Analysis of Combinatorial Dyson-Schwinger Equations With Several Physical Models

by

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Abstract

In this report we build on the work done in the previous term where we applied general equations for the flows of the the anomalous dimension and the beta function, were specialized to a massless ϕ^4 theory. One of the questions which we left unanswered in the previous term was when exactly are there Landau poles in ϕ^4 theory. In answering this question we find intimidate generalization to any massless theory. We next consider renormalization in the Wess-Zummino model and show how this can simplifies the equations we need to consider for the anomalous dimension and beta function fining only a single equation is needed for each of these functions. We then study the flows generated by these equations and find them to be very similar to the equations for the massless QED theory. Finally we give new Feynman rules for the vertices which result from the co-action relating to the renormalization of Cutkosky graphs and find that momentum conservation forces vertices to have a specific form.

Acknowledgements

I would first and foremost like to thank professor Karen Yeats for her support and guidance through the project and for taking me on as a student during these eight months I can think of no better person to have been a supervisor for this research project and to be an impromptu graduate school admissions coach for all of this she will have my gratitude as I continue my studies as a PhD student at the University of Toronto. I would also like to thank Professor Raffi Budakian for agreeing to be my co-supervisor for this project for without his willingness to do so, this project could not have taken place. I would also like to thank Paul Baldurf, for the valuable discussions he provided during his visit to the university of Waterloo.

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Chapter 1

Introduction and Background

1.1 Introduction

In this work our goal is to derive and investigate the renormalization group functions (the anomalous dimension and the beta function) for several different physical theories using the combinatorial methods developed in [21]. Renormalization has been an important part of quantum field theory nearly since the inception of the field. Radiative corrections, where the ideas of renormalization are seen in quantum electrodymanics are very important and many of these results are responsible for some of the most impressive results of quantum field theory, see for instance [15], [16], [14]. However for several years there was no known well defined mathematical prescription for the generalizations of these ideas. Eventually however the mathematical structure of renormalization was formalized and given a solid mathematical background [8]. With this more solid mathematical background it became possible to better understand the mathematical and specifically Hopf algebraic structure of renormalization[5],[11],[4],[2]. Studying and understanding the algebraic structures has lead to many physically relevant interesting results[13],[11],[3] and in this work we intend to make more progress in this direction by studying the ordinary differential equations which were derived in [21]. In my previous project relating to this we analysed the ordinary differential equations from ϕ^4 theory and proved a result relating to the existence of global solutions for the beta function in ϕ^4 theory analogous to the result in [7]. Our previous analysis however left open several important questions and the goal here is to answer these questions. Specifically we were interested in where ϕ^4 theory has Landau poles and how can we characterize the separating surfaces between solutions. To answer these questions we consider the Riccati equations associated to the non-linear differential equations which are derived in [21]. This reduces these first order non-linear differential equations into second order linear differential equations. This transformation gives an interesting relationship between the zeros of the transformed solution and the Landau poles of the anomalous dimension. We then apply this form of analysis to the massless version of Quantum electrodynamics as well as the super-symmetric Wess Zumino model. A lot of the work required to apply the analysis of [21] to the massless version of quantum electrodynamics and ϕ^4 theory has been done in previous works, see [20],[10], however to apply these methods to the Wess Zumino model there is some extra work we need to do and these questions are also addressed in our work. Specifically we address weather or not the sums of the insertion operators $\sum_i B_+^{k,i;r}$ as considered in [20] indeed form a Hochschild 1 cocycle and conversely what identities would need to be satisfied to have the insertion operators form a Hochschild 1 cocyle. We also derive the Dyson-Schwinger equations for the scalar and fermon self energies and show how the supergauge invariance of the lagrangian reduces these into a single equation. From this point on the equation in [21] can be applied directly. Finally, we investigate the co-action related to Cutcosky graphs developed in and how one can prescribe Feynman rules in a way consistent with momentum conservation imposed by the vertices.

The rest of the report is structured as follows, in chapter 2 I introduce the necessary combinatorial and physical background required to understand the report. In chapter 3 I develop the theory for proving the existence of Landau poles, and apply this to the specific case of ϕ^4 theory. In chapter 4 I present the Wess Zumino model as well as supergauge transformations and how the Lagrangian is invariant under these transformations as well as how this leads to the combinatorial Ward identities, and how these will lead to simplifying the equations derived in [21], which we then use to investigate the Wess Zumino model. In chapter 5 I discuss the new Feynman rules for the vertices generated by considering the co-action on Cutcosky graphs and conclude the report.

Chapter 2

Background

In this chapter I will present all of the necessary combinatorial and physics background for the project. The first section of this chapter will focus on the physics needed for the project, specifically renormalization and the different schemes for renormalization. We will also give an account of the different Feynman rules for different types of fields as we will use them frequently. From the combinatorics side I will introduce several key concepts from graph theory which will be of use in the entire report. There are several concepts which I will not cover in this report as they have been described in the previous report and therefore will not be elaborated on here. These concepts are Hopf algebras, the basic physics of renormalization, as well as the generalities on the derivation of the main equation in [21]. If one is interested in any more information on my topics they are referred to my previous report which may be found on professor Yeats' website.

2.1 Physics of Renormalization

2.1.1 Basics of Quantum Field Theory

In this section I will discuss the physics of renormalization, to begin I will give a brief summary of the standard Feynman rules for spin 0, spin 1/2 and spin 1 fields. I will then discuss the different commonly used renormalization schemes in physics. We begin by recalling the Feynman rules for a scalar field. First for a scalar field.

Type of Line	Diagram Representation	Mathematical Expression
External Line		1
Internal Line		$\frac{i}{k^2-m^2}$

Similarly for fermion (spin 1/2) lines we have,

Type of Line	Diagram Representation	Mathematical Expression
External Line		$u^s(p)$
External Line		$\bar{u}^s(p)$
External Line		$\bar{v}^s(p)$
External Line		$v^s(p)$
External Line		$\frac{i(\not{k}+m)}{k^2-m^2}$

Where s labels the spin of the fermion, and $\not{k} = \gamma^\mu k_\mu$. For spin 1 fields we have,

Type of Line	Diagram Representation	Mathematical Expression
External Line		$\epsilon_{\mu\nu}$
External Line		$\epsilon^*_{\mu\nu}$
Internal Line		$\frac{i(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2})}{k^2-m^2}$

These rules are general for any fields, but of course in a real physical theory one will also have interaction terms. These interaction terms will specify the theory and will of course be dependant on the theory. When we discuss specific theories I will provide the vertices, which specify the interaction terms. Next I want to introduce the idea of a functional derivative. Defining these formally in terms of mathematical distributions is beyond the scope of what we wish to discuss in this document. We will want to still do computations however, to this end we define.

$$\frac{\delta J(x)}{\delta J(y)} = \delta^4(x-y) \tag{2.1}$$

So that if for instance

$$R(x) = \int d^4y\, K(x,y)J(y)$$

Then

$$\frac{\delta R(x)}{\delta J(z)} = K(x,z)$$

These relations are all we will need to compute functional derivatives. Next I will define another useful tool we will use frequently called a(n) (n-point) Green's function, this is defined by

$$G(x_1, x_2, \ldots, x_n) = \langle 0 | \mathcal{T} \left(\hat{\phi}(x_1)\hat{\phi}(x_2) \cdots \hat{\phi}(x_n) \right) | 0 \rangle \tag{2.2}$$

And where $\mathcal{T}(\cdot)$ is the time ordering operator which we define by

$$\mathcal{T} \left(\hat{\phi}(x_1)\hat{\phi}(x_2) \cdots \hat{\phi}(x_n) \right) = \sum_{\sigma \in S_n} \left(\prod_{j=0}^{n-1} \theta \left(t_{\sigma(j)} - t_{\sigma(j+1)} \right) \right) \varepsilon(\sigma) \hat{\phi}(x_{\sigma(1)}) \hat{\phi}(x_{\sigma(2)}) \cdots \hat{\phi}(x_{\sigma(n)}) \tag{2.3}$$

Where as usual S_n is the permutation group on n elements, and $\varepsilon(\sigma) = 1$ if the operators are bosonic operators or $\varepsilon(\sigma) = \mathrm{sgn}(\sigma)$ if the operators are fermionic operators. Usually, when working with quantum field theory one would then relate these Greens functions to the S-matrix (or usually the other way around) using the LSZ reduction formula since this then allows one to calculate cross sections and decay rates. However since we won't be considering these quantities and I won't remark on that here other than saying that to compute these physical quantities it is enough to compute these n-point Green's functions. For different parts of this document it will be helpful to define the generating functional for the n-point Green's function to be

$$Z[J] = \sum_{n \geq 0} \frac{i^n}{n!} \int d^4x_1 \, d^4x_2 \, \cdots d^4x_n \, G(x_1, x_2, \ldots, x_n) J(x_1) J(x_2) \cdots J(x_n) \tag{2.4}$$

From this definition we can obtain the Green's function through functional as

$$G(x_1, x_2, \ldots, x_n) = \left. \frac{\delta^n Z[J]}{i\delta J(x_1) \, i\delta J(x_2) \cdots i\delta J(x_n)} \right|_{J=0} \tag{2.5}$$

This generating functional contains all information about the n-point Green's functions for all $n > 0$ and thus is computationally a powerful tool. However it is usually more common in physics to work with the the Fourier transform of each of these Green's functions. The next thing that we need is the concept of functional integrals. Once again delving into the mathematical details of these integrals (how they are defined and properties) is beyond the scope of the document and can be found in almost any standard quantum field theory textbook see for example [15]. What I will mention here is that for many quantum field theories for example in a spin-0 field we have

$$Z[J] = \frac{1}{Z[J=0]} \int \mathcal{D}\phi \, e^{i \int d^4x \, \mathcal{L} - J(x)\phi(x)} \tag{2.6}$$

And from these equations we can derive for example the propagators and vertices for the different theories. Two remarks are in order here. First is that when using these functional integrals with fermion fields, will use Grassman variable valued functions as the currents and not regular functions. I will comment on this more when we need to use them but for now I will move on. Secondly when deriving Green's functions or correlation function in this way one will get correct results but these will be in the "position representation". Although at first they might seem different these representations are equivalent because they are related by a Fourier transform which is invertible. We will often not use this and stick to the momentum representation.

Next I want to give a brief overview of Dyson-Schwinger equations which are generally speaking, (functional) differential equation for the generating functional and currents derived through symmetries. These are going to be an important part of what we are going to do with the Wess-Zumino model and so it is worth remarking on here. Since again these relations will depend on the theory we choose let's consider the specific Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$ then in this case we see using integration by parts we can write $\mathcal{L} = \frac{1}{2}\phi\left(\Box - m^2\right)\phi +$ Surface Terms and thus the generating functional is

$$Z[J] = \frac{1}{Z_0}\int \mathcal{D}\phi\, e^{i\int d^4x\, -\frac{1}{2}\phi(x)\left(\Box+m^2\right)\phi(x) - J(x)\phi(x)} \tag{2.7}$$

Now by the analogue of the fundamental theorem of calculus we have $\int \mathcal{D}\phi\frac{\delta F}{\delta\phi} = 0$ and so we find,

$$\left(\Box + m^2\right)\frac{1}{Z_0}\int \mathcal{D}\phi\, \phi e^{i\int d^4x\, -\frac{1}{2}\phi(x)\left(\Box+m^2\right)\phi(x) - J(x)\phi(x)} + J(x)Z[J] = 0 \tag{2.8}$$

Now we can use the fact that $-\frac{1}{Z_0}\int \mathcal{D}\phi\, \phi e^{i\int d^4x\, -\frac{1}{2}\phi(x)\left(\Box+m^2\right)\phi(x) - J(x)\phi(x)} = \frac{\delta Z}{\delta iJ(x)}$ to get

$$\left(\Box + m^2\right)\frac{\delta Z}{\delta iJ(x)} = J(x)Z[J] \tag{2.9}$$

This is a first example of a Dyson Swinger equation. One can see that it is indeed a differential equation for the generating functional. This is sometimes also called an equation of motion for the generating functional. This can also be combined with symmetries to yield what are usually called Ward identities. For instance by considering the transformation $\psi \longrightarrow \psi + ie\alpha(x)$ and using the fact that shifting the fields leaves the functional integrals and Fourier transforming the result one derives a standard QED Ward identity.

$$-ik^\mu\mathcal{M}_\mu = ie\left(\mathcal{M}(p+k;q) - \mathcal{M}(p;q-k)\right) \tag{2.10}$$

Of course this is a specific example, we can generalize this to different theories using these generating functionals which will come in useful while discussing the Wess-Zumino model and thus we will keep these ideas in mind for later. So far nothing we have discussed strictly speaking involves perturbation theory at all, all of these results are completely non pertubative. In the next section we will discuss perturbation theory and specifically renormalization.

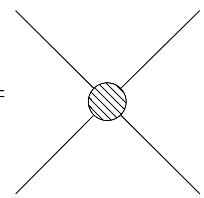
2.1.2 Renormalization

In this section I will describe more about renormalization and throughout this section I will be using the ϕ^4 theory which has Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{2.11}$$

Now as discussed in my last report and is well known the four-point function which we can represent diagrammatically as,

$$i\mathcal{M}(p_1, p_2 \to p_3, p_4) =$$



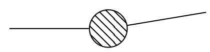
Is divergent. The goal of renormalization is thus to assign meaningful physical quantities to these divergent amplitudes which we can use in the computation of scattering cross sections for example. The way we do this is by using the fact that we actually have some freedom in the "constant" terms in the Lagrangian m and λ as well as an over all field strength renormalization Z_r. Let's denote the physical mass by m_p^2 and the physical coupling constant λ_p and the physical field Φ by $\sqrt{Z_r}\Phi = \phi$. Then the Lagrangian in terms of the physical field becomes

$$\mathcal{L} = \frac{Z_r}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{Z_r}{2}m^2\Phi^2 - \frac{Z_r^2\lambda}{4!}\Phi^4 \tag{2.12}$$

Now we want to write this in terms of only the physical quantities plus counter terms, we do this by noting $Z_r = 1 + (Z_r - 1)$ and $-Z_r m^2 = -m_p^2 - (Z_r m^2 - m_p^2)$ and $-Z_r^2\lambda = -\lambda_p - (Z_r^2\lambda - \lambda_p)$ now we can rewrite the Lagrangian in terms of these physical quantities and the quantities $\delta_Z = Z_r - 1$, $\delta_m = Z_r m^2 - m_p^2$ and $\delta_\lambda = Z_r^2\lambda - \lambda_p$ and this we can write

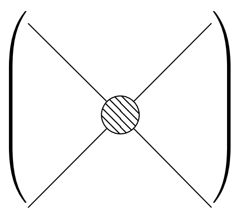
$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m_p^2\Phi^2 - \frac{\lambda_p}{4!}\Phi^4 + \left(\frac{\delta_Z}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{\delta_m}{2}\Phi^2 - \frac{\delta_\lambda}{4!}\Phi^4\right) = \mathcal{L}_p + \delta_\mathcal{L} \tag{2.13}$$

Note that the counter-terms (the δ terms) are unobservable since they are not part of the physical Lagrangian and thus can be infinite and in fact they must be infinite, their specific goal will be to absorb the infinite terms which come from the divergent diagrams. Of course this leads us to the question how to we determine what Z_r, m_p and λ_p are? The answer to these questions are called the renormalization conditions. Since the masses are location of the poles in the bare propagators and Z_r would be the value of the residue at these poles it makes sense to define m_p by the relation,



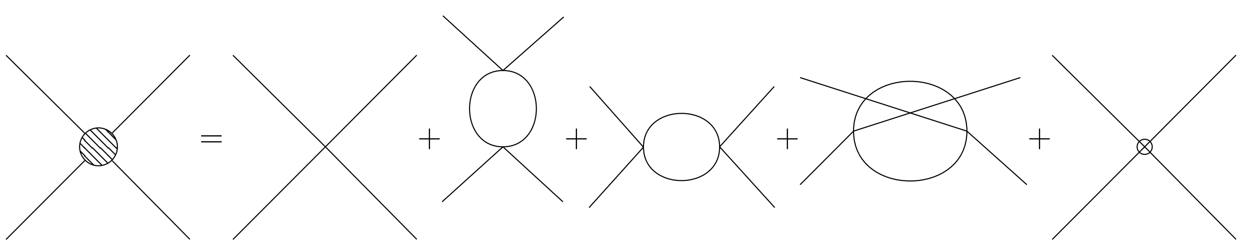
$$= \frac{i}{p^2 - m_p^2} + f(p^2) \tag{2.14}$$

Where $f(p^2)$ is a regular function at $p^2 = m_p^2$. This fixes both Z_r as the residue of the pole of the full 2-point function and the value of m_p as the location of the pole in the full two-point function. We can use the four-point function to define the physical coupling constant as



$$_{s=4m^2, t=u=0} = -i\lambda_p \tag{2.15}$$

Where, s, t and u are the standard mandelstam variables. With these conditions we can move on to calculate the different physical quantites of interest. Usually in physics this is done perturbatively order by order in the number of loops. For instance the one loop correction to the coupling constant is simply,



Where the last diagram comes from the vertex counter term in the Lagrangian. We could evaluate these diagrams and find the leading order values of λ_p for instance. We can so the same for the self energy diagrams as well. Now all of this would be good but a question is still remaining, can we do this in a consistent way i.e do we have enough counter terms to remove all the divergences we can generate at any loop order and get reasonable physical values for the coupling and mass.

We can get an idea of this by thinking about the structure of the diagrams, and anticipating what kind of integrals we will get. Based on the Feynman rules we know for every loop in the diagram we will have to do an integral over the loop momenta and (in n dimensions) this will be an be an integral with k^{n-1} in the numerator. We can see this by considering the n-dimensional version of spherical coordinates. Also for each propagator we will get a factor of k^{-p}. Using these two facts we can get an estimate for how badly our integrals will diverge when integrating over the loop momenta. Note that for an integral of the form k^m/k^n to converge we need to have $m < n-2$ since then the integral will go as $m-n+1 < -1$. Thus in order to estimate how the integrals will diverge, suppose we have in N space-time dimensions we have a diagram with L loops, additionally suppose our theory has r different types of propagators which are elements of a set Π with $|\Pi| = r$ to each propagator we assign a weight w_p which is the powers of momenta in the denominator of the propagator. Then for large momenta our integral will be of the form

$$\int dk_1 \cdots dk_L \, \frac{k_1^{N-1} \cdots k_L^{N-1}}{k^s} \to \frac{k^{(N-1)L+L}}{k^s} = k^D$$

Where, $D = NL - s$ and $s = \sum_{p \in \Pi} n_p w_p$ where n_p is the number of propagators of type p and w_p is the weight of the propagator of type p. This motivates us to define in N dimensions the superficial degree of divergence as

$$D = NL - \sum_{p \in \Pi} w_p n_p \tag{2.16}$$

For many diagrams $D < 0$ the integrals converge at large momenta, when $D = 0$ we will have a logarithmic divergence and when $D > 0$ we will have a polynomial divergence. This will not always be the case and many counter examples can be found but still this superficial degree of divergence is a useful litmus test for if the diagrams will lead to divergences.

Let's compute the value of D for our ϕ^4 in four dimensions theory to see how this can be useful. First we note that there is only one type of propagator with weight 2. We are only considering connected diagrams because when computing physical quantities these diagrams will be cancelled when we divide by an overall normalizing factor (the $Z[J=0]$) in equation 2.6. It is a combinatorial fact that $L = n_e - n_v + 1$ and since all of the vertices are four-valent we have $4n_v = 2n_e + r_e$ where n_e is the number of internal edges and r_e is the number of external edges. Thus $L = \frac{1}{2}n_e - \frac{1}{4}r_e + 1$ and hence

$$D = 2n_e - r_e + 4 - 2n_e = 4 - r_e \tag{2.17}$$

So we can see that only the zero, one, two, three and four point functions will be superficially divergent and hence need renormalizing, the zero point function is nothing more than an unobservable vacuum energy shift and so we don't need to worry about it, since the Lagrangian is symmetric under $\phi \to -\phi$ the one and three point functions have to vanish and thus only the two and four point functions need renormalizing. This is good news since for these two functions we have two parameters to play with the mass and coupling constant. It is also worth pointing out that this entire discussion is concerned with U.V divergences, there are also infrared divergences which occur in the limit of small k but this discussion does not apply to them.

A full and complete proof that ϕ^4 theory is fully renomalizable is known as the BPHZ theorem [8], of course the full details of this is beyond the scope of this document but it is nevertheless an amazing result. We can do this procedure to this for QED as well splitting up the Lagrangian in a similar way, this time with the mass of the electron and the electron charge, of course we will now have two field strength renormalzations on for the electron field and one of the electromagnetic field.

This is all we will need to know about the physics for the project to make sense, for a more in depth look into some of these topics the reader is referred to any one of [14][15] or [17]. In the next section I will describe some of the important combinatorial background needed to understand the project anything else not covered here is in the other report for 437A.

2.2 Combinatorics and Renormalization

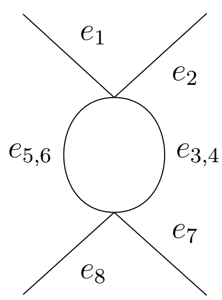
As was evident in the last section, there is a lot of utility in treating these diagrams as graphs, because there are a lot of theory for graphs which when needed can be applied to these diagrams. The goal of this section is to outline the basics of this which we will need for the project going forward. I have already discussed the idea of Hopf algebras and a small amount of graph theory in the previous report. Therefore I will not remark on the basics of Hopf algebras and graph theory here, I will however need to discuss some more advanced topics related to these for the last part of the report. Another set of important concepts which I will not be addressing in this report since again they have been discussed in detail in my first report is much of the theory covered in [21]. We will only be using the main results of [21] here and a lot of the detail is already covered in my first report.

2.2.1 Feynman Graphs

The first combinatorial concept we will use in this report is the idea of a Feynman graph. These are essentially the natural way of adding structure to Feynman diagrams which allow them to be seen as graphs. Strictly speaking the reason why these are not graphs, is due to the external edges. In fact if we remove all of the external edges then these would be perfectly good graphs. In addition we can't have arbitrary vertices, but this doesn't make these not graphs, it only makes them more restrictive graphs. Thus to be able to analyse these diagrams as graphs rather than considering full edges, as in standard graph theory it is better instead to consider half edges. Most of these definition and concepts are derived from [12]

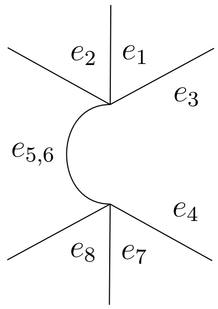
Definition 2.1.1 - 1:
A Feynman graph is a set of half edges H and a set of vertices V and a set of edges E, such that V is a partition of of H where each part of the partition has size at least three, and E is a partition of H where each part is has cardinality at most 2. The parts of E with size 2 are called internal edges and the parts of E with size 1 are called the external edges.

As an example consider the graph below,



In this case we can see that we can identify the half edge set, with $H = \{1, 2, 3, 4, 5, 6, 7, 8\}$ the partition $V = \{\{1, 2, 3, 5\}, \{4, 6, 7, 8\}\}$ and the edge set is $E = \{\{1\}, \{2\}, \{3, 4\}, \{5, 6\}, \{7\}, \{8\}\}$. One can see that the singleton sets in the edge partition indeed correspond to the external edges.

This notion of a Feynman graph is slightly different from the notion I had in the first report and for good reason, this notion of a Feynman graph allows us to more easily consider the standard graph theory operations of contraction and cutting. To see this suppose we want to cut the full edge $\{3, 4\}$ in the normal graph theory sense diagrammatically this would simply be the graph,



Now one can see that this corresponds to the sets, $H^{'} = H = \{1,2,3,4,5,6,7,8\}$ the partition $V^{'} = V = \{\{1,2,3,5\},\{4,6,7,8\}\}$ and the edge set partition is now $E^{'} = \{\{1\},\{2\},\{3\},\{4\},\{5,6\},\{7\},\{8\}\}$. One will immediately notice that all of the parts of E' are subsets of one of the parts of E. In the language of partitions we say that E' is a refinement of E. Thus, *the graph theory operation of cutting and edge corresponds to refining the partition of H by E.*

One can consider a similar set theoretic operation for the standard graph theory operation of contraction as well and one can also cut graphs by considering refinements of the V as well and graphically we would view these as cutting the vertex from a graph theory point of view. For the sake of keeping this report short, the reader is referred to [12].

One can define the notion of a pre-cut graph which for the purposes of this document we will consider as a regular Feynman graph with decorations on the edges and vertices specifying which edges are to be cut. This is a slight abuse as it might give one the impression that the pre-cut graph and the undecorated graph are the same. Strictly speaking this is not entirely accurate and for the slight subtlety related to this the reader is again referred to [12], for us it won't be very important. This allows us to define the concept of a cut graph.

Definition 2.1.1 - 2: A cut graph is a pre-cut graph where only the edge set partition is refined.

This definition allows us to define Cutkosky and pre-Cutkosky graphs.

Definition 2.1.1 - 3:

- A pre-Cutkosky graph is a pre-cut graph graph which has the property that has the property that every edge which is cut has each end in different components of the cut graph.

- A Cutkosky graph is a cut graph graph which has the property that has the property that every edge which is cut has each end in different components of the cut graph.

We will also be interested in the graph theory concepts of spanning trees and fundamental cycles. Recall that a tree is a connected graph with no cycles. A graph (with more than one connected component) with no cycles is called a forest. A spanning forest is a forest where each vertex in the graph is adjacent to some edge in the forest. Consider now a connected graph G[1] and a spanning tree $\mathcal{T}$ of G let $e \in G \setminus \mathcal{T}$ be an edge, and let $\mathcal{T}' = \mathcal{T} \cup \{e\}$ then $\mathcal{T}'$ contains a unique cycle which is called the fundamental cycle of G with respect to $\mathcal{T}$ and e.

Finally we are going to consider cycles in graph theory. These are precisely what a physicist would call a loop in a Feynman diagram. As discussed in the previous section these loops are very relevant for us since they in general cause worse convergences of the matrix elements which are parts of the S-matrix elements or Green's functions. This is the motivation for taking this slight graph theory detour. For the sake of brevity, and since nothing discussed here is novel in the slightest I will be not be providing proofs of many of the facts stated here, they can however be found in any text on linear algebra in graph theory such as [6]. First consider the Feynman graphs as we have defined them, for this discussion we call each part of the edge partition E an edge e, similarly we consider each part of the vertex partition V a vertex. To make these Feynman graphs an actual graph we will add a vertex to the end of all external edges. With this set up we may consider Feynman graphs as graphs in the regular sense. With this discussion out of the way let G be the graph derived from the Feynman graph in way just described. Let $E(G)$ be the edge set of G and let $\mathcal{P}(E(G))$ be the power set of the edge set of G[2]. There is a natural addition operation defined on $\mathcal{P}(E(G))$ which is given by symmetric difference of the two subsets in the power sets. There is also a natural multiplication by scalars in $\mathbb{Z}_2$ defined by $0 \cdot S = \emptyset$ and $1 \cdot S = S$. It is simple to verify that these operations are compatible and thus $(\mathcal{P}(E(G)), \mathbb{Z}_2)$ is a vector space with these two operations.

Now consider $\mathcal{C}(E(G)) \subset \mathcal{P}(E(G))$ be the set of all subset of $E(G)$ which are cycles. This is clearly a subset of $\mathcal{P}(E(G))$, but additionally it is vector subspace of $\mathcal{P}(E(G))$ which is called the cycle space of a graph G. The dimension of this space corresponds to what a physicist would call the number of loops of the graph so we define the loop number L of a Feynnam graph to be the dimension of the cycle space of G. Additionally a basis for the cycle space is given by the distinct fundamental cycles with respect to any spanning

[1] If G is not connected one can consider the same set up for each connected component.

[2] The power set is the set of all subsets of a set.

tree and edge. An explicit formula in terms of the number of vertices edges and connected components c is given by,

$$L = \dim\left(\mathcal{C}(E(G))\right) = |E(G)| - |V(G)| + c \tag{2.18}$$

One can also bi-partitions of the vertex set of G. That is ways to divide the vertex set into two disjoint subsets these are called (vertex) cuts of a graph. Any cut of a graph can be represented by a subset of the edge set $\mathcal{CU}(G)$ which when each of the edges in the sense of being split as discussed before splits the graph into two connected components where each the two parts of the vertex partition lie in different connected components. It turns out that $\mathcal{CU}(G)$ is also a vector subspace of $\mathcal{P}\left(E(G)\right)$ and is called the subspace, additionally it is the orthogonal complement of $\mathcal{C}(E(G))$. With all of this in mind we can discuss the incidence co-action which will be the last bit of set up needed for this report.

2.2.2 The Co-Action of Pairs

In this section I will discuss the ideas behind the co-action of pairs and specifically how one can compute the co-action of pairs. Before continuing however, I will mention that there are many different notions of Hopf algebras and co-actions associated to these Hopf algebras. All of these are discussed in [12] in greater detail than I can possibly go into here. In my last report I discussed briefly discussed the Hopf algebra of Feynman graph which in [12] is called the core Hopf algebra, there is a similarly defined Hopf algebra for pre-Cutkosky graph, and a Hopf algebra on the pairs of graphs and spanning forests of the graphs. All of these give rise to co-actions which are developed and studied in [12].

For this section I will focus on the co-action related to the Hopf algebra of pairs, and specifically how to compute them. Given a pair consisting of a Feynman graph G and a tree in the graph $\mathcal{T}$. We define the co-action of the pair to be,

$$\rho\left((G, \mathcal{T})\right) = \sum_{S \subseteq E(\mathcal{T})} \left(G/\left(\mathcal{T} \setminus S\right), S\right) \otimes \left(G_c(S), \mathcal{T} \setminus S\right) \tag{2.19}$$

Where $G_c(S)$ is the pre-cut graph, with cut defined by the set S. The set S defines uniquely a cut since $\mathcal{T} \setminus S$ is a forest and the vertices in the same connected component of the forest define a cut in the original graph. One can define then the co-action for a graph without reference to a spanning tree by

$$\rho(G) = \sum_{\mathcal{T}\text{a spanning tree}} \rho(G, \mathcal{T}) \tag{2.20}$$

In the next part I will lay out the models which we will be working with, as well as recall the important results from the last report which will be needed to understand the results of the report.

Chapter 3

Models and Set-Up

In this chapter the goal is to give a brief recap of the main results from last term's work as well as the models which we will be working with in this report, along with the necessary properties of these models we will use going forward.

3.1 The Models

In this report we will be primarily concerned with three different physical theories. The first is ϕ^4 theory with the Lagrangian we have already seen in the previous chapter, I will state it again here for completeness sake.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{3.1}$$

In the past a massless version of QED has also been studied using similar methods [7]. The Lagrangian for QED is simply,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\not{D} - m\right)\psi \tag{3.2}$$

Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$ is the gauge covariant derivative and ψ is a spinor field. This theory is invariant under the usual gauge transformations $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha(x)$ and $\psi \to e^{i\alpha(x)}\psi$. This gauge invariance leads to the QED Ward identities.

The Wess-Zummino model first described in [18] is given by the following Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\left[\partial_\mu A\partial^\mu A + \partial_\mu B\partial^\mu B + i\bar{\psi}\partial\!\!\!/\psi - F^2 - G^2\right] + m\left(FA + GB + \frac{i}{2}\bar{\psi}\psi\right) + g\left[F\left(A^2 - B^2\right) + 2GAB - ig\bar{\psi}\left(A - \gamma^5 B\right)\psi\right] \quad (3.3)$$

This Lagrangian was designed to be invariant under the following simultaneous infinitesimal transformations called superguage transformations[19].

$$\delta A = i\bar{\alpha}\psi \quad (3.4)$$

$$\delta B = i\bar{\alpha}\gamma^5\psi \quad (3.5)$$

$$\delta\psi = \partial_\mu\left(A - \gamma^5 B\right)\gamma^\mu\alpha + n\left(A - \gamma^5 B\right)\gamma^\mu\partial_\mu\alpha + F\alpha + G\gamma^5\alpha \quad (3.6)$$

$$\delta F = i\bar{\alpha}\gamma^\mu\partial_\mu\psi + i\left(n - \frac{1}{2}\right)\partial_\mu\bar{\alpha}\gamma^\mu\psi \quad (3.7)$$

$$\delta G = i\bar{\alpha}\gamma^5\gamma^\mu\partial_\mu\psi + i\left(n - \frac{1}{2}\right)\partial_\mu\bar{\alpha}\gamma^5\gamma^\mu\psi \quad (3.8)$$

Where n is called the weight of the supergauge transformation and α is a Majorana spinor field. These supergauge transformations make this a supersymmetric field theory [19].

These are the models to which we will be applying the results of [21]. Since the renormalization group equation as used in [21] requires that the fields be massless we will generally work in the massless limit which amounts to taking $m = 0$ in equations (3.1), (3.2) and (3.3). The massless QED theory has been studied in [7], and our last report we studied the massless ϕ^4 theory, so the only novel model we will be considering here is the Wess-Zummino model. We do however wish to present general results which apply to all of these models and so laying all of them out here will be helpful for later.

3.2 Main Equations

In this section I will lay out the main equations which we will be using in the body of the report. The derivation of these equations in its totality can be found in [21]. The first equation is an equation for the anomalous dimension,

$$\gamma_1^r(x) = P_r(x) - \mathrm{sgn}\left(s_r\right)\left(\gamma_1^r\right)^2 + \left(\sum_{j\in\mathcal{R}}|s_j|\gamma_1^j(x)\right)x\frac{\partial\gamma_1^r}{\partial x} \quad (3.9)$$

Where $\mathcal{R}$ is the set of residues (vertices and edges in the theory) and s_r is either 1 or -1. We also have an equation for the beta function as,

$$\beta(x) = x \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x) \tag{3.10}$$

The coupling constant x and the physical scale L are related by the following relationship

$$\frac{dx}{dL} = \beta(x) = x \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x) \tag{3.11}$$

Using this relationship we can rewrite equation (3.9) in terms of the physical scale to be, after rearranging

$$\frac{d\gamma_1^r}{dL} = \gamma_1^r(L) + \text{sgn}(s_r)\left(\gamma_1^r(L)\right)^2 - P_r(L) \tag{3.12}$$

It is equation (3.12) which will be of most use to us when we discuss Landau poles in these theories. Another important set of equations for the main results of the report will be the formula for the combinatorial invariant charge, this formula is discussed in detail in [21] and is given by

$$Q = \left(\frac{X^v}{\prod_i (X^{e_i})^{m_i}}\right)^{\frac{1}{\text{val}(v)-2}} \tag{3.13}$$

Where X^v and X^{e_i} are formal power series, where the coefficient of the k-th monomial are the sum of all the one particle irreducible Feynman graphs with k loops and external leg structure of v or e_i respectively. When the theory has more than one type of vertex such a quotient is formed for each vertex and in the case of QED and QCD respectively the Ward and Slavnov-Taylor identities ensure that each of these quotients agree. One other point to mention since we will be applying this to the Wess-Zumino model which has multiple vertices, this approach only works for theories with a single vertex. In theories with multiple vertices we need to make sure of sets of identities which ensure these quotients agree and this will allow us to apply the equations to these models.

Chapter 4

Results

In this chapter I will present the main results of my research term. There are three main results which I will be presenting here. The first is the theorems on the existence of Landau poles in general theories and then applying it specifically to our case of ϕ^4 theory. I will then discuss the renormalization and Ward identities for the Wess-Zumino model and how these results lead to identities which we can use to analyse the Wess-Zumino model using equations (3.9), (3.10) and (3.11). Lastly we discuss a method for giving a meaningful assignment of Feynman rules to the new vertices generated by co-action of pairs. We will start by discussing Landau poles in general theories.

4.1 Landau Poles

We begin by considering the general equation for the anomalous dimension found in [21]

$$\frac{d\gamma_1^r}{dL} = \gamma_1^r(L) + \mathrm{sgn}(s_r)\left(\gamma_1^r(L)\right)^2 - P_r(L)$$

And we consider the change of variables

$$\gamma_1^r = -\frac{\mathrm{sgn}(s_r)}{u^r}\frac{du^r}{dL} \tag{4.1}$$

And then the equation will become

$$\frac{d^2u^r}{dL^2} - \frac{du^r}{dL} + \mathrm{sgn}(s_r)P_r(L) = 0 \tag{4.2}$$

And this equation will be the focus of this section. Consider the case where $P_r(L)$ is a constant that is $P_r(L) = K$ then equation 4.2 becomes,

$$\frac{d^2u^r}{dL^2} - \frac{du^r}{dL} + \text{sgn}(s_r)K = 0 \tag{4.3}$$

We can use the ansatz $u^r(L) = e^{ikL}$ to find,

$$-k^2 - ik + \text{sgn}(s_r)K = 0$$

By the quadratic formula, we find

$$ik = \frac{1}{2} - \frac{i\sqrt{4\text{sgn}(s_r)K - 1}}{2}$$

Thus we can see that the solution will be oscillatory if $4\text{sgn}(s_r)K > 1$ and the solution will be of the form,

$$u^r(L) = e^{L/2}\left\{A\sin(\kappa L) + B\cos(\kappa L)\right\} \tag{4.4}$$

Where $\kappa = \frac{\sqrt{4\text{sgn}(s_r)K-1}}{2}$ is the frequency of oscillation, so the solution is periodic with period $T = 2\pi/\kappa$. This will be an important fact which will come up later. We now wish to study this equation in the more general case where $P_r(x)$ is not necessarily constant.

To continue we will first need a lemma which will connect the behaviour of $u^r(L)$ to the behaviour of $\gamma^r(L)$.

Lemma 1. *The zeros of $u^r(L)$ correspond to Landau poles of $\gamma_1^r(L)$*

Proof. First recall that by definition $\gamma_1^r = -\frac{\text{sgn}(s_r)}{u^r}\frac{du^r}{dL}$, note first that if $u^r \to 0$ then $\gamma_1^r(L) \to \infty$.

□

Thus the problem of finding Landau poles reduces to finding zeros of the function $u^r(L)$. Since $u^r(L)$ is the solution to a second order linear differential equation we can find zeros using Sturm Liouville theory. Let's begin by analysing the general case of equation (4.2).

Theorem 1. *Suppose $4sgn(s_r) > 1$ on some interval $I = [L_0, L^*]$ and let $\rho^r = \inf_{L\in I} P_r(L)$ then if $I_1 = \left[L_0, L_0 + \frac{2\pi}{k}\right] \subseteq I$ where $k = \frac{\sqrt{4sgn(s_r)\rho^r - 1}}{2}$ then $\gamma_1^r(L)$ has a Landau pole contained in I.*

Proof. We first note that equation (4.2) is equivalent to the equation

$$\frac{d}{dL}\left(e^{-L}\frac{du^r}{dL}\right) + e^{-L}\mathrm{sgn}(s_r)P_r(L) = 0 \tag{4.5}$$

This is the "self adjoint" form of equation (4.2). Now let $\tilde{u}(L)$ be the solution to the equation

$$\frac{d}{dL}\left(e^{-L}\frac{du^r}{dL}\right) + e^{-L}\mathrm{sgn}(s_r)\rho^r = 0 \tag{4.6}$$

We know the exact solution to (4.6) is

$$\tilde{u}(L) = e^{\frac{L-L_0}{2}}\left[\tilde{u}(L_0)\cos\left(k\left(L-L_0\right)\right) + \frac{1}{k}\frac{d\tilde{u}}{dL}(L_0)\sin\left(k\left(L-L_0\right)\right)\right] \tag{4.7}$$

Now by the Sturm comparison theorem since $P_r(L) > \rho^r$ and since $\tilde{u}(L)$ has two zeros in the interval $I = [L_0, L^*]$ there must be a zero of $u^r(L)$ in I as well since zeros of $\tilde{u}(L)$ are Landau poles we conclude there is at least one Landau pole of $u^r(L)$ in I also. □

The conditions on this theorem are rather strict, however the theorem also guarantees that the Landau pole is contained in a potentially very small interval. One might naturally then ask, if we weaken some of the conditions can we still guarantee the existence of Landau poles? The answer is yes and it is presented in the following theorem.

Theorem 2. *Suppose there is some L^* such that $\rho^\infty = \inf_{L\in[L^*,\infty)} P_r(L)$ has $4sgn(s_r)\rho^\infty > 1$ then $\gamma_1^r(L)$ has a Landau pole for some $L > L^*$.*

Proof. The proof follows directly by applying the Sturm comparison theorem on the with the same two functions and on the interval $I = \left[L^*, L^* + \frac{2\pi}{k}\right]$ where again $k = \frac{\sqrt{4\mathrm{sgn}(s_r)\rho^\infty}}{2}$. □

These two theorems are completely general and apply to any theory. Since we were initially were looking at these in the context of ϕ^4 theory, I will remark here that the condition $4\mathrm{sgn}(s_r) > 1$ is equivalent to $P^+(L) > \frac{1}{4}$ and $P^-(L) < -\frac{1}{4}$ respectively, so specializing these theorems to that case is immediate.

So far with these two theorems we have guarantees about the existence of Landau poles, it is also a worthwhile question to ask when we will not have Landau poles. This question is not as easy as we don't have the power of Sturm Liouville theory in this case, but there are some things we can say about it nonetheless. Specifically we have the following.

Theorem 3. *Suppose $\gamma_1^r(L_0) > 0$ and $sgn(s_r)P_r(L) < 0$ for all $L \in I = [L_0, L^*]$ then there are no Landau poles in I.*

Proof. For $L \in I$ we have by integrating equation (4.5)

$$\frac{du^r}{dL} = -\mathrm{sgn}(s_r) \int_{L'=L_0}^{L'=L} e^{L-L'} P_r(L') u^r(L') \, dL' \tag{4.8}$$

And by definition $\gamma(L_0) = \frac{1}{u^r(L_0)} \frac{du^r}{dL}(L_0)$.

Since $\gamma_0^r(L_0) > 0$ then both $u^r(L_0)$ and it's derivative have the same sign at L_0 since $\gamma_1^r(L_0)$ and $\mathrm{sgn}(s_r)P_r(L)$ have the opposite for all $L \in I =$ then by equation (4.8) the sign of the derivative and the sign of $u^r(L)$ are the same for all $L \in I$. Thus in this case $u^r(L)$ has no zeros in I and hence no Landau poles. The other case □

It should be noted that the theorem is not true if the anomalous dimension starts out negative since in these cases $u(L_0)$ can have zeros and depends entirely on if the second derivative of $u^r(L)$ is of large enough magnitude to cause the sign of the first derivative to swap before $u^(L)$ crosses 0. These set of three theorems characterize the Landau poles of ϕ^4 theory which is what we want to do. Next we will discuss analysing the Wess-Zumino model using the equations derived in [21].

4.2 The Wess-Zumino Model

As discusses in chapter three the Wess-Zumino model is a super-symmetric field theory with 5 scalar fields and a fermion field. Generally the fields F and G are not considered physical and they are usually eliminated using their equations of motion. However when considering renormalization it is more useful to keep these fields explicitly in the Lagrangian [9] and thus we will do the same here. The renormalization of the Wess-Zumino model has been studied in detail originally in [9], and they give a direct demonstration that to all orders in perturbation theory the mass and coupling constant renormalization are not needed and there is only a common wave function renormalization needed. For our purposes this means that there will be only one equation for us to consider.

Guided by this physical intuition, we should be able to show this combinatorially as well and to do this we will use the Ward identities generated by the currents to do so. Specifically, we will use the physical ward identities to prove the combinatorial ward identities which will do the work of eliminating the other equations in favour of a single anomalous dimension and beta function. First though it is instructive to look at the reasons for doing this. A priori to apply the methods of [21] to the Wess-Zumino model one would need to consider 10 coupled ordinary differential equations. Considering all of these however is unnecessary due to the symmetries present in the Wess-Zumino model which allows us to eliminate nine of these equations leaving only one. As pointed out in [9], although there are other ways to derive these Ward identities it is most convenient to discuss Ward identities by considering the generating functional

$$Z[J] = \int \mathcal{D}A\mathcal{D}B\mathcal{D}F\mathcal{D}G\mathcal{D}\bar{\theta}e^{i\int d^4x\, L+J_AA+J_BB+J_FF+J_GG-i\bar{\theta}\psi} \tag{4.9}$$

Where L is the Lagrangian which is given in section 3. Now the currents are meant to form a scalar multiplet with respect to supergauge transformations so that equation (4.9) is invariant under supergauge transformations. In order for this to work we need the currents to transform in the following way,

$$\delta J_A = -i\left(\partial_\mu\bar{\theta}\right)\gamma^\mu\alpha \tag{4.10}$$

$$\delta J_B = i\left(\partial_\mu\bar{\theta}\right)\gamma^5\gamma^\mu\alpha \tag{4.11}$$

$$\delta J_F = i\bar{\theta}\alpha \tag{4.12}$$

$$\delta J_G = i\bar{\theta}\gamma^5\alpha \tag{4.13}$$

$$\delta\theta = \partial_\mu\left(J_F - \gamma^5 J_G\right)\gamma^\mu\alpha + \left(J_A + \gamma_5 J_B\right)\alpha \tag{4.14}$$

By using these relationships and the fact that the full generating functional is invariant under supergauge transformations, it is easy to see

$$0 = \delta Z = \int d^4x\, \frac{\delta Z}{\delta A}\delta A + \frac{\delta Z}{\delta B}\delta B + \frac{\delta Z}{\delta G}\delta G + \frac{\delta Z}{\delta F}\delta F + \frac{\delta Z}{\delta\theta}\delta\theta \tag{4.15}$$

And inserting the expressions for the current variations equations (4.10) - (4.14) one gets an equation which can be repeatedly functionally differentiated to obtain different Ward identities. We could do this here obtaining a large number of Ward identities for the two and three point functions which will help us in our analysis. Fortunately this work is again done in [9] where they derive these Ward identities. We want to use the Ward identities to prove that the following quantities are equal.

$$\frac{X^{FAA}}{X^F\left(X^A\right)^2} = \frac{X^{FBB}}{X^F\left(X^B\right)^2} = \frac{X^{A\psi\psi}}{X^A\left(X^\psi\right)^2} = \frac{X^{B\psi\psi}}{X^B\left(X^\psi\right)^2} = \frac{X^{GAB}}{X^GX^AX^B} \tag{4.16}$$

These are easily verified using the Ward identities which are derived in the paper [9]. For instance the first equality is a result of the Ward identities on the three point function which says that $X^{FAA} = X^{FBB}$ and that $X^{B} = X^{A}$. Again this is all covered in the reference [9] and is not novel so I won't go into it here. What this allows us to do however is to reduce the system into a single equation since we only need to consider the field strength renormalization, for a single field since they are all the same and the vertex functions are all finite. Keeping this in mind it is easy to see that the equation for the anomalous dimension is simply.

$$\frac{d\gamma_1}{dx} = \frac{\gamma_1(x) + (\gamma_1(x))^2 - P(x)}{3x\gamma_1} \tag{4.17}$$

Which is remarkably similar to the equation for QED studied in [7]. In particular we see immediately that this is a special case of the equation studied there with $s = 3$ and thus we have the following theorem as a consequence of theorem (2.1) in [7].

Theorem 4. *Global solutions of the beta function for the Wess-Zumino model exist if and only if,*

$$\int_{x_0}^{\infty} \frac{P(z)}{z^{\frac{5}{3}}} < \infty \tag{4.18}$$

For some $x_0 > 0$.

Proof. The proof follows directly from specializing theorem 2.1 in [7] with $s = 3$. □

In addition to this all of the interesting theorems from [7] follow as well with $s = 3$ and I won't list them all here, but needless to say this can be done.

Before leaving the Wess-Zumino model I will analyse the asymptotic behaviour for the anomalous dimension and beta function with the specific choice of $P(x) = x$. This is an interesting case because it is the only case as far as we could find aside from the case of constant $P_r(x)$ where the anomalous dimension has a closed form solution. The closed form solution in this case has the form,

$$\gamma_1(L) = -\frac{L}{2} - \frac{\mathrm{Ai}'\left(L + \frac{1}{4}\right) + k\mathrm{Bi}'\left(L + \frac{1}{4}\right)}{\mathrm{Ai}\left(L + \frac{1}{4}\right) + k\mathrm{Bi}\left(L + \frac{1}{4}\right)} \tag{4.19}$$

Where $k = -\frac{\gamma_0 \mathrm{Ai}(\frac{1}{4}) + \mathrm{Ai}'(\frac{1}{4})}{\gamma_0 \mathrm{Bi}(\frac{1}{4}) + \mathrm{Bi}'(\frac{1}{4})}$ is a constant of integration. Although finding the beta function can't be done exactly, in closed form we can find it by considering well known asymptotic forms of the Airy and Bi-Airy function. Thus for large L we have

$$\gamma_1(L) \approx -\frac{1}{2}\left[L + \frac{1+\frac{21}{20}\frac{\Gamma(11/6)\Gamma(7/6)}{\Gamma(5/6)\Gamma(1/6)}\left(L+\frac{1}{4}\right)^{-3/2}}{1-\frac{3}{4}\frac{\Gamma(11/6)\Gamma(7/6)}{\Gamma(5/6)\Gamma(1/6)}\left(L+\frac{1}{4}\right)^{-3/2}}\right] \tag{4.20}$$

Which can be integrated to find the beta function easily. We are then interested with the growth of the coupling as a function of physical scale which again can be approximated asymptotically as the following.

$$\left(\frac{x_0}{x}\right)^{2/3} \approx e^{\frac{L^2}{2}+\frac{7L}{3}}\left[3\frac{\Gamma(11/6)\Gamma(7/6)}{\Gamma(5/6)\Gamma(1/6)}L + \frac{3}{4}\frac{\Gamma(11/6)\Gamma(7/6)}{\Gamma(5/6)\Gamma(1/6)} - 4\right]^{\frac{16}{30}\frac{\Gamma(5/6)\Gamma(1/6}{\Gamma(11/6)\Gamma(7/6)}} \tag{4.21}$$

Showing that the coupling eventually increases exponentially with the energy scale.

Finally we are interested in comparing our numerical results with the expressions derived in [1] where the anomalous dimension was computed to 200 loops using similar Hopf algebraic techniques. Using the recurrence relation

$$(k+1)\gamma_{k+1} = \sum_{n\geq 0}(\gamma_1 + \beta x\partial_x)\gamma_k \tag{4.22}$$

And using equation (4.17) to compute the anomalous dimension and beta function for the Wess-Zumino model with $P(x) = x$ and compare it to the anomalous dimension also computed in [1]. The results are shown in the figure below. The good agreement for a large range of different coupling values has given some validation for our choice of $P(x)$.

This is all we have considered for the Wess-Zumino model in the next section we will study the coaction of pairs and how we can assign in a reasonable way Feynman rules to the new vertices which are generated from it.

4.3 The Coaction of Pairs

The goal of this section is to develop a reasonable assignment of the Feynman rules for the new vertices which are created through contraction of edges in the co-action of pairs. We plan to study this more in the future but we will present our preliminary findings here for the sake of completeness. We already have quite a good understanding of what physically cutting the edges should mean it effectively just puts the particle on-shell and this is easy

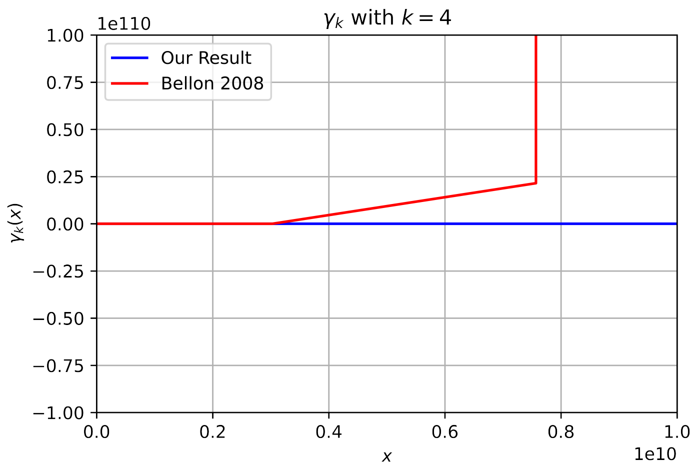


Figure 4.1: A comparison of the anomalous dimension computed both using our method along with the recursion and the exact solution from [1]

enough to give an assignment of a Feynman rule to. What is not so clear is what the physical meaning of contracting an edge should be.

Intuitively one can understand this as taking the limit of the virtual particles mass to be infinite so that it is larger than any momentum scale in the problem. This intuition however can break down for virtual particles since there momentum is not bounded. There is a better way to understand this however since contraction preserves momentum conservation but moreover it means there is no momentum transfer between the two vertices on either side of the edge which is being shrunk. This does give us a relatively meaningful way to assign values to the new vertices. Simply assign the value of the vertex to be the amplitude of the Feynman graph which produced it with delta functions enforcing zero momentum transfer for each contracted propagator. For instance if in QED we have the four point fermion vertex it would be associated to the fermion-fermion scattering diagram with the photon enforced to have zero momentum transfer. Another interesting thing to note is that (if we define things this way) for photons there is no difference between contracting them and cutting them.

If it is possible to do this in an unambiguous way then we will have a meaningful assignment of the Feynman rules to each new vertex. I will now show this is indeed possible to do in QED. We will assume each of the contracted vertices come from a tree. Thus letting m_f and m_p be the number of fermion and photon internal lines respectively and let

m_v be the number of QED vertices. Then we know that $m_f + m_p = m_v - 1$ and by double counting each of the different lines we find that $2m_v = 2m_f + N_f$ and $m_v = 2m_p + N_p$ where N_p and N_f are the number of external fermion and photon lines respectively. Thus for trees we must have

$$m_f = \frac{1}{2}N_f + N_p - 2 \tag{4.23}$$

and

$$m_p = \frac{1}{2}N_f - 1 \tag{4.24}$$

Thus for a vertex with s photon lines and r fermion lines it should be associated with the amplitude of the Feynman graph with all $\frac{r-2}{2}$ of the internal photon lines cut and with a delta function and the gamma matrix associated to the $(r+2s-4)/2$ fermion lines having zero momentum transfer.

4.4 Conclusion

During this term we have investigated several different aspects of several different models using the equations derived in [21]. We have found three novel things relating to finding Landau poles in general theories, the analysis of the Wess-Zumino model according to the methods of [21] and how to give meaningful Feynman amplitudes to the vertices which come from the co-action of pairs. We are interested in finding more out about the last subject specifically if the Ward identities are preserved and the other ways in which the co-action relates to the quantum field theory from which it is derived.

References

[1] Marc P. Bellon and Fidel A. Schaposnik. “Renormalization group functions for the Wess–Zumino model: Up to 200 loops through Hopf algebras”. In: *Nuclear Physics B* 800.3 (Sept. 2008), pp. 517–526. DOI: 10.1016/j.nuclphysb.2008.02.005. URL: https://doi.org/10.1016%2Fj.nuclphysb.2008.02.005.

[2] Christoph Bergbauer and Dirk Kreimer. *Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology*. 2006. arXiv: hep-th/0506190 [hep-th].

[3] D.J. Broadhurst and D. Kreimer. “Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality”. In: *Nuclear Physics B* 600.2 (Apr. 2001), pp. 403–422. ISSN: 0550-3213. DOI: 10.1016/s0550-3213(01)00071-2. URL: http://dx.doi.org/10.1016/S0550-3213(01)00071-2.

[4] Alain Connes and Dirk Kreimer. “Renormalization in quantum field theory and the Riemann-Hilbert problem”. In: *Journal of High Energy Physics* 1999.09 (Sept. 1999), pp. 024–024. ISSN: 1029-8479. DOI: 10.1088/1126-6708/1999/09/024. URL: http://dx.doi.org/10.1088/1126-6708/1999/09/024.

[5] Kurusch Ebrahimi-Fard and Dirk Kreimer. “The Hopf algebra approach to Feynman diagram calculations”. In: *Journal of Physics A: Mathematical and General* 38.50 (Nov. 2005), R385–R407. ISSN: 1361-6447. DOI: 10.1088/0305-4470/38/50/r01. URL: http://dx.doi.org/10.1088/0305-4470/38/50/R01.

[6] Jonathan L. Gross and Jay Yellen. *Graph Theory and Its Applications, Second Edition (Discrete Mathematics and Its Applications)*. 2005. ISBN: 158488505X.

[7] Guillaume van Baalen, Dirk Kreimer, David Uminsky and Karen Yeats. “The QED beta function from global solutions to Dyson–Schwinger equations”. In: *Annals of Physics* 324.1 (Jan. 2009), pp. 205–219. ISSN: 0003-4916. DOI: 10.1016/j.aop.2008.05.007. URL: http://dx.doi.org/10.1016/j.aop.2008.05.007.

[8] Klaus Hepp. “Proof of the Bogoliubov-Parasiuk theorem on renormalization”. In: *Communications in Mathematical Physics* 2.4 (1966), pp. 301–326. DOI: cmp/1103815087. URL: https://doi.org/.

[9] J. Iliopoulos and B. Zumino. “Broken supergauge symmetry and renormalization”. In: *Nuclear Physics B* 76.2 (1974), pp. 310–332. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(74)90388-5. URL: https://www.sciencedirect.com/science/article/pii/0550321374903885.

[10] K. Johnson, M. Baker, and R. Willey. “Self-Energy of the Electron”. In: *Phys. Rev.* 136 (4B Nov. 1964), B1111–B1119. DOI: 10.1103/PhysRev.136.B1111. URL: https://link.aps.org/doi/10.1103/PhysRev.136.B1111.

[11] D. Kreimer. “Anatomy of a gauge theory”. In: *Annals of Physics* 321.12 (Dec. 2006), pp. 2757–2781. ISSN: 0003-4916. DOI: 10.1016/j.aop.2006.01.004. URL: http://dx.doi.org/10.1016/j.aop.2006.01.004.

[12] Dirk Kreimer and Karen Yeats. *Algebraic Interplay between Renormalization and Monodromy*. 2021. DOI: 10.48550/ARXIV.2105.05948. URL: https://arxiv.org/abs/2105.05948.

[13] Dirk Kreimer and Karen Yeats. “An Étude in non-linear Dyson–Schwinger Equations”. In: *Nuclear Physics B - Proceedings Supplements* 160 (Oct. 2006), pp. 116–121. ISSN: 0920-5632. DOI: 10.1016/j.nuclphysbps.2006.09.036. URL: http://dx.doi.org/10.1016/j.nuclphysbps.2006.09.036.

[14] Franz Mandl and Graham Shaw. *QUANTUM FIELD THEORY*. 1985.

[15] Michael Edward Peskin and Daniel V. Schroeder. *An Introduction to Quantum Feild Theory*. Westview Press, 1995.

[16] Julian Schwinger. “On Quantum-Electrodynamics and the Magnetic Moment of the Electron”. In: *Phys. Rev.* 73 (4 Feb. 1948), pp. 416–417. DOI: 10.1103/PhysRev.73.416. URL: https://link.aps.org/doi/10.1103/PhysRev.73.416.

[17] Steven Weinberg. *Quantum Theory of Feilds*. Cambridge University Press, 1995.

[18] J. Wess and B. Zumino. “A lagrangian model invariant under supergauge transformations”. In: *Physics Letters B* 49.1 (1974), pp. 52–54. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(74)90578-4. URL: https://www.sciencedirect.com/science/article/pii/0370269374905784.

[19] Julius Wess and Bruno Zumino. “Supergauge Transformations in Four-Dimensions”. In: *Nuclear Physics* 70 (1974), pp. 39–50.

[20] Karen Yeats. *Growth estimates for Dyson-Schwinger equations*. 2008. arXiv: 0810.2249 [math-ph].

[21] Karen Yeats. *Rearranging Dyson-Schwinger Equations*. Vol. 211. Memoirs of the American Mathematical Society, 2011.