(L)$ then this system won't be as helpful for stability analysis in this case though we can still use the two equations (3.4) and (3.5) to analyse the equations.

3.2 Preliminary Analysis

Before delving into the both the numerical and analytical results we would like to have an idea of what we expect our solutions to look like and what we expect to find. First I consider the behaviour of $\beta(\lambda)$ close to $\beta(\lambda) = 0$. The reason for considering this specifically is that if $\beta(\lambda) \to 0$ then the denominators of equations (3.4) and (3.5) go to 0 which will cause problems. We would thus like to have some ideas about the solutions around this region which we can not find numerically. The main goal of this analysis is to rule out these solutions for $\beta(\lambda)$ as non-physical. The reason for this is the following lemma.

Lemma 1. [3]

Let $\lambda^* > 0$ be a point where $\beta(\lambda^*) = 0$ and suppose $Q(\lambda^*) > 0$ then λ^* is a maximum of $\lambda(L)$.

Proof. By equation (3.7) we have $d\lambda/dL|_{\lambda^*} = \beta(\lambda^*) = 0$ and by combining the chain rule with equation (3.11) we find

$$\left.\frac{d^2\lambda}{dL^2}\right|_{\lambda^*} = \beta\left.\frac{d\beta}{d\lambda}\right|_{\lambda^*} = -2\lambda^*\left(\gamma^+ + \gamma^-\right)^2 - \lambda^* Q(\lambda^*) < 0$$

Thus by elementary analysis, λ^* is a maximum of $\lambda(L)$ □

The importance of the lemma is the following, suppose we have a physical solution for $\beta(\lambda)$ which has a 0 for some λ^*. Then lemma 3.2.1 guarantees that this is a maximum for $\lambda(L)$ and therefore by continuity we must have at least for some interval (λ_a, λ_b) containing λ^* the function $\beta(\lambda)$ is multivalued and hence not a reasonable physical solution. One should also note that the fact that equations (3.4) and (3.5) give an infinite derivative is not to conclude that these solutions are not physically reasonable since the equations can be solved in terms of the physical scale L without issue and could therefore be valid solutions if $\beta(\lambda)$ were not multivalued. Thus we may assume that for any physical solutions we must have either $\beta(\lambda) > 0$ or $\beta(\lambda) < 0$ for all λ.

Having found problems with the solutions of $\beta(\lambda)$ which have zeros it is natural to ask, when do these solutions occur? In order to answer this investigate this question both numerically and analytically. Before diving into more of these preliminary results, it is instructive to solve the equations for the special case of $P^+(L) = K^+$ and $P^-(L) = K^-$ where K^+ and K^- are constants since this will give us something concrete to keep in mind as we discuss some more abstract results additionally we can use these for understanding asymptotic solutions if the functions are asymptotically constant which many interesting solutions will be. For this special case equations (3.8) and (3.9) can be integrated directly to get

$$\gamma^-(L) = \begin{cases} a_-\left(\tanh\left[b^- - a^-(L-L_0)\right]\right) - \frac{1}{2} & K^- > -\frac{1}{4} \\ \frac{1}{c^- - (L-L_0)} - \frac{1}{2} & K^- = -\frac{1}{4} \\ \tilde{a}^-\left(\tan\left[\tilde{a}^-(L-L_0) + \tilde{b}^-\right]\right) - \frac{1}{2} & K^- < -\frac{1}{4} \end{cases}$$

[3]Throughout this work I will assume in all results that both $\lambda(L)$ and $\beta(\lambda)$ are continuous functions of their arguments

$$\gamma^+(L) = \begin{cases} a^+\left(\tan\left[b^+ - a^+(L-L_0)\right]\right) + \frac{1}{2} & K^+ > \frac{1}{4} \\ \frac{1}{c^+ + (L-L_0)} + \frac{1}{2} & K^+ = \frac{1}{4} \\ \tilde{a}^+\left(\tanh\left[\tilde{b}^+ + \tilde{a}^+(L-L_0)\right]\right) + \frac{1}{2} & K^+ < \frac{1}{4} \end{cases}$$

Where $a^{\pm} = \sqrt{K^{\pm} \mp \frac{1}{4}}$, $\tilde{a}^{\pm} = \sqrt{-\left(K^{\pm} \mp \frac{1}{4}\right)}$, $b^+ = \text{arctan}\left(\frac{\gamma_0^+ - \frac{1}{2}}{a^+}\right)$,$b^- = \text{arctanh}\left(\frac{\gamma_0^- + \frac{1}{2}}{a^-}\right)$, $\tilde{b}^+ = \text{arctanh}\left(\frac{\gamma_0^+ - \frac{1}{2}}{\tilde{a}^+}\right)$,$\tilde{b}^- = \text{arctan}\left(\frac{\gamma_0^- + \frac{1}{2}}{\tilde{a}^-}\right)$ and $c^{\pm} = \frac{1}{\gamma_0^{\pm} \mp \frac{1}{2}}$. Plots of these solutions are shown in 3.1. These solutions are simple but also quite illustrative of some general concepts which we will see later on. Now however I will point out that for $K^+ < \frac{1}{4}$ and for $K^- > -\frac{1}{4}$ we find global solutions without Landau poles for both of the anomalous dimensions and for all initial conditions where as for $K^{\pm} = \pm\frac{1}{4}$ we find that there are Landau poles if $\gamma_0^+ < \frac{1}{2}$ or if $\gamma_0^- > -\frac{1}{2}$. For $K^+ > \frac{1}{4}$ and $K^- < -\frac{1}{4}$ we have Landau poles for any initial conditions. There are 9 different cases for both $\beta(\lambda)$ and for $\lambda(L)$ depending on the values of K^+ and K^- and enumerating them in their totality is not overly interesting so I will not present them here but they all have closed functional forms which can be solved for in this case. In the main results section we will see how this is a special case of a form for the anomalous dimension.

Another question of interest is about when solutions cross. The reason for this is knowledge that the solutions can't cross along with a demonstration of two solutions with different limits as $\lambda \to \infty$ guarantees a separating solution similar to what was found in [2]. Of course for a single equation the solutions will not cross and so the argument for the QED system is slightly easier than in this case. In fact it is quite easy to find numerically solutions of $\beta(\lambda)$ which cross each other, see 3.2 for an example of this, meaning this method will unfortunately not work. Of course in the three dimensional phase space we will have this non-crossing property and additionally we can quantify exactly the way the difference between two solutions changes as a function of λ and this again is another one of our results.

To conclude this preliminary analysis we will present a simple stability analysis of the equations so we will have more of an idea of how to make sense of the phase portraits which will be displayed in the results section and so we can better understand what to expect. Assuming that $\lambda > 0$ we see the critical points of the system occur

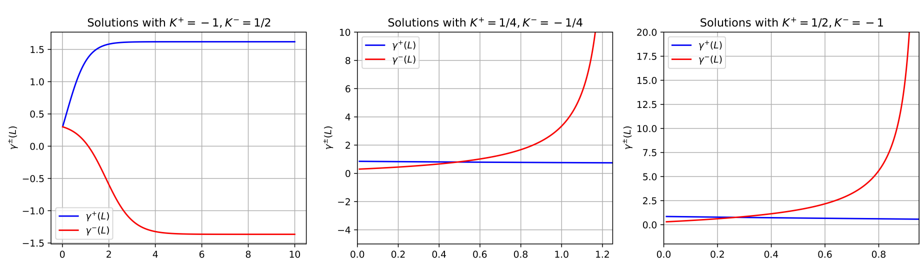


Figure 3.1: The different types of solutions in the case of a constant $P^+(L)$,$P^-(L)$. The leftmost plot shows two global solutions with no Landau pole, meaning $\beta(L)$ or equivalently either $\gamma^{\pm}(L)$ diverge for finite L, the center solution shows one solution γ^- with a Landau pole and one solution γ^+ without a Landau pole, the rightmost solution shows a solution where both solutions have Landau poles, although only one is shown. The initial conditions for all plots were $\gamma_0^+ = 0.85$ and $\gamma_0^- = 0.30$.

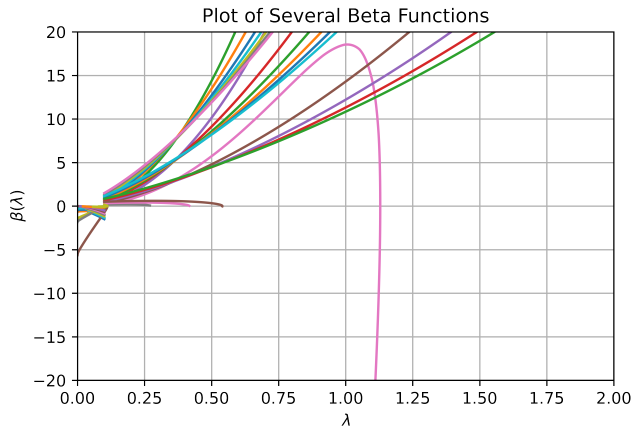


Figure 3.2: Several different numerical solutions of $\beta(\lambda)$. One can easily see the pink solution crossing several other solutions.

when $-2\gamma^+ = \gamma^-$ and

$$\gamma^+ = \frac{1}{2} \pm \frac{\sqrt{1-4P^+(L)}}{2}$$

$$\gamma^- = -\frac{1}{2} \pm \frac{\sqrt{1+4P^-(L)}}{2}$$

Linearising the system (3.13) in general will be difficult due to the fact that the system is non autonomous in general and since it is not our main goal we will simply analyse the simple case where the system is autonomous in this case we see by simply linearizing the system there are indeed no stable solutions. We could see this directly from integrating the solutions in this case as well but it is a fine thing to cross check for this analysis.

3.3 Methods of Numerical Analysis

In this section I describe the methods used to numerically analyse equations (3.4)-(3.9). To solve the equations I used the scipy package *odeint* using the Runge-Kutta-Fehlberg method for solving odes [12]. Using this method equations (3.8), (3.9) and (3.7) are integrated together with L as the evolution variable and the system $\{\lambda, \gamma^+, \gamma^-\}$ being treated as dependant on L alone. The system of equations (3.7), (3.8) and (3.9) are chosen over (3.4) and (3.5) to avoid numerical problems which arise when attempting to integrate the latter equations near $\beta = 0$. When we integrate equations (3.9) and (3.8) we can see this points indeed correspond to vertical tangent lines of the parametric curves for $\beta(\lambda)$. We can then numerically compute $\beta(\lambda)$ by the appropriate linear combination of the anomalous dimensions numerically. We also use python's *matplotlib* and *matplotlib3d* to generate all plots in this work.

4. Results and Main Analysis

In this chapter I will present the main results of the work. The main goal of this analysis is to determine when solutions exist globally and conversely when the solutions have Landau poles. For this section I will present 3 different rearrangements of the equations presented in chapter 3 and what we can analytically determine from these rearrangements. I will also present the numerical solutions to these equations and what we can empirically determine from these numerical solutions.

4.1 The First Rearrangement

In this section I present our first rearrangement of equations (3.8) and (3.9). First with equation (3.8) we can divide by $\gamma^- + \left(\gamma^-\right)^2$ to get

$$\frac{d\gamma^-}{\gamma^- + \left(\gamma^-\right)^2} = \left(1 - \frac{P^-(L)}{\gamma^- + \left(\gamma^-\right)^2}\right) dL \tag{4.1}$$

We then integrate both side of (4.1) from γ_0^- to $\gamma^-(L)$ on the left hand side and from L_0 to L on the right hand side, so that $\gamma^-(L) = \gamma_0^-$. We can evaluate the left hand integral using a partial fraction expansion to write

$$\frac{1}{\gamma^- + \left(\gamma^-\right)^2} = \frac{1}{\gamma^-} - \frac{1}{1+\gamma^-}$$

Thus we have, assuming $0 \notin [\gamma^-, \gamma_0^-]$ and $1 \notin [\gamma^-, \gamma_0^-]$ since otherwise the division to get equation (4.1) is not permitted.

$$\int_{\gamma_0^-}^{\gamma^-} \frac{d\gamma^-}{\gamma^- + \left(\gamma^-\right)^2} = \log\left(\gamma^-\right) - \log\left(1+\gamma^-\right) - \left[\log\left(\gamma_0^-\right) - \log\left(1+\gamma_0^-\right)\right] = \log\left(\frac{\gamma^-}{1+\gamma^-}\frac{1+\gamma_0^-}{\gamma_0^-}\right)$$

The right hand side integrates to

$$\int_{L_0}^{L} 1 - \frac{P^-(L)}{\gamma^- + \left(\gamma^-\right)^2}\, dL = (L - L_0) - \int_{L_0}^{L} \frac{P^-(L)}{\gamma^- + \left(\gamma^-\right)^2}\, dL$$

Therefore the integrated version of equation (4.1) becomes

$$\log\left(\left|\frac{\gamma^-}{1+\gamma^-}\right|\left|\frac{1+\gamma_0^-}{\gamma_0^-}\right|\right) = (L-L_0) - \int_{L_0}^{L}\frac{P^-(L)}{\gamma^- + (\gamma^-)^2}\,dL$$

We can exponentiate both sides to get

$$\left|\frac{\gamma^-}{1+\gamma^-}\right| = C\left[e^{L-L_0}e^{-f(L,L_0)}\right]$$

Where $C = \left|\frac{\gamma_0^-}{1+\gamma_0^-}\right|$ and $f(L,L_0) = \int_{L_0}^{L}\frac{P^-(L')}{\gamma^-+(\gamma^-)^2}\,dL'$ have been introduced for ease of notation. We can solve this equation for $\gamma^-(L)$ to find

$$\gamma^-(L) = \pm\frac{e^{(L-L_0)-f(L,L_0)}}{\frac{1}{C}\mp e^{(L-L_0)-f(L,L_0)}} \tag{4.2}$$

We can perform a similar rearrangement for $\gamma^+(L)$ and it is similar but is different in a very important way due to the sign of the terms in the partial fraction expansion. The details are given in appendix B, but the final result is

$$\gamma^+(L) = \pm\frac{e^{(L-L_0)-g(L,L_0)}}{\frac{1}{D}\pm e^{(L-L_0)-g(L,L_0)}} \tag{4.3}$$

Where in a similar way $D = \left|\frac{\gamma_0^+}{1-\gamma_0^+}\right|$ and $g(L,L_0) = \int_{L_0}^{L}\frac{P^+(L')}{\gamma^+-\left(\gamma^+\right)^2}\,dL'$. It is worth noting that strictly speaking equations (4.2) and (4.3) are not "solutions" to the differential equation in the way one normally thinks of since the solutions depend on the solutions at all previous L values and not functions of L alone. They are quite useful in some respects. In particular they will allow us to find where the solution exists and specifically where there are Landau poles.

4.1.1 Conditions on γ^+

Here I will examine equation (4.3) and what we can derive about the solutions based on the functional form. Before considering this however, we need to be careful about how we use equation (4.3). The reason for this is that if one looks at equation (4.3) it would seem that all of the solutions for $\gamma^+(L)$ are positive if D is positive. However we know from the example in chapter 3 that indeed this need not be the case. Consider for example $K^+ = 1/4$ and $c^+ = -0.3$ then $\gamma_0^+ = 0.2$ and so $D > 0$ however we see that indeed the solution diverges to $-\infty$. This is also a slight bug of the presented solution for $\gamma^+(L)$ and the problem is that the solution for $\gamma^+(L)$ crosses 0 and so doing the integration is not formally valid since it crosses a singularity. It is for this reason we will need to make some extra assumptions in the propositions for both of these rearrangements. There are specifically problems when using equation (4.3) if $\gamma(L)$ crosses either 0 or 1 since dividing by $\gamma^+ - \left(\gamma^+\right)^2$ is not defined there. We can always construct full solutions using (4.3) in each of the three regions with different initial conditions. As a final disclaimer I also mention that a lot of conditions will be in terms of $P^{\pm}(L)$ which can not necessarily be applied to $P^{\pm}(\lambda)$ additionally a global solution for $\gamma^{\pm}(L)$ does not give necessarily a global solution for $\gamma^{\pm}(\lambda)$[1]. Nevertheless studying this is still interesting since if $\gamma^{\pm}(L)$ fails to exist for some L then so too does $\gamma^{\pm}(\lambda)$ since

$$\lambda = \lambda_0 + \int_{L_0}^{L} \beta\left(L'\right) dL'$$

So if either of $\gamma^{\pm}(L)$ fail to exist so too does $\beta(L)$ and hence λ does also. Note that the sign choice of equation (4.3) is not arbitrary if $\gamma_0^+ \in (0,1)$ then we must choose the negative sign. Otherwise, we must choose the positive sign. This leads to two different cases, if $\gamma_0 > 1$ then $1/D < 1$ and additionally if $\gamma_0^+ < 0$ then $1/D > 1$. This leads to natural conditions on γ^+ and $g(L, L_0)$. A simple thing we can see is that for instance the following result.

Proposition 1. *Let $\gamma^+(L) > 1$ then if $g(L, L_0) < 0$, $\gamma^+(L)$ is a global solution to equation (3.9)*

Proof. By the first assumption we have that the solution to $\gamma^+(L)$ is given by (4.3) with $1/D < 1$ since the exponential is positive the term in the denominator is positive and hence will never go to 0. □

We can get a similar condition if we are in the other direction. That is,

Proposition 2. *Let $\gamma^+(L) < 1$ then if $g(L, L_0) > 0$ and $g(L, L_0) > L - L_0$ for all L, $\gamma^+(L)$ is a global solution to equation (3.9)*

Proof. By the first assumption we have that the solution to $\gamma^+(L)$ is given by (4.3) with $1/D > 1$ since g grows faster than L the exponential is less than 1 for all L and so this solution can't have a pole. □

These are simple propositions but are not particularly helpful since they have complicated conditions which are not easy to verify.

4.1.2 Conditions on γ^-

Before continuing I will reiterate a point from the previous section which is that equation (4.2) is only valid assuming the solutions don't cross 0 or -1. But again we can always construct piecewise solutions for which these solutions are valid for the entire range of solutions. Here I will present the analogues of the two propositions in the previous section. Note that here we have different conditions, if $C \in (-1, 0)$ then we need to choose the positive sign in the denominator and otherwise we need to choose the minus sign for the denominator. With this in mind and slightly abbreviated since it has been covered in the last section.

Proposition 3. *Suppose that $\gamma^- < -1$ for all L, then if $f(L, L_0) < 0$ there are global solutions to equation (3.8).*

Proof. The proof is the same as the analogous proposition for γ^+ since in this case $C < 1$. □

[1]An example of this is given in section 4.3

Proposition 4. *Suppose that $\gamma^- > 0$ for all L, then if $f(L, L_0) > 0$ and $f(L, L_0) > L - L_0$ for all L there are global solutions to equation (3.8).*

Proof. The proof is the same as the analogous proposition for γ^+ since in this case $C > 1$. □

These results are true, but not overly useful since the hypothesises are difficult to verify. In the next section I show another analysis which has significantly easier to verify hypotheses.

4.2 The Second Rearrangement

While everything done in the previous section is perfectly valid, the thing I am most interested in is $\beta(\lambda)$ Here and throughout the section we take $Q(\lambda)$ to be a C^2 positive function. In order to find conditions where global solutions of $\beta(\lambda)$ exist for all λ we rewrite equation (3.12) using the integrating factor $1/\lambda$ to write

$$\gamma^+ + 2\gamma^- = \left(\frac{\lambda}{\lambda_0}\right)\left[\gamma_0^+ + 2\gamma_0^- + 1\right] - 1 - \lambda \int_{\lambda_0}^{\lambda} \frac{2(\gamma^+ + \gamma^-)^2 + Q(z)}{z\,(\gamma^+ + 2\gamma^-)} \tag{4.4}$$

Now suppose that $\gamma^+ + 2\gamma^- > 0$, the reason behind this assumption is the purpose of lemma 1 any solutions which cross 0 become multivalued. We therefore have,

$$\gamma^+ + 2\gamma^- \leq \left(\frac{\lambda}{\lambda_0}\right)\left[\gamma_0^+ + 2\gamma_0^- + 1\right] \tag{4.5}$$

These two equations will be key in the next theorem which is the main goal of this work.

Theorem 1. *Let $Q(\lambda)$ be a C^2 positive function with $\gamma^- > 0$ and $2\gamma^+ + 3\gamma^-$ having the same sign, then global solutions to (3.12) exist if there is some λ_0 such that*

$$\int_{\lambda_0}^{\infty} zQ(z)\,dz < \infty \tag{4.6}$$

This theorem gives an integrability condition for global solutions to exist. The proof will follow very closely [2].

Proof. First, let λ_0 be as in the theorem statement, $\gamma_0^{\pm} = \gamma^{\pm}(\lambda_0)$ and $\epsilon > 0$ choose

$$\gamma_0^+ + 2\gamma_0^- = \frac{1}{\lambda_0}\left(2\int_{\lambda_0}^{\infty} zQ(z)\,dz + \epsilon^2\right)^{\frac{1}{2}} \tag{4.7}$$

Note that for global solutions to not exist we must have that for some $\lambda^* < \infty$ we have either $\gamma^+(\lambda^*) + 2\gamma^-(\lambda^*) \to \infty$ or $\gamma^+(\lambda^*) + 2\gamma^-(\lambda^*) = 0$. The solution can't have a pole since equation (4.5) bounds the solution,thus assume for a contradiction that $\gamma^+(\lambda^*) + 2\gamma^-(\lambda^*) = 0$ so the third term is positive too, now consider,

$$\frac{1}{2}\frac{d}{d\lambda}\left(\gamma^+ + 2\gamma^-\right)^2 = \frac{\gamma^+ + 2\gamma^-}{\lambda} + \frac{\left(\gamma^+ + 2\gamma^-\right)^2}{\lambda} - \frac{2\left(\gamma^+ + \gamma^-\right)^2}{\lambda} - \frac{Q(\lambda)}{\lambda}$$

Simplifying we have

$$\frac{1}{2}\frac{d}{d\lambda}\left(\gamma^+ + 2\gamma^-\right)^2 = \frac{\gamma^+ + 2\gamma^-}{\lambda} - \frac{\left(\gamma^+ + 2\gamma^-\right)^2}{\lambda} + \frac{2\gamma^-\left(2\gamma^+ + 3\gamma^-\right)}{\lambda} - \frac{Q(\lambda)}{\lambda}$$

Or,

$$\frac{1}{2}\frac{d}{d\lambda}\left(\gamma^+ + 2\gamma^-\right)^2 \geq -\frac{\left(\gamma^+ + 2\gamma^-\right)^2}{\lambda} - \frac{Q(\lambda)}{\lambda}$$

Where the inequality follows since both terms removed are positive rearranging gives,

$$\lambda^2\frac{d}{d\lambda}\left(\gamma^+ + 2\gamma^-\right)^2 + 2\lambda\left(\gamma^+ + 2\gamma^-\right)^2 \geq -2\lambda Q(\lambda)$$

Or,

$$\frac{d}{d\lambda}\left(\lambda^2\left[\gamma^+ + 2\gamma^-\right]^2\right) \geq -2\lambda Q(\lambda) \tag{4.8}$$

Integrating equation (4.8) on $[\lambda_0, \lambda^*]$ and using (4.7) gives

$$\gamma^+ + 2\gamma^- \geq \frac{1}{\lambda^2}\left(\lambda_0^2\left[\gamma_0^+ + 2\gamma_0^-\right] - \int_{\lambda_0}^{\lambda^*} zQ(z)\,dz\right) > \left(\frac{\epsilon}{\lambda_0}\right)^2 > 0 \tag{4.9}$$

Contradicting our assumption that $\gamma^+(\lambda^*) + 2\gamma^-(\lambda^*) = 0$. This shows that given our assumptions (4.7) is enough to give global solutions. □

theorem 1 is the main result of our work during the term. The converse to theorem 1 is false and an example of this is provided in section 4.3. As an example of the theorem I show a global solution of $\beta(\lambda)$ which is guaranteed by theorem 1 in 4.1

There is one more thing worth remarking regarding this theorem. Notice that from [19] $Q(z)$ should be a polynomial in z and not a Laurent series. This is important because it means the integrability condition in theorem 1 can't possibly be satisfied. This is something we would expect because indeed our massless ϕ^4 model on it's own is not a physical theory.

In this section we also consider how solutions grow apart or converge to each other. This is related to equation to equation (10) in [2] which quantifies exactly how fast solutions grow apart. Unsurprisingly they find the solutions

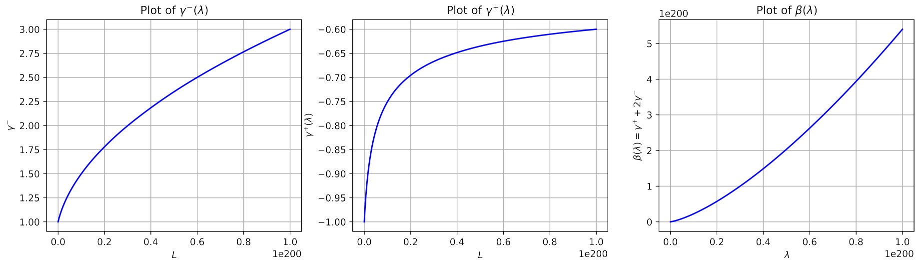


Figure 4.1: An example of a global solution to $\beta(\lambda)$ with existence guaranteed by theorem 1 the functions are $P^{+}(\lambda) = P^{-}(\lambda) = \frac{4}{\lambda^3}$ and $\gamma_0^{+} = -0.6$ and $\gamma_0^{-} = 3$.

grow apart; since the system in that case is one dimensional we know solutions can't cross. This question is slightly more interesting in our case since we only have this property in the full three dimensional $\{\gamma^{+}, \gamma^{-}, \lambda\}$ space and does not in general hold for $\beta(\lambda)$ as discussed in chapter 3. Here I will quantify and explain this relationship. Suppose we have two solutions for let $\alpha_i := \gamma_i^{+} + 2\gamma_i^{-}$ for $i = 1, 2$ then we can write.

$$\frac{d(\alpha_1 - \alpha_2)}{d\lambda} = \frac{\alpha_1 - \alpha_2}{\lambda} - 2\frac{\left(\gamma_1^{+} + \gamma_1^{-}\right)^2 - \left(\gamma_2^{+} + \gamma_2^{-}\right)^2}{\lambda} \tag{4.10}$$

We can write an implicit solution to this equation as,

$$\alpha_1 - \alpha_2 = \frac{\lambda}{\lambda_0}\left[\alpha_{1,0} - \alpha_{2,0}\right] - 2\lambda \int_{\lambda_0}^{\lambda} \frac{\left(\gamma_1^{+} + \gamma_1^{-}\right)^2 - \left(\gamma_2^{+} + \gamma_2^{-}\right)^2}{z^2}\, dz \tag{4.11}$$

Equation (4.11) gives some idea of how the difference between two solutions change as a function of λ. There are two interesting observations to make. First note that it is very possible for solutions cross as long as the first term in the integral is larger than the second term and is not damped too much by the z^2 in the denominator. Conversely if either of the two previous conditions fail we will have solutions crossing eventually. Another interesting point to make is that if the first term in the integral is grater than the second for all $\lambda > \lambda^*$ as λ increases it is more likely that solutions will cross.

4.3 Numerical Results

In this section I will discuss some of the numerical results which were obtained throughout the semester. First a numerical example with the constant $P^{+}(L)$ and $P^{-}(L)$, which illustrates that even if global solutions to γ^{+} and γ^{-} exist this does not necessarily give a global solution for $\beta(\lambda)$. The converse however is true, if global solutions to $\beta(\lambda)$ exist we must have global solutions to both γ^{+} and γ^{-}

In order to understand the solutions I also generated the slope fields for several different types of solutions. Below I present an example of this with global solutions. This is another way of analysing equations which in some cases is operationally more useful than the analytical arguments presented. For instance in 4.3 one can see that in the case of the given $P^{\pm}$ at least for the plotted range for $\gamma_0^+ \geq\approx -0.5$ there are solutions which are attracted to one and below this are Landau poles. Similarly for γ^- with the opposite signs and directions.

Of course we can also analyse only $\beta(\lambda)$ on its own using the same slope field code. This is shown in

While examples like these are easy to visualize and quite illustrative, ultimately they are not perfect since ultimately this is a three dimensional system and thus we would like to consider this full three dimensional system. Thankfully this system is still 3 dimensional system we can visualize this numerically an example of this is shown in 4.5.

We can also demonstrate the non crossing property of solutions in this three dimensional phase space, we of course already know this has to be true due to existence and uniqueness theorems as in [15] but it is a good test of my numerical code to ensure this. For example 4.6 shows several solutions sampled from different initial conditions, one can see directly the fact that the solutions don't cross.

Next I performed some numerical analysis on what initial conditions lead to the $\beta(\lambda)$ having zeros. This is an interesting question because as discussed in chapter 3 given certain conditions these solutions can be ruled out as non-physical because of being multivalued. An example of the regions of initial conditions which lead to different initial conditions is shown in 4.7. One interesting thing to note is that the boundary between the two types of solutions follows a line on one side which in the figure is shown in red. It is also worth noting that many of the solutions in purple which are not ruled out through having zeros, will still not be physical solutions, they may still have Landau poles for example. Determining more about where these solutions have Landau poles is something we plan to look at in the future. The zeros are counted by counting the number of points which have neighbours of an opposite sign. This means that each zero gets counted twice once from the point before the sign change and once after, this leads to a numerical artefact when the integration ends after a zero and this is seen through the green line of each of the plots. For the purposes of zeros, this should be counted as a single zero.

Lastly I present an example of a global solution for $\beta(\lambda)$ found numerically showing that, indeed global solutions are possible this is shown in 4.8. This is also incidentally an example where the converse of theorem 1 fails.

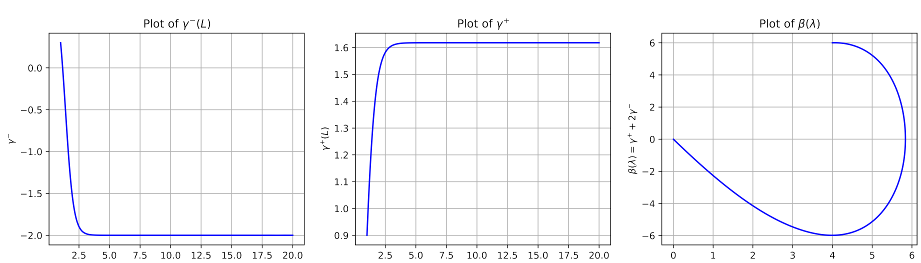


Figure 4.2: A Numerical Example of the solutions for $P^+(L) = -1$ and $P^-(L) = 2$ with the initial conditions $\gamma_0^+ = 0.9$ and $\gamma_0^- = 0.3$. This shows even though both γ^+ and γ^- have global solutions this doesn't imply that $\beta(\lambda)$ does as well.

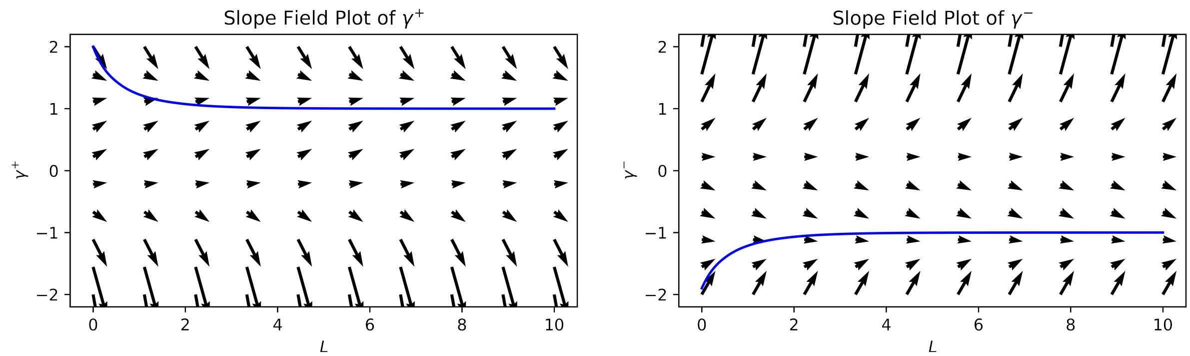


Figure 4.3: An example of the two dimensional slope fields generated with a solution curve super imposed on each. The functions in this case were $P^{+}(\lambda) = 1.5\lambda + 1.8336\lambda^2 - 3.728\lambda^3$ and $P^{-}(\lambda) = 0.0833\lambda^2 + 0.1874\lambda^3$ motivated by the ϕ^4 perturbation expansion. This system is difficult to analyse through the methods of section 4.1 since the functions are not defined in terms of L

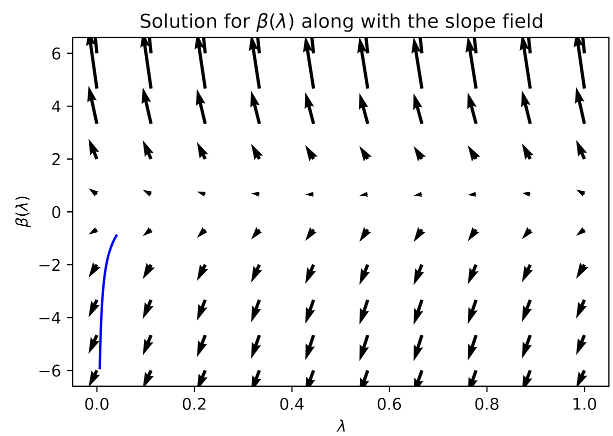


Figure 4.4: An example of the two dimensional slope field of $\beta(\lambda)$ with a solution curve super imposed. The functions are the same as in 4.3 with the initial conditions of the solution curve $\beta(\lambda_0) = -0.9$ with $\lambda_0 = 0.04$.

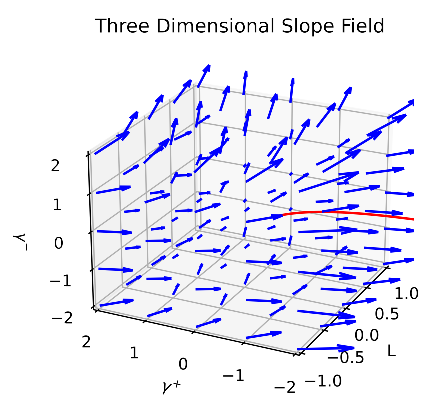


Figure 4.5: An Example of the three dimensional slope field for $P^{+}(L) = 1.5L + 1.8336L^2 - 3.728L^3$ and $P^{-}(L) = 0.0833L^2 + 0.1874L^3$ along with the solution curve superimposed.

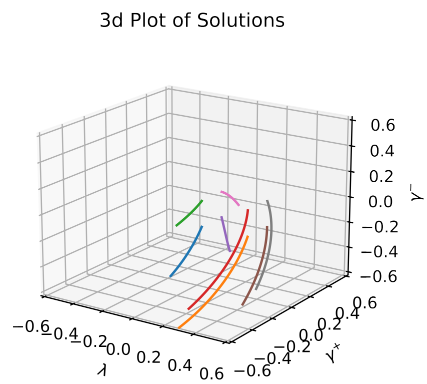


Figure 4.6: An example of several solutions sampled from different initial conditions where the non crossing property can be clearly seen.

4.4 Conclusion

In this work we have analysed the differential equations governing the beta function and anomalous dimension in massless ϕ^4 theory. We have presented the background and used these background concepts to derive these differential equations, this is based heavily on previous work. We then analysed these equations both numerically and analytically. On the analytic side we present a condition which guarantees the existence global solution for the massless ϕ^4 beta function. In particular we see that for all forms of $Q(\lambda)$ which we would obtain from the physical theory these global solutions are not guaranteed to exist. This is something we would expect because massless ϕ^4 theory is not a physical theory on it's own and this gives some mathematical insight as to why. On the numerical side we use standard integration software to plot different forms of solutions and their slope fields. This gives us insights into the different behaviour of the solutions with different initial conditions. We also used the integrator to determine where the solutions have zeros. This is interesting because as discussed these solutions can not be physical solutions and can be ruled out immediately. Solutions not crossing zero doesn't mean the solutions are necessarily physical only that they might be, since they can still have Landau poles for instance. For the next part of this project we plan to use a different rearrangement to give more insight on where the Landau poles exist and apply these combinatorial ideas to a super-symmetric physical theory.

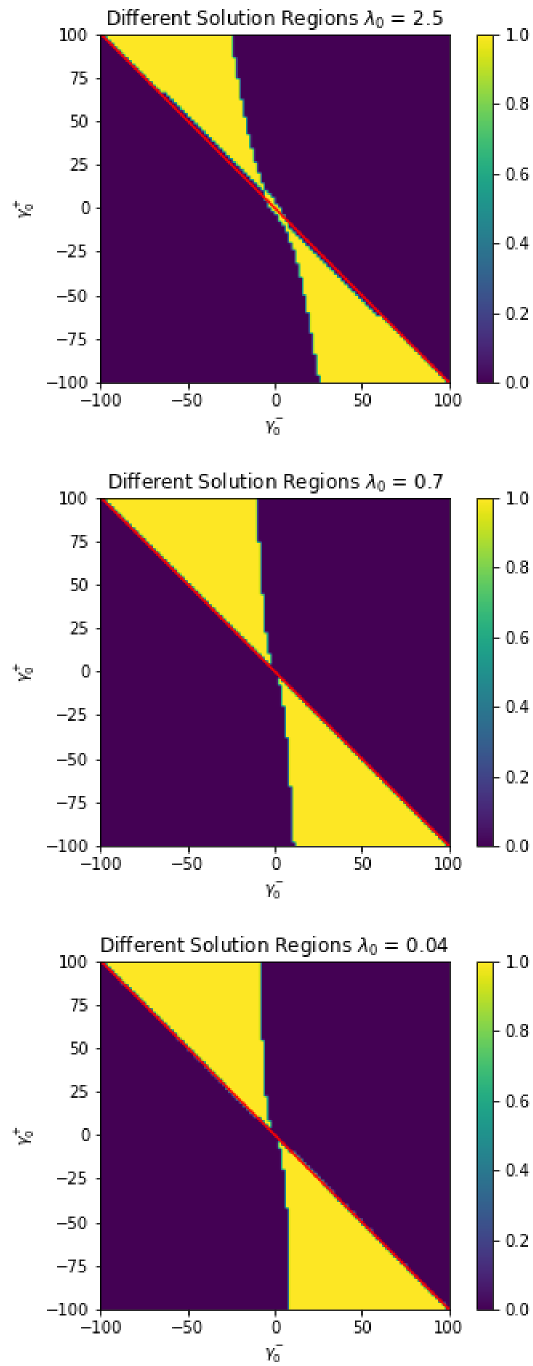


Figure 4.7: An example of the regions generated by the different initial conditions. The Yellow region has solutions with one non-trivial zero of the beta functioneta function and the purple region has no zeros. The putple region is the solutions with no non trivial zero of the beta function. The green line is a numerical artefact which essential comes from the way the zeros are counted and the red line is the line $\gamma_0^+ = -\gamma_0^-$

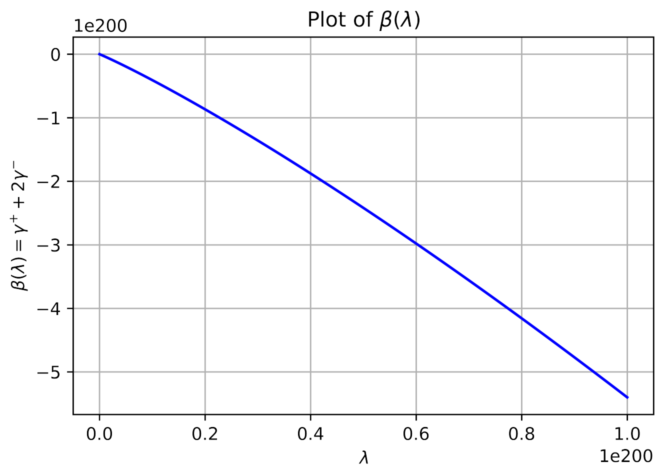


Figure 4.8: A numerical example of a global solution for $\beta(\lambda)$, the functions were $P^{+}(\lambda) = -1$ and $P^{-}(\lambda) = 3$, with $\gamma_0^{+} = 0.6$ and $\gamma_0^{-} = -3$.

Bibliography

[1] Alain Connes and Dirk Kreimer. *Hopf algebras, renormalization and noncommutative geometry*. Springer, 1999, pp. 59–109.

[2] Guillaume van Baalen et al. "The QED beta function from global solutions to Dyson–Schwinger equations". In: *Annals of Physics* 324.1 (Jan. 2009), pp. 205–219. ISSN: 0003-4916. DOI: 10.1016/j.aop.2008.05.007. URL: http://dx.doi.org/10.1016/j.aop.2008.05.007.

[3] Marc P. Bellon and Fidel A. Schaposnik. "Renormalization group functions for the Wess–Zumino model: Up to 200 loops through Hopf algebras". In: *Nuclear Physics B* 800.3 (Sept. 2008), pp. 517–526. ISSN: 0550-3213. DOI: 10.1016/j.nuclphysb.2008.02.005. URL: http://dx.doi.org/10.1016/j.nuclphysb.2008.02.005.

[4] D.J. Broadhurst and D. Kreimer. "Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality". In: *Nuclear Physics B* 600.2 (Apr. 2001), pp. 403–422. ISSN: 0550-3213. DOI: 10.1016/s0550-3213(01)00071-2. URL: http://dx.doi.org/10.1016/S0550-3213(01)00071-2.

[5] Kurusch Ebrahimi-Fard and Dirk Kreimer. "The Hopf algebra approach to Feynman diagram calculations". In: *Journal of Physics A: Mathematical and General* 38.50 (Nov. 2005), R385–R407. ISSN: 1361-6447. DOI: 10.1088/0305-4470/38/50/r01. URL: http://dx.doi.org/10.1088/0305-4470/38/50/R01.

[6] Darij Grinberg and Victor Reiner. *Hopf Algebras in Combinatorics*. 2020. arXiv: 1409.8356 [math.CO].

[7] Peter W. Higgs. "Broken Symmetries and the Masses of Gauge Bosons". In: *Phys. Rev. Lett.* 13 (16 Oct. 1964), pp. 508–509. DOI: 10.1103/PhysRevLett.13.508. URL: https://link.aps.org/doi/10.1103/PhysRevLett.13.508.

[8] K. Johnson, M. Baker, and R. Willey. "Self-Energy of the Electron". In: *Phys. Rev.* 136 (4B Nov. 1964), B1111–B1119. DOI: 10.1103/PhysRev.136.B1111. URL: https://link.aps.org/doi/10.1103/PhysRev.136.B1111.

[9] D. Kreimer. "Anatomy of a gauge theory". In: *Annals of Physics* 321.12 (Dec. 2006), pp. 2757–2781. ISSN: 0003-4916. DOI: 10.1016/j.aop.2006.01.004. URL: http://dx.doi.org/10.1016/j.aop.2006.01.004.

[10] Dirk Kreimer and Karen Yeats. “An Étude in non-linear Dyson–Schwinger Equations”. In: *Nuclear Physics B - Proceedings Supplements* 160 (Oct. 2006), pp. 116–121. ISSN: 0920-5632. DOI: 10.1016/j.nuclphysbps.2006.09.036. URL: http://dx.doi.org/10.1016/j.nuclphysbps.2006.09.036.

[11] Michael Edward Peskin and Daniel V. Schroeder. *An Introduction to Quantum Feild Theory*. Westview Press, 1995.

[12] William H. Press et al. *Numerical Recipes 3rd Edition: The Art of Scientific Computing*. 3rd ed. USA: Cambridge University Press, 2007. ISBN: 0521880688.

[13] Julian Schwinger. “On Quantum-Electrodynamics and the Magnetic Moment of the Electron”. In: *Phys. Rev.* 73 (4 Feb. 1948), pp. 416–417. DOI: 10.1103/PhysRev.73.416. URL: https://link.aps.org/doi/10.1103/PhysRev.73.416.

[14] M.E Sweedler. *Hopf Agebras*. W.A. Benjamin, 1969.

[15] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*. Graduate studies in mathematics. American Mathematical Society, 2012. ISBN: 9780821883280. URL: https://books.google.ca/books?id=FZ0CAQAAQBAJ.

[16] Steven Weinberg. *Quantum Theory of Feilds*. Cambridge University Press, 1995.

[17] Karen Yeats. *A Combinatorial Perspective on Quantum Field Theory*. Springer Briefs in Mathematical Physics, 2017.

[18] Karen Yeats. *Growth estimates for Dyson-Schwinger equations*. 2008. arXiv: 0810.2249 [math-ph].

[19] Karen Yeats. *Rearranging Dyson-Schwinger Equations*. Memoirs of the American Mathematical Society, 2011.

A. Algebras and Hopf Algebras

The point of this appendix is to give the reader a brief introduction to the mathematical structure known as a Hopf algebra. Since the core audience for this report is physicists who don't necessarily work often (if at all) with Hopf algebras this is included so that these readers can be made familiar with the subject quickly within this document.

In order to define the notion of a Hopf algebra we need to first define the notion of an algebra. Let V be a vector space over a field F and let $\nabla : V \otimes V \longrightarrow V$ be a linear map[1] and let η be a linear map $\eta : \mathbb{F} \longrightarrow V$ called the unit with the following two conditions, $\nabla \circ (\nabla \otimes \mathbb{1}_V) = \nabla \circ (\mathbb{1}_V \otimes \nabla)$ and $\nabla \circ (\mathbb{1}_V \otimes \eta) \circ \mu_R = \nabla \circ (\eta \otimes \mathbb{1}_V) \circ \mu_L = \mathbb{1}_V$, where $\mu_L : V \longrightarrow F \otimes V$ is the map $\mu_L(v) = 1_F \otimes v$ and $\mu_R : V \longrightarrow V \otimes F$ is the map $\mu_R(v) = v \otimes 1_F$. With these two conditions the structure (V, F, ∇, η) is called a unital, associative algebra. In order to define the idea of a Hopf algebra we also need to define coalgebra. Let V be a vector space over a field F and let $V \otimes V$ by the tensor space of V with itself. Let $\Delta : V \longrightarrow V \otimes V$ be a map which we call the coproduct and let $\epsilon : V \longrightarrow F$ be a map we call the counit, if the maps Δ and ϵ satisfy $(\mathbb{1}_V \otimes \Delta) \circ \Delta = (\Delta \otimes \mathbb{1}_V) \circ \Delta$ and $(\mathbb{1}_V \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathbb{1}_V) \circ \Delta$ then the tuple (V, F, Δ, ϵ) is called a unital associative coalgebra.

Lastly we will need to define a bialgebra, the rough idea behind a bialgebra is that it is an algebra and coalgebra together with some compatibility conditions. Formally a bialgeba is a tuple $(V, F, \nabla, \eta, \Delta, \epsilon)$ such that (V, F, ∇, η) is a unital associative algebra and (V, F, Δ, ϵ) is a coalgebra and the following compatibility conditions are satisfied.

1. $\Delta \circ \nabla = \nabla \otimes \nabla \circ \mathbb{1}_V \otimes \tau \otimes \mathbb{1}_V \circ \Delta \otimes \Delta$.
2. $\epsilon \circ \nabla = \epsilon \otimes \epsilon$.
3. $\Delta \circ \eta = \eta \otimes \eta$.
4. $\epsilon \circ \eta = 1_F$.

Where $\tau : V \otimes V \longrightarrow V \otimes V$ is the linear operator which satisfies $\tau(v \otimes w) = w \otimes v$ for all $v, w \in V$ and extended to all of $V \otimes V$ by linearity.

Given all of this I can finally give the definition of a Hopf algebra. A Hopf algebra is a bialgebra together with an antiautomorphism called the anipode $S : V \longrightarrow V$ which satisfies the following compatibility condition. $\nabla \circ S \otimes \mathbb{1}_V \circ \Delta = \nabla \circ \mathbb{1}_V \otimes S \circ \Delta = \eta \circ \epsilon$. There are a lot of different uses for the notions of Hopf algebras which

[1] This is equivalent to a bilinear map from $V \times V \longrightarrow V$ by the universal property

arise in combinatorics see for example [6] for a large review of these concepts. For us the only use will be to note that the 2-edge connected divergent Feynman graphs can be made into a Hopf algebra structure.

B. Auxiliary Calculations

In this appendix I will provide some of the calculations which are necessary for the main report, but would cause the flow of the main report to be broken. Beginning with equation (3.12) in the main text by considering $2\times(3.4)+(3.5)$ in the main text I get

$$\frac{d\left(\gamma^{+}+2\gamma^{-}\right)}{d\lambda}=\frac{\left(\gamma^{+}+2\gamma^{-}\right)+\left\{-\left(\gamma^{+}\right)^{2}+2\left(\gamma^{-}\right)^{2}\right\}-\left(P^{+}(\lambda)+2P^{-}(\lambda)\right)}{\lambda\left(\gamma^{+}+2\gamma^{-}\right)}$$

We then define $Q(\lambda):=P^{+}(\lambda)+2P^{-}(\lambda)$ and we get adding 0 inside the brackets we have

$$\frac{d\left(\gamma^{+}+2\gamma^{-}\right)}{d\lambda}=\frac{\left(\gamma^{+}+2\gamma^{-}\right)+\left\{-2\left(\gamma^{+}\right)^{2}+\left(\gamma^{+}\right)^{2}+4\left(\gamma^{-}\right)^{2}-2\left(\gamma^{-}\right)^{2}+4\gamma^{+}\gamma^{-}-4\gamma^{+}\gamma^{-}\right\}-Q(\lambda)}{\lambda\left(\gamma^{+}+2\gamma^{-}\right)}$$

We can collect terms and simplify to get

$$\frac{d\left(\gamma^{+}+2\gamma^{-}\right)}{d\lambda}=\frac{\left(\gamma^{+}+2\gamma^{-}\right)+\left\{\left(\gamma^{+}\right)^{2}+4\gamma^{+}\gamma^{-}+4\left(\gamma^{-}\right)^{2}-2\left(\gamma^{+}\right)^{2}-4\gamma^{+}\gamma^{-}-2\left(\gamma^{-}\right)^{2}\right\}-Q(\lambda)}{\lambda\left(\gamma^{+}+2\gamma^{-}\right)}$$

We can now write the terms in the curly brackets as a sum of square terms to get,

$$\frac{d\left(\gamma^{+}+2\gamma^{-}\right)}{d\lambda}=\frac{\left(\gamma^{+}+2\gamma^{-}\right)+\left(\gamma^{+}+2\gamma^{-}\right)^{2}-2\left(\gamma^{+}+\gamma^{-}\right)^{2}-Q(\lambda)}{\lambda\left(\gamma^{+}+2\gamma^{-}\right)}$$

And dividing through the denominator we get

$$\frac{d\left(\gamma^{+}+2\gamma^{-}\right)}{d\lambda}=\frac{1}{\lambda}+\frac{\gamma^{+}+2\gamma^{-}}{\lambda}-\frac{2\left(\gamma^{+}+\gamma^{-}\right)^{2}+Q(\lambda)}{\lambda(\gamma^{+}+2\gamma^{-})}\tag{B.1}$$

Which is equation (3.12) in the main text. In order to derive equation (3.11) we simply write

$$\frac{d\beta}{d\lambda}=\frac{d}{d\lambda}\left\{\lambda\left(\gamma^{+}+2\gamma^{-}\right)\right\}=\gamma^{+}+2\gamma^{-}+\lambda\frac{d\left(\gamma^{+}+2\gamma^{-}\right)}{d\lambda}$$

Therefore using (B.1) we have

$$\frac{d\beta}{d\lambda} = 1 + 2\left(\gamma^{+} + 2\gamma^{-}\right) - \frac{2\left(\gamma^{+} + \gamma^{-}\right)^{2} + Q(\lambda)}{\gamma^{+} + 2\gamma^{-}}$$

Using $\gamma^{+} + 2\gamma^{-} = \beta(\lambda)/\lambda$ we finally get

$$\frac{d\beta}{d\lambda} = 1 + \frac{2\beta(\lambda)}{\lambda} - \lambda\frac{2(\gamma^{+} + \gamma^{-})^{2} + Q(\lambda)}{\beta(\lambda)} \tag{B.2}$$

This is equation (3.11) in the main text. Next I will derive equation (4.3) in the main text. First dividing through $\gamma^{+} - \left(\gamma^{+}\right)^{2}$ in equation (3.9) we get

$$\frac{d\gamma^{+}}{\gamma^{+} - \left(\gamma^{+}\right)^{2}} = \left[1 - \frac{P^{+}(\lambda)}{\gamma^{+} - \left(\gamma^{+}\right)^{2}}\right] dL$$

Again using a partial fraction expansion on the left hand side we have

$$\left[\frac{1}{\gamma^{+}} + \frac{1}{1-\gamma^{+}}\right] d\gamma^{+} = \left[1 - \frac{P^{+}(\lambda)}{\gamma^{+} - \left(\gamma^{+}\right)^{2}}\right] dL$$

Integrating using the initial condition $\gamma^{+}(L_0) = \gamma_0^{+}$. We get

$$\log\left(\left|\frac{\gamma^{+}}{1-\gamma^{+}}\frac{1-\gamma_0^{+}}{\gamma_0^{+}}\right|\right) = (L - L_0) - \int_{L_0}^{L} \frac{P^{+}(\lambda)}{\gamma^{+} - \left(\gamma^{+}\right)^{2}} dL$$

Defining $D = \left|\frac{\gamma_0^{+}}{1-\gamma_0^{+}}\right|$ and $g(L, L_0) = \int_{L_0}^{L} \frac{P^{+}(\lambda)}{\gamma^{+} - \left(\gamma^{+}\right)^{2}} dL$ we get

$$\frac{\gamma^{+}}{1-\gamma^{+}} = \pm D e^{(L-L_0) - g(L, L_0)}$$

Rearranging and simplifying we have

$$\gamma^{+}(L) = \pm\frac{e^{(L-L_0) - g(L, L_0)}}{\frac{1}{D} \pm e^{(L-L_0) - g(L, L_0)}} \tag{B.3}$$

Which is equation (4.3) in the main text.