Rooted Graph Minors and Reducibility of Graph Polynomials

by

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Abstract

In 2009, Brown gave a set of conditions which when satisfied imply that a Feynman integral evaluates to a multiple zeta value. One of these conditions is called reducibility, which loosely says there is an order of integration for the Feynman integral for which Brown's techniques will succeed. Reducibility can be abstracted away from the Feynman integral to just being a condition on two polynomials, the first and second Symanzik polynomials. The first Symanzik polynomial is defined from the spanning trees of a graph, and the second Symanzik polynomial is defined from both spanning forests of a graph and some edge and vertex weights, called external momenta and masses. Thus reducibility is a property of graphs augmented with certain weights. We prove that for a fixed number of external momenta and no masses, reducibility is graph minor closed, correcting the previously claimed proofs of this fact. A computational study of reducibility was undertaken by Bogner and Lüders who found that for graphs with 4-on-shell momenta and no masses, K_4 with momenta on each vertex is a forbidden minor. We add to this and find that when we restrict to graphs with four on-shell external momenta the following graphs are forbidden minors: K_4 with momenta on each vertex, W_4 with external momenta on the rim vertices, $K_{2,4}$ with external momenta on the large side of the bipartition, and one other graph. We do not expect that these minors characterize reducibility, so instead we give structural characterizations of the graphs not containing subsets of these minors. We characterize graphs not containing a rooted K_4 or rooted W_4 minor, graphs not containing rooted K_4 or rooted W_4 or rooted $K_{2,4}$ minors, and also a characterization of graphs not containing all of the known forbidden minors. Some comments are made on graphs not containing $K_{3,4}$, K_6 or a graph related to Wagner's graph as a minor.

Keywords: Graph Minors; Rooted Minors; Reducibility; Feynman integrals; Symanzik polynomials

Acknowledgements

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Introduction

1.1 Motivation

This thesis looks at when graphs have a property called reducibility. Reducibility is closely related to when Feynman integrals evaluate to multiple zeta values. Feynman integrals arise naturally in perturbative quantum field theory and have been the focus of a large amount of research.

One advance in calculating Feynman integrals was obtained by Brown in [12], where he gave sufficient conditions for when a Feynman integral evaluates to a multiple zeta value. He also showed that these conditions hold for an infinite family of Feynman integrals, leading to one of the largest known families of graphs for which it is known how to calculate the Feynman integral. One of Brown's conditions is reducibility, which essentially tells you a suitable order of integration for Brown's techniques. However, reducibility can be abstracted away from Feynman integrals into simply being a property of polynomials. As the polynomials relevant for Feynman integrals come from graphs, reducibility can be viewed as a property of graphs. Furthermore, it can be shown that reducibility is graph minor closed, and thus one can use graph minor theory to try and understand reducibility. This thesis looks at reducibility from a graph minor point of view. The outline of this thesis is as follows. First, in chapter one, we give an overview of the Symanzik polynomials. In chapter 2, we give a proof that reduciblity is graph minor closed (Corollary 46), correcting the proofs given in [26] and [7]. We also give a new forbidden minor, which we denote L, for reducibility for graphs with four on-shell external momenta, and rewrite the non-reducible graphs in [26] in terms of forbidden minors. In chapter 3, we give two different excluded minor theorems for rooted K_4 and rooted W_4 minors (Theorem 82 and Theorem 74). We also give the complete class of graphs not containing any of a rooted K_4, rooted W_4, or rooted $K_{2,4}$-minor (Theorem 97). Additionally, we give the complete class of graphs not containing any of a rooted K_4, rooted W_4, rooted $K_{2,4}$ or rooted L-minors (although here we suspect a

nicer characterization should be able to be found). We also make some observations about $K_{3,4}$-minors, K_6-minors and minors of a graph related to Wagner's graph.

1.2 Graph Theory Basics

Graphs will be the fundamental objects throughout this thesis. This section will give basic definitions of graph theory terms which will be used throughout. A good textbook for basic graph theory is [10].

A graph G is a pair $(V(G), E(G))$ where $E(G)$ is a 2-element (multi)set of $V(G)$. The elements of $V(G)$ are called the *vertices* of the G, and the elements of $E(G)$ are the edges of G. Two vertices u and v in a graph are said to be adjacent if $\{u, v\} \in E(G)$, which we abbreviate to $uv \in E(G)$. If $e = uv \in E(G)$, we say that u and v are *neighbours* and that e has *endpoints* u and v. The set of all neighbors of a vertex v is called the *neighborhood* of v, denoted $N(v)$. Given a set of vertices X, we similarly define $N(X)$ to be the set of neighbours for the vertices in X, not including vertices in X. Given a graph $G = (V(G), E(G))$, and sets $V' \subseteq V(G)$ and $E' \subseteq E(G)$ such that $e = uv \in E'$ implies $u, v \in V'$, then the graph $G' = (V', E')$ is a *subgraph* of G. If we have a graph $G = (V(G), E(G))$ and $X \subseteq V(G)$, we let $G[X]$ be the graph where $V(G[X]) = X$ and $E(G[X]) = \{uv|\ u \in X \text{ and } v \in X\}$. We say that $G[X]$ is the *graph induced by* X. If $e_1 = uv \in E(G)$ and $e_2 = uv \in E(G)$, but $e_1 \neq e_2$ then we say that e_1 and e_2 are *parallel edges*. If for some edge $e = uv \in E(G)$, we have $u = v$, then we say e is a *loop edge*. A graph is *simple* if there are no parallel or loop edges.

Given a graph G, a vertex $v \in V(G)$ is a *dominating vertex* if the neighbourhood of v is $V(G) \setminus \{v\}$. Some authors refer to this as an *apex vertex*.

Given a graph G, its *line graph* $L(G)$ is a graph such that for each edge e in G, we create a vertex v in $L(G)$ and given two vertices $x, y \in V(L(G))$, we have $xy \in E(L(G))$ if and only if the corresponding edges in G share an endpoint.

Given an edge $e = xy \in E(G)$, the graph obtained by *contracting* e, denoted G/e, is the graph where we delete the vertices x and y, and replace them with a new vertex z where for each vertex u adjacent to x, we have an edge ux, and for each edge adjacent to y we have an edge uy. Given a graph G, to *delete an edge*, for some edge $e \in E(G)$, is to create a graph $G \setminus e$ where $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$. It is an easy exercise to show that deletion and contraction commute.

A graph G has a H-*minor* for some graph H, if a graph isomorphic to H can be obtained from G through the contraction and deletion of some edges, and possibly the removal of some isolated vertices. It is easy to see that if G is connected, then we never need to delete isolated vertices. If for some H, a graph G does not contain an H-minor, then we say that G is H-*minor-free*. If J is a minor of a graph G, then J is *proper* if J is not isomorphic to G. A set of graphs, $\mathcal{G}$, is *graph minor closed* if for every graph $G \in \mathcal{G}$, all proper minors of

G are in $\mathcal{G}$. Let $\mathcal{G}$ be a set of graphs which is graph minor closed. A *forbidden minor* of $\mathcal{G}$ is a graph H such that $H \notin \mathcal{G}$ but all proper minors of H are in $\mathcal{G}$.

Given a graph G and an edge $e = xy \in E(G)$, to *subdivide an edge e* is to create a new graph G' where $V(G') = V(G) \cup \{z\}$ and $E(G') = E(G) \setminus \{xy\} \cup \{zx, zy\}$. A graph G has a graph H as a *topological minor* if there is a subgraph of G which is isomorphic to a iterated subdivision of H in G.

Given a graph G, a *k-separation* of G is a pair (A, B) such that $A \subseteq V(G)$, $B \subseteq V(G)$, $A \cup B = V(G)$, $|A \cap B| \leq k$, and if $v \in B \setminus A$, and $u \in A \setminus B$, then $uv \notin E(G)$. The vertices in $A \cap B$ are called the *vertex boundary* of the separation. We say a k-separation is *proper* if $A \setminus (A \cap B) \neq \emptyset$ and $B \setminus (A \cap B) \neq \emptyset$. A proper k-separation is *tight* if for all subsets $X \subsetneq A \cap B$, the set X is not the vertex boundary of a separation. In a proper 1-separation (A, B), the vertex in $A \cap B$ is called a *cut-vertex*. In general, a set of vertices, $\{v_1, \ldots, v_k\}$, is a *k-vertex-cut* if there is a proper k-separation (A, B) such that $A \cap B = \{v_1, \ldots, v_k\}$.

We will run into a few common graph families throughout the thesis. Let $n \in \mathbb{N}$. The *complete graph on n vertices*, denoted K_n, is the graph where we can label the vertices $\{v_0, \ldots, v_{n-1}\}$ such that $E(G) = \{v_i v_j | \forall\ i, j \in \{0, \ldots, n-1\}, i \neq j\}$. If a graph G has a complete graph on k vertices as a subgraph, the subgraph will be called a *k-clique*.

Let G_1 and G_2 be graphs a k-cliques as subgraphs. A *k-clique-sum* or just *k-sum* of G_1 and G_2 is a bijective identification of pairs of vertices in the two k-cliques with, if desired, removal of some edges from the new k-clique. We note sometimes authors enforce that all edges are removed in a k-sum. In practice, under the assumption we can have parallel edges, this is equivalent to the above definition, as one simply adds parallel edges as desired.

A graph $G = (V, E)$ is *bipartite* if there is a bipartition (A, B) of V such that if $e = uv \in E$, then exactly one of u and v is in A and exactly one of u or v is in B. Let $n, m \in \mathbb{N}$. A *complete bipartite graph* with partition sizes n and m, denoted $K_{n,m}$, is a bipartite graph with a bipartition (A, B) such that $|A| = n$, $|B| = m$ and every vertex in A is adjacent to every vertex in B.

A graph P is a *path on n vertices* if we can label the vertices of P, $v_0, \ldots, v_{n-1}$ such that $E(P) = \{v_i v_{i+1} | \forall i \in \{0, \ldots, n-2\}\}$. Given a path P, the vertices v_0 and v_{n-1} are the *endpoints* of P. An *(a, b)-path* is a path with endpoints a and b. Two paths are *disjoint* if their vertex sets are disjoint. Two (a, b)-paths are *internally disjoint* if the intersection of their vertex sets is $\{a, b\}$. Given two sets of vertices X and Y, an *(X, Y)-path* is a path with one endpoint in X and one endpoint in Y. A graph is *connected* if for every pair of vertices, x, y, it contains an (x, y)-path. A graph G is *k-connected* if $|V(G)| \geq k + 1$ and for every pair of vertices x, y, there are k internally disjoint (x, y)-paths. Menger's Theorem is a very important and well known result on k-connectivity ([32]).

Menger's Theorem. *Let G be a graph and x, y be non-adjacent vertices. Then the maximum number of pairwise internally disjoint (x, y)-paths is equal to the minimum order of a separation (A, B) where $x \in A$ and $y \in B$.*
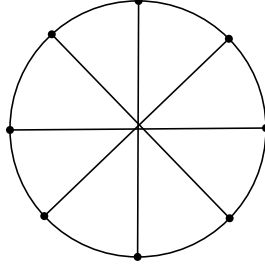
Figure 1.1: The Wagner graph, V_8.

A useful immediate implication of Menger's Theorem is:

Corollary 1. *Suppose G is k-connected. Let $X, Y \subseteq V(G)$ and $|X|, |Y| \geq k$. Then there is a family of k disjoint (X, Y)-paths.*

A *cycle on n vertices*, denoted C_n, is a graph where $V(C_n) = \{v_0, \ldots, v_{n-1}\}$ and $E(C_n) = \{v_i v_{i+1}\} \cup \{v_0 v_{n-1}\}$ for $i \in \{0, \ldots, n-2\}$. A *tree* is a connected graph with no cycles. Given a graph G, a *spanning tree of G*, T, is a subgraph of G where T is a tree and $V(T) = V(G)$.

A *wheel on n spokes*, denoted W_n, is a graph where $V(W_n) = \{v_0, \ldots, v_n\}$ such that $\{v_0, \ldots, v_{n-1}\}$ induces a cycle on n vertices, and v_n is adjacent to every other vertex. The vertices $\{v_0, \ldots, v_{n-1}\}$ are called *rim vertices* and v_n is called the *hub*.

The *Wagner graph*, denoted V_8, is defined to have vertex set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ and edge set $\{01, 12, 23, 34, 45, 56, 67, 70, 04, 15, 26, 37\}$.

A graph G is *planar* if G can be embedded in the Euclidean plane, where embedded means the graph can be drawn in the Euclidean plane without edges crossing. For 2-connected graphs which are embedded in the plane, a *face* is a region bounded by a cycle C such that all vertices and edges not in C are drawn outside of C. A cycle which bounds a face is a *facial cycle*. A *facial path* between two vertices on a facial cycle C is a path P between the two vertices such that $V(P) \subseteq V(C)$. We will use the following well known result.

Jordan Curve Theorem. *Any continuous simple closed curve in the plane separates the plane into two disjoint regions, the interior and the exterior.*

Any other undefined graph theory terminology can be found in Bondy and Murty's graph theory textbook ([10]).

1.3 The Symanzik Polynomials

This section will give an overview of the *Symanzik polynomials*, which are graph polynomials relevant to Feynman integrals.

To define the Symanzik polynomials, we first need to augment our graph with additional labels. Let G be a graph. First, for each edge $e \in E(G)$ we create a variable α_e, called a *Schwinger parameter*. Additionally, for each edge $e \in E(G)$, we associate with e an edge weight, $m_e \in \mathbb{R}$, called the *mass* of edge e. We note that some authors' (see [7]) allow for complex masses to allow for particles such as tachyons, however we restrict to real masses. An edge e is *massive* if $m_e \neq 0$. Also, for each vertex $v \in V(G)$, we will associate a vector $\rho_v \in \mathbb{R}^4$, called the *external momentum at v*. We will enforce $\rho_v = 0$ for most vertices throughout this thesis. For the vertices where we allow $\rho_v \neq 0$, we will say there is an *external edge* at v. In figures, we will draw external edges as a edge with only one endpoint (see Figure 1.2).

We will enforce that the external momenta of our graph will satisfy conservation of momentum. Namely, for a given graph G,

$$\sum_{v \in V(G)} \rho_v = 0.$$

Definition 2. *Let G be a graph with associated external momenta, masses, and Schwinger parameters. Let $\mathcal{T}$ be the set of spanning trees of G. Then the* first Symanzik polynomial *of G is*

$$\psi_G = \sum_{T \in \mathcal{T}} \prod_{e \notin E(T)} \alpha_e.$$

In the special case where G is disconnected, then $\psi_G = 0$. Although we do not do so here, it is common to derive the above definition from the Laplacian matrix by using the Matrix-tree Theorem ([8]). We note that some authors refer to the first Symanzik polynomial as the *Kirchoff polynomial* ([3]), or occasionally as the *dual Kirchoff polynomial* ([8]).

Example 3. *Consider any cycle, C_n. Then $\psi_{C_n} = \sum_{e \in E(C_n)} \alpha_e$.*

To see this, notice that deleting any edge of C_n results in a path and every spanning tree of C_n is such a path.

Before defining the second Symanzik polynomial we need some definitions. Given a graph G, a *spanning 2-forest* of G is a pair (T_1, T_2) where T_1 and T_2 are trees such that $V(T_1) \cup V(T_2) = V(G)$, and $V(T_1) \cap V(T_2) = \emptyset$.

Let G be a graph and H a subgraph of G. We will say the momentum flowing into H is

$$\sum_{v \in V(H)} \rho_v.$$

We will denote the momentum flowing into H as ρ^H. Now we can define the second Symanzik polynomial.
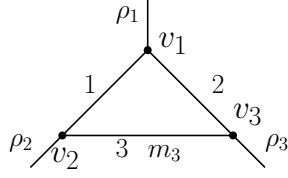
Figure 1.2: A small graph. Edges are labelled 1, 2, 3. Edge 3 has mass m_3 and each vertex has external momenta incoming at the vertex.

Definition 4. *Let G be a graph with associated external momenta, masses, and Schwinger parameters. Let $\mathcal{T}$ be the set of spanning 2-forests of G. Then the* second Symanzik polynomial *is*

$$\phi_G = \sum_{(T_1,T_2)\in\mathcal{T}} (\rho^{T_1})^2 \prod_{e\notin T_1\cup T_2} \alpha_e + \psi_G \sum_{i=1}^{|E(G)|} \alpha_i m_i^2.$$

Here we note that the two forests are unordered, as in $(T_1, T_2) = (T_2, T_1)$. Also, $(\rho^{T_1})^2$ is taken to mean take a norm squared, under an appropriate norm (we define the Minkowski metric later). In the literature, it is common to take just the terms involving the momenta to be the second Symanzik polynomial, and include the masses as a separate polynomial (see [13], [8]). Additionally, as with the first Symanzik polynomial, it is possible to derive the second Symanzik polynomial from a matrix. See [8] for an exposition. As a special case, we note that $\phi_G = 0$ when G has more than 2 connected components.

As an example, we calculate the second Symanzik polynomial of the graph in Figure 1.2. Notice every spanning 2-forest of K_3 is an edge combined with an isolated vertex. By example 3, $\psi_{K_3} = \alpha_1 + \alpha_2 + \alpha_3$. Therefore,

$$\phi_{K_3} = (\rho_1 + \rho_2)^2 \alpha_2 \alpha_3 + (\rho_2 + \rho_3)^2 \alpha_1 \alpha_2 + (\rho_3 + \rho_1)^2 \alpha_3 \alpha_1 + \psi_{K_3}(\alpha_3 m_3^2).$$

Additionally, by conservation of momenta, we have $\rho_1 + \rho_2 + \rho_3 = 0$, so one can rewrite the above equation as,

$$\phi_{K_3} = \rho_3^2 \alpha_2 \alpha_3 + \rho_1^2 \alpha_1 \alpha_2 + \rho_2^2 \alpha_3 \alpha_1 + \psi_{K_3} \alpha_3 m_3^2.$$

With these, we define the Feynman integral of a graph G in parametric space to be:

$$\frac{\Gamma(\nu - LD/2)}{\prod_{j=1}^m \Gamma(\nu_j)} \int_{\alpha_j\geq 0} \delta(1 - \sum_{i=1}^m \alpha_i)(\prod_{j=1}^m (d\alpha_j)\alpha_j^{\nu_j-1})\frac{\psi^{\nu-(L+1)D/2}}{\phi^{\nu-LD/2}}.$$

Here, m is the number of edges in G, D is the dimension of the physical theory, Γ is the the gamma function, δ is the Dirac delta function, L is the rank of the cycle space of G (where the rank of the cycle space of G is $|E(G)| - |V(G)| + 1$), and α_i is the Schwinger

parameter for edge i. The variable ν_j is a parameter which can be used to regularize the integral (converting a divergent integral into an expansion with poles in the regularization parameters) or it can be used for keeping track of how subgraph insertions into the original graph affect the integral. It is noted in [26] that for most applications $\nu_j = 1$, for all $j \in \{1, \ldots, m\}$, however in general that might not be the case. Lastly, $\nu = \sum_{j=1}^{m} \nu_j$. Other techniques have been created to deal with divergent Feynman integrals, such as renormalization, however this thesis will not look at these tools (see, for example, [20]).

We will now go through some basic properties of the Symanzik polynmials. An easy observation is that ψ_G is a homogeneous polynomial linear in all variables, and if there are no massive edges, then ϕ_G is a homogeneous polynomial linear in all variables ([8]). A second observation which is well known (see [7], [8], for example) is that the Symanzik polynomials behave nicely under deletion and contraction.

Lemma 5. *Let G be a graph. Let $e \in E(G)$ such that e is not massive and e is not a loop. Then the following identities hold:*

$$\psi_G = \psi_{G \backslash e} \alpha_e + \psi_{G/e},$$

$$\phi_G = \phi_{G \backslash e} \alpha_e + \phi_{G/e}.$$

Lemma 6. *Let G be a graph. Let $e \in E(G)$ such that e is not massive and e is not a loop. Then the following identities hold:*

$$\frac{\partial}{\partial \alpha_e} \phi_G = \phi_{G \backslash e}, \ \ \phi_G|_{\alpha_e = 0} = \phi_{G/e},$$

$$\frac{\partial}{\partial \alpha_e} \psi_G = \psi_{G \backslash e}, \ \ \psi_G|_{\alpha_e = 0} = \psi_{G/e}.$$

Lemma 7. *Let G be a graph and consider any edge e which is a loop. Then $\psi_G = \alpha_e \psi_{G \backslash e}$, and $\phi_G = \phi_{G \backslash e} \alpha_e$.*

Of course, the fact that this thesis deals with graph minors and the Symanzik polynomials exhibit nice deletion contradiction identities is no coincidence. Throughout this thesis, we will impose a natural condition on the external momenta which will simplify the second Symanzik polynomial. Before defining the condition, we will abuse notation slightly and introduce the Minkowski metric. Let $\rho \in \mathbb{R}^4$. Then under the Minkowski metric we define $\rho^2 = \rho_1^2 + \rho_2^2 + \rho_3^2 - \rho_4^2$. Here physically we have ρ_1, ρ_2 and ρ_3 corresponding to the three spacial dimensions, and ρ_4 corresponding to time. This does not actually define a metric since there are non-zero vectors such that $\rho_1^2 + \rho_2^2 + \rho_3^2 - \rho_4^2 = 0$, but since quantum field theory is formulated in Minkowski space, it is used by physicists and thus we abuse notation and call it a metric. Sometimes these types of functions are called pseudometrics. Now we introduce a natural physics condition which we will impose on our momenta.

Definition 8. *Let G be a graph let $v \in V(G)$. The momentum ρ_v is* on-shell *if $\rho_v^2 = m^2$, where m is the mass of the external edge corresponding to ρ_v. For the purposes of this thesis, we will always assume $m^2 = 0$, so a momentum vector ρ_v is on-shell if $\rho_v^2 = 0$.*

Observe that under the Euclidean norm, if a momenta ρ is on-shell, then $\rho = \vec{0}$. Thus if for some graph G, all of the momenta were on-shell, the terms involving the external momenta in the second Symanzik polynomial would be zero. However, under the Minkowski metric we can have on-shell momenta and still have a non-trivial dependence on external momenta in the second Symanzik polynomial. Notice under the Minkowski metric, we have $\rho_1^2 + \rho_2^2 + \rho_3^2 = \rho_4^2$, which is a cone, so being on-shell means that the momentum vector is lying on the shell of the cone. From a physics perspective, being on-shell means that the momenta is satisfying Einstein's energy and momentum relationship, and in our restricted notion of on-shell, this is saying that the external momenta are behaving in a light-like fashion. Therefore it is reasonable to study graphs whose momenta are on-shell.

As we are interested in studying reducibility, which we define in chapter two, we will mainly be interested in graphs where there are exactly four vertices with external edges and all momenta are on-shell or graphs with no external momenta. To justify this seemingly arbitrary choice of external momenta, we can show that the cases with 1, 2 or 3 on-shell external momenta reduce to the case with zero external momenta. If a graph G had one external momentum, ρ, then by conservation of momenta ρ is the zero vector, so we actually have no dependence on the external momenta in the second Symanzik polynomial. If G has two external momenta and they are both on-shell, then again by conservation of momenta and the on-shell condition, there is no dependence on external momenta in the second Symanzik polynomial. With two external momenta, even if they are not on-shell, in [11] it was shown that for a wide class of graphs, we can reduce the problem to a different graph with no external momenta. In essence what is found in [11] is a relationship between the second Symanzik polynomial of graphs with two external momenta, and the first Symanzik polynomial of a specifically constructed graph. That result encourages looking at the case where we have zero external momenta instead of two.

Now suppose we have three on-shell external momenta for some graph G. If for some spanning 2-forest, all of the momenta are in one of the trees, then conservation of momenta implies that the 2-forest contributes nothing to the second Symanzik polynomial. If we have exactly one momentum in one of the trees, then the on-shell condition implies that two forest contributes nothing to the second Symanzik polynomial. Therefore, at least for graphs with no massive edges, having three on-shell momenta is the same as having zero external momenta. Therefore, graphs with four on-shell external momenta are the smallest case where there is a non-trivial dependence on the external momenta. However, we can still simplify the second Symanzik polynomial in this case, though not to the point of triviality.

Lemma 9. *Let G be a graph and let $X = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ be the set of vertices with external momenta in G such that for all $i \in \{1, 2, 3, 4\}$, ρ_{x_i} is on-shell. Let (T_1, T_2) be a spanning 2-forest of G. If $|X \cap V(T_1)| \neq 2$, then*

$$(\rho_1^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e = 0.$$

Proof. We will analyze the four cases individually.

Case 1: Suppose $|X \cap V(T_1)| = 0$. Then $\rho_1^{T_1} = \sum_{v \in V(T_1)} \rho_v = \sum_{v \in V(T_1)} 0 = 0$ and thus $(\rho_1^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e = 0$.

Case 2: Suppose $|X \cap V(T_1)| = 4$. As $V(T_1) \cup V(T_2) = V(G)$ and $V(T_1) \cap V(T_2) = \emptyset$ by conservation of momentum we have that $\rho^{T_1} = -\rho^{T_2}$. By the argument from case 1, we have that $-\rho^{T_2} = 0$ and thus $(\rho_1^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e = 0$.

Case 3: Suppose that $|X \cap V(T_1)| = 1$. Without loss of generality, suppose $X \cap V(T_1) = \{x_1\}$. Then $(\rho^{T_1})^2 = \rho_{x_1}^2 = 0$ by the on-shell condition. Thus $(\rho_1^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e = 0$.

Case 4: Suppose that $|X \cap V(T_1)| = 3$. Then $|X \cap V(T_2)| = 1$. Thus by the argument in case 3, we have that $(\rho^{T_2})^2 = 0$, and thus by conservation of momentum we have $(\rho^{T_1})^2 = 0$. Thus $(\rho_1^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e = 0$. $\qquad \square$

Corollary 10. *Let G be a graph and $X = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ such that for $i \in \{1, 2, 3, 4\}$, ρ_{x_i} is on-shell. Suppose that for all vertices $v \in V(G) \setminus X$, we have $\rho_v = \vec{0}$. Let $\mathcal{T}$ be the set of spanning 2-forests (T_1, T_2) such that $|X \cap T_1| = 2$. Then*

$$\phi_G = \sum_{(T_1, T_2) \in \mathcal{T}} (\rho^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e + \psi_G \sum_{i=1}^{|E(G)|} \alpha_i m_i^2.$$

Proof. Follows immediately from Lemma 9. $\qquad \square$

This simplified version of the second Symanzik polynomial will prove useful. We end this section by proving some identities for the first and second Symanzik polynomials in graphs with cut vertices. These lemmas are going to allow us to do some connectivity reductions when looking at reducibility. The first lemma is well-known (see, for example [12]).

Lemma 11. *Let G be a graph such that G has a 1-separation (A, B). Then $\psi_G = \psi_{G[A]} \psi_{G[B]}$.*

Proof. Let T_A, T_B be spanning trees of $G[A]$, $G[B]$ respectively. Notice that $T_A \cup T_B$ is a spanning tree of G. Similarly, for any spanning tree T of G, the graphs $G[V(T) \cap A]$ and $G[V(T) \cap B]$ are spanning trees of $G[A]$ and $G[B]$ respectively. The result follows. $\qquad \square$

Unlike the first Symanzik polynomial, we have to be careful where the external edges are in the graph relative to the 1-separation and restrict the masses to obtain similar factorization results.

Lemma 12. *Let G be a graph with no massive edges and let $X = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ be the set of vertices with external momenta. Suppose G has a 1-separation (A, B) such that $A \cap B = \{v\}$. Suppose $X \subseteq A$. Then $\phi_G = \phi_{G[A]}\psi_{G[B]}$.*

Proof. By Lemma 9 we restrict our attention to spanning 2-forests where each of the trees contain exactly two vertices of X. Let $\mathcal{T}$ denote the set of spanning two forests for G, $\mathcal{T}_1$ the set of spanning 2-forests in $G[A]$, and $\mathcal{T}_2$ the set of spanning trees in $G[B]$. Before doing the calculation, we make the easy observation that if (T_1, T_2) is a spanning 2-forest of G, and $v \in V(T_2)$, then $G[B \cap V(T_2)]$ is a spanning tree of $G[B]$. Up to relabelling, we will assume that $v \in T_2$. Then,

$$
\begin{aligned}
\phi_G &= \sum_{(T_1, T_2) \in \mathcal{T}} (\rho^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e \\
&= \sum_{\substack{(T_1, T_2) \in \mathcal{T}}} (\rho^{T_1})^2 \Big(\prod_{\substack{e \notin T_1 \cup T_2 \\ e \in G[A]}} \alpha_e \cdot \prod_{\substack{e \notin T_2 \\ e \in G[B]}} \alpha_e \Big) \\
&= \Big(\sum_{\substack{(T_1, T_2) \in \mathcal{T}_1}} (\rho^{T_1})^2 \prod_{\substack{e \notin T_1 \cup T_2 \\ e \in G[A]}} \alpha_e \Big) \cdot \Big(\sum_{T \in \mathcal{T}_2} \prod_{\substack{e \notin T \\ e \in G[B]}} \alpha_e \Big) \\
&= \phi_{G[A]}\psi_{G[B]},
\end{aligned}
$$

which completes the claim. $\qquad\square$

Lemma 13. *Let G be a graph with no massive edges and let $X = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ be the set of vertices with external momenta, and furthermore suppose all external momenta are on-shell. Suppose G has a 1-separation (A, B) such that $A \cap B = \{v\}$. Suppose $x_1, x_2, x_3 \in A$ and $x_4 \in B \setminus \{v\}$. Let $G[A]$ have the same external momenta, masses and Schwinger parameters as A in G with the exception that $\rho_{x_4} = \rho_v$. Then $\phi_G = \phi_{G[A]}\psi_{G[B]}$.*

Proof. First note that $G[A]$ satisfies conservation of momentum as we have

$$
\sum_{x \in V(G[A])} \rho_x = \rho_{x_1} + \rho_{x_2} + \rho_{x_3} + \rho_v = \rho_{x_1} + \rho_{x_2} + \rho_{x_3} + \rho_{x_4} = \sum_{x \in V(G)} \rho_x = 0.
$$

By Lemma 9 we restrict our attention to spanning 2-forests which contain exactly two vertices of X. Let (T_1, T_2) be a spanning 2-forest of G and without loss of generality suppose that $|T_1 \cap X \cap A| = 2$. Then $|T_2 \cap A \cap X| = 1$. Thus $|T_2 \cap B \cap X| = 1$ which implies $v \in T_2$. As (T_1, T_2) is a spanning 2-forest of G, we have that $T_2 \cap B$ is a spanning tree of B. Therefore,

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$$(p^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e = (p^{T_1})^2 \Big(\prod_{\substack{e \notin T_1 \cup T_2 \\ e \in G[A]}} \alpha_e \cdot \prod_{\substack{e \notin T_2 \\ e \in G[B]}} \alpha_e \Big).$$

Now let (T_1, T_2) be a spanning 2-forest of $G[A]$ and without loss of generality, suppose that $v \in T_2$. We can extend (T_1, T_2) to a spanning 2-forest of G by considering any spanning tree of B, say T_3 and noticing $(T_1, T_2 \cup T_3)$ is a spanning 2-forest of G. Also notice that for $(T_1, T_2 \cup T_3)$, we have $|T_1 \cap X| = 2$. Also, as $\rho_v = \rho_w$ in $G[A]$, we have $(\rho^{T_1})^2$ is the same in $G[A]$ as in G.

Let $\mathcal{T}$ denote the set of spanning 2-forests in G, $\mathcal{T}_1$ denote the set of spanning 2-forests in $G[A]$, and $\mathcal{T}_2$ denote the set of spanning trees in B. Then we have

$$
\begin{aligned}
\phi_G &= \sum_{(T_1,T_2) \in \mathcal{T}} (\rho^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e \\
&= \sum_{(T_1,T_2) \in \mathcal{T}} (\rho^{T_1})^2 \Big(\prod_{\substack{e \notin T_1 \cup T_2 \\ e \in G[A]}} \alpha_e \cdot \prod_{\substack{e \notin T_2 \\ e \in G[B]}} \alpha_e \Big) \\
&= \Big(\sum_{(T_1,T_2) \in \mathcal{T}_1} (\rho^{T_1})^2 \prod_{\substack{e \notin T_1 \cup T_2 \\ e \in G[A]}} \alpha_e \Big) \cdot \Big(\sum_{T \in \mathcal{T}_2} \prod_{\substack{e \notin T \\ e \in G[B]}} \alpha_e \Big) \\
&= \phi_{G[A]} \psi_{G[B]},
\end{aligned}
$$

which completes the claim.

$\square$

Lemma 14. *Let G be a graph with no massive edges and let $X = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ be the set of vertices with external momenta, and furthermore suppose all external momenta is on-shell. Suppose G has a 1-separation (A, B) such that $A \cap B = \{v\}$. Suppose and that $X \cap (A \setminus \{v\}) = \{x_1, x_2\}$ and that $X \cap (B \setminus \{v\}) = \{x_3, x_4\}$. Let $A' = G[A]$ have the same external momenta and Schwinger parameters as in G with the exception that we let $\rho_v = \rho_{x_1} + \rho_{x_2}$. Similarly, Let $B' = G[B]$ have the same external momenta and Schwinger parameters as in G with the exception that we let $\rho_v = \rho_{x_3} + \rho_{x_4}$. Then*

$$\phi_G = \phi_{A'} \psi_{B'} + \phi_{B'} \psi_{A'}.$$

Proof. First we claim that both A' and B' satisfy conservation of momentum. For A' we have

$$\sum_{x \in V(A')} \rho_x = \rho_{x_1} + \rho_{x_2} + \rho_v = \rho_{x_1} + \rho_{x_2} + \rho_{x_3} + \rho_{x_4} = \sum_{x \in V(G)} \rho_x = 0.$$

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Applying a similar argument gives the result for B'. Now consider a spanning 2-forest (T_1, T_2) of G. By Lemma 9 we restrict our attention to spanning two forests where $|X \cap T_1| = 2$. If $v \in V(T_2)$, then $T_2 \cap B'$ is a spanning tree of B'. Otherwise, $v \notin V(T_2)$ and $T_1 \cap B'$ is a spanning tree of B'.

Now let (T_1, T_2) be a spanning 2-forest of A'. Note that ρ_v may not be on-shell. However, as $\rho_v = \rho_{x_3} + \rho_{x_4}$, and both ρ_{x_1} and ρ_{x_2} are on-shell, applying the arguments from Lemma 9, we may assume that either $x_1, x_2 \in V(T_1)$ and $v \notin V(T_1)$ or that $v \in V(T_1)$ and $x_1, x_2 \notin V(T_1)$. Let T_3 be a spanning tree of B. If $v \in T_1$, then $(T_1 \cup T_3, T_2)$ is a spanning 2-forest of G such that $|T_2 \cap X| = 2$. If $v \in V(T_2)$, then we have that $(T_1, T_2 \cup T_3)$ is a spanning 2-forest of G such that $|T_1 \cap X| = 2$. Similar statements hold for B'.

Let $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ be the set of spanning 2-forests for G, A' and B' respectively. Let $\mathcal{T}_3, \mathcal{T}_4$ be the set of spanning trees of A' and B' respectively. Throughout the upcoming equations, we will assume when $(T_1, T_2) \in \mathcal{T}$, that $v \in V(T_1)$. Then we have:

$$
\begin{aligned}
\phi_G &= \sum_{\substack{(T_1,T_2)\in\mathcal{T}}} (\rho^{T_1})^2 \prod_{e\notin T_1\cup T_2} \alpha_e \\[2mm]
&= \sum_{\substack{(T_1,T_2)\in\mathcal{T} \\ (T_1\cap A')\in\mathcal{T}_3}} (\rho^{T_1})^2 \prod_{e\notin T_1\cup T_2} \alpha_e + \sum_{\substack{(T_1,T_2)\in\mathcal{T} \\ (T_1\cap B')\in\mathcal{T}_4}} (\rho^{T_1})^2 \prod_{e\notin T_1\cup T_2} \alpha_e \\[2mm]
&= \sum_{\substack{(T_1,T_2)\in\mathcal{T} \\ (T_1\cap A')\in\mathcal{T}_3}} (\rho^{T_1})^2 \prod_{\substack{e\notin T_1\cup T_2 \\ e\in E(A')}} \alpha_e \cdot \prod_{\substack{e\notin T_1 \\ e\in E(B')}} \alpha_e + \sum_{\substack{(T_1,T_2)\in\mathcal{T} \\ (T_1\cap B')\in\mathcal{T}_4}} (\rho^{T_1})^2 \prod_{\substack{e\notin T_1\cup T_2 \\ e\in E(B')}} \alpha_e \cdot \prod_{\substack{e\notin T_2 \\ e\in E(A')}} \alpha_e \\[2mm]
&= \Big(\sum_{\substack{(T_1,T_2)\in\mathcal{T}_1}} (\rho^{T_1})^2 \prod_{e\notin T_1\cup T_2} \alpha_e \Big) \cdot \Big(\sum_{T\in\mathcal{T}_4} \prod_{e\notin T} \alpha_e \Big) + \Big(\sum_{\substack{(T_1,T_2)\in\mathcal{T}_2}} (\rho^{T_1})^2 \prod_{e\notin T_1\cup T_2} \alpha_e \Big) \cdot \Big(\sum_{T\in\mathcal{T}_3} \prod_{e\notin T} \alpha_e \Big) \\[2mm]
&= \phi_{A'}\psi_{B'} + \phi_{B'}\psi_{A'}.
\end{aligned}
$$

$\square$

Since in Lemma 14 the momenta at v is not necessarily on-shell, the equation we get is not quite as nice as one might hope for. For reducibility, this is going to mean that we are not going to be able to reduce the problem to 2-connected graphs in all of the cases. Another well known equation for the Symanzik polynomials is:

Proposition 15. *Let H be a graph and consider any edge e. Let G be the graph obtained by subdividing edge e into two edges e_1 and e_2 with no external momenta on the vertex obtained from the subdivision. Furthermore, suppose G has no massive edges and four on-shell momenta. Then $\phi_G = \phi_{H\backslash e}(\alpha_{e_1} + \alpha_{e_2}) + \phi_{H/e}$, and $\psi_G = \psi_{H\backslash e}(\alpha_{e_1} + \alpha_{e_2}) + \psi_{H/e}$.*

Proof. Let T be a spanning tree of H. Then either $e \in E(T)$ or $e \notin E(T)$. If $e \notin E(T)$, then T can be extended to a spanning tree of G by using exactly one of edge e_1 or e_2. Otherwise,

T can be extended to a spanning tree of G by using both edges e_1 and e_2. It is easy to see that these are all the spanning trees of G. Thus $\psi_G = \psi_{H \setminus e}(\alpha_{e_1} + \alpha_{e_2}) + \psi_{H/e}$.

Now consider a spanning two forest (T_1, T_2) of H. If $e \notin E(T_1) \cup E(T_2)$, we can extend this to a spanning 2-forest of G by either adding edge e_1 or e_2 to one of the trees. Otherwise without loss of generality we have that $e \in T_1$. Then we can extend (T_1, T_2) to a spanning 2-forest of G by adding e_1 and e_2 to T_1. Now notice that this covers all spanning 2-forests of G except for the ones that have the subdivided vertex by itself as a tree. But by Lemma 9, that spanning 2-forest does not have exactly two momenta in each tree, and thus the term goes to zero. The result follows. $\qquad\square$

Our final Lemma is an observation of how the second Symanzik polynomial behaves in graphs with exactly two connected components.

Lemma 16. *Let G be a graph with no massive edges and let $X = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ be the set of vertices with external momenta, and furthermore suppose all external momenta is on-shell. Let G have exactly two connected components, G_1 and G_2. If $|X \cap V(G_1)| \neq 2$, then $\psi_G = 0$. Otherwise, let x_1 and x_2 be in $V(G_1)$. Then $\psi_G = (\rho_{x_1} + \rho_{x_2})^2(\psi_{G_1}\psi_{G_2})$.*

Proof. Let (T_1, T_2) be any spanning 2-forest of T. Then up to relabelling, T_1 is a spanning tree of G_1 and T_2 is a spanning tree of G_2. Then it is immediate from Lemma 9 that if $|X \cap V(G_1)| \neq 2$, then $\psi_G = 0$. Therefore we assume that x_1 and x_2 are in $V(G_1)$. Notice that since G is disconnected that $\psi_G = 0$, and so the terms involving masses in the second Symanzik polynomial go to zero. Let $\mathcal{T}$ denote the spanning two forests of G, $\mathcal{T}_1$ denote the spanning trees of G_1, and $\mathcal{T}_2$ denote the spanning trees of G_2. Thus we have

$$
\begin{aligned}
\psi_G &= \sum_{(T_1,T_2)\in\mathcal{T}} (\rho^{T_1})^2 \prod_{e\notin T_1\cup T_2} \alpha_e + \psi_G \sum_{i=1}^{|E(G)|} \alpha_i m_i^2 \\
&= \sum_{(T_1,T_2)\in\mathcal{T}} (\rho^{T_1})^2 \prod_{e\notin T_1\cup T_2} \alpha_e \\
&= (\rho_{x_1} + \rho_{x_2})^2 \left(\sum_{T\in\mathcal{T}_1} \prod_{e\notin T} \alpha_e\right)\left(\sum_{T\in\mathcal{T}_2} \prod_{e\notin T} \alpha_e.\right) \\
&= (\rho_{x_1} + \rho_{x_2})^2 \psi_{G_1}\psi_{G_2}
\end{aligned}
$$

$\qquad\square$

Lastly, Erik Panzer has implemented functions in his program HyperInt ([33]) which can compute the Symanzik polynomials for a given graph and given external momenta. All Symanzik polynomial calculations done throughout this thesis are done with HyperInt.

Chapter 2

Reducibility

This chapter introduces the notion of reducibility, and in particular will look at reducibility of graphs with respect to the Symanzik polynomials.

Francis Brown introduced the notion of reducibility which tells you if there is a suitable order of integration for his integration algorithm ([12]). The notion of reducibility has gone through numerous iterations, with each refinement extending the number of graphs which are reducible. As the definitions build on each other, it is instructive to see how the definitions are modified over time. We note that the actual integration algorithm always remains the same, but the different notions of reducibility get closer to correctly representing when the integration algorithm can be used.

Polynomial reduction algorithms are at the heart of reducibility. If we focus on the Symanzik polynomials, what a polynomial reduction algorithm does is decide if there is an admissible order of integration for Brown's integration technique. A nice property of reduction algorithms is they do not require looking at the integral at all, they just act on the polynomials. Since the Symanzik polynomials come from graphs, what we will see is that we can decide when the reduction algorithm will succeed solely by looking at the structure of the graph. The most simplistic of the reduction algorithms was introduced in [11], and is aptly named the simple reduction algorithm.

2.1   Reduction Algorithms

2.1.1   The Simple Reduction Algorithm

Let S be a set of polynomials in the polynomial ring $\mathbb{Q}[\alpha_1, \alpha_2, \ldots, \alpha_r]$. Let σ be a permutation of $\{\alpha_1, \ldots, \alpha_r\}$. As an abuse of notation, for this thesis we will let $\sigma(i) = \sigma(\alpha_i)$. The idea of the algorithm will be to create a sequence of new sets of polynomials with rational coefficients $S, S_{(\sigma(1))}, S_{(\sigma(1),\sigma(2))}, \ldots, S_{(\sigma(1),\ldots,\sigma(r))}$ and check that each polynomial in the set $S_{(\sigma(1),\ldots,\sigma(i))}$ is linear in $\sigma(i+1)$ for each $i \in \{1, \ldots, r-1\}$. Now for the definition; suppose we are at the kth iteration of the algorithm. If $k > 1$, we have a set of polynomials with

rational coefficients $S_{(\sigma(1),...,\sigma(k-1))} = \{f_1, \ldots, f_n\}$. Otherwise $k = 1$ and we use S. We then do the following:

1. If there is a polynomial $f \in S_{(\sigma(1),...,\sigma(k-1))}$ such that f is not linear in $\sigma(k)$, we end the algorithm. Otherwise continue.

2. Then for all $i \in \{1, \ldots, n\}$, we write $f_i = g_i\sigma(k) + h_i$, where $g_i = \frac{\partial f_i}{\partial \sigma(k)}$ and $h_i = f_i|_{\sigma(k)=0}$.

3. Let $S^1 = \{g_i | i \in \{1, \ldots, n\}$. Let $S^2 = \{h_i | i \in \{1, \ldots, n\}$. Let $S^3 = \{g_ih_j - h_ig_j | i, j \in \{1, \ldots, n\}, i \neq j\}$. Let $S^4 = S^1 \cup S^2 \cup S^3$.

4. Let $f \in S^4$ and let $f_1, f_2, f_3, \ldots, f_m$ be the polynomials such that $\prod_{i=1}^m f_i = f$, where each f_i has coefficients in $\mathbb{Q}$ and each f_i is irreducible over $\mathbb{Q}$. Let $\tilde{S}$ be the set of polynomials $f_1, \ldots, f_m$ for each $f \in S^4$.

5. We define $S_{(\sigma(1),...,\sigma(k))}$ to be $\tilde{S}$. Now repeat the above steps with $S_{(\sigma(1),...,\sigma(k))}$ in place of $S_{(\sigma(1),...,\sigma(k-1))}$ for $r - 1$ iterations or the until first step ends the algorithm.

Definition 17. *Let S be a set of polynomials in the polynomial ring $\mathbb{Q}[\alpha_1, \ldots \alpha_r]$ and let σ be a permutation of $\{\alpha_1, \ldots, \alpha_r\}$. Then S is simply reducible with respect to σ if for all $i \in \{1, \ldots, r-1\}$, all polynomials in $S_{(\sigma(1),...,\sigma(i))}$ are linear in $\sigma(i+1)$. If there exists a permutation σ of $\{\alpha_1, \ldots, \alpha_r\}$ such that S is simply reducible with respect to σ, then we say that S is simply reducible. If for some graph G, the set $S \subseteq \{\psi_G, \phi_G\}$ is simply reducible, then we will say G is* simply reducible *with respect to S.*

Throughout this section and the other reduction algorithms, if we say a graph is reducible without mentioning what set of polynomials, reducibility is assumed to be with respect to both Symanzik polynomials.

Remark 18. *If $S_{(\sigma(1),...,\sigma(k))}$ contains any monomial of the form α_i or any constant, we may remove them as they do not affect whether the set is simply reducible. As in, if $S = [S', c]$ for some constant c, and some set of polynomials S', then S is simply reducible if and only if S' is simply reducible. The same statement holds if $S = [S', \alpha]$ for some variable α.*

Proof. Suppose S' is a set of polynomials in the polynomial ring $\mathbb{Q}[\alpha_1, \ldots, \alpha_r]$ which is simply reducible for some permutation σ of $\alpha_1, \ldots, \alpha_r$. Now let $S = [S', c]$, where c is some constant. For $\sigma(1)$, the polynomial c forces $c \in S^2$ and $0 \in S^1$. Thus the polynomial $g_ih_j - h_ig_j = cg_j$ when $f_i = c$. Then after factoring over $\mathbb{Q}$, we get $\tilde{S} = \{c, S'_{\sigma(1)}\}$. Notice that this happens at every step of the algorithm. Then since c is linear in every variable, by induction S is simply reducible with respect to σ.

Similarly, suppose we have $S = [S', \alpha]$ for some variable α. Let S' and σ be as above. First suppose $\sigma(1) = \alpha$. Then from α, we have $1 \in S^1$ and $0 \in S^2$. Thus the polynomial

$g_i h_j - h_i g_j = h_j$, when $f_i = \alpha$. Thus $\tilde{S} = \tilde{S}' = S'_{\sigma(1)}$. Then since S' is reducible, S is reducible.

Now consider when $\sigma(1) \neq \alpha$. Then from α, we have $0 \in S^1$ and $\alpha \in S^2$. Thus the polynomial $g_i h_j - h_i g_j = g_j \alpha$, when $f_i = \alpha$. Thus after factoring we get $S_{\sigma(1)} = \{\alpha, S'_{\sigma(1)}\}$. Noticing that the above argument holds for every step of the algorithm, by induction we have that S is simply reducible with respect to α. See Lemma 37 for the proof of the converse. $\square$

Then if we are given a polynomial which is just a monomial, we can ignore it in the reduction algorithm, since a monomial factors into single monomials of the form α_i, and then we can apply Remark 18. Before continuing we give some examples.

Example 19. *Let $n \in \mathbb{N}$. Let $S = \{\Sigma_{i=1}^n x_i\}$. Then S is simply reducible.*

Proof. We proceed by induction on n. If $n = 1$ this is immediate. Now assume $n \geq 2$. Let σ be the identity permutation on $\{x_1, \ldots, x_n\}$. Then applying the simple reduction algorithm to $\sigma(1) = x_1$, we get $S^4 = \{1, \Sigma_{i=2}^n x_i\}$. As these polynomials are already irreducible over $\mathbb{Q}$, we have $\tilde{S} = S^4$. Thus $S_{\sigma(1)} = \{1, \Sigma_{i=2}^n x_i\}$. By Remark 18, we may remove the constant and continue the reduction with $S_{\sigma(1)} = \{\Sigma_{i=2}^n x_i\}$. By induction, $S_{\sigma(1)}$ is simply reducible with the identity permutation, therefore S is reducible with the identity permutation. $\square$

From example 3, $\psi_{C_n} = \Sigma_{i=1}^n \alpha_i$. So the above example says that all cycles are simply reducible with respect to the first Symanzik polynomial. Also, notice that the identity permutation was not critical in the above example, with some slight modifications, any permutation would have sufficed. So cycles are simply reducible with respect to the first Symanzik polynomial for any permutation.

As a remark, notice that the second Symanzik polynomial may not have coefficients in $\mathbb{Q}$ since it depends on masses and external momenta. As we want to test reducibility on the second Symanzik polynomial, we will allow coefficients of polynomials to be algebraic functions of the masses and external momenta or rational numbers, following what is done in [7]. While we will generally assume all our graphs have no massive edges, for the next example we will consider massive edges.

Example 20. *Let G be a 2-connected graph. Suppose every edge is massive. Then G is not simply reducible with respect to ψ.*

Proof. As G is 2-connected, for every edge $e \in E(G)$ there exists a spanning tree T such that $e \notin E(T)$. Thus for every $e \in E(G)$, there is a term of ϕ_G which has α_e in it. Then, as each edge is massive,

$$\psi_G \sum_{i=1}^{|E(G)|} \alpha_i m_i^2$$

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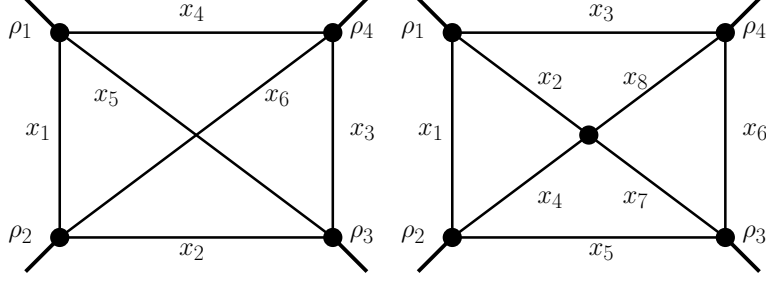
Figure 2.1: A labelling of K_4 and W_4.

is quadratic in every variable. Therefore ϕ_G is quadratic in every variable. Thus for every permutation of the variables of ϕ_G, the simple reduction algorithm will end at the first step. □

The next example was first calculated in Martin Lüders masters thesis ([26]), although we give a slightly different argument, but one with the same spirit. It will become a forbidden minor for reducibility, so it is worthwhile to work out all the details.

Example 21 ([26])**.** *The graph K_4 is not simply reducible with respect to both $\{\psi, \phi\}$ when each vertex has on-shell external momenta and no edges are massive.*

Proof. Let σ be a permutation of $\{x_1, \ldots, x_6\}$. Using the labels as in Figure 2.1, one can calculate that,

$$\psi_{K_4} = x_1 x_3 x_2 + x_1 x_4 x_2 + x_1 x_5 x_2 + x_1 x_3 x_4 + x_3 x_1 x_5 + x_1 x_3 x_6 + x_1 x_4 x_6 + x_1 x_5 x_6$$
$$+ x_2 x_3 x_4 + x_2 x_3 x_6 + x_2 x_4 x_5 + x_2 x_4 x_6 + x_2 x_5 x_6 + x_3 x_4 x_5 + x_3 x_5 x_6 + x_4 x_5 x_6.$$

When calculating ϕ_{K_4}, since the four momenta are on-shell, by Lemma 9 there are only three relevant spanning 2-forests. Therefore we have:

$$\phi_{K_4} = (p_1 + p_3)^2 (x_1 x_2 x_3 x_4) + (p_1 + p_2)^2 (x_4 x_2 x_5 x_6) + (p_1 + p_4)^2 (x_1 x_3 x_5 x_6).$$

For notational ease, we will let $(p_1 + p_2)^2 = s$, $(p_1 + p_3)^2 = t$, $(p_1 + p_2)^2 = u$. Due to the symmetry of K_4, without loss of generality we may assume that $\sigma(1) = x_1$.

Then we calculate:

$S^1 = \{x_2 x_3 + x_2 x_4 + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_3 x_6 + x_4 x_6 + x_5 x_6, s x_2 x_3 x_4 + u x_3 x_5 x_6\}$,

$S^2 = \{x_2 x_3 x_4 + x_2 x_3 x_6 + x_2 x_4 x_5 + x_2 x_4 x_6 + x_2 x_5 x_6 + x_3 x_4 x_5 + x_3 x_5 x_6 + x_4 x_5 x_6, t x_2 x_4 x_5 x_6\}$,

$S^3 = \{(x_2 x_3 + x_2 x_4 + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_3 x_6 + x_4 x_6 + x_5 x_6)(t x_2 x_4 x_5 x_6) - (s x_2 x_3 x_4 + u x_3 x_5 x_6)(x_2 x_3 + x_2 x_4 + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_3 x_6 + x_4 x_5 + x_5 x_6)\}$.

We claim that the polynomial in S^3 does not factor into linear polynomials in any variable. Let f denote the polynomial in S^3. By the symmetry of K_4, it suffices to show

that f does not factor into polynomials linear in x_2 or x_3. If we collect the coefficients of f in terms of x_2, we get $f = Ax_2^2 + Bx_2 + C$ where

$A = -sx_3x_4x_5x_6 - sx_4{}^2x_5x_6 - sx_4x_5{}^2x_6 + tx_3{}^2x_4{}^2 + tx_3{}^2x_4x_6 + tx_3x_4{}^2x_5 + tx_3x_4{}^2x_6 + tx_3x_4x_5x_6$,

$B = -sx_3x_4{}^2x_5x_6 - sx_3x_4x_5{}^2x_6 - sx_3x_4x_5x_6{}^2 - sx_4{}^2x_5x_6{}^2 - sx_4x_5{}^2x_6{}^2 + tx_3{}^2x_4{}^2x_5 + tx_3{}^2x_4x_5x_6 + tx_3x_4{}^2x_5x_6 + ux_3{}^2x_4x_5x_6 + ux_3{}^2x_5x_6{}^2 + ux_3x_4x_5{}^2x_6 + ux_3x_4x_5x_6{}^2 + ux_3x_5{}^2x_6{}^2$,

$C = ux_3{}^2x_4x_5{}^2x_6 + ux_3{}^2x_5{}^2x_6{}^2 + ux_3x_4x_5{}^2x_6{}^2$.

Suppose $f = f_1f_2$ for some polynomials f_1 and f_2 where $\deg(f_i, x_2) = 1$, $i \in \{1, 2\}$. Then by the quadratic formula, $B^2 - 4AC$ is a perfect square. If we treat s, t and u as integers, then setting $s = t = u = 1$ and $x_i = 1$, for all $i \in \{3, 4, 5, 6\}$, we get $B^2 - 4AC = -15$ which is not a square. Therefore $B^2 - 4AC$ is not a perfect square and thus f does not factor into two polynomials linear in x_2.

Applying the same strategy to x_3, we collect the coefficients of f in terms of x_3, and we get $f = Ax_3^2 + Bx_3 + C$ where

$A = tx_2{}^2x_4{}^2 + tx_2{}^2x_4x_6 + tx_2x_4{}^2x_5 + tx_2x_4x_5x_6 + ux_2x_4x_5x_6 + ux_2x_5x_6{}^2 + ux_4x_5{}^2x_6 + ux_5{}^2x_6{}^2$,

$B = -sx_2{}^2x_4x_5x_6 - sx_2x_4{}^2x_5x_6 - sx_2x_4x_5{}^2x_6 - sx_2x_4x_5x_6{}^2 + tx_2{}^2x_4{}^2x_5 + tx_2{}^2x_4{}^2x_6 + tx_2{}^2x_4x_5x_6 + tx_2x_4{}^2x_5x_6 + ux_2x_4x_5{}^2x_6 + ux_2x_4x_5x_6{}^2 + ux_2x_5{}^2x_6{}^2 + ux_4x_5{}^2x_6{}^2$,

$C = -sx_2{}^2x_4{}^2x_5x_6 - sx_2{}^2x_4x_5{}^2x_6 - sx_2x_4{}^2x_5x_6{}^2 - sx_2x_4x_5{}^2x_6{}^2$.

Now setting $x_i = 1$ for all $i \in \{2, 3, 4, 6\}$, $x_5 = 2$, and $s = t = u = 1$ we get $B^2 - 4AC = 985$, which is not a perfect square. Thus $B^2 - 4AC$ is not a perfect square so f does not factor into two polynomials linear in x_3. With this we have shown that for all x_i, $i \in \{2, \ldots, 6\}$, f does not factor into two polynomials linear in x_i. Thus $f \in \tilde{S}$ and thus $S_{(\sigma(1))}$ has a polynomial not linear in $\sigma(2)$, completing the claim.

$\square$

The same technique used above can be used to prove that W_4 is not simply reducible when we add external momenta to the rim vertices. As with K_4, this example will end up becoming a forbidden minor for reducibility, and thus it is worthwhile to work out the details.

Example 22. *The graph W_4 is not simply reducible with respect to both $\{\psi, \phi\}$ when the rim vertices have on-shell external momenta and no edges are massive.*

Proof. Let σ be any permutation of $\{x_1, x_2, \ldots, x_8\}$. Using the labels in Figure 2.1, one can calculate that:

$\psi_{W_4} = x_1x_2x_4x_6 + x_1x_2x_4x_7 + x_1x_2x_4x_8 + x_1x_2x_5x_6 + x_1x_2x_5x_7 + x_1x_2x_5x_8 + x_1x_2x_6x_7 + x_1x_2x_7x_8 + x_1x_3x_4x_6 + x_1x_3x_4x_7 + x_1x_3x_4x_8 + x_1x_3x_5x_6 + x_1x_3x_5x_7 + x_1x_3x_5x_8 + x_1x_3x_6x_7 + x_1x_3x_7x_8 + x_1x_4x_6x_8 + x_1x_4x_7x_8 + x_1x_5x_6x_8 + x_1x_5x_7x_8 + x_1x_6x_7x_8 + x_2x_3x_4x_6 + x_2x_3x_4x_7 + x_2x_3x_4x_8 + x_2x_3x_5x_6 + x_2x_3x_5x_7 + x_2x_3x_5x_8 + x_2x_3x_6x_7 + x_2x_3x_7x_8 + x_2x_4x_5x_6 + x_2x_4x_5x_7 + $

$x_2x_4x_5x_8+x_2x_4x_6x_7+x_2x_4x_6x_8+x_2x_5x_6x_8+x_2x_5x_7x_8+x_2x_6x_7x_8+x_3x_4x_5x_6+x_3x_4x_5x_7+$
$x_3x_4x_5x_8+x_3x_4x_6x_7+x_3x_4x_7x_8+x_4x_5x_6x_8+x_4x_5x_7x_8+x_4x_6x_7x_8.$

As the momenta are on-shell, appealing to Lemma 9, we calculate that:

$\phi_{W_4} = sx_1x_3x_5x_7x_8 + sx_2x_3x_4x_5x_6 + sx_2x_3x_4x_5x_7 + sx_2x_3x_4x_5x_8 + sx_2x_3x_5x_7x_8 +$
$sx_3x_4x_5x_7x_8+ux_1x_2x_4x_6x_7+ux_1x_2x_4x_6x_8+ux_1x_2x_5x_6x_8+ux_1x_2x_6x_7x_8+ux_1x_3x_4x_6x_7+$
$ux_1x_4x_6x_7x_8,$

where $u = (p_1 + p_4)^2$ and $s = (p_1 + p_3)^2$.

By symmetry of W_4, without loss of generality we may assume that $\sigma(1) = x_1$ or $\sigma(1) = x_2$. We first consider the case where $\sigma(1) = x_1$. Then we have

$S^1 = \{x_2x_4x_6 + x_2x_4x_7 + x_2x_4x_8 + x_2x_5x_6 + x_2x_5x_7 + x_2x_5x_8 + x_2x_6x_7 + x_2x_7x_8 +$
$x_3x_4x_6 + x_3x_4x_7 + x_3x_4x_8 + x_3x_5x_6 + x_3x_5x_7 + x_3x_5x_8 + x_3x_6x_7 + x_3x_7x_8 + x_4x_6x_8 +$
$x_4x_7x_8 + x_5x_6x_8 + x_5x_7x_8 + x_6x_7x_8, sx_3x_5x_7x_8 + ux_2x_4x_6x_7 + ux_2x_4x_6x_8 + ux_2x_5x_6x_8 +$
$ux_2x_6x_7x_8 + ux_3x_4x_6x_7 + ux_4x_6x_7x_8\},$

$S^2 = \{x_2x_3x_4x_6+x_2x_3x_4x_7+x_2x_3x_4x_8+x_2x_3x_5x_6+x_2x_3x_5x_7+x_2x_3x_5x_8+x_2x_3x_6x_7+$
$x_2x_3x_7x_8+x_2x_4x_5x_6+x_2x_4x_5x_7+x_2x_4x_5x_8+x_2x_4x_6x_7+x_2x_4x_6x_8+x_2x_5x_6x_8+x_2x_5x_7x_8+$
$x_2x_6x_7x_8+x_3x_4x_5x_6+x_3x_4x_5x_7+x_3x_4x_5x_8+x_3x_4x_6x_7+x_3x_4x_7x_8+x_4x_5x_6x_8+x_4x_5x_7x_8+$
$x_4x_6x_7x_8, sx_2x_3x_4x_5x_6 + sx_2x_3x_4x_5x_7 + sx_2x_3x_4x_5x_8 + sx_2x_3x_5x_7x_8 + sx_3x_4x_5x_7x_8\},$

$S^3 = \{(x_2x_4x_6 + x_2x_4x_7 + x_2x_4x_8 + x_2x_5x_6 + x_2x_5x_7 + x_2x_5x_8 + x_2x_6x_7 + x_2x_7x_8 +$
$x_3x_4x_6+x_3x_4x_7+x_3x_4x_8+x_3x_5x_6+x_3x_5x_7+x_3x_5x_8+x_3x_6x_7+x_3x_7x_8+x_4x_6x_8+x_4x_7x_8+$
$x_5x_6x_8 + x_5x_7x_8 + x_6x_7x_8)(sx_2x_3x_4x_5x_6 + sx_2x_3x_4x_5x_7 + sx_2x_3x_4x_5x_8 + sx_2x_3x_5x_7x_8 +$
$sx_3x_4x_5x_7x_8)-(sx_3x_5x_7x_8+ux_2x_4x_6x_7+ux_2x_4x_6x_8+ux_2x_5x_6x_8+ux_2x_6x_7x_8+ux_3x_4x_6x_7+$
$ux_4x_6x_7x_8)(x_2x_3x_4x_6+x_2x_3x_4x_7+x_2x_3x_4x_8+x_2x_3x_5x_6+x_2x_3x_5x_7+x_2x_3x_5x_8+x_2x_3x_6x_7+$
$x_2x_3x_7x_8+x_2x_4x_5x_6+x_2x_4x_5x_7+x_2x_4x_5x_8+x_2x_4x_6x_7+x_2x_4x_6x_8+x_2x_5x_6x_8+x_2x_5x_7x_8+$
$x_2x_6x_7x_8+x_3x_4x_5x_6+x_3x_4x_5x_7+x_3x_4x_5x_8+x_3x_4x_6x_7+x_3x_4x_7x_8+x_4x_5x_6x_8+x_4x_5x_7x_8+$
$x_4x_6x_7x_8)\}.$

Let f be the polynomial in S^3. We claim that f does not factor into polynomials linear in any of x_4, x_5, x_6 or x_8. By the symmetry of W_4, this suffices to show that $S_{\sigma(1)}$ is not linear in $\sigma(2)$. For x_5, we assume that $f = f_1 f_2$ where f_1 and f_2 are both linear in x_5 and we write $f = Ax_5^2 + Bx_5 + C$. Then from the quadratic formula, $\sqrt{B^2 - 4AC}$ is a polynomial. However by evaluating $B^2 - 4AC$ at 1 for all s, t, u and all $x_i, i \in \{2, 3, 4, 6, 7, 8\}$, we get $B^2 - 4AC = 3564$ which is not a perfect square, a contradiction.

Applying the same strategy to x_4, evaluating $B^2 - 4AC$ at one for s, t, u and $x_i = 1$ for $i \in \{2, 3, 5, 6, 7, 8\}$ we get -1008 which is not perfect square, a contradiction.

For x_8, evaluating $B^2 - 4AC$ at one for s, u and $x_i, i \in \{2, \ldots, 7\}$ we get 585 which is not a perfect square, a contradiction.

For x_6, evaluating $B^2 - 4AC$ at one for s, u and $x_i, i \in \{2, 3, 4, 5, 7, 8\}$ we get 10368 which is not a perfect square, a contradiction.

With that, we may assume that $\sigma(1) \neq x_1$. Therefore $\sigma(1) = x_2$. Then we calculate:

$$S^1 = \{x_1x_4x_6 + x_1x_4x_7 + x_1x_4x_8 + x_1x_5x_6 + x_1x_5x_7 + x_1x_5x_8 + x_1x_6x_7 + x_1x_7x_8 +$$
$$x_3x_4x_6 + x_3x_4x_7 + x_3x_4x_8 + x_3x_5x_6 + x_3x_5x_7 + x_3x_5x_8 + x_3x_6x_7 + x_3x_7x_8 + x_4x_5x_6 +$$
$$x_4x_5x_7 + x_4x_5x_8 + x_4x_6x_7 + x_4x_6x_8 + x_5x_6x_8 + x_5x_7x_8 + x_6x_7x_8, sx_3x_4x_5x_6 + sx_3x_4x_5x_7 +$$
$$sx_3x_4x_5x_8 + sx_3x_5x_7x_8 + ux_1x_4x_6x_7 + ux_1x_4x_6x_8 + ux_1x_5x_6x_8 + ux_1x_6x_7x_8\},$$

$$S^2 = \{x_1x_3x_4x_6 + x_1x_3x_4x_7 + x_1x_3x_4x_8 + x_1x_3x_5x_6 + x_1x_3x_5x_7 + x_1x_3x_5x_8 + x_1x_3x_6x_7 +$$
$$x_1x_3x_7x_8 + x_1x_4x_6x_8 + x_1x_4x_7x_8 + x_1x_5x_6x_8 + x_1x_5x_7x_8 + x_1x_6x_7x_8 + x_3x_4x_5x_6 + x_3x_4x_5x_7 +$$
$$x_3x_4x_5x_8 + x_3x_4x_6x_7 + x_3x_4x_7x_8 + x_4x_5x_6x_8 + x_4x_5x_7x_8 + x_4x_6x_7x_8, sx_1x_3x_5x_7x_8 + sx_3x_4x_5x_7x_8 +$$
$$ux_1x_3x_4x_6x_7 + ux_1x_4x_6x_7x_8\},$$

$$S^3 = \{(x_1x_4x_6 + x_1x_4x_7 + x_1x_4x_8 + x_1x_5x_6 + x_1x_5x_7 + x_1x_5x_8 + x_1x_6x_7 + x_1x_7x_8 + x_3x_4x_6 +$$
$$x_3x_4x_7 + x_3x_4x_8 + x_3x_5x_6 + x_3x_5x_7 + x_3x_5x_8 + x_3x_6x_7 + x_3x_7x_8 + x_4x_5x_6 + x_4x_5x_7 + x_4x_5x_8 +$$
$$x_4x_6x_7 + x_4x_6x_8 + x_5x_6x_8 + x_5x_7x_8 + x_6x_7x_8)(sx_1x_3x_5x_7x_8 + sx_3x_4x_5x_7x_8 + ux_1x_3x_4x_6x_7 +$$
$$ux_1x_4x_6x_7x_8) - (sx_3x_4x_5x_6 + sx_3x_4x_5x_7 + sx_3x_4x_5x_8 + sx_3x_5x_7x_8 + ux_1x_4x_6x_7 + ux_1x_4x_6x_8 +$$
$$ux_1x_5x_6x_8 + ux_1x_6x_7x_8)(x_1x_3x_4x_6 + x_1x_3x_4x_7 + x_1x_3x_4x_8 + x_1x_3x_5x_6 + x_1x_3x_5x_7 + x_1x_3x_5x_8 +$$
$$x_1x_3x_6x_7 + x_1x_3x_7x_8 + x_1x_4x_6x_8 + x_1x_4x_7x_8 + x_1x_5x_6x_8 + x_1x_5x_7x_8 + x_1x_6x_7x_8 + x_3x_4x_5x_6 +$$
$$x_3x_4x_5x_7 + x_3x_4x_5x_8 + x_3x_4x_6x_7 + x_3x_4x_7x_8 + x_4x_5x_6x_8 + x_4x_5x_7x_8 + x_4x_6x_7x_8)\}.$$

As above, let f be the polynomial in S^3. We will show that f does not factor into polynomials linear in any of x_1, x_4 or x_5. By the symmetry of W_4, this suffices to show that f does not factor into polynomials linear in any of the variables.

As before, we assume that $f = f_1 f_2$ where f_1 and f_2 are linear in x_1. Then we write f as $f = Ax_1^2 + Bx_1 + C$, and evaluate $B^2 - 4AC$ at one for all s, t, u and x_i, $i \in \{3, \ldots, 8\}$ gives -1008 which is not a perfect square, a contradiction.

For x_4, writing f as $f = Ax_4^2 + Bx_1 + C$, and evaluating $B^2 - 4AC$ at one for s, u and x_i, $i \in \{1, 3, 5, 6, 7, 8\}$ we get -567 which is not a square, a contradiction.

For x_5, writing f as $f = Ax_5^2 + Bx_5 + C$, and evaluating $B^2 - 4AC$ at one for s, u and x_i, $i \in \{1, 3, 4, 6, 7, 8\}$ we get 585 which is not a perfect square, a contradiction.

Therefore, $S_{(\sigma(1))}$ is not linear in $\sigma(2)$ for any permutation σ, and thus W_4 with on-shell external momenta on the rim vertices is not simply reducible. $\qquad\square$

The last example in this section first appeared in [11] and is an example of a non-trivial graph which is simply reducible with respect to the first Symanzik polynomial.

Example 23. *The graph K_4 is simply reducible with respect to the first Symanzik polynomial.*

Proof. Using the labels from Figure 2.1 recall that

$$\psi_{K_4} = x_1x_3x_2 + x_1x_4x_2 + x_1x_5x_2 + x_1x_3x_4 + x_3x_1x_5 + x_1x_3x_6 + x_1x_4x_6 + x_1x_5x_6$$
$$+ x_2x_3x_4 + x_2x_3x_6 + x_2x_4x_5 + x_2x_4x_6 + x_2x_5x_6 + x_3x_4x_5 + x_3x_5x_6 + x_4x_5x_6.$$

Consider the permutation σ where $\sigma(1) = x_1, \sigma(2) = x_2, \sigma(3) = x_5, \sigma(4) = x_6, \sigma(5) = x_3$. One can calculate that:

$$S_{\sigma(1)} = \{x_2x_3 + x_2x_4 + x_2x_5 + x_3x_4 + x_5x_3 + x_6x_3 + x_4x_6 + x_5x_6, x_2x_3x_4 + x_2x_3x_6$$
$$+ x_2x_4x_5 + x_2x_4x_6 + x_2x_5x_6 + x_3x_4x_5 + x_3x_5x_6 + x_4x_5x_6\}.$$

Now for $\sigma(2)$ we calculate that:

$S^1 = \{x_3 + x_4 + x_5, x_3x_4 + x_6x_3 + x_5x_4 + x_4x_6 + x_5x_6\}$,

$S^2 = \{x_3x_4 + x_5x_3 + x_6x_3 + x_4x_6 + x_5x_6, x_3x_4x_5 + x_3x_5x_6 + x_4x_5x_6\}$,

$S^3 = \{-x_3{}^2x_4{}^2 - 2\,x_3{}^2x_4x_6 - x_3{}^2x_6{}^2 - 2\,x_3x_4{}^2x_6 - 2\,x_3x_4x_5x_6 - 2\,x_3x_4x_6{}^2 - 2\,x_3x_5x_6{}^2 - x_4{}^2x_6{}^2 - 2\,x_4x_5x_6{}^2 - x_5{}^2x_6{}^2\}$.

Now one can check that the polynomial in S^3 can be expressed as $-\left(x_3x_4 + x_3x_6 + x_4x_6 + x_5x_6\right)^2$, and that $x_3x_4x_5 + x_3x_5x_6 + x_4x_5x_6 = x_5\left(x_3x_4 + x_3x_6 + x_4x_6\right)$. Ignoring the monomial, we have

$S_{(\sigma(1),\sigma(2))} = \{x_3 + x_4 + x_5, x_3x_4 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6, x_3x_4 + x_3x_5 + x_3x_6 + x_4x_6 + x_5x_6, x_3x_4 + x_3x_6 + x_4x_6, x_3x_4 + x_3x_6 + x_4x_6 + x_5x_6\}$.

Then for $\sigma(3)$, ignoring monomials and constants we calculate that:

$S_{(\sigma(1),\sigma(2),\sigma(3))} = \{x_3 + x_4, x_4 + x_6, x_3x_4 + x_3x_6 + x_4x_6, x_3 + x_6, x_3 - x_4\}$.

Then for $\sigma(4)$ we calculate: $S_{(\sigma(1),\sigma(2),\sigma(3),\sigma(4))} = \{x_3 + x_4, x_3 - x_4\}$.

Then finally we see that $S_{(\sigma(1),...,\sigma(5))}$ after removing monomials is the empty set and thus K_4 is simply reducible with respect to the first Symanzik polynomial. $\qquad\square$

We can relate the simple reduction algorithm back to Francis Brown's integration algorithm for Feynman integrals. The set S at the start is the Symanzik polynomials appearing the Feynman integral and the permutation σ is corresponding to a choice of order of integration. The polynomials we create at each iteration are an overestimate of the polynomials which appear at that step of the integration under that ordering. So a graph being simply reducible says that there is an order such that at each step the polynomials appearing will not cause a problem for the integration algorithm.

The reason I say we are only approximating the polynomials appearing at each step during the intergration is because with the above algorithm, we possibly get spurious polynomials. By this I mean, we possibly get polynomials in the simple reduction which do not appear in the integration. Clearly, the more polynomials we have in the reduction sets the more likely we are to say the graph is not simply reducibile. Thus a set of Symanzik polynomials may be amenable to Brown's integration algorithm but not simply reducible. Given that the point of the reduction algorithm is to decide if we can use Brown's integration algorithm, it would be advantageous to reduce the number of spurious polynomials. An easy improvement comes from noting that the simple reduction algorithm fundamentally relied on an ordering of the variables, but by Fubini's Theorem, the order of integration

does not change the outcome of the integration. The next reduction algorithm reflects this, and was introduced by Brown in [11].

2.1.2 The Fubini Reduction Algorithm

As in the simple reduction algorithm, let S be a set of polynomials in the polynomial ring $Q[\alpha_1, \alpha_2, \ldots, \alpha_r]$. Let σ be a permutation of $\{\alpha_1, \ldots, \alpha_r\}$. Recall that $S_{(\sigma(1),\ldots,\sigma(k))}$ is the set obtained by running the simple reduction algorithm on the variables $\sigma(1), \ldots, \sigma(k)$ in that order.

We define $S_{[\sigma(1)]} = S_{(\sigma(1))}$ and $S_{[\sigma(1),\sigma(2)]} = S_{(\sigma(1),\sigma(2))} \cap S_{(\sigma(2),\sigma(1))}$. In general, we have:

$$S_{[\sigma(1),\ldots,\sigma(k)]} = \bigcap_{1 \leq i \leq k} S_{[\sigma(1),\ldots,\widehat{\sigma(i)},\ldots,\sigma(k)](\sigma(i))}$$

Where $\widehat{\sigma(i)}$ means we are omitting $\sigma(i)$. As stated this definition is not well defined. It is entirely possible that in the intersection one of the sets $S_{[\sigma(1),\ldots,\widehat{\sigma(i)},\ldots,\sigma(k)](\sigma(i))}$ is undefined. In this case, we simply ignore that set and continue on. In the event $S_{[\sigma(1),\ldots,\widehat{\sigma(i)},\ldots,\sigma(k)](\sigma(i))}$ is undefined for all i, the algorithm stops. Additionally, here when we take intersections, we do so up to constants. Thus if two polynomials differ by a constant factor, we do not remove them in the intersection.

Definition 24. *Let S be a set of polynomials in the polynomial ring $\mathbb{Q}[\alpha_1, \ldots, \alpha_r]$. Let σ be a permutation of $\{\alpha_1, \ldots, \alpha_r\}$. We say S is* Fubini reducible with respect to σ *if for all $1 \leq i \leq r - 1$, all polynomials in the set $S_{[\sigma(1),\ldots,\sigma(i)]}$ are linear in $\sigma(i+1)$. If there exists a permutation σ such that S is Fubini reducible with respect to σ, we say that S is* Fubini reducible. *Given a graph G, and $S \subseteq \{\psi_G, \phi_G\}$, if S is Fubini reducible, then we say G is reducible with respect to S.*

Remark 25. *If S is simply reducible with respect to some permutation σ, then S is Fubini reducible with respect to σ.*

Proof. It follows immediately from the definitions that for any integer k, $S_{[\sigma(1),\ldots,\sigma(k)]} \subseteq S_{(\sigma(1),\ldots,\sigma(k))}$. As S is simply reducible, $S_{(\sigma(1),\ldots,\sigma(k))}$ is linear in $\sigma(k+1)$ for all k. Therefore, $S_{[\sigma(1),\ldots,\sigma(k)]}$ is linear in $\sigma(k+1)$ for all k, completing the proof. $\square$

Example 26. *The graphs K_4 and W_4 with external momenta as in example 21 and example 22 are not Fubini reducible with respect to the first and second Symanzik polynomials.*

Proof. Notice that given any set of polynomials S, and any permutation σ of the Schwinger parameters, $S_{[\sigma(1)]} = S_{(\sigma(1))}$. From example 21 and example 22 both W_4 and K_4 have $S_{(\sigma(1),\sigma(2))}$ undefined for all permutations σ. Thus we can conclude that K_4 and W_4 are not Fubini reducible with respect to the first and second Symanzik polynomials. $\square$

Example 27 ([11]). *The graph K_4 is Fubini reducible with respect to the first Symanzik polynomial, and the Fubini reduction produces less polynomials than the simple reduction.*

Proof. The fact that K_4 is Fubini reducible is immediate by Remark 25 as K_4 is simply reducible with respect to the first Symanzik polynomial. We partially go through the calculation so as to show that the Fubini reduction algorithm does indeed remove some spurious polynomials. As before, we use the labelling in Figure 2.1. Recall that

$$\psi_{K_4} = x_1x_3x_2 + x_1x_4x_2 + x_1x_5x_2 + x_1x_3x_4 + x_3x_1x_5 + x_1x_3x_6 + x_1x_4x_6 + x_1x_5x_6$$
$$+ x_2x_3x_4 + x_2x_3x_6 + x_2x_4x_5 + x_2x_4x_6 + x_2x_5x_6 + x_3x_4x_5 + x_3x_5x_6 + x_4x_5x_6.$$

Let σ be a permutation of $\{x_1, x_2, x_3, x_4, x_5\}$ such that $\sigma(1) = x_1, \sigma(2) = x_2, \sigma(3) = x_5, \sigma(4) = x_6$, and $\sigma(5) = x_3$. We have already calculated that:

$S_{[\sigma(1)]} = \{x_2x_3 + x_2x_4 + x_2x_5 + x_3x_4 + x_5x_3 + x_6x_3 + x_4x_6 + x_5x_6, x_2x_3x_4 + x_2x_3x_6 + x_2x_4x_5 + x_2x_4x_6 + x_2x_5x_6 + x_3x_4x_5 + x_3x_5x_6 + x_4x_5x_6\}$.

Similarly, we have that,

$S_{[\sigma(2)]} = \{x_1x_3 + x_1x_4 + x_1x_5 + x_3x_4 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6, x_1x_3x_4 + x_1x_3x_5 + x_1x_3x_6 + x_1x_4x_6 + x_1x_5x_6 + x_3x_4x_5 + x_3x_5x_6 + x_4x_5x_6\}$.

Then we calculate that:

$S_{[\sigma(1)](\sigma(2))} = \{x_3 + x_4 + x_5, x_3x_4 + x_3x_5 + x_3x_6 + x_4x_6 + x_5x_6, x_3x_4 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6, x_3x_4 + x_3x_6 + x_4x_6 + x_5x_6, x_3x_4 + x_3x_6 + x_4x_6\}$.

Similarly we can calculate that $S_{[\sigma(2)](\sigma(1))} = S_{[\sigma(1)](\sigma(2))}$. Therefore $S_{[\sigma(1),\sigma(2)]} = S_{[\sigma(1)](\sigma(2))}$.

Since $S_{[\sigma(1),\sigma(2)]} = S_{[\sigma(1)](\sigma(2))}$, we have that $S_{[\sigma(1),\sigma(2)](\sigma(3))} = S_{(\sigma(1),\sigma(2),\sigma(3))}$ so we have that:

$S_{[\sigma(1),\sigma(2)](\sigma(3))} = \{x_3 + x_4, x_4 + x_6, x_3x_4 + x_3x_6 + x_4x_6, x_3 + x_6, x_3 - x_4\}$.

Now we calculate $S_{[\sigma(1),\sigma(3)](\sigma(2))}$ and we get that:

$S_{[\sigma(1),\sigma(3)]} = \{x_2 + x_4 + x_5 + x_6, x_4 + x_6, x_2 + x_5, x_4 + x_5, x_2 + x_6, x_4x_2 - x_5x_6\}$, and that

$S_{[\sigma(1),\sigma(3)](\sigma(2))} = \{x_4 + x_6, x_4 + x_5, x_4 + x_5 + x_6\}$.

Now notice that $S_{[\sigma(1),\sigma(3)](\sigma(2))} \cap S_{[\sigma(1),\sigma(2)](\sigma(3))} = \{x_4 + x_6\}$. This set has significantly fewer polynomials than $S_{(\sigma(1),\sigma(2),\sigma(3))}$. Therefore $S_{[\sigma(1),\sigma(2),\sigma(3)]}$ has fewer polynomials than $S_{(\sigma(1),\sigma(2),\sigma(3))}$ so the Fubini reduction algorithm does reduce the number of polynomials in the reduction. We do not finish the calculation as we already know that W_4 is Fubini reducible with respect to ψ. $\square$

We define the *loop order* of a graph to be $E(G) - E(T)$ where T is a spanning tree of G. The loop order is also known as the dimension of the cycle space or the *Betti number* of G. It turns out, if G has loop order less than 6, then G is Fubini reducible with respect to the first Symanzik polynomial ([34]). Furthermore, most massless graphs with four on-shell momenta and loop order less than 3 are Fubini reducible [34]. A computational study of Fubini reducibility for graphs with no massive edges and four on-shell momenta was

undertaken by Lüders in [26]. For 3-loop graphs, he tested 109 graphs and found that 39 of them were not Fubini reducible. For 4-loop graphs, 600 graphs were tested and around 200 were found to not be Fubini reducible. Largely, the graphs that he found to not be reducible can be explained by having W_4 as a minor with the external momenta as in example 22. The other non-Fubini reducible graphs all will turn out to be reducible by the next polynomial reduction algorithm we introduce. For those that would like to actually run the Fubini algorithm to do tests as in [26], Christian Bogner has implemented the Fubini algorithm in Maple and his program MPL can be freely downloaded ([6]).

However despite the improvement the Fubini algorithm offers, this algorithm still produces spurious polynomials. Analysing Francis Brown's integration algorithm, it turns out that the polynomials $h_i g_j - h_j g_i$ only appear in the integration sometimes. Therefore one can create a bookkeeping method which keeps track of which keeps track of when we want to consider $h_i g_j - h_j g_i$ in our reduction. Additionally, we note that under computational tests, the Fubini reduction algorithm fares much better than the simple reduction algorithm for the number of graphs which are determined to be reducible. However, it is hard to prove infinite families of graphs are Fubini reducible as one needs to have a strong understanding of how the polynomials factor throughout the reduction. The compatibility graph algorithm deals with both concerns.

2.1.3 Compatibility Graph Reduction Algorithm

Compatibility graphs are essentially a book-keeping method for keeping track of when we want to consider the polynomial $h_i g_j - h_j g_i$ in our reduction algorithms. The notion first arose in [12], and was essential in proving an infinite family of graphs have Feynman integrals which evaluate to multiple zeta values. Let S be a set of polynomials. A *compatibility graph* for S, C_S, is a graph where $V(C_S) = \{f | f \in S\}$. If $fg \in E(C_S)$, then we will say the polynomials f and g are *compatible*. First, we describe the compatibility graph reduction modification for the simple reduction algorithm.

Let S be a set of polynomials in the polynomial ring $Q[\alpha_1, \alpha_2, \ldots, \alpha_r]$. Let σ be a permutation of $\{\alpha_1, \ldots, \alpha_r\}$. Let C_S be the initial compatibility graph for S. The idea will be to create a sequence of new sets of polynomials with rational coefficients and compatibility graphs $(S_{(\sigma(1))}, C_{(\sigma(1))}), (S_{(\sigma(1),\sigma(2))}, C_{(\sigma(1),\sigma(2))}), \ldots, (S_{(\sigma(1),\ldots,\sigma(r))}, C_{(\sigma(1),\ldots,\sigma(r))})$ and check that each polynomial in the set $S_{(\sigma(1),\ldots,\sigma(i))}$ is linear in $\sigma(i+1)$ for all $i \in \{1, \ldots, r-1\}$. Suppose we are at the kth iteration of the algorithm. If $k \geq 2$, then we have a set of polynomials with rational coefficients $S_{(\sigma(1),\ldots,\sigma(k-1))} = \{f_1, \ldots, f_n\}$ and compatibility graph $C_{(\sigma(1),\ldots,\sigma(k-1))}$. Otherwise $k = 1$ and we use S and C_S. We then do the following:

1. If there is a polynomial $f \in S_{(\sigma(1),\ldots,\sigma(k))}$ such that f is not linear in $\sigma(k)$, we end the algorithm, otherwise continue.

2. Then for all $i \in \{1, \ldots, n\}$, given a polynomial $f_i \in S_{(\sigma(1),\ldots,\sigma(k))}$, we write $f_i = g_i\sigma(k) + h_i$, where $g_i = \frac{\partial f_i}{\partial \sigma(k)}$ and $h_i = f_i|_{\sigma(k)=0}$.

3. Let $S^1 = \{g_i | i \in \{1, \ldots, n\}$. Let $S^2 = \{h_i | i \in \{1, \ldots, n\}\}$. Let $S^3 = \{g_ih_j - h_ig_j | i, j \in \{1, \ldots, n\}, i \neq j$ where f_i and f_j are compatible$\}$. Let $S^4 = S^1 \cup S^2 \cup S^3$.

4. Let $f \in S^4$ and let $f_1, f_2, f_3, \ldots, f_N$ be the polynomials such that $\prod_{i=1}^N f_i = f$, and f_i is an irreducible polynomial over $\mathbb{Q}$. Let $\tilde{S}$ be the set of polynomials $f_1, \ldots, f_N$ for each $f \in S^4$.

5. Let $S_{(\sigma(1),\ldots,\sigma(k))} = \tilde{S}$ and construct a new compatibility graph $C_{(\sigma(1),\ldots,\sigma(k))}$ with some rule set (see below).

6. Repeat the above steps with $S_{(\sigma(1),\ldots,\sigma(k))}$ in place of $S_{(\sigma(1),\ldots,\sigma(k-1))}$ and $C_{(\sigma(1),\ldots,\sigma(k))}$ in place of $C_{(\sigma(1),\ldots,\sigma(k-1))}$.

Observe that this algorithm is equivalent to the simple reduction algorithm if we set the compatibility graphs to be the complete graph at each step. Brown noticed that there is a scheme for constructing compatability graphs which results in removing non-trivial polynomials, while still getting an output which gives a suitable ordering for his integration algorithm. However, while his method for constructing compatability graphs works for the Symanzik polynomials, if one abstracts Feynman integrals to a more general class of integrals, it is unclear if Brown's compatability method still works to give a suitable order of integration for the integrals ([34]). In [34], Panzer describes an alternative construction which holds for more general integrals. We outline both constructions below.

2.1.3.1 First Compatibility Graph Construction

Before describing Francis Brown's method for constructing compatibility graphs, we give some intuition for the construction. Essentially, at some iteration of the reduction, we look back at the genealogy of the polynomials in our current set. If far back enough, the polynomials do not share a common ancestor then we will say those two polynomials are not compatible.

Now for the definition. Suppose we are at the kth iteration of the reduction algorithm under a given permutation σ of $\{\alpha_1, \ldots, \alpha_r\}$. Then we have the set $S_{(\sigma(1),\ldots\sigma(k-1))}$ and at the end of the iteration we obtain the set $S_{(\sigma(1),\ldots,\sigma(k))}$.

We now define $C_{(\sigma(1),\ldots,\sigma(k))}$. Recall, $V(C_{(\sigma(1),\ldots,\sigma(k))}) = \{f \mid f \in S_{(\sigma(1),\ldots\sigma(k-1))}\}$. To determine the edges of $C_{(\sigma(1),\ldots,\sigma(k))}$ we associate a set of 2-tuples to each vertex of $C_{(\sigma(1),\ldots,\sigma(k))}$ to keep track of how the polynomials were created. Let $m \in V(C_{(\sigma(1),\ldots,\sigma(k))})$.

2-tuple assignments

- If m is an irreducible factor of a polynomial $g_i \in S^1$ we associate the set $\{0, i\}$ with m.

- If m is an irreducible factor of some polynomial $h_i \in S^2$ then we associate the set $\{i, \infty\}$ with m. Additionally, if $h_i = f_i$ where $f_i \in S_{(\sigma(1),\ldots\sigma(k-1))}$ then we associate the set $\{0, i\}$ to m as well as $\{i, \infty\}$.

- If m is the irreducible polynomial of some polynomial $g_i h_j - h_i g_j \in S^3$, then we associate the set $\{i, j\}$ to m.

Now let $m, n \in V(C_{(\sigma(1),\ldots,\sigma(k))})$. The edge $mn \in E(C_{(\sigma(1),\ldots,\sigma(k))})$ if and only if there exists an set associated with m and an set associated with n such that their intersection is non-empty.

As an example, suppose a polynomial f is associated with the sets $\{1, \infty\}$ and $\{3, 4\}$, and polynomial g is associated with the sets $\{2, \infty\}$ and $\{1, 3\}$. Then since $\{3, 4\} \cap \{1, 3\} = \{3\}$, the polynomials f and g are compatible. However if g instead was associated with the sets $\{5, 6\}$ and $\{0, 2\}$, then f and g would not be compatible.

Note that this construction does indeed remove sone non-trivial polynomials. For the first Symanzik polynomial one must go at least three iterations before you can see the effects. We will see later in the chapter that when looking at both Symanzik polynomials we see the effect of compatibility graphs much sooner. The general principle is, under the above construction, if two polynomials are compatible, then they share an ancestor two iterations back ([12]). With this, we can now state one version of the compatibility graph reduction.

2.1.4 Brown's Compatibility Graph Reduction

Throughout this algorithm, all compatibility graphs will be constructed using Brown's technique outlined above. Let S be a set of polynomials in the polynomial ring $Q[\alpha_1, \alpha_2, \ldots, \alpha_r]$. Initialize C_S to be the complete graph on $|S|$ vertices and let σ be a permutation of $\{\alpha_1, \ldots, \alpha_r\}$.

We define $S_{[\sigma(1)]} = S_{(\sigma(1))}$ where we calculate $S_{(\sigma(1))}$ using the simple compatibility graph algorithm outlined at the start of the section. We define $C_{[\sigma(1)]} = C_{(\sigma(1))}$. We define $S_{[\sigma(1),\sigma(2)]} = S_{(\sigma(1),\sigma(2))} \cap S_{(\sigma(2),\sigma(1))}$ and $C_{[\sigma(1),\sigma(2)]}$ to be a graph where $fg \in E(C_{[\sigma(1),\sigma(2)]})$ if and only if $fg \in E(C_{(\sigma(1),\sigma(2))}) \cap E(C_{(\sigma(2),\sigma(1))})$. We define

$$S_{[\sigma(1),\ldots,\sigma(k)]} = \bigcap_{1 \leq i \leq k} S_{[\sigma(1),\ldots,\sigma\hat{(i)},\ldots,\sigma(k)](\sigma(i))},$$

and we define $C_{[\sigma(1),\ldots,\sigma(k)]}$ to be the compatibility graph for $S_{[\sigma(1),\ldots,\sigma(k)]}$ such that $fg \in C_{[\sigma(1),\ldots,\sigma(k)]}$ if and only if

$$fg \in \bigcap_{1 \leq i \leq k} E(C_{[\sigma(1),\ldots,\sigma\hat{(i)},\ldots,\sigma(k)](\sigma(i))}).$$
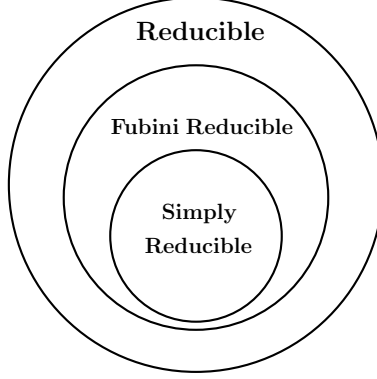
Figure 2.2: The relationship between the different versions of reducibility.

Definition 28. *Let S be a set of polynomials in the polynomial ring $\mathbb{Q}[\alpha_1, \ldots, \alpha_r]$. Let σ be a permutation of $\{\alpha_1, \ldots, \alpha_r\}$. We say S is compatibility graph reducible with respect to σ if for all $1 \leq i \leq r - 1$, all polynomials in the set $S_{[\sigma(1),\ldots,\sigma(i)]}$ are linear in $\sigma(i+1)$. If there exists a permutation σ such that S is compatibility graph reducible with respect to σ, we say that S is compatibility graph reducible. Given a graph G, and $S \subseteq \{\psi_G, \phi_G\}$, if S is compatibility graph reducible, then we say G is compatibility graph reducible with respect to S.*

As this will be the notion of reducibility which will be used for the rest of the thesis, we will abbreviate compatibility graph reducibility to just reducibility.

Remark 29. *If S is Fubini reducible with respect to σ, then S is reducible with respect to σ.*

Proof. Observe that the Fubini reduction algorithm is the compatibility graph reduction algorithm described in this section where at each step the compatibility graph is a complete graph. Let $S'_{[\sigma(1),\ldots,\sigma(k)]}$ denote the set obtained by the Fubini reduction algorithm at iteration k and $S_{[\sigma(1),\ldots,\sigma(k)]}$ the set obtained from the compatibility graph reduction algorithm at iteration k. Then we have $S_{[\sigma(1),\ldots,\sigma(k)]} \subseteq S'_{[\sigma(1),\ldots,\sigma(k)]}$ for every k. Since S is Fubini reducible, $S'_{[\sigma(1),\ldots,\sigma(k)]}$ is linear in $\sigma(k+1)$, and thus $S_{[\sigma(1),\ldots,\sigma(k)]}$ is linear in $\sigma(k+1)$, completing the proof. $\square$

Example 30. *The graphs W_4 and K_4 with external momenta as in example 21 and example 22 are not reducible with respect to the first and second Symanzik polynomials.*

Proof. Let σ be a permutation. In the compatibility graph reduction algorithm, the initial compatibility graph is the complete graph. Therefore, $S_{[\sigma(1)]}$ is the same under the Fubini algorithm as the compatibility graph algorithm. As seen in example 21 and example 22, $S_{[\sigma(1)]}$ is not linear in $\sigma(2)$, and thus W_4 and K_4 are not reducible. $\square$

There is an nice infinite family of graphs which are known to be reducible with respect to the first Symanzik polynomial.

Definition 31. *Let G be a graph with m edges. The* width *of an ordering $e_1, e_2, \ldots, e_m$ of $E(G)$ is the maximum order of a separation of the form $(\{e_1, \ldots, e_l\}, \{e_{l+1}, \ldots, e_m\})$ for $l \in \{1, \ldots, m\}$. To be consistent with the given definition of separation, here we are viewing $\{e_1, \ldots, e_l\}$ as a set of vertices induced by the edges $e_1, \ldots, e_l$. The* vertex width *of G is the minimum width among all edge orders of G.*

We note that the notion of vertex width is similar to the more well known width variant pathwidth.

Theorem 32 (Brown, [12]). *All graphs with vertex width ≤ 3 are reducible with respect to the first Symanzik polynomial.*

At this point, all infinite families of reducible graphs for the first Symanzik polynomial reside in the family of graphs with vertex width less than 3. Also it is important to note that in the proof of Theorem 32, compatability graphs are essential. A nice result of Black, Crump, Deos, and Yeats ([3]) is that a 3-connected graph G has vertex width ≤ 3 if and only if G does not contain one of the cube, octahedron, K_5, $K_{3,3}$ and one other graph as a minor (see Figure 2.3). This result also appears in Crump's masters thesis ([16]).

Note there are graphs which are do not satisfy vertex width ≤ 3 but are reducible with respect to the first Symanzik polynomial. For example, in [12], it is shown that both $K_{3,3}$ and K_5 are reducible with respect to the first Symanzik polynomial, but it is easy to see that both $K_{3,3}$ and K_5 have vertex width greater than 3, and they are in fact both forbidden minors for graphs having vertex width 3.

We note that in [26], a computational study of reducibility for graphs with four on-shell momenta and no massive edges was undertaken. From Lüder's tests, he found 34 graphs which are not reducible with respect to both Symanzik polynomials where the graph is massless and there are four on-shell momenta. Compared to the 39 graphs he found for Fubini reducibility, this is a slight improvement. This tells us there are graphs which are reducible but not Fubini reducible. See Figure 2.4 for a graph which is reducible but not Fubini reducible. We note that every single graph that he found was not reducible has either a rooted K_4-minor or a rooted W_4-minor (defined in chapter 3) which will explain all the graphs he found to be non-reducible. We end this section with an example of a compatibility graph reduction.

While it takes at least three iterations to start to see the usefulness of compatibility graphs for the first Symanzik polynomial, their usefulness is easy to see for reductions involving both Symanzik polynomials. Take any 3-connected graph G with some external momenta and no massive edges such that $\phi_G \neq 0$. Consider $\{\psi_G, \phi_G\}$. Let σ be any permutation of the variables in ψ_G and ϕ_G. Then in $S_{\sigma(1)}$, we have the polynomials
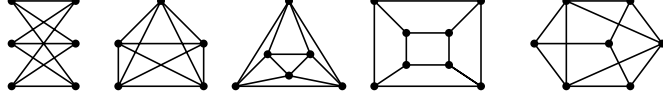
Figure 2.3: Forbidden minors for 3-connected graphs with vertex width 3.

$\psi_{G/e}, \psi_{G\backslash e}, \phi_{G/e}$, and $\phi_{G\backslash e}$, for some edge e corresponding to the edge associated with the Schwinger variable $\sigma(1)$. We note that those polynomials do not factor as $G \backslash e$ and G/e are both at least 2-connected since G is 3-connected. Assume that $\psi_{G/e}\phi_{G\backslash e} - \psi_{G\backslash e}\phi_{G/e}$ does not factor into any of $\psi_{G/e}, \psi_{G\backslash e}, \phi_{G/e}$, and $\phi_{G\backslash e}$. Let $\psi_G = f_1$ and $\phi_G = f_2$. Then the polynomial $\phi_{G/e}$ only is associated with the set $\{1, \infty\}$ and the polynomial $\phi_{G\backslash e}$ is only associated with the set $\{0, 2\}$. Thus $\phi_{G/e}$ and $\phi_{G\backslash e}$ are not compatible. So as quickly as the second step of the reduction algorithm we likely see fewer polynomials in the compatibility graph reduction versus the Fubini reduction.

For those who would like to test if sets are reducible, Panzer in [34] implemented this algorithm in his program HyperInt ([33]) and it is freely available for download. Additionally, Christian Bogner's MPL can also be used to test if sets are reducible ([6]).

2.1.4.1 Second Compatibility Graph Construction

Recall that the point of the reduction algorithms is to find a suitable order of integration for Brown's integration technique. The technique was created initially for Feynman integrals, but in principle it can be applied to a huge class of integrals far more general than just Feynman integrals. Up until the previous reduction algorithm, the reduction algorithms could be used to find an order for an arbitrary integral satisfying the correct properties and not necessarily just the Feynman integrals. However, in the last reduction algorithm the idea of intersecting compatibility graphs from the given construction to reduce the number of spurious polynomials may not guarantee that Brown's integration technique will still succeed for integrals not defined from the Symanzik polynomials ([34]). In [12], the compatability graph reduction given above was proven to maintain the necessary properties for Feynman integrals, but no proof was given for arbitrary integrals. In [34], there is some evidence that intersecting compatibility graphs as in the previous algorithm may still satsify the properties we want to use the integration algorithm, but no proof has been given. In [34], Panzer gave a different construction which gets around these problems. Following the notation of [34] we have:

Definition 33. *Let f_1 and f_2 be polynomials linear in α. Then $f_1 = g_1\alpha + h_1$ and $f_2 = g_2\alpha + h_2$. Then we define $[f_1, 0]_\alpha = g_1$, $[f_1, f_2]_\alpha = g_1h_2 - g_2h_1$ and $[f_1, \infty]_\alpha = h_1$ if $h_1 \neq 0$, otherwise $[f_1, \infty]_\alpha = g_1$.*

Let (S, C) be a set of polynomials in $\mathbb{Q}[\alpha_1, \ldots, \alpha_n]$ and its corresponding compatibility graph. Suppose all polynomials in S are linear in α_i. We define $(S_{\alpha_i}, C_{\alpha_i})$ as follows. We
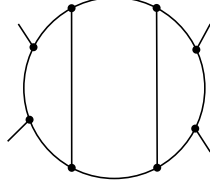
Figure 2.4: A graph which is not Fubini reducible, but is reducible with respect to both Symanzik polynomials. Here all edges are massless, and the external momenta are on-shell. Calculated in [26].

let S_{α_i} = irreducible factors of $\{[f, 0]_{\alpha_i} \ [f, g]_{\alpha_i} \ [f, \infty]_{\alpha_i}\}$ for $f \in S$ and $f, g \in S$ such that $fg \in E(C)$.

Then we say two polynomials p, q are compatible, or $pq \in E(C_{\alpha_i})$, if there exists $f, g, h \in S \cup \{0, \infty\}$ such that $p \mid [f, g]_{\alpha_i}$, $q \mid [g, h]_{\alpha_i}$ and each pair $fg, gh, hf \in E(C)$. Here we consider that 0 and ∞ are compatible with each other and compatible with all other polynomials.

Applying the above construction for compatibility graphs to the simple reduction algorithm gives an alternative reduction algorithm which in [34] can be used to find admissible orderings for a large class of integrals. Notice that this construction generates fewer polynomials per iteration, but by not intersecting off compatibility graphs, we possibly end up with extra polynomials relative to the other compatibility graph construction.

Definition 34. *Let S be a set of polynomials in $\mathbb{Q}[\alpha_1, \ldots, \alpha_n]$. Let C be the complete graph on $|S|$ vertices. If there exists a permutation σ of $\{\alpha_1, \ldots, \alpha_n\}$ such that $(S_{(\sigma(1),\ldots,\sigma(i))}, C_{(\sigma(1),\ldots,\sigma(i))})$ exist and are linear in $\sigma(i+1)$ for all $i \in \{1, \ldots, n-1\}$, then we will say S is reducible with respect to Panzer's method.*

Aside from proving that this reduction holds for more general integrals, this reduction algorithm was also used to proof a generalization of Theorem 32 to graphs with three not necessarily on-shell external momenta.

Definition 35. *Let G be a graph and $e_1, \ldots, e_{|E(G)|}$ an ordering of the edges of G. This ordering induces a sequence of graphs of G, $\emptyset \subsetneq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_{|E(G)|} = G$ such that G_i differs from G_{i+1} by edge e_{i+1}. Similarly, the ordering induces a sequence of graphs of G, $G = G^1 \supsetneq G^2 \supsetneq \ldots \supsetneq \emptyset$ where G^i differs from G^{i+1} by exactly one edge for all $i \in \{1, \ldots, |E(G)| - 1\}$. Define $A_i = V(G_i) \cap V(G^{i+1})$. The graph G is k-constructable if for all A_i, we have that $|A_i| = k$.*

We note that in the above definition, additional vertices are added to the graph if the edge being added has that vertex as an endpoint. Thus the notion of *adding a vertex* in the construction is well defined. Observe that if we restrict ourselves to 3-constructable graphs, they are precisely the graphs with vertex width less than 3.

Theorem 36 ([34]). *Let G be 3-constructable and let v_1, v_2, v_3 be the last three vertices added in the construction. Suppose v_1, v_2, v_3 are the only vertices with external momenta. Suppose there are no massive edges. Then G is reducible with respect to Panzer's method.*

While this reduction algorithm is seemingly weaker than the technique given by Brown, in [34] it is shown to prove most of the known results for reducible graphs. Additionally, a large number of computational results on reduciblity using this variant are covered in [34] which we do not include here.

Erik Panzer has implemented this reduction alogrithm as well as the reduction algorithm for Brown's construction in his software package HyperInt, which is freely available ([33]).

2.2   Some facts about Reducibility

Now that we have gone through the reduction algorithms, we are ready to prove some facts about reducibility. First we will prove some reducibility facts for arbitrary sets of polynomials, and then move into reducibility specifically for the Symanzik polynomials, with the main result giving a correct proof that reducibility is graph minor closed. We note that throughout this section, most of these observations have been noted in other papers or theses but not always explicitly written down.

Additionally, to avoid specifying which type of reducibility all the time, the word reducibility will refer to Brown's compatibility graph algorithm exclusively. That being said, most of the proofs apply to all the algorithms. We start this section by noting that reducibility is well behaved under subsets.

Lemma 37. *Let $S = \{f_1, \ldots, f_n\}$ be a set of polynomials in the polynomial ring $\mathbb{Q}[\alpha_1, \ldots, \alpha_r]$ and let σ be a permutation of $\{\alpha_1, \ldots, \alpha_r\}$. Suppose S is reducible with respect to σ. Then any subset $L \subseteq S$ is reducible with respect to σ.*

Proof. Consider the sequence of sets and compatibility graphs

$$(S, C_S), (S_{[\sigma(1)]}, C_{[\sigma(1)]}), \ldots, (S_{[\sigma(1), \ldots, \sigma(r-1)]}, C_{[\sigma(1), \ldots, \sigma(r-1)]}).$$

We claim that that for all $i \in \{0, 1, \ldots, r-1\}$, the set $L_{[\sigma(1), \ldots, \sigma(i)]} \subseteq S_{[\sigma(1), \ldots, \sigma(i)]}$ and that $C_{L_{[\sigma(1), \ldots, \sigma(i)]}}$ is a subgraph of $C_{S_{[\sigma(1), \ldots, \sigma(i)]}}[L_{[\sigma(1), \ldots, \sigma(i)]}]$. When $i = 0$ we consider the sets (S, C_S) and (L, C_L). We proceed by induction on i.

The base case follows trivially. Now consider the set $L_{[\sigma(1), \ldots, \sigma(i-1)]}$ and the compatibility graph $L_{C_{[\sigma(1), \ldots, \sigma(i-1)]}}$. By induction we have $L_{[\sigma(1), \ldots, \sigma(i-1)]} \subseteq S_{[\sigma(1), \ldots, \sigma(i-1)]}$ and that $C_{L_{[\sigma(1), \ldots, \sigma(i-1)]}}$ is a subgraph of $C_{S_{[\sigma(1), \ldots, \sigma(i-1)]}}[L_{[\sigma(1), \ldots, \sigma(i-1)]}]$. We will now apply one step of the reduction algorithm to these sets.

We claim that for all $1 \leq j \leq i$, we have $L_{[\sigma(1), \ldots, \hat{\sigma}(j), \ldots, \sigma(i)](\sigma(j))} \subseteq S_{[\sigma(1), \ldots, \hat{\sigma}(j), \ldots, \sigma(i)](\sigma(j))}$ when the set $S_{[\sigma(1), \ldots, \hat{\sigma}(j), \ldots, \sigma(i)](\sigma(j))}$ exists. We know that S is reducible with respect to

σ, so there exists a $j \in \{1, \ldots, i\}$ such that $S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}$ exists. Fix any such $j \in \{1, \ldots, i\}$. Then by induction we have

$$L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]} \subseteq S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]},$$

and $C_{L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}}$ is a subgraph of $C_{S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}}[L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}]$.

Note as $S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}$ exists, all polynomials in $L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}$ are linear in $\sigma(j)$. We will denote the set L^4 to be the set obtained in the second step of the reduction algorithm when starting with $L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}$, and S^4 will be the set obtained in the second step of the reduction algorithm applied to $S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}$. Then $L^4 \subseteq S^4$ as $L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]} \subseteq S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}$ and since if two polynomials in $L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}$ are compatible then they are compatible in $S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}$.

Let $\tilde{S}$ and $\tilde{L}$ be the sets obtained from step four of the reduction algorithm in for $S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}$ and $L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)]}$ respectively. Then as $L^4 \subseteq S^4$, we have that $\tilde{L} \subseteq \tilde{S}$. Now consider the compatibility graph constructed $C_{L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}}$. If $mn \in E(C_{L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}})$, then there is an 2-tuple associated to m and a 2-tuple associated to n such that their intersection is non-empty. By construction, these 2-tuples are derived from some polynomials in $L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}$. But we know that $L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))} \subseteq S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}$ and thus the polynomials which gave m and n the associated 2-tuples in $L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}$ exist in $S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}$. Thus $mn \in C_{S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}}$. Therefore we have $C_{L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}}$ is a subgraph of $C_{S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}}[L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))}]$. Therefore,

$$L_{[\sigma(1),\ldots,\sigma(i)]} = \bigcap_{1 \leq j \leq k} L_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))} \subseteq \bigcap_{1 \leq j \leq k} S_{[\sigma(1),\ldots,\sigma(\hat{j}),\ldots,\sigma(i)](\sigma(j))} = S_{[\sigma(1),\ldots,\sigma(i)]},$$

and $C_{L_{[\sigma(1),\ldots,\sigma(i)]}}$ is a subgraph of $C_{S_{[\sigma(1),\ldots\sigma(i)]}}[L_{[\sigma(1),\ldots,\sigma(i)]}]$, which completes the claim.
$\square$

Definition 38. *Let $S = \{f_1, \ldots, f_n\}$ be a set of polynomials in the polynomial ring $\mathbb{Q}[\alpha_1, \ldots, \alpha_r]$. We will denote the set of reducible orders, O_S, to be the permutations σ of $\{\alpha_1, \ldots, \alpha_r\}$ such that S is reducible with respect to σ.*

A possibly more useful restatement of Lemma 37 is:

Corollary 39. *Let $S = \{f_1, \ldots, f_n\}$ be a set of polynomials in the polynomial ring $\mathbb{Q}[\alpha_1, \ldots, \alpha_r]$. Let $L_1, \ldots, L_n$ be subsets of S. If*

$$\bigcap_{i=1}^{n} O_{L_i} = \emptyset,$$

then S is not reducible.

We would like to assume that in our initial set of polynomials, all of the polynomials are all irreducible with respect to $\mathbb{Q}$. Essentially we can, as long as we are able to get past the first step of the reduction algorithm.

Lemma 40. *Let $S = \{f_1, \ldots, f_n\}$ be a set of polynomials in the polynomial ring $\mathbb{Q}[\alpha_1, \ldots, \alpha_r]$. Let σ be a permutation of $\{\alpha_1, \ldots, \alpha_r\}$. Let S^I be the set of irreducible factors of S with rational coefficients. If S is reducible with respect to σ then S^I is reducible with respect to σ. Furthermore, if S^I is reducible with respect to σ, and all polynomials in S are linear in $\sigma(1)$, then S is reducible with respect to σ.*

Proof. Suppose S is reducible with respect to σ. Let $p \in S$ and $p = p_1 p_2$ where p_1, p_2 are polynomials in $\mathbb{Q}[\alpha_1, \ldots, \alpha_r]$. Let $L = S \setminus \{p\} \cup \{p_1, p_2\}$. We claim L is reducible with respect to σ. Note this completes the first claim as one can repeatedly apply this fact. We will show that $S_{[\sigma(1)]} = L_{[\sigma(1)]}$ and that their compatibility graphs are isomorphic. We consider two cases.

Case 1: Suppose $\deg(p, \sigma(1)) = 0$. Then $\deg(p_i, \sigma(1)) = 0$ for $i \in \{1, 2\}$. We perform the first iteration of the reduction algorithm on S and L. Let L^4 and S^4 be the sets obtained in step three of the reduction algorithm for $\sigma(1)$ for L and S respectively. As $\deg(p, \sigma(1)) = 0$, we have that $p_1, p_2 \in L^4$ and $p \in S^4$. Notice that the polynomial $g_i h_j - g_j h_i = p g_1$ when one of the polynomials is p. Similarly, when one of the polynomials is p_1 or p_2, we get $p_1 g_i$ or $p_2 g_i$. Then after factoring, we have $\tilde{L} = \tilde{S}$, and so $L_{[\sigma(1)]} = S_{[\sigma(1)]}$. Thus it suffices to show $C_{L_{\sigma(1)}}$ is isomorphic to $C_{S_{\sigma(1)}}$.

As $L = S \setminus \{p\} \cup \{p_1, p_2\}$, the graphs $C_{L_{[\sigma(1)]}}$ and $C_{S_{[\sigma(1)]}}$ have the same vertex set, and we may restrict our attention to compatibilities caused by p, p_1 and p_2. Notice the irreducible factors of p, p_1 and p_2 all contain an associated 2-tuple containing ∞ and 0, thus we may restrict our attention to 2-tuples not containing ∞ or 0.

Suppose an irreducible factor of p, say p', is adjacent to a vertex g' in $S_{C_{\sigma(1)}}$, where g' is an irreducible factor of some g_i. Since we are assuming that the compatibility did not arise from a 2-tuple containing $\{0\}$ or $\{\infty\}$ we may assume that compatibilities comes from the polynomial $p g_j$. Then without loss of generality, p' is an irreducible factor of p_1 and so $p_1 g_j$ would provide the desired index. The same argument works in the other direction as well, so we have that $C_{L_{[\sigma(1)]}}$ is isomorphic to $C_{S_{[\sigma(1)]}}$. As $S_{[\sigma(1)]} = L_{[\sigma(1)]}$ and their compatibility graphs are the same, since S is reducible, L is reducible.

Case 2: Suppose $\deg(p, \sigma(1)) = 1$. Notice that as p is linear in $\sigma(1)$, exactly one of p_1 or p_2 is linear in $\sigma(1)$. Without loss of generality, we assume $\deg(p_1, \sigma(1)) = 1$.

As before let $S^4, \tilde{S}$ and $L_4, \tilde{L}$ denote the sets obtained from the third and fourth step of the reduction for S and L respectively. Let $p_1 = g_1 \sigma(1) + h_1$. Then $p = p_2(g_1 \sigma(1) + h_1)$. Then $p_2, h_1, g_1 \in L^4$, and $p_2 g_1, p_2 h_1 \in S^4$. Notice that the polynomial $g_i h_j - h_i g_j = p_2(g_1 h_j - h_1 g_j)$ when one of the polynomials is p. Thus $S_{[\sigma(1)]}$ will contain all the irreducible factors obtained

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from p_1 and p_2. Similarly, $L_{[\sigma(1)]}$ will contain all the irreducible factors obtained from p, so $S_{[\sigma(1)]} = L_{[\sigma(1)]}$.

For the compatibility graphs, as before all irreducible factors of p are adjacent in $C_{S_{[\sigma(1)]}}$ as they share an 2-tuple from p. In $C_{L_{[\sigma(1)]}}$, all irreducible factors of either p_1 or p_2 are adjacent. This follows since $\deg(p_2, \sigma(1)) = 0$, all irreducible factors of p_2 have a 2-tuple with 0 and a 2-tuple with ∞, and every irreducible factor of p_1 has a 2-tuple containing either 0 or ∞. If f and g are adjacent in $C_{S_{[\sigma(1)]}}$ and the 2-tuples which make them adjacent came from a polynomial $p_2(g_1 h_j - h_1 g_j)$, then the desired 2-tuples exist for $C_{L_{[\sigma(1)]}}$ as the polynomials $p_2 g_1 \in L^4$ and $g_1 h_j - h_1 g_j \in L^4$. A similar statement holds for two adjacent polynomials in $C_{L_{[\sigma(1)]}}$. Therefore one can see that $C_{L_{[\sigma(1)]}}$ is isomorphic to $C_{S_{[\sigma(1)]}}$, and thus as S is reducible with respect to σ, L is reducible with respect to σ. Notice for the partial converse, the same argument works, as the only obstruction is if the reduction algorithm stops immediately. $\square$

As a remark, in the above proof we did not rely on the fact that the initial compatibility graph is complete. As we will see later in the chapter, it can be convenient to apply the lemma in the middle of the a reduction.

Lemma 41. *Let $S = \{f_1, \ldots, f_n\}$ be a set of irreducible polynomials in $\mathbb{Q}[\alpha_1, \ldots, \alpha_r]$. Let σ be a permutation of $\{\alpha_1, \ldots, \alpha_r\}$. Suppose we have sets $L' \subseteq S$ and $L'' \subseteq S$ such that $L' \cap L'' = \emptyset$, $L' \cup L'' = S$, and up to relabeling, there exists an $l \in \{1, \ldots, r-1\}$ such that all polynomials in L' are in the polynomial ring $\mathbb{Q}[\alpha_1, \ldots, \alpha_l]$ and all polynomials in L'' are in the polynomial ring $\mathbb{Q}[\alpha_{l+1}, \ldots, \alpha_r]$. Then S is reducible if and only if both of the sets L' and L'' are reducible.*

Proof. If S is reducible with respect to σ, then by Lemma 37 both L' and L'' are reducible with respect to σ.

Now suppose L' and L'' are reducible with respect to σ_1 and σ_2 respectively. Let $\sigma = \sigma_1 \sigma_2$ represent the concatenation of σ_1 and σ_2. We claim S is reducible with respect to σ. It suffices to show that $S_{[\sigma(1),\ldots,\sigma(l)]} = L''$, and that $C_{[\sigma(1),\ldots,\sigma(l)]} = K_{|L''|}$. By induction, we assume at the kth step, $k \leq l$, we have $S_{[\sigma(1),\ldots,\sigma(k-1)]} = \{L'_{[\sigma(1),\ldots,\sigma(k-1)]}, L''\}$ and that the compatibility graph is the compatibility graph for $L'_{[\sigma(1),\ldots,\sigma(k-1)]}$ with every vertex of L'' being a dominating vertex. We note that the base case calculation is the same as the inductive step, and thus we ignore the base case. Then we want to show that

$$S_{[\sigma(1),\ldots,\sigma(k)]} = \bigcap_{1 \leq i \leq k+1} S_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))} = \{L'_{[\sigma(1),\ldots,\sigma(k-1)]}, L''\}.$$

So it suffices to show that for any i, $1 \leq i \leq k+1$, that

$$S_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))} = \{L'_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))}, L''\}.$$

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Fix $i \in \{1, \ldots, k\}$. Notice that $S_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))}$ exists if and only if $L'_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))}$ exists, and thus we may assume $L'_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))}$ exists.

Let $p \in L'_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))}$. Then p is the irreducible factor for some p' where either $p' = g_m$, $p' = h_m$ or $p' = g_m h_j - h_m g_j$ for some $f_m, f_j \in L'_{[\sigma(1),\ldots,\sigma(k)]}$. By our induction hypothesis, $f_m, f_j \in S_{[\sigma(1),\ldots,\sigma(k)]}$, and thus $p \in S'_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))}$. Also, all polynomials from L'' are in $S_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))}$ since all polynomials in L'' are irreducible and do not contain $\sigma(i)$ as a variable.

Therefore, it suffices to show that in the compatibility graph for $S_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))}$, all polynomials in L'' are dominating vertices. Let $f \in L''$. By assumption $\deg(f, \alpha(i)) = 0$ so f has receives a 2-tuple containing 0 and a 2-tuple containing ∞. Consider any polynomial $f_z \in S_{[\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)](\sigma(i))}$. Then the polynomial $f_z f \in S^4$ gives f a 2-tuple containing z. Therefore for every possible value one of the associated 2-tuples can have, f has a 2-tuple containing that value. Thus f is adjacent to every polynomial, completing the claim. $\square$

Now we will prove an important relationship between reducibility, leading coefficients and setting variables to zero, which is going to give us our desired graph minor closed property.

Theorem 42 ([7]). *Let $S = \{P_1, \ldots, P_N\}$ be a set of polynomials which is Fubini reducible in the order $(\alpha_1, \ldots, \alpha_n)$. Fix some $l \in \{1, \ldots, n\}$ and suppose that each polynomial $P_i \in S$ is linear in α_l for all $1 \leq i \leq N$. Then the sets $S^l = \{\frac{\partial P_1}{\partial \alpha_l}, \frac{\partial P_2}{\partial \alpha_l}, \ldots, \frac{\partial P_n}{\partial \alpha_l}\}$ and $S_l = \{P_1|_{\alpha_l=0}, \ldots, P_N|_{\alpha_l=0}\}$ are Fubini reducible in the order $(\alpha_1, \ldots, \hat{\alpha}_l, \ldots, \alpha_n)$.*

The proof given in [7] for Theorem 42 is flawed as printed, however it appears that their proof can be fixed with some effort, but we do not go through the details (the suggested fix pointed was out by Christian Bogner via email correspondence ([5])). Instead, we prove essentially the same theorem for reducibility (split into two theorems as the arguments are slightly different).

Theorem 43. *Let $S = \{P_1, \ldots, P_N\}$ be a set of polynomials which is reducible in the order $(\alpha_1, \ldots, \alpha_n)$. Fix some $l \in \{1, \ldots, n\}$. Then the set $S^l = \{P_1|_{\alpha_l=0}, \ldots, P_N|_{\alpha_l=0}\}$ is reducible in the order $(\alpha_1, \ldots, \alpha_n)$.*

Proof. Let S and S^l be a counterexample with n minimized, as in, S is reducible with order $(\alpha_1, \ldots, \alpha_n)$, but S^l is not reducible with order $(\alpha_1, \ldots, \alpha_n)$.

First suppose that $l = 1$. Then for all polynomials $P \in S$, we have that $\deg(P, \alpha_l) \leq 1$. Then notice that $S^l = S^2$ where S^2 is the set obtained by applying the reduction algorithm to S for α_l. Then $S^l_{[\alpha_l]} \subseteq S_{[\alpha_1]}$. Then by Lemma 37, we have that $S^l_{[\alpha_l]}$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$, which implies that S^l is reducible with order $(\alpha_1, \ldots, \alpha_n)$, a contradiction.

Therefore we assume that $l \neq 1$ and consider one step of the reduction algorithm. As S is reducible with order $(\alpha_1, \ldots, \alpha_n)$, we have that $S_{[\alpha_1]}$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$. Consider the set $S^{l'}_{[\alpha_1]} = \{f|_{\alpha_l=0} | f \in S_{[\alpha_1]}\}$. Then since S is a minimal counterexample

with respect to n, we have that $S^{l'}_{[\alpha_1]}$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$. Additionally, by Lemma 40, the set of irreducible polynomials of $S^{l'}_{[\alpha_1]}$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$. Notice from the definitions we have:

$$S^{l'}_{[\alpha_1]} = \{f|_{\alpha_l=0}|f \in S_{[\alpha_1]}\}$$
$$= \{f|_{\alpha_l=0}|f \in \text{irreducible factors of } S^4\}.$$

For notational convenience, we will say S^{jl} will be the set S^j obtained by starting with S^l for $j \in \{1, 2, 3, 4\}$. Suppose we have a polynomial f, and $f = f_1 f_2$ for some polynomials f_1 and f_2. Then fix any variable α and notice that $f|_{\alpha=0} = f_1|_{\alpha=0} f_2|_{\alpha=0}$. This follows since any term which contains α in f is generated by a pair of terms in f_1 and f_2, where at least one of these terms contains α. Now, notice that $S^l_{[\alpha_1]}$ exists and,

$$
\begin{aligned}
S^l_{[\alpha_1]} &= \text{irreducible factors of } S^{4l} \\
&= \text{irreducible factors of } (S^{1l} \cup S^{2l} \cup S^{3l}) \\
&= \text{irreducible factors of } (\{\frac{\partial f}{\partial \alpha_1}|f \in S^l\} \cup \{f|_{\alpha_1=0}|f \in S^l\} \cup \\
&\quad \{\frac{\partial f_1}{\partial \alpha_1} f_2|_{\alpha_1=0} - \frac{\partial f_2}{\partial \alpha_1} f_1|_{\alpha_1=0}|f_1, f_2 \in S^l\}) \\
&= \text{irreducible factors of } (\{\frac{\partial f|_{\alpha_l=0}}{\partial \alpha_1}|f \in S\} \cup \{f|_{\alpha_l,\alpha_1=0}|f \in S\} \cup \\
&\quad \{\frac{\partial f_1|_{\alpha_l=0}}{\partial \alpha_1} f_2|_{\alpha_l,\alpha_1=0} - \frac{\partial f_2|_{\alpha_l=0}}{\partial \alpha_1} f_1|_{\alpha_l,\alpha_1=0}|f_1, f_2 \in S\}) \\
&= \text{irreducible factors of } (\{\frac{\partial f}{\partial \alpha_1}|_{\alpha_l=0}|f \in S\} \cup \{f|_{\alpha_l,\alpha_1=0}|f \in S\} \cup \\
&\quad \{\frac{\partial f_1}{\partial \alpha_1}|_{\alpha_l=0} f_2|_{\alpha_l,\alpha_1=0} - \frac{\partial f_2}{\partial \alpha_1}|_{\alpha_l=0} f_1|_{\alpha_l,\alpha_1=0}|f_1, f_2 \in S\}) \\
&= \text{irreducible factors of } (\{\frac{\partial f}{\partial \alpha_1}|_{\alpha_l=0}|f \in S\} \cup \{f|_{\alpha_l,\alpha_1=0}|f \in S\} \cup \\
&\quad \{(\frac{\partial f_1}{\partial \alpha_1} f_2|_{\alpha_1=0} - \frac{\partial f_2}{\partial \alpha_1} f_1|_{\alpha_1=0})|_{\alpha_l=0}|f_1, f_2 \in S\}) \\
&\subseteq \text{irreducible factors of } \{f|_{\alpha_l=0}|f \in \text{irreducible factors of } S^4\} \\
&= \text{irreducible factors of } S^{l'}_{[\alpha_1]}.
\end{aligned}
$$

By previous observations, the set of irreducible factors of $S^{l'}_{[\alpha_1]}$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$, so by Lemma 37, we get that $S^l_{[\alpha_1]}$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$, completing the proof. $\square$

Lemma 44. *Let f_1 and f_2 be polynomials linear in α_1. Fix a variable α_l such that $\alpha_l \neq \alpha_1$. For a polynomial f, let $lc(f)$ be the leading coefficient of f with respect to α_l. Then either*

1. $\frac{\partial lc(f_1)}{\partial \alpha_1} lc(f_2)|_{\alpha_1=0} - \frac{\partial lc(f_2)}{\partial \alpha_1} lc(f_1)|_{\alpha_1=0} = 0$ *or*

2. $\frac{\partial lc(f_1)}{\partial \alpha_1} lc(f_2)|_{\alpha_1=0} - \frac{\partial lc(f_2)}{\partial \alpha_1} lc(f_1)|_{\alpha_1=0} = lc(\frac{\partial f_1}{\partial \alpha_1} f_2|_{\alpha_1=0} - \frac{\partial f_{|}2}{\partial \alpha_1} f_1|_{\alpha_1=0})$.

Proof. Let $f_1 = \sum_{i=0}^{k}(f_{1,i,1}\alpha_1 + f_{1,i,2})\alpha_l^i$ and $f_2 = \sum_{j=0}^{t}(f_{2,j,1}\alpha_1 + f_{2,j,2})\alpha_l^j$. Then

$$\frac{\partial lc(f_1)}{\partial \alpha_1} lc(f_2)|_{\alpha_1=0} - \frac{\partial lc(f_2)}{\partial \alpha_1} lc(f_1)|_{\alpha_1=0} = f_{1,k,1}f_{2,t,2} - f_{2,t,1}f_{1,k,2},$$

and

$$lc(\frac{\partial f_1}{\partial \alpha_1} f_2|_{\alpha_1=0} - \frac{\partial f_2}{\partial \alpha_1} f_1|_{\alpha_1=0}) = lc((\sum_{i=0}^{k} f_{1,i,1}\alpha_l^i)(\sum_{j=0}^{t} f_{2,j,2}\alpha_l^j) - (\sum_{j=0}^{t} f_{2,j,1}\alpha_l^j)(\sum_{i=0}^{k} f_{1,i,2}\alpha_l^i)).$$

Now we consider various cases. If $f_{1,k,1}f_{2,t,2} - f_{2,t,1}f_{1,k,2} = 0$, then the claim immediately holds. Therefore we assume that $f_{1,k,1}f_{2,t,2} - f_{2,t,1}f_{1,k,2} \neq 0$. Suppose each of $f_{1,k,1}, f_{2,t,2}, f_{2,t,1}$ and $f_{1,k,2}$ are non-zero, then the largest power of α_l in $(\sum_{i=0}^{k} f_{1,i,1}\alpha_l^i)(\sum_{j=0}^{t} f_{2,j,2}\alpha_l^j) - (\sum_{j=0}^{t} f_{2,j,1}\alpha_l^j)(\sum_{i=0}^{k} f_{1,i,2}\alpha_l^i)$ is α_l^{k+t}, and thus

$$lc((\sum_{i=0}^{k} f_{1,i,1}\alpha_l^i)(\sum_{j=0}^{t} f_{2,j,2}\alpha_l^j) - (\sum_{j=0}^{t} f_{2,j,1}\alpha_l^j)(\sum_{i=0}^{k} f_{1,i,2}\alpha_l^i)) = f_{1,k,1}f_{2,t,2} - f_{2,t,1}f_{1,k,2}.$$

Now suppose that $f_{1,k,1} = 0$ and $f_{2,t,2}, f_{2,t,1}$, and $f_{1,k,2}$ are not zero. Then the highest power of α_l in $(\sum_{i=0}^{k} f_{1,i,1}\alpha_l^i)(\sum_{j=0}^{t} f_{2,j,2}\alpha_l^j) - (\sum_{j=0}^{t} f_{2,j,1}\alpha_l^j)(\sum_{i=0}^{k} f_{1,i,2}\alpha_l^i)$ is α_l^{k+t} which only occurs in $(\sum_{j=0}^{t} f_{2,j,1}\alpha_l^j)(\sum_{i=0}^{k} f_{1,i,2}\alpha_l^i)$ so,

$$lc((\sum_{i=0}^{k} f_{1,i,1}\alpha_l^i)(\sum_{j=0}^{t} f_{2,j,2}\alpha_l^j) - (\sum_{j=0}^{t} f_{2,j,1}\alpha_l^j)(\sum_{i=0}^{k} f_{1,i,2}\alpha_l^i)) = -f_{2,t,1}f_{1,k,2}.$$

Now suppose that $f_{1,k,2} = 0$ and $f_{2,t,2}, f_{2,t,1}$, and $f_{1,k,2}$ are not zero. Then the highest power of α_l in $(\sum_{i=0}^{k} f_{1,i,1}\alpha_l^i)(\sum_{j=0}^{t} f_{2,j,2}\alpha_l^j) - (\sum_{j=0}^{t} f_{2,j,1}\alpha_l^j)(\sum_{i=0}^{k} f_{1,i,2}\alpha_l^i)$ is α_l^{k+t} so,

$$lc((\sum_{i=0}^{k} f_{1,i,1}\alpha_l^i)(\sum_{j=0}^{t} f_{2,j,2}\alpha_l^j) - (\sum_{j=0}^{t} f_{2,j,1}\alpha_l^j)(\sum_{i=0}^{k} f_{1,i,2}\alpha_l^i)) = f_{1,k,1}f_{2,t,2}.$$

The case where $f_{2,t,1} = 0$ and $f_{2,t,2}, f_{1,k,1}$, and $f_{1,k,2}$ are non-zero and the case where $f_{2,t,2} = 0$ and $f_{2,t,1}, f_{1,k,1}, f_{1,k,2}$ are non-zero follow similarly.

Now consider the case where $f_{1,k,1} = 0$, $f_{2,t,1} = 0$, and $f_{1,k,2}, f_{2,t,1}$ are not zero. Then $f_{1,k,1}f_{2,t,2} - f_{2,t,1}f_{1,k,2} = 0$, satisfying the claim.

Now consider the case where $f_{1,k,2} = 0, f_{2,t,2} = 0$, and $f_{1,k,1}, f_{2,t,1}$ are not zero. Then $f_{1,k,1}f_{2,t,2} - f_{2,t,1}f_{1,k,2} = 0$, satisfying the claim.

Now consider the case where $f_{1,k,1} = 0$, $f_{2,t,2} = 0$ and $f_{1,k,2}, f_{2,t,1}$ are non-zero. Then the highest power of α_l in $(\sum_{i=0}^{k} f_{1,i,1}\alpha_l^i)(\sum_{j=0}^{t} f_{2,j,2}\alpha_l^j) - (\sum_{j=0}^{t} f_{2,j,1}\alpha_l^j)(\sum_{i=0}^{k} f_{1,i,2}\alpha_l^i)$ is α_l^{k+t} so,

$$lc((\sum_{i=0}^{k} f_{1,i,1}\alpha_l^i)(\sum_{j=0}^{t} f_{2,j,2}\alpha_l^j) - (\sum_{j=0}^{t} f_{2,j,1}\alpha_l^j)(\sum_{i=0}^{k} f_{1,i,2}\alpha_l^i)) = -f_{2,t,1}f_{1,k,2}.$$

The case where $f_{1,k,2} = 0$, $f_{2,t,1} = 0$ and $f_{1,k,1}, f_{2,t,1}$ are non-zero follows similarly. Notice that these are all possible cases, so the claim holds. $\qquad\square$

Theorem 45. *Let $S = \{P_1, \ldots, P_N\}$ be a set of polynomials which is reducible in the order $(\alpha_1, \ldots, \alpha_n)$. Fix $l \in \{1, \ldots, n\}$. For all $i \in \{1, \ldots, N\}$, Let $S^l = \{lc(P)|P \in S\}$. Then S^l is reducible with the order $(\alpha_1, \ldots, \alpha_n)$.*

Proof. We proceed by induction on n. If $n = 1$, then all polynomials in S are monomials, and the result is immediate. Therefore we assume $n > 1$.

First we suppose that $l = 1$. Then for all $P \in S$, we have $\deg(P, \alpha_l) \leq 1$. Then $S^l = S^1$ where S^1 is the set obtained in the reduction algorithm applied to S and α_l. Then $S^l_{[\alpha_l]} \subseteq S_{[\alpha_l]}$. Since $S_{[\alpha_l]}$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$, by Lemma 37 we have that $S^l_{[\alpha_l]}$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$, and thus S^l is reducible with order $(\alpha_1, \ldots, \alpha_n)$.

Therefore we can assume that $l \neq 1$. Now consider the set $(S_{[\alpha_1]})^l$, which we define to be the set of leading coefficients of polynomials in $S_{[\alpha_1]}$ with respect to α_l. Since S is reducible with order $(\alpha_1, \ldots, \alpha_n)$, we have that $S_{[\alpha_1]}$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$ and thus by induction, $(S_{[\alpha_1]})^l$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$. Notice from the definitions we have:

$$\begin{aligned}(S_{[\alpha_1]})^l &= \{lc(f)|f \in S_{[\alpha_1]}\} \\ &= \{lc(f)|f \in \text{irreducible factors of } S^4\}.\end{aligned}$$

For notational convenience, we will say S^{jl} will be the set S^j obtained by starting with S^l for $j \in \{1, 2, 3, 4\}$. Suppose we have a polynomial f, and $f = f_1 f_2$ for some polynomials f_1 and f_2. Now, notice that $S^l_{[\alpha_1]}$ exists and,

$$\begin{aligned}S^l_{[\alpha_1]} &= \text{irreducible factors of } S^{4l} \\ &= \text{irreducible factors of } (S^{1l} \cup S^{2l} \cup S^{3l}) \\ &= \text{irreducible factors of } (\{\frac{\partial f}{\partial \alpha_1}|f \in S^l\} \cup \{f|_{\alpha_1=0}|f \in S^l\} \cup \\ &\quad \{\frac{\partial f_1}{\partial \alpha_1}f_2|_{\alpha_1=0} - \frac{\partial f_2}{\partial \alpha_1}f_1|_{\alpha_1=0}|f_1, f_2 \in S^l\}) \\ &= \text{irreducible factors of } (\{\frac{\partial lc(f)}{\partial \alpha_1}|f \in S\} \cup \{lc(f)|_{\alpha_1=0}|f \in S\} \cup \\ &\quad \{\frac{\partial lc(f_1)}{\partial \alpha_1}lc(f_2)|_{\alpha_1=0} - \frac{\partial lc(f_2)}{\partial \alpha_1}lc(f_1)|_{\alpha_1=0}|f_1, f_2 \in S\}) \\ &\subseteq \text{irreducible factors of } (\{lc(\frac{\partial f}{\partial \alpha_1})|f \in S\} \cup \{lc(f|_{\alpha_1=0})|f \in S\} \cup \\ &\quad \{lc(\frac{\partial f_1}{\partial \alpha_1}f_2|_{\alpha_1=0} - \frac{\partial f_2}{\partial \alpha_1}f_1|_{\alpha_1=0})|f_1, f_2 \in S\})\end{aligned}$$

$$= \text{irreducible factors of } \{lc(f)|f \in \text{irreducible factors of } S^4\}$$
$$= \text{irreducible factors of } (S_{[\alpha_1]})^l$$

We note that the subset relationship between lines 4 and 5 above holds by appealing to Lemma 44 and assuming that if $\frac{\partial lc(f_1)}{\partial \alpha_1}lc(f_2)|_{\alpha_1=0} - \frac{\partial lc(f_2)}{\partial \alpha_1}lc(f_1)|_{\alpha_1=0} = 0$, then we remove the 0 from the set. This does not affect reducibility since 0 is a constant (see Remark 18).

Then since $(S_{[\alpha_1]})^l$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$, the set of irreducible factors of $(S_{[\alpha_1]})^l$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$ by Lemma 40. Then by Lemma 37, we have that $S^l_{[\alpha_1]}$ is reducible with order $(\alpha_2, \ldots, \alpha_n)$, completing the claim. $\qquad\square$

Corollary 46. *Let r be a fixed positive integer. Let $\mathcal{G}$ be the set of graphs which have r external momenta, no massive edges, and are reducible with respect to $\{\phi, \psi\}$. Then if $G \in \mathcal{G}$, all minors of G are in $\mathcal{G}$.*

Proof. Let $G \in \mathcal{G}$. Pick any edge $e \in E(G)$. We consider two cases.

Case 1: Suppose e is a loop. Notice for loops, deletion and contraction are the same. Then by Lemma 7, we have $\{\phi_G, \psi_G\} = \{\phi_{G\backslash e}\alpha_e, \psi_{G\backslash e}\alpha_e\}$. By Lemma 40, $\{\phi_{G\backslash e}\alpha_e, \psi_{G\backslash e}\alpha_e\}$ is reducible if and only if $\{\phi_{G\backslash e}, \psi_{G\backslash e}, \alpha_e\}$ is reducible. Since α_e is a monomial, we may remove it and thus we get that $\{\phi_G, \psi_G\}$ is reducible if and only if $\{\phi_{G\backslash e}, \psi_{G\backslash e}\}$ is reducible. Therefore $G \backslash e \in \mathcal{G}$ and thus by induction, all minors of $G \backslash e$ are in $\mathcal{G}$.

Case 2: Suppose that e is not a loop edge and consider $G\backslash e$ and G/e. By Lemma 6 the Symanzik polynomials for $G \backslash e$ are $\{\frac{\partial}{\partial \alpha_e}\phi_G, \frac{\partial}{\partial \alpha_e}\psi_G\}$. Similarly by Lemma 6 the Symanzik polynomials for G/e are $\{\psi_{G|_{\alpha_e=0}}, \phi_{G|_{\alpha_e=0}}\}$. Notice that since we have no massive edges, both of the Symanzik polynomials are linear in α_l, thus $\frac{\partial}{\partial \alpha_e}\phi_G = lc(\phi_G)$ and $\frac{\partial}{\partial \alpha_e}\phi_G = lc(\phi_G)$. Thus both sets are reducible by appealing to Theorem 45 and Theorem 43 and thus in $\mathcal{G}$. Therefore by induction, all minors of G are in $\mathcal{G}$, completing the claim. $\qquad\square$

Therefore we have that reducibility with respect to the Symanzik polynomials when we have a fixed set of external momenta is graph minor closed. This fact motivates chapter three. Notice that by similar arguments, reducibility with respect to the first (or second) Symanzik polynomial is also graph minor closed. We note it is interesting that in the above proof, that leading coefficients were the key to prove reducibility is closed under contraction, whereas in the literature contraction is usually viewed as taking a derivative. Now we can give some easy connectivity reductions for reduciblity of graphs with respect to the Symanzik polynomials.

Lemma 47. *Let G be a graph with 1-separation (A, B). Then G is reducible with respect to the first Symanzik polynomial if and only if $G[A]$ and $G[B]$ are reducible with respect to the first Symanzik polynomial.*

Proof. By Lemma 11 we have $\psi_G = \psi_{G[A]}\psi_{G[B]}$. So the set $\{\psi_G\}$ is reducible if and only if $\{\psi_{G[A]}, \psi_{G[B]}\}$ is reducible by Lemma 40. Observe that there are no shared variables

between $\psi_{G[A]}$ and $\psi_{G[B]}$. Then by Lemma 41, $\{\psi_{G[A]}, \psi_{G[B]}\}$ is reducible if and only if both $\{\psi_{G[A]}\}$ and $\{\psi_{G[B]}\}$ are reducible. By definition, that occurs exactly when both $G[A]$ and $G[B]$ are reducible with respect to the first Symanzik polynomial. $\qquad\square$

Lemma 48. *Let G be a graph with where $X = \{a, b, c, d\} \subseteq V(G)$ is the set of vertices with momenta, and assume that all external momenta is on-shell. Furthermore, suppose G has no massive edges. Suppose G has a 1-separation (A, B). Suppose $a, b, c \in A \setminus (A \cap B)$ and $d \in B \setminus (A \cap B)$. Then G is reducible with respect to both Symanzik polynomials if and only if $G[A]$ is reducible with respect to both Symanzik polynomials and $G[B]$ is reducible with respect to the first Symanzik polynomial.*

Proof. By Lemma 11 and Lemma 13, we have $\phi_G = \phi_{G[A]}\psi_{G[B]}$ and $\psi_G = \psi_{G[A]}\psi_{G[B]}$. Then $\{\phi_G, \psi_G\}$ is reducible if and only if $\{\phi_{G[A]}, \psi_{G[A]}, \psi_{G[B]}\}$ is reducible by Lemma 40. By Lemma 41, $\{\phi_{G[A]}, \psi_{G[A]}, \psi_{G[B]}\}$ is reducible if and only if both $\{\phi_{G[A]}, \psi_{G[A]}\}$ and $\{\psi_{G[B]}\}$ are reducible. By definition, $\{\phi_{G[A]}, \psi_{G[A]}\}$ and $\{\psi_{G[B]}\}$ are reducible exactly when $G[A]$ is reducible with respect to both the first and second Symanzik polynomials, and $G[B]$ is reducible with respect to the first Symanzik polynomial. $\qquad\square$

Lemma 49. *Let G be a graph with where $X = \{a, b, c, d\} \subseteq V(G)$ is the set of vertices with momenta, and assume that all external momenta are on-shell. Furthermore, suppose G has no massive edges. Suppose G has a 1-separation (A, B) and that $X \subseteq A$. Then G is reducible with respect to both Symanzik polynomials if and only $G[A]$ is reducible with respect to both Symanzik polynomials and $G[B]$ is reducible with respect to the first Symanzik polynomial.*

Proof. By Lemma 11 and Lemma 13 we have that $\phi_G = \phi_{G[A]}\psi_{G[B]}$ and $\psi_G = \psi_{G[A]}\psi_{G[B]}$. Then by $\{\phi_G, \psi_G\}$ is reducible if and only if $\{\phi_{G[A]}, \psi_{G[A]}, \psi_{G[B]}\}$ is reducible by Lemma 40. By Lemma 41, $\{\phi_{G[A]}, \psi_{G[A]}, \psi_{G[B]}\}$ is reducible if and only if both $\{\phi_{G[A]}, \psi_{G[A]}\}$ and $\{\psi_{G[A]}\}$ are reducible. By definition, $\{\phi_{G[A]}, \psi_{G[A]}\}$ and $\{\psi_{G[B]}\}$ are reducible exactly when $G[A]$ is reducible with respect to both the first and second Symanzik polynomials, and $G[B]$ is reducible with respect to the first Symanzik polynomial. $\qquad\square$

One cannot extend the above lemmas to when we have a 1-separation (A, B) such that two of the vertices with external momenta are in $A \setminus (A \cap B)$ and two of the momenta are in $B \setminus (A \cap B)$. This follows since the second Symanzik polynomial does not factor nicely to allow a reduction to the 2-connected case (see Lemma 14).

2.3   Graphs which are not reducible

A celebrated result is that if a graph class is graph minor closed, then it is characterized by a finite set of forbidden minors ([38]). Then by Corollary 46, reducibility with a fixed number
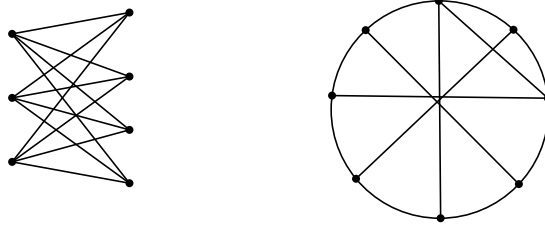
Figure 2.5: Forbidden minors for reducibility with respect to ψ.

of momenta is characterized by a finite set of forbidden minors. Therefore it is of interest to try and exactly determine that set. We do not claim to have anywhere near a complete characterization, but we list some of the graphs which are known to be non-reducible here.

Quite a bit of work has been done on reducibility with respect to the first Symanzik polynomial. In [12], the graph $K_{3,4}$ was shown to be non-reducible with respect to ψ and that, in particular, for any permutation σ, the set $S_{[\sigma(1),...,\sigma(7)]}$ (if it exists) is quadratic in $\sigma(8)$. Furthermore, by direct computation, one can show that all proper minors of $K_{3,4}$ are reducible. Therefore $K_{3,4}$ is a forbidden minor for the first Symanzik polynomial ([12]).

Also in [12], Brown notes that the graph $V_8 \cup \{02\}$ (see Figure 2.5, also known as $Q48$ in [39], V_8 is Wagner's graph as defined in chapter 1) is not reducible with respect to ψ. Again by direct computation, it is possible to show that $V_8 \cup \{02\}$ is a forbidden minor with respect to ψ ([12]).

A census of periods of Feynman integrals has been constructed in [39]. Since reducibility is tied to Brown's integration algorithm, and we know that when Brown's algorithm succeeds that the Feynman integral is a multiple zeta value, all graphs in [39] whose Feynman integral is not a multiple zeta value are not reducible. There are some graphs which are known to not have periods which are multiple zeta values, but they are almost certainly not forbidden minors. However, it is quite difficult computationally to try and find the forbidden minor from these graphs. For example, we attempted to show K_6 was not reducible by running it on the Westgrid servers for seven days but it was only around halfway through the reduction, and seemingly slowing down as the reduction continued.

One graph which I suspect to be a forbidden minor for reducibility with respect to the first Symanzik polynomial is K_6. While I leave it open on whether or not K_6 is reducible, we will see in Corollary 119, that if K_6 is not reducible it does not change the class of reducible graphs much.

Now we shift focus to reducibility with respect to both Symanzik polynomials, where the underlying graph has four on-shell external momenta, and G has no massive edges. From example 21 and 22 we have that K_4 and W_4 are not reducible with four on-shell external momenta. Furthermore, one can directly compute that those examples are minor minimal. Using HyperInt, one can show that $K_{2,4}$ and the other graph in Figure 2.6 which
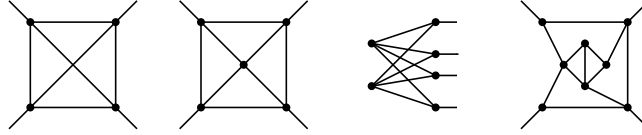
Figure 2.6: Forbidden minors for reducibility with respect to both Symanzik polynomials. The vertices with the on-shell momenta are denoted by lines with adjacent to only one vertex.

we will call L, are forbidden minors. The graph L was found by doing a brute force test for reducibility on small graphs. We note that by Lemma 37, all graphs which are not reducible with respect to the first Symanzik polynomial are also not reducible with respect to both Symanzik polynomials.

So to give a list of known forbidden minors, $K_{3,4}$ and $V_8 \cup \{02\}$ (see Figure 2.5) are forbidden minors for the first Symanzik polynomial. The graphs $K_{2,4}$, W_4, K_4 and the graph L (see Figure 2.6) are forbidden minors for both Symanzik polynomials. We note that we almost surely do not have the full list of forbidden minors. In the next chapter, we look at graphs not containing the above minors and see what structure those graphs must have.

Chapter 3

Graph Minors

Recall that a graph G has a *H-minor* for some graph H, if a graph isomorphic to H can be obtained from G through the contraction and deletion of some edges, and possibly the removal of some isolated vertices. It is easy to see that if G is connected, then we never need to delete isolated vertices. Before jumping into new results on graph minors, we give a brief survey.

Graph minors have been heavily studied in the literature. The most notable contribution is the graph minors project by Neil Robertson and Paul Seymour. The brunt of the project is dedicated to approximately describing the structure of graphs without a K_t-minor. The theorem which accomplishes this is usually known as the Graph Structure Theorem, which says if a graph G does not have a K_t-minor, then G "almost embeds" on a surface of low Euler genus relative to t ([37]). From this theorem one obtains one of the most well known and celebrated results of the graph minors project.

Theorem 50 ([38])**.** *Graphs are well quasi ordered with respect to the minor operation.*

In other words, if $\mathcal{G}$ is a minor closed class of graphs, then $\mathcal{G}$ is characterized by a finite set of forbidden minors. Of course, this theorem is meaningless without some examples of classes of graphs which are minor closed, but fortunately there are lots. Before listing some examples, a few definitions are needed. For some integer k, a graph G has treewidth $< k$ if G can be decomposed into clique sums of graphs with less than $k + 1$ vertices. We note that is not the standard definition of treewidth but can easily be shown to be equivilant (see, for example, [4]). A graph G is *outerplanar* if G is planar and all of its vertices lie on the outerface. Some examples of graph minor closed classes are:

1. Planar Graphs.

2. Outerplanar graphs.

3. Graphs with treewidth less than k for a fixed positive integer k.

4. Graphs with vertex width less than k for a fixed positive integer k.

5. Graphs which embed in a fixed surface.

6. Graphs which do not contain k pairwise disjoint cycles, for a fixed positive integer k.

7. Graphs which are reducible with respect to the Symanzik polynomials for a fixed number of external momenta.

8. Graphs with no path of length k.

9. For a fixed set graph H, graphs not containing an H-minor.

Notice for the classes $1, 2, 3, 4, 5$, and 7, it is not obvious what the complete set of forbidden minors are. But for some of these classes the complete set of forbidden minors is known. For example, a graph is planar if and only if it does not contain a K_5 or $K_{3,3}$ as a minor ([42]). A graph is outerplanar if and only if it does not contain K_4 or $K_{2,3}$ as a minor ([15]). On the other hand, it is an open problem to determine the full set of forbidden minors for graphs with treewidth less than k for all values of $k \geq 4$ ([4], [2]). Also, with the exception of the plane ([42]) and the projective plane ([1]), the full forbidden minor set for graphs which embed in some fixed surface is unknown ([25]). In general the size of the list of forbidden minors can get rather large, for instance Chambers and Myrvold have found that there are at least $16,629$ forbidden minors for graphs which embed on the torus ([14]). As an even more extreme example, Wye-Delta-Wye reducibility currently has at least 69 billion known forbidden minors ([47]). So a natural question is, given a minor closed class $\mathcal{G}$, what are the forbidden minors for $\mathcal{G}$?

However for the classes $6, 8$, and 9, that question is essentially useless since the class is being defined from the set of forbidden minors. So for those classes, it would be nice to have a theorem describing the structure of graphs in these classes in some meaningful way. These type of theorems are usually called excluded minor theorems. They generally outline some small set of graphs which do not contain an H-minor for some topological reason and then give a way to glue the small graphs together in a way which prevents an H-minor. Examples of such theorems include:

Theorem 51 ([43]). *A graph G is $K_{3,3}$-minor free if and only if G can be constructed from a series of 0-sums, 1-sums, and 2-sums of planar graphs and K_5.*

Theorem 52 ([42]). *A graph G is K_5-minor free if and only if G can be constructed from a series of 0-sums, 1-sums, 2-sums, and 3-sums of planar graphs and V_8 (as defined in the graph theory basics section).*

Theorem 53 ([19]). *A graph G is K_4-minor free if and only if G can be constructed from a series of 0-sums, 1-sums and 2-sums of K_1, K_2 and K_3.*

Similar theorems are known for the cube ([28]), prism ([24]), octahedron ([27]), V_8 ([29]), and all 3-connected graphs with less than 11 edges ([18]). One might notice that all the graphs I have listed are 3-connected, and this is due to the usefulness of Seymour's Splitter Theorem (see Theorem 118). That being said, there are non-trivial excluded minor theorems known for graphs which are not 3-connected, for instance graphs not containing $K_{2,4}$-minors were completely characterized in [21].

This chapter is dedicated to looking at excluded minor theorems for graphs which are forbidden minors for reducibility. First let us reformulate the notion of a minor in a way which will be easy to generalize.

Proposition 54. *Let G and H be graphs. The graph G has an H-minor if and only if we can find a set $\{G_x : x \in V(H)\}$ of pairwise disjoint connected subgraphs of G indexed by the vertices of H, such that if $xy \in E(H)$ then there exists a vertex $v \in V(G_x)$ and a vertex $u \in V(G_y)$ such that $uv \in E(G)$.*

Proof. Suppose G has an H-minor. For each $x \in V(H)$, let $V(G_x)$ be the set of vertices in G which were contracted to obtain x. Then let $G_x = G[V(G_x)]$. From the definition of contraction it follows that G_x is connected. Additionally, since we started with an H-minor, if $xy \in E(H)$ there exists a vertex in $V(G_x)$ which is adjacent to a vertex in $V(G_y)$.

Conversely suppose we are given a set of disjoint subgraphs $\{G_x : x \in V(H)\}$. Then contract each subgraph to a distinct vertex. All other edges can be contracted arbitrarily and, if needed, other isolated vertices can be deleted. It is easy to see this results in an H-minor. $\qquad\square$

We will refer to the set of subgraphs $\{G_x : x \in V(H)\}$ as a H-*model of G* and the subgraph G_x is the *branch set* of x. Note for a connected graph G and an H-model of G, without loss of generality we can assume that $\bigcup_{x \in V(H)} V(G_x) = V(G)$.

Now we can naturally generalize our notion of graph minors. Let $X \subseteq V(G)$ and let $\pi : X \to V(H)$ be a one-to-one mapping. We say G has a *rooted H-minor on X with respect to π* if there is an H-model of G such that for all $v \in X$, we have $v \in G_{\pi(v)}$. If G has a rooted H-minor on X with respect to π, we will say G has an $H(X)$-minor with respect to π, and we will call the vertices in X the *roots* or *terminals* of the H-minor. For the application of reducibility, the terminals are taking the role of the 4 on-shell external momenta.

We note this definition is a little stricter than what we want from the point of view of reducibility. In general we will allow families of maps from X to H, say $\pi_1, \pi_2, \ldots, \pi_n$, such that if a graph G has an $H(X)$-minor with respect to π_i, for any $i \in \{1, \ldots, n\}$, we will say that G has an $H(X)$-minor with respect to $\pi_1, \ldots, \pi_n$. For ease of notation, once the family of maps is defined for some graph H and X, we will just say that G has an $H(X)$-minor.

Now we can try and understand excluded minor theorems for rooted minors. So the question we are trying to solve is: if the size of X is fixed, the graph H is fixed, and the

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family of mappings $\pi_i : X \to V(H)$ is fixed, what is the structure of graphs without an $H(X)$-minor? We note that such problems have been studied in the literature, and even show up in the graph minors project ([35]), but they appear to be much less studied than standard excluded minor theorems.

We will survey some examples of excluded rooted minor theorems. Consider a $K_3(X)$-minor where $X = \{a, b, c\}$ and where the family of maps is the family of surjective maps from X to $V(K_3)$. Then we have the following theorem.

Theorem 55 ([46]). *For distinct vertices a, b, c in a graph G, there is a $K_3(X)$-minor on a, b, c unless for some vertex $v \in V(G)$, at most one of a, b, c are in each component of $G - v$.*

Jung's 2-linkage Theorem states that given four vertices (s_1, t_1, s_2, t_2) in a 4-connected graph G, there is a (s_1, s_2)-path which is disjoint from a (t_1, t_2)-path unless G is planar and s_1, t_1, s_2, t_2 lie on a face in that cyclic order ([23]). This can be viewed as a partial characterization for the rooted minor problem on a graph $K_2 + K_2$ where $V(K_2 + K_2) = \{s_1, s_2, t_1, t_2\}$ and $E(K_2 + K_2) = \{s_1 s_2, t_1 t_2\}$, $X = \{a, b, c, d\}$, and the family of maps we are considering is the family where $\{a, b\}$ gets mapped surjectively to $\{t_1, t_2\}$ and $\{c, d\}$ gets mapped subjectively to $\{s_1, s_2\}$. Generalizations of the 2-linkage theorem with a rooted minor flavour have been studied in [44]. Various results have been found on the following question: Given a graph G with n vertices, what is the maximium number of edges G can have before it must have an $H(X)$-minor ([22], [44])?

This chapter will give some characterizations to assorted $H(X)$-minor classes which are relevant to reducibility, namely graphs without $K_4(X)$-minors, $W_4(X)$-minors, $K_{2,4}(X)$-minors and $L(X)$-minors (see Figure 2.6 for a picture of the graphs, Figure 3.11 for a picture just of the graph L). As mentioned before, from the point of view of reducibility, the set X represents the external momenta of a graph. First we outline a characterization of $K_4(X)$-minors which was proved in [45].

For the purposes of this thesis, given a graph G and $X = \{a, b, c, d\} \subseteq V(G)$, G has a $K_4(X)$ minor if and only if G has a $K_4(X)$-minor with respect to π, where π is any surjective map from X to $V(K_4)$. Thus G has a $K_4(X)$-minor if G has a K_4-minor where each vertex of X ends up in a distinct branch set. Alternatively, G has a $K_4(X)$ minor if G has a K_4-minor where none of a, b, c or d get contracted together (here we are enforcing that when we contract a, b, c or d, that the resulting vertex is called a, b, c or d, respectively). Before we can state Monroy and Wood's result on $K_4(X)$-minors, we need some definitions.

Let H be a graph. The graph H^+ is defined in the following way: For each triangle $T = \{x, y, z\}$ in H, let F_T be a clique of arbitrary size such that all vertices in F_T are adjacent to every vertex in T and no other vertices of H. Let F be the disjoint union of F_T for all triangles T. Then we define $V(H^+) = V(H) \cup V(F)$, and $E(H^+) = E(H) \cup E(F) \cup \{xy | x \in V(T), \text{T a triangle}, y \in F_T\}$. Note that given H and F, the graph H^+ is uniquely defined

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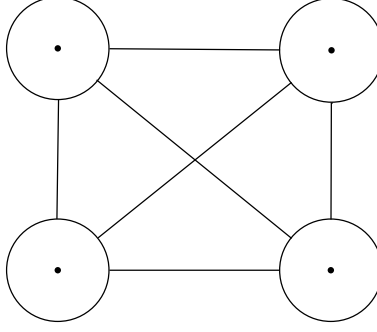
Figure 3.1: A graph G that has a model of a $K_4(X)$-minor. The circles represent connected subgraphs. The vertices represent vertices from X.

so we will frequently use the notation $H^+ = (H, F)$. Note if H is a planar graph, it is said that the graph H^+ admits a *flat embedding*.

Now, consider a planar graph H where the outerface is a 4-cycle which we will call C_4, and every internal face of H is a triangle, and every triangle is a face. Let $V(C_4) = \{x_1, x_2, x_3, x_4\}$. Then we call the graph H^+ an $\{x_1, x_2, x_3, x_4\}$-*web*. Now we can state the excluded $K_4(X)$-minor Theorem.

Theorem 56 (Monroy, Wood, [45])**.** *Let G be a graph and $X = \{a, b, c, d\} \subseteq V(G)$. Then either G has a $K_4(X)$-minor or G is the spanning subgraph of a graph belonging to one of the following six classes of graphs:*

- *Class $\mathcal{A}$: Let H be the graph with vertex set $V(H) = \{a, b, c, d, e\}$ and with edge set $E(H) = \{ae, ad, be, bd, ce, cd, de\}$. Class $\mathcal{A}$ is the set of all graphs H^+.*

- *Class $\mathcal{B}$: Let H be the graph with vertex set $V(H) = \{a, b, c, d, e, f\}$ and with edge set $E(H) = \{ae, af, be, bf, ce, cf, de, df\}$. Class $\mathcal{B}$ is the set of all graphs H^+.*

- *Class $\mathcal{C}$: Let H be the graph such that $V(H) = \{a, b, c, d, e, f, g\}$ and with edge set $E(H) = \{ae, ag, be, bg, cf, cg, df, dg, ef, eg, fg\}$. Class $\mathcal{C}$ is the set of all graphs H^+.*

- *Class $\mathcal{D}$: The set of all $\{a, b, c, d\}$-webs.*

- *Class $\mathcal{E}$: Let H' be a planar graph with a 4-cycle on the outerface and every internal face is a triangle, and every triangle is a face. Let $V(C) = \{c, d, e, f\}$ and suppose they appear in that order on C. Let H be the graph with vertex set $V(H) = V(H') \cup \{a, b\}$ and edge set $E(H) = E(H') \cup \{ae, af, be, bf\}$. Class $\mathcal{E}$ is the set of all graphs H^+.*

- *Class $\mathcal{F}$: Let H' be a planar graph with a 4-cycle on the outerface and every internal face is a triangle, and every triangle is a face. Let $V(C) = \{e, f, g, h\}$ and suppose they appear in that order in H'. Let H be the graph with vertex set $V(H) = V(H') \cup$*
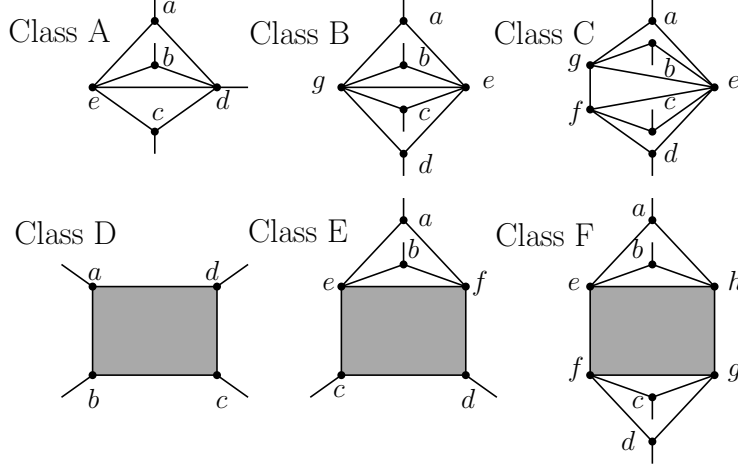
Figure 3.2: The graphs H which appear in Theorem 56. The cliques in the triangles are omitted. The shaded sections are webs.

$\{a, b, c, d\}$ and edge set $E(H) = E(H') \cup \{ae, af, be, bf, cg, ch, dg, dh\}$. Class $\mathcal{E}$ is the set of all graphs H^+.

Figure 3.2 gives a pictorial representation of the graphs H in the above classes. It is easily seen that the result simplifies significantly when we restrict to 3-connected graphs.

Corollary 57 ([45]). *Let G be a 3-connected graph and $X = \{a, b, c, d\} \subseteq V(G)$. Then either G has a $K_4(X)$-minor or G is a spanning subgraph of a Class $\mathcal{D}$ graph. In other words, G is a spanning subgraph of an $\{a, b, c, d\}$-web.*

When we have 3-connected planar graphs G, this result can be simplified even further.

Corollary 58 ([45]). *Let G be a 3-connected planar graph and $X = \{a, b, c, d\} \subseteq V(G)$. Then G does not have a $K_4(X)$-minor if and only if all the vertices of X lie on the same face.*

We will use this characterization as a starting point for most of the upcoming minor characterizations.

3.1 Connectivity reductions

In this section we will show that if I have a graph G, and I want to determine an $H(X)$-minor, then in some sense, I can assume that the connectivity of G is the same as the connectivity of H. Throughout this section, if we do not say what the underlying family of maps $\pi_1, \pi_2, \ldots, \pi_n$ is, it is assumed that we have an arbitrary family of maps. We make the following easy observations.

Observation 59. *Let H be a connected graph. Let G be a graph and $X \subseteq V(G)$. Then G has an $H(X)$-minor if and only if X is contained in a connected component of G, and the connected component has an $H(X)$-minor.*

Proof. If X is contained in a connected component of G and that component has an $H(X)$-minor then immediately G has an $H(X)$-minor.

Conversely, suppose $\{G_x | x \in V(H)\}$ is a model of an $H(X)$-minor in G. Let $u, v \in X$ such that u and uv lie in different connected components of G. Let $u \in G_x$ and $v \in G_y$ respectively. Since H is connected, there is a (x, y)-path in H. Then there is a sequence of branch sets $G_x, G_{x_1}, G_{x_2}, \ldots, G_{x_n}, G_y$ such that G_x has a vertex which is adjacent to a vertex in G_{x_1}, G_{x_i} has a vertex which is adjacent to a vertex in $G_{x_{i+1}}$ for all $i \in \{1, \ldots, n-1\}$ and G_{x_n} has a vertex adjacent to G_y. Then since each branch set is connected, each of $G_x, G_{x_1} \ldots, G_y$ are in the same connected component. But this contradicts that we assumed x and y were in different components. Therefore all vertices of X lie in the same component. Call this component C. Notice that if there is a branch set G_z which is not contained in C, then it has no vertices in C since branch sets are connected. By essentially the same argument as above, this implies that there is no path from x to z in H, which contradicts H being connected, and so C has an $H(X)$-minor. $\square$

Observation 60. *Let H be a simple graph. Let G be a graph and $X \subseteq V(G)$. Then G has an $H(X)$-minor if and only if the graph obtained from G by removing all parallel edges and loop edges has an $H(X)$-minor.*

Proof. Let $\{G_x | x \in V(H)\}$ be a model of an H-minor in G. Since H is simple, all the required adjacencies between branch sets are single edges. Therefore $\{G_x | x \in V(H)\}$ is a model of an H-minor in G', where G' is the graph obtained from G by deleting all parallel and loop edges. Conversely, if G' has an $H(X)$-model, this immediately implies that G has an $H(X)$-model. $\square$

Then since all of our forbidden minors we will discuss are simple and connected, we will assume all graphs are simple and connected. Notice that in observation 59, since H was connected, we could restrict our attention to the connected components of G. There is a more general principle at work here. If H is k-connected, then essentially we can reduce the problem of determining if G has an H-minor to looking at the "k-connected-components" of G. The following lemma is a well known example of this.

Lemma 61. *Let H be a 3-connected graph. Let G be a k-sum of G_1 and G_2 where $k \in \{0, 1, 2\}$. Then G has an H-minor if and only if G_1 or G_2 has an H-minor.*

The next two sections give an extension of this lemma to rooted minors when we have four roots, which the author could not find written down anywhere.

3.1.1 Cut vertices

For the following sequence of lemmas suppose that we have a simple 2-connected graph H, and a connected graph G where G has a 1-separation (A, B) such that $A \cap B = \{v\}$. Furthermore, let $X = \{a, b, c, d\} \subseteq V(G)$, and without loss of generality suppose that $|A \cap X| \geq |B \cap X|$. Let $\mathcal{F}$ be an arbitrary family of maps from X to $V(H)$. Figure 3.3 gives a picture representation for the upcoming lemmas.

Lemma 62. *If $X \subseteq A$, then G has an $H(X)$-minor if and only if $G[A]$ has an $H(X)$-minor.*

Proof. First suppose $G[A]$ has an $H(X)$-minor. Then G has an $H(X)$-minor by extending the branch set containing v to contain all of $G[B]$.

Conversely, suppose G has an $H(X)$-minor and let $\{G_x | x \in V(H)\}$ be a model of an $H(X)$-minor in G. As $X \subseteq A$, and the graph H is 2-connected, and v is a cut vertex, there is no branch set which is contained inside $B \setminus \{v\}$. Notice if all of the branch sets are contained inside $G[A]$, then we are done since $\{G[V(G_x) \cap A] | x \in V(H)\}$ would be the desired $H(X)$-model in $G[A]$. Therefore we assume at least one branch set contains vertices from B. Furthermore, as v is a cut vertex and each branch set is a connected subgraph, we have at most one branch set with vertices in B. Let G_z, $z \in V(H)$, be such branch set. Notice that $G_z \cap G[A]$ is a connected subgraph of $G[A]$, and as G_z was the only branch set containing vertices in B, $\{G_x \cap G[A] | x \in V(H)\}$ is an $H(X)$-model in $G[A]$. $\qquad\square$

Lemma 63. *Suppose $a, b, c \in A \setminus \{v\}$, and $d \in B \setminus \{v\}$. Let $X_A = \{a, b, c, v\}$ and for each $\pi \in \mathcal{F}$, define $\pi' : X_A \to V(H)$ such that $\pi' = \pi$ except that $\pi'(v) = \pi(d)$. Then G has an H-minor if and only if $G[A]$ has an $H(X_A)$-minor.*

Proof. Let $\{G_x | x \in V(H)\}$ be an $H(X_A)$-model in $G[A]$. Suppose that G_d is the branch set where $v \in V(G_d)$. Then we obtain an $H(X)$-model of G by extending G_d to contain all of $G[B]$. By construction, the now extended G_d is connected, and thus we have an $H(X)$-model in G.

Conversely, let $\{G_x | x \in V(H)\}$ be a model of an $H(X)$-minor in G. Let G_d be the branch set where $d \in V(G_d)$. As H is 2-connected, and v is a cut vertex, and since we may assume that the branch sets partition $V(G)$, $B \subseteq V(G_d)$, and in particular, $v \in G_d$. Therefore all other branch sets are contained inside $G[A]$, and thus all required adjacencies for the $H(X)$-minor exist in $G[A]$. Therefore $\{G_x \cap G[A] \mid x \in V(H)\}$ is an $H(X_A)$-minor of $G[A]$. $\qquad\square$

Lemma 64. *Suppose $v = a$. Then G has an $H(X)$-minor if and only if $X \subseteq A$ and $G[A]$ has an $H(X)$-minor.*

Proof. Sufficiency follows from Lemma 62.

Conversely, let $\{G_x | x \in V(H)\}$ be a model of an $H(X)$-minor in G. Towards a contradiction, we consider the case where $d, c \in A \setminus \{v\}$ and $b \in B \setminus \{v\}$. Let G_{v_1}, G_{v_n}, G_z be
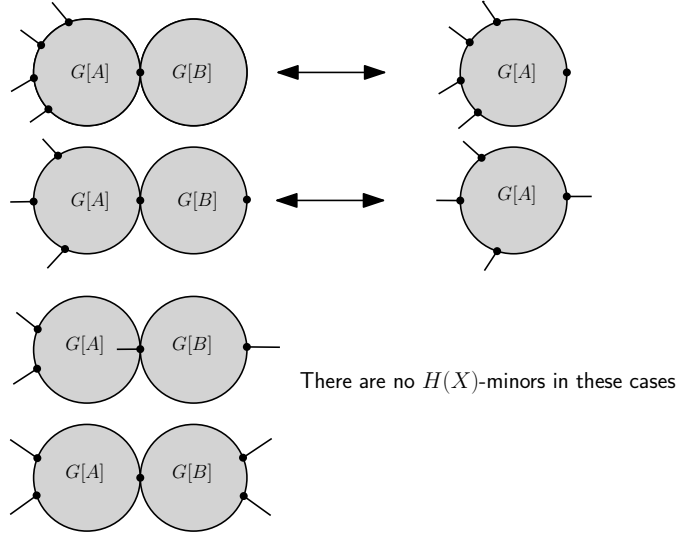
Figure 3.3: Cut vertex reductions. Edges with only one endpoint represent vertices of X.

the branch sets for which $b \in V(G_{v_1})$, $c \in G_{v_n}$, and $a \in G_z$. As H is 2-connected, there is a path $P = v_1, v_2, \ldots, v_n$ in H such that $z \notin V(P)$. Then there is a sequence of branch sets, $G_{v_1}, \ldots, G_{v_n}$ such that G_{v_i} has a vertex which is adjacent to a vertex in $G_{v_{i+1}}$ for all $i \in \{1, \ldots, n-1\}$. But $G_{v_1} \subseteq G[A - v]$, $G_{v_n} \subseteq G[B - v]$, $G_z \neq G_{v_i}$ for any $i \in \{1, \ldots, n\}$ and $a \in V(G_z)$, a contradiction. The other cases follow similarly. $\square$

Lemma 65. *If exactly two vertices of X are in $B \setminus \{v\}$ and exactly two vertices of X are in $A \setminus \{v\}$, then G does not have an $H(X)$-minor.*

Proof. Let $\{G_x \mid x \in V(H)\}$ be an H-model of G. Let G_y be the branch set containing v. Suppose $a \in A \setminus \{v\}$ and $b \in B \setminus \{v\}$ and $a \in G_a$ and $b \in G_b$. Then if we contract each branch set to a vertex to obtain the H-minor, all (a, b)-paths in H contain y since $v \in G_y$ and v is a cut vertex. But then H is not 2-connected, a contradiction. $\square$

3.1.2   2-connected reductions

For this section suppose that H is a 3-connected simple graph and that G is a 2-connected graph. Furthermore, assume that G has a 2-separation (A, B) such that $A \cap B = \{u, v\}$, and let $X = \{a, b, c, d\} \subseteq V(G)$. We define $G_A = G[A] \cup \{uv\}$ and $G_B = G[B] \cup \{uv\}$ (if this results in the graph having a parallel edge, we simply remove the parallel edge). Let $\mathcal{F}$ be a family of maps from X to $V(H)$. By the discussion in the previous section, there is no loss of generality by assuming G is 2-connected.

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Lemma 66. *Let $L \subseteq X$ such that $|L| = 2$ and suppose $L \subseteq A \setminus \{u, v\}$. Furthermore, suppose that $X \setminus L \subseteq B \setminus \{u, v\}$. Let $X_A = \{L, u, v\}$ and $X_B = \{(X \setminus L), u, v\}$. Suppose that for any $\pi_1 \in \mathcal{F}$, there exists a $\pi_2 \in \mathcal{F}$ such that $\pi_1(c) = \pi_2(d)$ and $\pi_1(d) = \pi_2(c)$. For each $\pi \in \mathcal{F}$, define π_A and π_B in the natural way so that u, v replace the vertices in L and $X \setminus L$ respectively. Then G has an $H(X)$-minor if and only if either G_A has an $H(X_A)$-minor or G_B has an $H(X_B)$-minor.*

Proof. Suppose that G_A contains an $H(X_A)$-minor and suppose that $c, d \in B \setminus \{u, v\}$. By Corollary 1, there are two disjoint paths between $\{u, v\}$ and $\{c, d\}$. Since $c, d \in B \setminus \{u, v\}$, these paths are contained inside of $G[B]$. Thus we can contract $G[B]$ to $\{u, v\}$ in such a way that c and d do not get identified together. Since we supposed that for any $\pi_1 \in \mathcal{F}$, there exists a $\pi_2 \in \mathcal{F}$ such that $\pi_1(c) = \pi_2(d)$ and $\pi_1(d) = \pi_2(c)$, the graph G has an $H(X)$-minor. The other cases follow similarly.

Conversely, let $\{G_x | x \in V(H)\}$ be a model of an $H(X)$-minor. Suppose $a, b \in A \setminus \{u, v\}$ and $c, d \in B \setminus \{u, v\}$. Let $a \in G_a$, $b \in G_b$, $c \in G_c$, and $d \in G_d$. First, suppose there are branch sets G_y and G_z such that $G_y \subseteq G[A - \{u, v\}]$ and $G_z \subseteq G[B - \{u, v\}]$. Then if we contract each branch set down to a vertex, there would be at most two internally disjoint (y, z)-paths, contradicting that H is 3-connected.

Therefore we can assume that either $u \in G_a$, $v \in G_b$ and $A \subseteq V(G_a) \cup V(G_b)$ or $u \in G_c$, $v \in G_d$ and $B \subseteq V(G_c) \cup V(G_d)$. Suppose $u \in G_a$ and $v \in G_b$ and $A \subseteq V(G_a) \cup V(G_b)$. Then all other branch sets are contained in $G[B - \{u, v\}]$. Then since $uv \in E(G_b)$, $\{G'_x = G_B[V(G_x) \cap V(G_B)] \, | \, x \in V(H)\}$ is an $H(X)$-model in G_B. The other case follows similarly. $\square$

We remark that all rooted graph minors we will see throughout this thesis satisfy the technical condition in the above lemma.

Lemma 67. *Suppose $a \in A \setminus \{u, v\}$ and $b, c, d \in B \setminus \{u, v\}$. Let $X_1 = \{u, b, c, d\}$ and $X_2 = \{v, b, c, d\}$. For each $\pi \in \mathcal{F}$, let π_1 satisfy $\pi_1 = \pi$ on $X_2 \setminus \{u\}$ and $\pi_1(u) = \pi(a)$. Let $\pi_2 = \pi$ on $X_1 \setminus \{v\}$ and $\pi_2(v) = \pi(a)$. Then G has an $H(X)$-minor if and only if either G_B has an $H(X_1)$-minor or an $H(X_2)$-minor.*

Proof. Suppose G_B has an $H(X_1)$-minor. By Menger's Theorem there exists a path from a to u which does not contain v, and therefore we can contract $G[A]$ to $\{u, v\}$ in such a way that a gets contracted onto u. Therefore G has an $H(X)$-minor. The case where G_B has an $H(X_2)$-minor follows similarly.

Conversely, let $\{G_x | x \in V(H)\}$ be a model of an $H(X)$-minor in G. Let $a \in G_a$, $b \in G_b, c \in G_c$, and $d \in G_d$. First, suppose for some $y \in V(H)$, $G_y \subseteq G[A - \{u, v\}]$. If $yb \in E(H)$, then one of u or v is in $V(G_b)$. Then at least two of the following occur: $V(G_a) \subseteq A \setminus \{u, v\}$, $V(G_c) \subseteq B \setminus \{u, v\}$ or $V(G_d) \subseteq B \setminus \{u, v\}$. If $V(G_a) \subseteq A \setminus \{u, v\}$ and $V(G_c) \subseteq B \setminus \{u, v\}$, then if we contract all the branch sets down to a vertex, there

are at most two internally disjoint (a, c)-paths contradicting that H is 3-connected. The other case when $V(G_a) \subseteq A \setminus \{u, v\}$ follows similarly, and thus we can assume that G_a contains one of u or v. But then, contracting all branch sets to a vertex there are at most two internally disjoint (c, y)-paths contracting that H is 3-connected.

Thus we may assume that $yb \notin E(H)$. Then by Menger's Theorem there are three internally disjoint (b, y)-paths. If either u or v is in $V(G_b)$, then the above argument can be applied to derive a contradiction. Therefore we assume that $u, v \notin V(G_b)$. But then contradicting all the branch sets down to a vertex, every (b, y)-path uses the vertex which was obtained by contracting the branch sets that u or v were in down to a single vertex. But that implies there are at most two internally disjoint (b, y)-paths, a contradiction. Therefore for every $y \in V(H)$, $G_y \nsubseteq G[A - \{u, v\}]$.

Then since $a \in A \setminus \{u, v\}$, at least one of u or v is contained in G_a. Therefore at most one other branch set contains vertices from A. Then since $uv \in G_B$, $\{G'_x = G_B[V(G_x) \cap V(G_B)] \mid x \in V(H)\}$ is a model for either an $H(X_1)$ or $H(X_2)$-minor in G_B, depending on which of u and v is in G_a. □

Lemma 68. *Suppose that $X \subseteq A$. Then G has an $H(X)$-minor if and only if G_A has an $H(X)$-minor.*

Proof. Suppose G_A has an $H(X)$-minor. Then contracting $G[B]$ onto $\{u, v\}$ gives G_A, and thus G has a $W_4(X)$-minor.

Let $\{G_x \mid x \in V(H)\}$ be a model of an $H(X)$-minor. Let $a \in G_a, b \in G_b, c \in G_c$ and $d \in G_d$. Suppose there is a $y \in V(H)$ such that G_y is contained in $G[B \setminus \{u, v\}]$. Note that $y \neq a, b, c$ or d. Then since $\{u, v\}$ is a 2-vertex cut, at least two of the following occur: $G_a \in G[A \setminus \{u, v\}]$, $G_b \in G[A \setminus \{u, v\}]$, $G_c \in G[A \setminus \{u, v\}]$ and $G_d \in G[A \setminus \{u, v\}]$. Without loss of generality, suppose that $G_a \in G[A \setminus \{u, v\}]$. But then if we contract each branch set down to a vertex, there is at most two internally disjoint (a, y)-paths in H, contradicting that H is 3-connected. Therefore there are no branch sets contained in $G[B \setminus \{u, v\}]$. Then since $\{u, v\}$ is a 2-vertex cut, there are at most two branch sets using vertices in B. If there is only one branch set using vertices from B, then easily $\{G'_x = G[V(G_x) \cap V(G_A)] \mid x \in V(H)\}$ is an $H(X)$-minor of G_A. If two branch sets contain vertices from B, then since $uv \in E(G_A)$, $\{G'_x = G_A[V(G_x) \cap V(G_A)] \mid x \in V(H)\}$ is a model of an $H(X)$-minor of G_A. □

Lemma 69. *Suppose $L \subseteq X$ where $L = \{u, v\}$. Furthermore, suppose that there is a vertex of $X \setminus L$ in $A \setminus \{u, v\}$ and a vertex of $X \setminus L$ in $B \setminus \{u, v\}$. Then G does not have an $H(X)$-minor.*

Proof. Consider any $L \subseteq X$. Suppose for a contradiction that $\{G_x \mid x \in V(H)\}$ is a model of an $H(X)$-minor. Consider the case where $b, c \in L$ and $a \in A \setminus \{u, v\}$, $d \in B \setminus \{u, v\}$, and $a \in G_a$, and $d \in G_d$. Then $V(G_a) \subseteq A \setminus \{u, v\}$ and $V(G_d) \subseteq B \setminus \{u, v\}$. But then
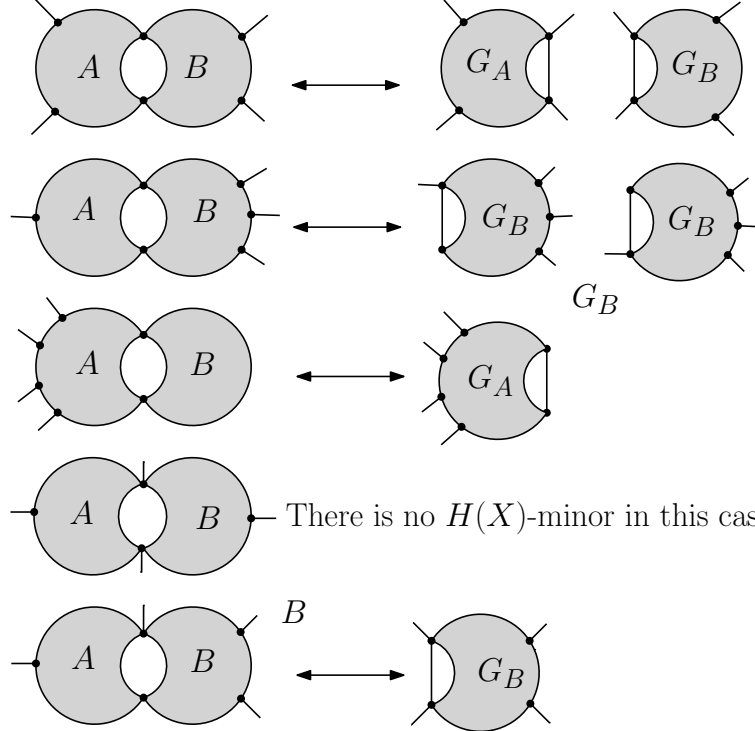
Figure 3.4: The 2-connected reductions. Edges with only one endpoint represent vertices from X.

contracting each branch set down to a vertex, there are at most two internally disjoint (a, d)-paths, contradicting that H is 3-connected. The other cases follow similarly. $\square$

Lemma 70. *Suppose $u = a$, and that there is exactly one vertex of X in $A \setminus \{u, v\}$ and two vertices of X in $B \setminus \{u, v\}$. Let $X_A = \{u, v, (X \cap B \setminus \{u, v\})\}$. For each $\pi \in \mathcal{F}$, define $\pi_A = \pi$ on $X_A \setminus \{v\}$ and let $\pi_A(v) = \pi(x)$ where $x \in X \setminus \{u, (X \cap A \setminus \{u, v\})\}$. Then G has an $H(X)$-minor if and only if G_A has an $H(X_A)$-minor.*

Proof. Suppose G_A has an $H(X_A)$-minor. Then by Menger's Theorem there is a path from the vertex of $X \cap A \setminus \{u, v\}$ to v not containing u. Thus we can contract $G[A]$ to $\{u, v\}$ such that we obtain the edge uv and the vertex of X in $A \setminus \{u, v\}$ is contracted to v. The resulting graph is isomorphic to G_A and thus G has an $H(X)$-minor.

Conversely, let $\{G_x | x \in V(H)\}$ be a model of an $H(X)$-minor in G. Let $b \in A \setminus \{u, v\}$ and suppose $a \in G_a$, $b \in G_b$, $c \in G_c$ and $d \in G_d$. First, suppose for some $y \in V(H)$, $V(G_y) \subseteq A \setminus \{u, v\}$. As $a = u$, either $V(G_c) \subseteq B \setminus \{u, v\}$ or $V(G_d) \subseteq B \setminus \{u, v\}$. Without loss of generality, suppose that $V(G_c) \subseteq B \setminus \{u, v\}$. Then if we contract all of the branch sets to a vertex, there are at most two internally disjoint (y, c)-paths, contradicting that H is 3-connected. Thus the only two branch sets with vertices in A are G_a and G_b. Since

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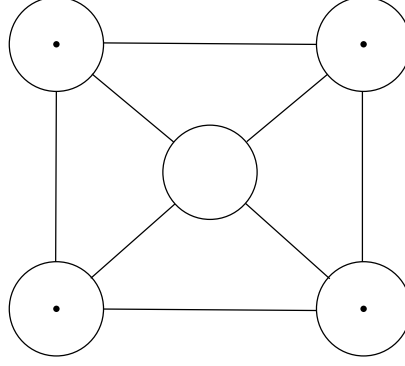
Figure 3.5: A model of a $W_4(X)$-minor in some graph G. Each circle represents a connected subgraph of G. The vertices represent vertices from X

$uv \in E(G_B)$, we have that $\{G'_x = G_B[V(G_x) \cap V(G_B)] \mid x \in V(H)\}$ is a model of an $H(X)$-minor of G_B.

$\square$

3.2 A characterization of graphs without a $K_4(X)$ or $W_4(X)$ minors

First we define what we mean by $W_4(X)$-minors. Let G be a graph and $X = \{a, b, c, d\} \subseteq V(G)$. Let $\mathcal{F}$ be the family of maps from X to $V(W_4)$ such that each vertex of X is mapped to a distinct rim vertex of W_4. For the purposes of this thesis, a $W_4(X)$-minor refers to the X and $\mathcal{F}$ given above (see Figure 3.5).

In this section we will prove that all planar 3-connected graphs have either a $W_4(X)$-minor or a $K_4(X)$-minor, and then we will use that result to show that every 3-connected graph has either a $W_4(X)$-minor or a $K_4(X)$-minor (see Theorem 74). Since the graphs W_4 and K_4 are 3-connected, by the previous sections discussion, this completely characterizes graphs with no $W_4(X)$ or $K_4(X)$-minor.

Lemma 71. *Let G be a 3-connected planar graph and let C be a facial cycle of G. Suppose $v, w \in V(C)$ and $vw \notin E(G)$. Then there is a (v, w)-path P such that $V(P) \cap V(C) = \{v, w\}$.*

Proof. Let P_1, P_2, P_3 be internally disjoint (v, w)-paths, and let F_1, F_2 be the two facial walks of C from v to w. If any two of P_1, P_2, P_3 are the facial walks then we are done.

Therefore we assume at most one of P_1, P_2, P_3 is a facial walk. First we claim that $i \in \{1, 2, 3\}$, either $V(P_i) \cap V(F_1) = \{v, w\}$ or $V(P_i) \cap V(F_2) = \{v, w\}$. If not, then there exists an $i \in \{1, 2, 3\}$ such that we have two vertices $x_1 \in V(F_1) \setminus \{v, w\}$ and $x_2 \in V(F_2) \setminus \{u, v\}$ where $x_1, x_2 \in V(P_i)$. Then consider the (x_1, x_2)-subpath on P_i which we denote P_{x_1, x_2}. We may assume that this subpath has no additional vertices from F_1 or F_2 (if it did, then

we change the subpath to one with fewer vertices from F_1 and F_2). Note that this subpath partitions C into two cycles which separate v and w. But then as C is a facial cycle, and G is planar, and G is 3-connected any path from u to v intersects the subpath P_{x_1,x_2}, contradicting that P_1, P_2, P_3 were three internally disjoint (v,w)-paths.

Now suppose $\{v,w\} \subsetneq V(P_1) \cap V(F_1)$. We claim that both $V(P_2) \cap V(F_1) = \{v,w\}$ and $V(P_3) \cap V(F_1) = \{v,w\}$. By the above argument, $V(P_1) \cap V(F_2) = \{v,w\}$. Therefore every vertex of F_1 either belongs to P_1 or lies in a cycle created from some subpath of P_1 and a subpath of F_1. But then since C is a facial cycle, G is planar, and P_2 and P_3 are internally disjoint from P_1, P_2 and P_3 both are internally disjoint from F_1. By similar arguments, at most one of P_2, P_3 contains a vertex in F_2 which is not v,w. Therefore at least one of P_1, P_2, and P_3 has no vertex from C except for v and w, completing the claim. $\square$

Lemma 72. *Let G be a 3-connected planar graph and let $X = \{a,b,c,d\} \subseteq V(G)$. Then G has either a $K_4(X)$-minor or a $W_4(X)$-minor.*

Proof. We assume G does not contain a $K_4(X)$-minor. Then by Corollary 58, a, b, c and d lie on a facial cycle, F in that order.

First, suppose that $ac \in E(G)$. Without loss of generality we may assume ac lies in the interior of F. Notice either the edge $bd \in E(G)$ or by Menger's Theorem there is a (b,d)-path P where $a, c \notin V(P)$. In either case, this would contradict F being a facial cycle, and G not having a $K_4(X)$-minor.

Therefore we can assume that $ac \notin E(G)$. Then by Lemma 71 there is an (a,c)-path, $P_{a,c}$, which is internally disjoint from F. Without loss of generality we may assume that $P_{a,c}$ lies in the interior of F. Let $F_{a,c}$ be a facial walk from a to c. Notice $C = P_{a,c} \cup F_{a,c}$ partitions the interior of F into two regions, and that b and d lie in distinct regions. By Lemma 71, we also have a (b,d)-path, $P_{b,d}$, which is disjoint from F. If $P_{b,d}$ lies on the exterior of F, then this would contradict that F is a face. Thus $P_{b,d}$ lies in the interior of F. As b and d lie on differing sides of the partition of C, by the Jordan Curve Theorem, $P_{a,c}$ intersects $P_{b,d}$ at some vertex v. Then, contracting F down to a four cycle on a, b, c, d and contracting the subpaths from v to a, b, c, d on $P_{a,c}$ and $P_{b,d}$ internal vertices of the paths $P_{a,c}$ and $P_{b,d}$ to a vertex give a $W_4(X)$-minor. $\square$

Now that we have shown that all planar 3-connected graphs have a $W_4(X)$ or a $K_4(X)$-minor, we will prove that all 3-connected graphs have a $W_4(X)$ or $K_4(X)$-minor by reducing to the planar case.

Lemma 73. *Let G be a 3-connected graph and $X = \{a,b,c,d\} \subseteq V(G)$. Suppose G is a spanning subgraph of an $\{a,b,c,d\}$-web, $H^+ = (H,F)$. Then there is a 3-connected planar graph K such that K is a minor of G.*

Proof. For every triangle $T \in H$, consider the graph $G[V(F_T)]$ and let $C_1, \ldots, C_n$ be the connected components of $G[V(F_T)]$. Now for every triangle $T \in H$, contract C_1 down to

a vertex, which we will call v_T, and contract all of $C_2, \ldots, C_n$ to any of the vertices of T, removing any parallel edges or loops created. Let K be the graph obtained from G after applying the above construction. We claim K is planar and 3-connected.

First we show G is 3-connected. Notice for each T where $V(T) = \{x_1, x_2, x_3\}$, there is a vertex $v \in V(C_1)$ such that $vx_i \in E(G)$ for all $i \in \{1, 2, 3\}$. Thus v_T is adjacent to x_i for all $i \in \{1, 2, 3\}$. Then v_T has 3 internally disjoint paths to any other vertex of K as G is 3 connected. Now let $x, y \in V(H)$ and consider 3 internally disjoint (x, y)-paths P_1, P_2 and P_3 in G. Now notice that for every triangle $T \in H$, at most one of P_1, P_2 or P_3 uses vertices from $V(F_T)$, as $|V(T)| = 3$. Therefore in K, if necessary, we can reroute the path using vertices from $V(F_T)$ to use v_T, and thus we still have 3 internally disjoint (x, y)-paths. Therefore K is 3-connected.

So it suffices to show that K is planar. Notice that if given a some planar embedding of H, to all the faces bounded by a triangle, we can add a vertex to the interior of the face and make the vertex adjacent to every vertex in the triangle and remain planar. The graph from that construction contains K as a subgraph, so K is planar, completing the proof. $\square$

Theorem 74. *Let G be a 3-connected graph and $X = \{a, b, c, d\} \subseteq V(G)$. Then G either has a $K_4(X)$-minor or a $W_4(X)$-minor.*

Proof. We may assume G does not have a $K_4(X)$-minor. By Corollary 57, G is a spanning subgraph of an $\{a, b, c, d\}$-web. Then by Lemma 73, G has a 3-connected planar minor K. By Lemma 72, K has $W_4(X)$-minor and thus G has a $W_4(X)$-minor. $\square$

3.2.1 An alternative characterization of 2-connected graphs without $W_4(X)$ and $K_4(X)$-minors

We have a characterization of graphs without $W_4(X)$ and $K_4(X)$ minors, given by Theorem 74, however it relies on reducing to the 3-connected case, which will not be useful as $K_{2,4}$ and L are both 2-connected (and as every 3-connected graph either has a $K_4(X)$ or $W_4(X)$-minor). Therefore it would be nice to have a characterization in terms of spanning subgraphs as in Theorem 56. This section gives such a characterization (see Theorem 82).

Lemma 75. *Let G be 2-connected and a spanning subgraph of a class $\mathcal{A}, \mathcal{B}$ or $\mathcal{C}$ graph from Theorem 56. Then G does not have a $W_4(X)$-minor.*

Proof. We treat each case separately.

If G is the spanning subgraph of a class $\mathcal{A}$ graph, then $\{d, e\}$ is a 2-vertex cut. Applying Lemma 70 and Lemma 69 successively to the separation induced by $\{d, e\}$, we see G does not have a $W_4(X)$-minor.

If G is the spanning subgraph of a class $\mathcal{B}$ graph, then $\{e, f\}$ is a 2-vertex cut. Applying Lemma 66 and Lemma 69 successively to the separation induced by $\{e, f\}$, we see G does not have a $W_4(X)$-minor.

If G is the spanning subgraph of a class $\mathcal{C}$ graph, then $\{g, f\}$ is a 2-vertex cut. Apply Lemma 66 to the separation induced by $\{g, f\}$ and let G_1 and G_2 be the graphs obtained from Lemma 66. Without loss of generality, let G_1 be the graph such that $\{f, g\}$ induces a separation satisfying Lemma 69. Then G_1 does not have a $W_4(X)$-minor. Then in G_2, to the separation induced by the 2-vertex cut $\{g, e\}$, apply Lemma 70 to obtain a graph G_3. Then in G_3, notice that Lemma 69 applies, thus G_3 does not have a $W_4(X)$-minor, and then by working back through the lemmas, we get that G does not have a $W_4(X)$-minor. $\qquad\square$

Notice that by applying Lemma 66 to 2-connected spanning subgraphs of class $\mathcal{E}$ and $\mathcal{F}$ graphs, we see that one of these graphs has a $W_4(X)$-minor if and only if the corresponding web from class $\mathcal{D}$ has a $W_4(X)$-minor. So now we restrict ourselves to looking at $\{a, b, c, d\}$-webs. First we show that $\{a, b, c, d\}$-webs always have a cycle which contains $\{a, b, c, d\}$.

Observation 76. *Suppose G is a planar spanning subgraph of some $\{a, b, c, d\}$-web $H^+ = (H, F)$. Then the graph G' defined by $V(G') = V(G)$ and $E(G') = E(G) \cup \{ab, bc, cd, da\}$ is planar, and furthermore the cycle C with edge set ab, bc, cd, da is a face in G'.*

Proof. Fix a planar embedding $\tilde{H}$ of H. As G is planar, for each triangle $T \in H$ the graph $F_T \cup T$ is planar. For each triangle $T \in H$, fix a planar embedding of $F_T \cup T$ where T is the outerface. Then we can combine the planar embedding of H with the planar embeddings of $F_T \cup T$ by joining $F_T \cup T$ to the appropriate triangle. This implies that the graph H^+ is planar, and thus G' is planar. Additionally, notice in H^+, the cycle with edge set ab, bc, cd, da is a face in H^+, and thus the cycle with edge set ab, bc, cd, da in G' is a face in G'. $\qquad\square$

Lemma 77. *Let G be a 2-connected planar graph and let $X = \{a, b, c, d\} \subseteq V(G)$. If G is the spanning subgraph of an $\{a, b, c, d\}$-web, then there is a cycle, C, such that $X \subseteq V(C)$.*

Proof. Let $G_1, \ldots, G_n$ be a sequence of graphs where $G_1 = H^+$, $G_n = G$ and $G_{i+1} = G_i \backslash \{e\}$ where e is some edge of G_i. We proceed by induction on i. When $i = 1$, $G_1 = H^+$ and the 4-cycle on a, b, c, d in H is our desired cycle.

Now consider G_i, $i \geq 2$ and let $e = xy \in E(G)$ be the edge such that $G_i = G_{i-1} \setminus \{e\}$. By induction, G_{i-1} contains a cycle C containing X. We may assume that $e \in E(C)$ as otherwise C completes the claim. Let $P = C \setminus \{e\}$. Without loss of generality, suppose that a, b, c, d appear in that order in C, and that x and y lie on the (a, d)-path, $P_{a,d}$, in C in G_{i-1} which does not contain c and d, such that a, x, y, d appear in that order. Similarly define paths $P_{a,b}, P_{b,c}$ and $P_{d,a}$. Additionally define $P_{a,x}$ to be the (a, x)-subpath on $P_{a,d}$ and $P_{y,d}$ to be the (y, d)-subpath on $P_{a,d}$.

By observation 76, in the graph G'_{i-1}, we have that $P_{a,d} \cup \{ad\}$, $P_{a,b} \cup \{ab\}$, $P_{b,c} \cup \{bc\}$ and $P_{c,d} \cup \{cd\}$ are cycles. Furthermore, we may assume that in a planar embedding of G'_{i-1}, no edges cross ad, ab, bc, or cd. We define the interior of C to be the component of $G'_{i-1} - C$ which does not contain any of ab, bc, cd or ab, and the exterior is the component which is
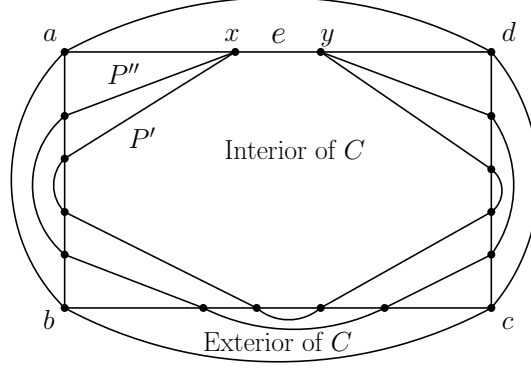
Figure 3.6: The situation in Lemma 77. The edges ab, bc, cd and ab exist only in G_i'. The path P''' is obtained by rerouting along paths in the exterior of C which are subpaths of P''.

not the interior. We abuse notation and will refer also interior and exterior of C in G_{i-1}. Notice that if we have a path whose two endpoints are on $P_{a,d}$ and whose vertices only use exterior vertices, then that path does not contain any vertices from $P_{a,b} \cup P_{b,c} \cup P_{c,d} \setminus \{a, d\}$.

As G_i is 2-connected, there is an (x, y)-path, P' such that $P' \neq P$. If $V(P') \cap V(P) = \{x, y\}$, then $P' \cup P$ is our desired cycle. Therefore we may assume that every (x, y)-path intersects P. If there is any path P'' from a vertex $x' \in V(P_{a,x})$ to a vertex $y' \in V(P_{y,d})$ using only vertices from the exterior, then $P_{a,x'} \cup P'' \cup P_{y',d} \cup P_{c,d} \cup P_{b,c} \cup P_{a,d}$ is a cycle, since no edges cross the edge $ad \in E(G_{i-1}')$. Here $P_{a,x'}$ is the (a, x')-subpath on $P_{a,x}$ and $P_{y',d}$ is the (y', d)-subpath on $P_{y,d}$. Therefore we assume no such path of that form exists. By essentially the same argument, we can assume no path of that form exists with vertices in the interior which does not intersect any of $P_{a,b}, P_{b,c}$ and $P_{c,d}$.

Since G_i is 2-connected, there are two internally disjoint (x, y)-paths, say P' and P''. By our previous discussion, we may assume that both P' and P'' are not P, and that both P' and P'' intersect P. Suppose that P' intersects all of $P_{b,c}$, $P_{c,d}$ and $P_{a,b}$. Notice that by planarity, these paths cannot cross, so it is well defined to say that one of $P' - \{x, y\}$ or $P'' - \{x, y\}$ lies on the interior of the cycle $P' \cup \{xy\}$ or $P'' \cup \{xy\}$. Without loss of generality, suppose that $P' - \{x, y\}$ lies on the interior of $P'' \cup \{xy\}$. Then by our previous discussion, and planarity, the only way for P'' to be an internally disjoint (x, y)-path is for P'' to intersect $P_{a,b}$, go through the exterior of C and intersect $P_{a,b}$ again, do this some finite number of times, then intersect $P_{b,c}$, go through the exterior of C and intersect $P_{b,c}$ again, do this some finite number of times, then intersect $P_{c,d}$, go through the exterior of C and intersect $P_{c,d}$ again. Then we can reroute P along paths in the exterior of C along P'' to get a new path P''' such that P''' contains all of a, b, c, d and $V(P''') \cap V(P') \setminus \{x, y\} = \emptyset$. But then $P''' \cup P'$ a cycle satisfying the claim. We note the same strategy holds if P' intersects any subset of $P_{b,c}$, $P_{c,d}$ and $P_{a,b}$. Therefore there is a cycle containing $a, b, c,$ and d in G. $\square$

Corollary 78. *Let G be a 2-connected graph and let $X = \{a, b, c, d\} \subseteq V(G)$. If G is the spanning subgraph of an $\{a, b, c, d\}$-web, $H^+ = (H, F)$, then there is a cycle, C, such that $X \subseteq V(C)$.*

Proof. By Lemma 77, we may assume that G is non-planar. For each triangle $T \in H$, consider the graph $G[V(F_T)]$ and let $M_1, \ldots, M_n$ be the connected components of $G[V(F_T)]$. Now for each triangle $T \in H$ let G' be the graph obtained by contracting each connected component, M_i, down to a vertex, call it v_T^i, $i \in \{1, \ldots, n\}$.

Now consider some triangle $T \in H$ such that $V(T) = \{x_1, x_2, x_3\}$. First suppose there exists an $i \in \{1, \ldots, n\}$ such that v_T^i is adjacent to x_j for all $j \in \{1, 2, 3\}$. Then for all v_T^k, $k \in \{1, \ldots, n\}, k \neq i$, contract v_T^k to any vertex of T.

Now suppose that there was no $i \in \{1, \ldots, n\}$ such that v_T^i is adjacent to x_j for all $j \in \{1, 2, 3\}$. Since G is 2-connected, that means that for all $i \in \{1, \ldots, n\}$, v_T^i is adjacent to exactly two of x_1, x_2 and x_3. Let v_T^i, v_T^j, v_T^k be vertices such that v_T^i is adjacent to x_1, x_2, and v_T^j is adjacent to x_1, x_3, and v_T^k is adjacent to x_2, x_3 for $i, j, k \in \{1, \ldots, n\}, i \neq j \neq k$. Then for all v_T^l, $l \neq i, j, k$, contract v_T^l to an arbitrary vertex of T. Let G' be the resulting graph after applying the above procedure to every triangle $T \in H$ to G. We note that some subset of the vertices v_T^i, v_T^j, v_T^k may not exist, but in this case we just do not have that subset of vertices in G'. We claim that G' is planar and 2-connected.

First we show that G' is 2-connected. Notice that since G is 2-connected, for every $T \in H$, all of the v_T^i have 2 internally disjoint paths to every other vertex. Now consider two vertices $x, y \in V(H)$ and let P_1, P_2 be two internally disjoint (x, y)-paths in G. Notice that for each triangle $T \in H$, at most one of P_1 or P_2 uses vertices from any connected component of $G[V(F_T)]$, so these paths exist in G' by possibly augmenting them to the appropriate vertex v_T^i, v_T^j or v_T^k. Therefore G' is 2-connected.

So it suffices to show that G' is planar. Take any planar embedding of H, and for every face bounded by a triangle T, either add a vertex adjacent to all of the vertices of T to the interior of the face, or add three vertices v_1, v_2, v_3 to the interior of the face such that for all $i \in \{1, 2, 3\}$, v_i is adjacent to two vertices of T, and $N(v_i) \neq N(v_j)$ if $i \neq j$. Note that the resulting graph is planar. Furthermore, using this construction, we can obtain a planar graph K such that G' is a subgraph of K, and thus G' is planar.

Now since G' is 2-connected, planar, and by construction we did not contract any of a, b, c or d together, we can apply Lemma 77. Thus G' has a cycle C' containing X. But then G has a cycle C containing X, obtained by extending C' along the contracted edges, if necessary. $\qquad\square$

We note that we cannot extend the above lemma to the non-web $K_4(X)$-free classes, as none of them have a cycle containing X.

Lemma 79. *Let G be a 2-connected graph and $X = \{a, b, c, d\} \subseteq V(G)$. Suppose G is the spanning subgraph of a class $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{E}$ or $\mathcal{F}$ graph. Then there is no cycle C in G which contains X.*

Proof. Suppose G is the spanning subgraph of a class $\mathcal{A}$ graph. Then notice that $G - \{d, e\}$ has at least 3 components, say $A, B,$ and C such that $a \in A$, $b \in B$, and $c \in D$. Since $|\{d, e\}| = 2$, there is no cycle containing a, b, c in G. Therefore there is no cycle containing a, b, c and d in G.

Suppose G is the spanning subgraph of a class $\mathcal{B}$ graph. Notice $G - \{e, f\}$ has at least four components and each vertex of X lies in a distinct component. Then there is no cycle C in G which contains X.

Suppose G is the spanning subgraph of a class $\mathcal{C}$ graph. Then notice $G - \{e, f\}$ has at least 3 components and at least 3 components contain vertices from X. Then G does not have a cycle containing X.

Suppose G is the spanning subgraph of a class $\mathcal{E}$ or $\mathcal{F}$ graph. Then notice $G - \{e, f\}$ has at least 3 components and at least 3 components contain vertices from X. Then G does not have a cycle containing X. $\qquad\square$

Now we will give a characterization of graphs not containing $K_4(X)$ and $W_4(X)$-minors of a different flavour that from the previous sections. First we give some definitions.

A common idea which appears in the study of graph minors is the notion of a k-dissection, which is simply a sequence of nested k-separations. Formally, a sequence $((A_1, B_1), \ldots, (A_n, B_n))$ is a k-*dissection* if for all $i \in \{1, \ldots, n\}$, (A_i, B_i) is a k-separation, and for all $i \neq n$, $A_i \subseteq A_{i+1}$, and $B_{i+1} \subseteq B_i$. We will use special types of k-dissections.

Definition 80. *Let G be a 2-connected graph and $X = \{a, b, c, d\} \subseteq V(G)$. Let n be any positive integer. Let $((A_1, B_1), \ldots, (A_n, B_n))$ be a 2-dissection. If $A_i \cap B_i \cap A_{i+1} \cap B_{i+1} \neq \emptyset$ for all $i \in \{1, \ldots, n-1\}$, then we will say the 2-dissection is a 2-chain. Let $((A_1, B_1), \ldots, (A_n, B_n))$ be a 2-chain. Suppose both $A_1 \cap B_1$ and $A_n \cap B_n$ contain at least one vertex of X, and there is exactly one vertex of X in $A_1 \setminus (A_1 \cap B_1)$, and exactly one vertex of X in $B_n \setminus (A_n \cap B_n)$. Then we say $((A_1, B_1), \ldots, (A_n, B_n))$ is a* terminal separating 2-chain.*

Definition 81. *Let G be a 2-connected graph and $X = \{a, b, c, d\} \subseteq V(G)$. Suppose there are three distinct 2-separations $(A_1, B_1), (A_2, B_2), (A_3, B_3)$. We say these separations form a* triangle *if $A_1 \cap B_1 = \{x, y\}$, $A_2 \cap B_2 = \{x, v\}$ and $A_3 \cap B_3 = \{v, y\}$ for distinct vertices $x, y, v \in V(G)$. For notational convenience, we will enforce that in a triangle, $(A_i \setminus (A_i \cap B_i)) \cap (A_j \setminus (A_j \cap B_j)) = \emptyset$ for any $i, j \in \{1, 2, 3\}$, $i \neq j$. We say a triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$, is* terminal separating *if exactly two vertices of X are contained in A_1, exactly one vertex of X is contained in $A_2 \setminus (A_2 \cap B_2)$ and exactly one vertex of X is contained in $A_3 \setminus (A_3 \cap B_3)$. Two triangles $((A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1))$,*
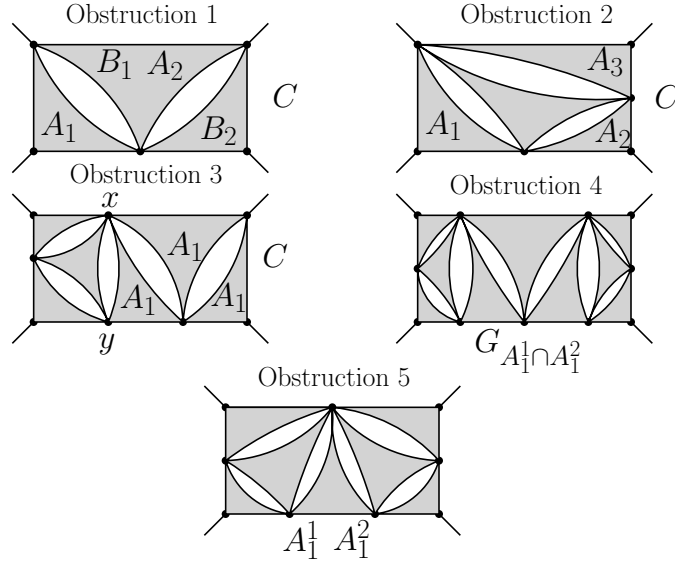
Figure 3.7: The obstructions for Theorem 82. The vertices of X are represented by vertices which have an edge not adjacent to a vertex. Curved lines represent a 2-separation. Shaded sections are spanning subgraphs of webs.

$((A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2))$, are distinct if there exists an $i \in \{1, 2, 3\}$, such that for A_i^2, $A_j^1 \cap B_j^1 \subseteq A_i^2$ for all $j \in \{1, 2, 3\}$ and there exists an $i \in \{1, 2, 3\}$ such that for A_i^1, $A_j^2 \cap B_j^2 \subseteq A_i^1$ for all $j \in \{1, 2, 3\}$.

Theorem 82. *Let G be a 2-connected graph and $X = \{a, b, c, d\} \subseteq V(G)$. Suppose G is a spanning subgraph of an $\{a, b, c, d\}$-web. Then G is $W_4(X)$-minor free if and only if for every cycle C where $X \subseteq V(C)$, we have one of the following obstructions (See Figure 3.7).*

1. *There is a terminal separating 2-chain $((A_1, B_1), \ldots, (A_n, B_n))$ such that $A_i \cap B_i \subseteq V(C)$ for all $i \in \{1, \ldots, n\}$.*

2. *There is a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$, such that $A_1 \cap B_1$ contains a vertex from X and $A_i \cap B_i \subseteq V(C)$, for all $i \in \{1, 2, 3\}$.*

3. *There is a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ such that $A_1 \cap B_1 = \{x, y\}$, where $x, y \notin X$. Furthermore, the graph $G_{A_1} = G[A_1] \cup \{xy\}$ and $C_A = G[V(C) \cap A] \cup \{xy\}$ has a terminal separating 2-chain $(A_1', B_1'), \ldots, (A_n', B_n')$ where we let x and y replace the two vertices in X from G not in G_{A_1}. Additionally, $A_i \cap B_i \subseteq V(C)$, for all $i \in \{1, 2, 3\}$, and $A_i' \cap B_i' \subseteq V(C_A)$ for all $i \in \{1, \ldots, n\}$.*

4. *There are two distinct terminal separating triangles $((A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1))$, and $((A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2))$ where for all $i \in \{1, 2, 3\}$, $A_i^1 \cap B_i^1 \subseteq A_1^2$ and $A_i^2 \cap B_i^2 \subseteq$*

A_3^1. *Furthermore, consider the graph* $G_{A_1^2 \cap A_1^1} = G[A_1^2 \cap A_1^1] \cup \{xy | x, y \in A_1^i \cap B_1^i, i \in \{1, 2\}\}$ *and the cycle* $C' = G[V(C) \cap A_1^2 \cap A_1^1] \cup \{xy | x, y \in A_1^i \cap B_1^i, i \in \{1, 2\}\}$. *Let* X' *be defined to be the vertices* $A_1^2 \cap B_1^2$ *and* $A_3^2 \cap B_3^2$. *Then there is a terminal separating 2-chain with respect to* X', $((A_1, B_1), \ldots, (A_n, B_n))$, *in* $G_{A_1^2 \cap A_1^1}$ *such that* $A_i \cap B_i \subseteq V(C')$.

5. *There are 2 distinct terminal separating triangles* $((A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1))$, $((A_1^2, B_i^2), (A_2^2, B_2^2), (A_3^2, B_3^2))$ *where for all* $i \in \{1, 2, 3\}$, $A_i^1 \cap B_i^1 \subseteq A_1^2$ *and* $A_i^2 \cap B_i^2 \subseteq A_1^1$, *the set* $A_1^2 \cap B_1^2 \cap A_1^1 \cap B_1^1$ *is not empty and* $A_i^j \cap B_i^j \subseteq V(C)$ *for all* $i \in \{1, 2, 3\}$, *and* $j \in \{1, 2\}$.

Before proving this, we prove some lemmas to make the proof cleaner.

Lemma 83. *Let* G *be a 2-connected graph,* $X = \{a, b, c, d\} \subseteq V(G)$. *Let* C *be a cycle in* G *such that* $X \subseteq V(C)$. *If any of the obstructions in Theorem 82 occur, then* G *does not have a* $W_4(X)$*-minor.*

Proof. We deal with each case separately. In each case suppose G is a minimal counterexample with respect to the number of vertices.

Case 1: Suppose we have a terminal separating 2-chain $((A_1, B_1), \ldots, (A_n, B_n))$. If $n = 1$, then (A_1, B_1) is a 2-separation satisfying the conditions in Lemma 69 and thus G does not have a $W_4(X)$-minor. Therefore we assume $n \geq 2$. Then (A_1, B_1) satisfies the conditions in Lemma 70. Let G' be the graph obtained after applying Lemma 70 to (A_1, B_1). Then G has a $W_4(X)$-minor if and only if G' has a $W_4(X_1)$-minor, where X_1 is defined from Lemma 70. Notice in G', $((A_2, B_2), \ldots, (A_n, B_n))$ is a terminal separating 2-chain satisfying the properties of obstruction 1 for the cycle $G'[V(C) \cap B_1]$ when we replace the vertex of X in $A_1 \setminus (A_1 \cap B_1)$ with the vertex in $(A_1 \cap B_1) \setminus X$. Since G is a vertex minimal counterexample, G' has no $W_4(X)$-minor, and thus G has no $W_4(X)$-minor.

Case 2: Suppose there is a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfying properties in the second obstruction. Then we apply Lemma 70 to (A_1, B_1) giving a new graph G' which has a $W_4(X)$-minor if and only if G' has a $W_4(X_1)$-minor, where X_1 is defined from Lemma 70. In G', apply Lemma 70 to (A_2, B_2) giving a graph G''. Then G'' has a $W_4(X_2)$-minor if and only if G has a $W_4(X)$-minor. Observe in G'', (A_3, B_3) is a separation satisfying Lemma 69 and thus G'' does not have a $W_4(X_2)$-minor. But G'' has a $W_4(X)$-minor if and only if G has a $W_4(X)$-minor, so therefore G has no $W_4(X)$-minor.

Case 3: Suppose there is a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ and a terminal separating 2-chain $((A_1', B_1'), \ldots (A_n', B_n'))$ in the graph G_{A_1}, as in the third obstruction. Apply Lemma 67 to (A_2, B_2) to obtain two new reduced graphs G_1' and G_2'. Now in one of G_1' and G_2', we can apply Lemma 70 to (A_1, B_1) to obtain a graph G'', and in one of G_1' and G_2', we can apply Lemma 70 twice to (A_3, B_3) and (A_1, B_1) to obtain the

graph G'' (note that the graph G'' obtained from both G_1' and G_2' is indeed the same graph). Then G'' has a $W_4(X)$-minor if and only if G has a $W_4(X)$-minor. Notice that in the graph G'', $((A_1', B_1'), \ldots, (A_n', B_n'))$ is a terminal separating 2-chain satisfying obstruction 1. Then by case one, G'' has no $W_4(X)$-minor, and thus G has no $W_4(X)$-minor.

Case 4: Suppose we have the fourth obstruction in Theorem 82 and let (A_1, B_1), $(A_2, B_2), (A_3, B_3)$ be one of the terminal separating triangles. If we apply Lemma 67 and Lemma 70 to $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ as we did in case 3, we obtain a graph G' which has a $W_4(X)$-minor if and only if G does. Furthermore, in the graph G', the other terminal separating triangle and terminal separating 2-chain given by the fourth obstruction for G and C satisfy the properties of obstruction 3. But then by case 3, G' has no $W_4(X)$-minor, and thus G has no $W_4(X)$-minor.

Case 5: Let $(A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1)$ and $(A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2)$ be the two distinct triangles satisfying the properties in obstruction five. Applying Lemma 67 and Lemma 70 to $(A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1)$ as in case 3, we obtain a graph G' which has a $W_4(X)$-minor if and only if G has a $W_4(X)$-minor. If $A_1^1 \cap B_1^1 = A_1^2 \cap B_1^2$, then in G', the separations (A_2^2, B_2^2) and (A_3^2, B_3^2) are a terminal separating 2-chain in G' satisfying obstruction 1 on the cycle $C' = G'[V(C) \cap B_1^1]$. Then by case 1, G' has no $W_4(X)$-minor. Therefore we assume that $A_1^1 \cap B_1^1 \neq A_1^2 \cap B_1^2$. Then in G' on the cycle C', $(A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2)$ is a terminal separating triangle, so by case 2, G' has no $W_4(X)$-minor, and thus G has no $W_4(X)$-minor. $\qquad\square$

Lemma 84. *Let G be a 2-connected graph, $X = \{a, b, c, d\} \subseteq V(C)$. Let C be a cycle in G such that $X \subseteq V(C)$. Suppose that a, b, c, d appear in that order on C, and suppose that $cd \in E(C)$. Then*

- *if there is a terminal separating triangle satisfying the properties of obstruction 2, then $A_1 \cap B_1$ contains exactly one of vertices c or d.*

- *If C has a terminal separating triangle and a terminal separating 2-chain $(A_1, B_1), \ldots, (A_n, B_n)$ as in obstruction 3, then either $A_1 \cap B_1$ contains c or d or $A_n \cap B_n$ contains c or d.*

- *Obstructions 4 and 5 do not occur on C.*

Proof. Let $P_{a,b}$ be the (a, b)-path on C such that $c, d \notin V(P_{a,b})$. Similarly define $P_{b,c}$, $P_{c,d}$ and $P_{d,a}$. Then by construction, $P_{a,b} \cup P_{b,c} \cup P_{c,d} \cup P_{d,a} = C$.

We first show that if we have a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ as in obstruction 2, that $A_1 \cap B_1$ contains one of c or d.

If $A_1 \cap B_1$ contains c or d we are done. Therefore we assume that $A_1 \cap B_1$ contains a. Let $(A_1 \cap B_1) \setminus \{a\} = v$. Then to satisfy the definition of a terminal separating triangle we have that $v \in V(P_{b,c})$ or $v \in V(P_{c,d})$. This follows since if v belonged to either of $P_{a,b} \setminus \{a, b\}$ or $P_{d,a} \setminus \{d, a\}$, then either there would not be two exactly two vertices of X contained in A_1,

or we could not satisfy the condition that $A_j \not\subseteq A_i$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$ and maintain that $A_2 \setminus (A_2 \cap B_2)$ and $A_3 \setminus (A_3 \cap B_3)$ both contain a vertex of X.

If $v = b$, then notice that the vertices of $A_2 \cap B_2$ and $A_3 \cap B_3$ that are not a or b lie on $P_{c,d}$. Since $cd \in E(C)$, without loss of generality $A_2 \cap B_2 = \{b, c\}$ and $A_3 \cap B_3 = \{a, c\}$, but this is not a terminal separating triangle, a contradiction.

Therefore $v \neq b$. If $v = c$ or $v = d$ we are done, and since $cd \in E(C)$ it suffices to consider the case when $v \in P_{b,c} \setminus \{b, c\}$. Suppose $A_2 \cap B_2 = \{v, x\}$. If $x \notin V(P_{a,d})$, then either $A_2 \setminus (A_2 \cap B_2)$ does not contain a vertex of X, or $A_3 \setminus (A_3 \cap B_3)$ does not contain a vertex of X which is a contradiction. If $x = d$, then $A_3 \cap B_3 = \{a, d\}$ which implies that $A_3 \setminus (A_3 \cap B_3)$ does not contain a vertex from X, a contradiction. Therefore $A_1 \cap B_1$ cannot contain a. Mirroring the above argument, $A_1 \cap B_1$ cannot contain b, and thus since by definition $A_1 \cap B_1$ contains a vertex of X it contains c or d.

Now suppose we have a terminal separating triangle $(A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1)$ and a terminal separating 2-chain $(A_1, B_1), \ldots, (A_n, B_n)$ in G_{A_1} satisfying the properties of obstruction 3. We will show that either $A_1 \cap B_1$ or $A_n \cap B_n$ contains c or d.

Since $A_1^1 \cap B_1^1 = \{x, y\}$ and $x, y \notin X$, to be a terminal separating triangle, for all $i \in \{1, 2, 3\}$, $A_i^1 \cap B_i^1$ does not contain a vertex from X. This follows from the definition of terminal separating triangle as two vertices of X lie in $A_1^1 \setminus (A_1^1 \cap B_1^1)$, so if one of $A_i^1 \cap B_i^1$, for $i \in \{2, 3\}$ contained a vertex of X, then at least one of $A_2^1 \setminus (A_2^1 \cap B_2^1)$ or $A_3^1 \setminus (A_3^1 \cap B_3^1)$ does not contain a vertex from X, a contradiction. Now notice that since $A_1^1 \setminus \{x, y\}$ contains two vertices of X, and $cd \in E(C)$, either $c, d \in A_1^1 \setminus \{x, y\}$ or $a, b \in A_1^1 \setminus \{x, y\}$. If $a, b \in A_1^1 \setminus \{x, y\}$ then to remain a terminal separating triangle, one of $A_2^1 \cap B_2^1$ or $A_3^1 \cap B_3^1$ contains a vertex from $P_{c,d}$. But since $cd \in E(C)$, and we know that for all $i \in \{1, 2, 3\}$, no vertex of X is contained in $A_i^1 \cap B_i^1$, a contradiction. Therefore $c, d \in A_1^1 \setminus \{x, y\}$. Then by definition of terminal separating 2-chain and since a, b, c, d appear in that order on C, the terminal separating 2-chain in G_{A_1} contains either d or c in $A_n \cap B_n$ or $A_1 \cap B_1$.

Now suppose we have two distinct terminal separating triangles $(A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1)$ and $(A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2)$, and a terminal separating 2-chain in $G[A_1 \cap A_2]$ satisfying the properties of obstruction 4. We will show that this obstruction does not exist since $cd \in E(C)$.

Notice from the assumptions that $A_1^i \cap B_1^i$ does not contain any vertices from X for $i \in \{1, 2\}$. Then since $cd \in E(C)$, without loss of generality we may assume that $a, b \in A_1^1$ and $c, d \in A_1^2$. But then by the previous discussion, one of $A_2^2 \cap B_2^2$ and $A_3^2 \cap B_3^2$ contains a vertex from $P_{c,d}$. But then one of $A_2^2 \cap B_2^2$ and $A_3^2 \cap B_3^2$ contains c or d, which implies that $(A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2)$ is not a terminal separating triangle, a contradiction.

Finally, suppose that the fifth obstruction occurred. Then we have two distinct terminal separating triangles $((A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1)), ((A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2))$ satisfying the properties of obstruction 5. Then from our assumptions and since $cd \in E(C)$, without loss of generality we may assume $c, d \in A_1^1 \setminus (A_1^1 \cap B_1^1)$ and $a, b \in A_1^2 \setminus (A_1^2 \cap B_1^2)$.

By the same argument as for the obstruction 4 case, this gives a contradiction, completing the proof. $\square$

We will extensively make use of the following sub-modularity lemma for separations.

Proposition 85. *Let G be a graph with 2-separations (A_1, B_1) and (A_2, B_2). Let $A_1 \cap B_1 = \{u, v\}$ and $A_2 \cap B_2 = \{x, y\}$ where x, y, u and v are distinct vertices. Furthermore, suppose that $x \in A_1 \setminus (A_1 \cap B_1)$ and $y \in B_1 \setminus (A_1 \cap B_1)$. Then let $u \in A_2 \setminus (A_2 \cap B_2)$ and $v \in B_2 \setminus (A_2 \cap B_2)$. Then there is a separation (A', B') such that $A' \cap B' = \{u, x\}$ and $A' \subseteq A_1$ and $B_1 \subseteq B'$.*

Proof. Let $A' = A_1 \cap A_2$ and $B' = V(G) \setminus (A_1 \cap A_2)$. Let z be any vertex in $A_1 \cap A_2$ and consider a path P from z to any vertex not in $A_1 \cap A_2$. Consider the first vertex in P which is not in $A_1 \cap A_2$. If this vertex is in $B_1 \cap A_2$, then since $u \in A_2 \setminus (A_2 \cap B_2)$, this vertex is u, and thus $u \in V(P)$. If this vertex is in $A_1 \cap B_2$ then since $x \in A_1 \setminus (A_1 \cap B_1)$ this vertex is x and thus $x \in V(P)$. Notice that these are the only options, and thus (A', B') is a 2-separation with $A' \cap B' = \{u, x\}$. Additionally, it is immediate that $A' \subseteq A_1$ and $B_1 \subseteq B'$. $\square$

Now we prove the theorem. Throughout the proof we will abuse the notation of separations slightly. Given a graph G and a 2-separation (A, B), if in the graph G_B, there is a 2-separation (A', B'), then we will refer to the 2-separation $(A' \cup A, B)$ in G as (A', B') to avoid notational clutter. Thus essentially if a separation (A', B') in G_B induces a natural separation in G, then we refer to the separation in G as (A', B').

Proof of Theorem 82. Lemma 83 proves one direction of the theorem.

For the other direction, consider a graph G which is a minimal counterexample with respect to $|V(G)|$. That is, we consider a graph G such that there exists a cycle C such that $X \subseteq V(C)$ where none of the five above obstructions exist on the cycle C, and G is $W_4(X)$-minor free. Note such cycle always exists by Corollary 78. Without loss of generality let a, b, c, d appear in that order on C. Let $P_{a,b}$ be the (a, b)-path on C such that $c, d \notin V(P_{a,b})$. Similarly define $P_{b,c}, P_{c,d}$ and $P_{d,a}$. The goal will be to show that since G is a vertex minimal counterexample, G is 3-connected, but that contradicts that all 3-connected graphs either have a $K_4(X)$ or $W_4(X)$-minor (Theorem 74). We go through all the different possibilities for where the vertices of X can be in relation to a 2-separation.

Claim 1: There is no 2-separation (A, B) such that $X \subseteq A$.

Suppose we had such separation and first suppose $V(C) \subseteq A$. By applying Lemma 68 to (A, B), the graph G has a $W_4(X)$-minor if and only if G_A has a $W_4(X)$-minor. Since G is a vertex minimal counterexample, G_A has one of the obstructions on C. But then the obstruction exists in G, a contradiction.

Therefore we can assume that there are vertices of C in $B \setminus (A \cap B)$. Since $X \subseteq V(C)$ and $X \subseteq A$, all of the vertices in $V(C) \cap B$ lie in exactly one of $P_{a,b}$, $P_{b,c}$, $P_{c,d}$ or $P_{d,a}$.

Without loss of generality, suppose all the vertices in $V(C) \cap B$ lie on $P_{a,b}$. Furthermore, since there is a vertex in $B \setminus (A \cap B)$, this implies that $A \cap B \subseteq V(P_{a,b})$. Then after applying Lemma 68 to (A, B) notice that in the graph G_A, that $C_A = G_A[V(C) \cap A]$ is a cycle. As G is a minimal counterexample, C_A has one of the five obstructions. Notice that regardless of the obstruction, since all the vertices of C that were in B were on $P_{a,b}$, the obstruction for C_A in G_A exists in G for C. But this is a contradiction.

Claim 2: There is no 2-separation (A, B) such that two vertices of X lie in $A \setminus (A \cap B)$ and two vertices of X that lie in $B \setminus (A \cap B)$.

Suppose such separation existed and let $A \cap B = \{x, y\}$. Notice that for such a separation to exist we have that $x \in V(C)$ and $y \in V(C)$. Consider the graphs G_A and G_B inherited from Lemma 66 and let C_A and C_B be the cycles where $C_A = G_A[V(C) \cap A]$ and $C_B = G_B[V(C) \cap A]$. By minimality, both C_A and C_B have one of the five obstructions. We consider the various cases.

Case 1: The cycle C_B has a terminal separating 2-chain as in obstruction 1, say $((A_1^2, B_1^2), \ldots, (A_n^2, B_n^2))$.

Subcase 1: The cycle C_A has a terminal separating 2-chain as in obstruction 1, $((A_1^1, B_1^1), \ldots, (A_n^1, B_n^1))$.

Then if $A_n^1 \cap B_n^1 \cap A_1^2 \cap B_1^2 \neq \emptyset$ we may concatenate the two terminal separating 2-chains giving a terminal separating 2-chain of G on C satisfying obstruction 1. Therefore we can assume $A_n^1 \cap B_n^1$ does not share a vertex with $A_1^2 \cap B_1^2$. But then the terminal separating 2-chain $((A_1^1, B_1^1), \ldots, (A_n^1, B_n^1), (A, B), (A_1^2, B_1^2), \ldots, (A_n^2, B_n^2))$ satisfies obstruction 1 on C, a contradiction.

Subcase 2: The cycle C_A has a terminal separating triangle satisfying obstruction two, $(A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1)$.

Since $xy \in E(C_A)$, the terminal vertex in $A_1^1 \cap B_1^1$ is either x or y, by Lemma 84. But then $(A_1^2, B_1^2), (A_2^2, B_2^2)$, (A_3^2, B_3^2) combines with either the terminal separating 2-chain in G_B or the terminal separating 2 chain plus the separation (A, B) to form the third obstruction in G, a contradiction.

Subcase 3: The cycle C_A has a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ and a terminal separating 2-chain $((A_1^1, B_1^1), \ldots, (A_n^1, B_n^1))$ as in obstruction 3.

By Lemma 84, we can assume that either x or y is in $A_1^1 \cap B_1^1$. But then either we can concatenate the two terminal separating 2-chains such that there resulting terminal separating 2-chain and the terminal separating triangle satisfy obstruction 3, or we can add in the separation (A, B) to the terminal separating 2-chains such that the terminal separating 2-chains plus (A, B) plus the terminal separating triangle satisfy obstruction 3.

Subcase 4: Obstruction 4 or 5 occurs on C_A.

By Lemma 84 the fourth and fifth obstructions do not occur in G_A, since $xy \in E(C_A)$. Now we consider cases where G_B does not have a terminal separating 2-chain.

Case 2: The cycle C_B has a terminal separating triangle $(A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2)$ satisfying obstruction 2.

Notice by Lemma 84 the terminal vertex contained in $A_1^2 \cap B_1^2$ is x or y.

Subcase 1: The cycle C_A has a terminal separating triangle $(A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1)$ satisfying obstruction 2.

Again, by Lemma 84 the terminal vertex contained in $A_1^1 \cap B_1^1$ is x or y. If the terminal vertex in $A_1^i \cap B_1^i$ is x for both $i = 1, 2$, then notice in G the two triangles form obstruction five. A similar statement holds if they are both y. Then we must have that one of the triangles contains x as the terminal vertex in $A_1^i \cap B_1^i$, $i \in \{1, 2\}$ and the other contains y. But then the two triangles plus the separation (A, B) form obstruction 4 in G, a contradiction.

Subcase 2: The cycle C_A has a terminal separating triangle and a terminal separating 2-chain as in obstruction 3.

By Lemma 84 the terminal separating 2-chain contains one of the vertices x or y. Then by possibly adding in the separation (A, B) to the existing terminal separating 2-chains, we can extend this to a triangle plus a terminal separating 2-chain and another terminal separating triangle, as in obstruction 4, a contradiction.

Case 3: Both of the cycles C_A and C_B have obstructions 3.

But then by Lemma 84 the terminal separating 2-chains in both C_A and C_B both contain x or y, and thus after possibly adding in (A, B), we get in G, the fourth obstruction exists on C, a contradiction.

We note these are all of the cases, and therefore there is no 2-separation (A, B) such that two vertices of X lie in $A \setminus (A \cap B)$ and two vertices of X that lie in $B \setminus (A \cap B)$, and $A \cap B \subseteq V(C)$.

Claim 3: There is no 2-separation (A, B) such that $b \in A \cap B$, $a \in A \setminus (A \cap B)$, and $c, d \in B \setminus (B \cap A)$.

Suppose there is such a separation, (A, B), and let $A \cap B = \{v, b\}$. First notice that since C is a cycle containing X, for such separation to exist $v \in V(C)$. Applying Lemma 70 to (A, B) we get that G has a $W_4(X)$-minor if and only if G_B has a $W_4(X)$-minor. Since we picked G to be a minimal counterexample, G_B has one of the five obstructions occuring on the cycle $C_B = G_B[V(C) \cap B]$. We consider the various cases.

Case 1: The cycle C_B has a terminal separating 2-chain $((A_1, B_1), \ldots, (A_n, B_n))$ satisfying obstruction 1.

If $A_1 \cap B_1$ or $A_n \cap B_n$ contains b then it is a terminal separating 2-chain in G for C. Otherwise without loss of generality we have that $A_1 \cap B_1$ contains v. But then $((A, B), (A_1, B_1), \ldots, (A_n, B_n))$ is a terminal separating 2-chain in G for C, contradicting that G is a minimal counterexample.

Case 2: The cycle C_B has a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfying obstruction 2.

First suppose that $b \in A_1 \cap B_1$. Then the terminal separating triangle exists in G, contradicting that we have a minimal counterexample. Then $v \in A_1 \cap B_1$. But then in G, the separation (A, B) plus $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfies obstruction 3, contradicting that G is a minimal counterexample.

Case 3: The cycle C_B has a terminal separating triangle plus a terminal separating 2-chain as in obstruction 3.

Then by Lemma 84, then the terminal separating 2-chain contains either v or b. In the case where the terminal separating 2-chain contains v, then by adding in the separation (A, B), we get a terminal separating 2-chain plus terminal separating triangle in G. In the case where the terminal separating 2-chain contains b, the terminal separating 2-chain and terminal separating triangle were already an obstruction in G, a contradiction.

Case 4: Obstructions 4 or 5 occur on C_B.

By Lemma 84, these obstructions do not occur, a contradiction. We have considered all possible cases, so therefore we can assume there is no 2-separation (A, B) such that $b \in A \cap B$, $a \in A \setminus (A \cap B)$, and $c, d \in B \setminus (B \cap A)$.

Claim 4: There are no 2-separations of the form (A, B) such that $A \cap B = \{u, v\}$, $a \in A \setminus \{u, v\}$, and $b, c, d \in B \setminus \{u, v\}$.

Without loss of generality, we may assume that $u \in V(P_{a,b})$ and $v \in V(P_{a,d})$. By applying Lemma 67 we get that G has a $W_4(X)$-minor if and only if G_B has either a $W_4(X_1)$-minor or a $W_4(X_2)$-minor, where $X_1 = \{b, c, d, u\}$ and $X_2 = \{b, c, d, v\}$. Since we have a minimal counterexample with respect to the number of vertices, we get that $C_B = G_B[V(C) \cap B]$ has one of the obstructions when we consider X_1 and when we consider X_2. First notice that if under either X_1 or X_2, we get obstruction 4 or 5, then the obstruction exists in G, a contradiction. We consider the various other cases.

Case 1: Under X_2, we get a terminal separating 2-chain $((A_1, B_1), \ldots, (A_n, B_n))$.

Notice that we may assume that $v \in A_1 \cap B_1$ or $v \in A_n \cap B_n$, as otherwise $((A_1, B_1), \ldots, (A_n, B_n))$ exists in G for C, a contradiction. Therefore without loss of generality, we suppose that $v \in A_1 \cap B_1$.

Subcase 1: Under X_1, we get a terminal separating 2-chain $((A'_1, B'_1), \ldots, (A'_{n'}, B'_{n'}))$.

By similar reasoning as above, we may assume that $u \in A'_1 \cap B'_1$. Notice that the vertex in $A'_1 \cap B'_1 \setminus \{u\}$ lies in $P_{b,c} \setminus \{b\}$ or in $P_{d,c} \setminus \{d\}$ and and the vertex in $A_1 \cap B_1 \setminus \{v\}$ lies in $P_{b,c} \setminus \{b\}$ or $P_{d,c} \setminus \{d\}$. Consider the case where both the vertex in $A'_1 \cap B'_1 \setminus \{u\}$ and the vertex in $A_1 \cap B_1 \setminus \{v\}$ lie on $P_{b,c} \setminus \{b\}$.

We consider various subcases. Suppose the vertex in $A'_1 \cap B'_1 \setminus \{u\}$ is the same vertex as $A_1 \cap B_1 \setminus \{v\}$. If this vertex is c, then $(A, B), (B_1, A_1), (A'_1, B'_1)$ is a terminal separating triangle in G satisfying obstruction 2, a contradiction. Thus we assume the vertex in $A'_1 \cap B'_1 \setminus \{u\}$ is not c. But then $(A, B), (A_1, B_1), (A'_1, B'_1)$ plus $(A_2, B_2), \ldots, (A_n, B_n)$ is a terminal separating triangle and terminal separating 2-chain satisfying obstruction 3, a contradiction.

Therefore we can assume that $A_1' \cap B_1' \setminus \{u\} \neq A_1 \cap B_1 \setminus \{v\}$. First suppose that the vertex in $A_1' \cap B_1' \setminus \{u\}$ lies in A_1. Notice that one of the vertices in $A_2' \cap B_2'$ lies on $P_{u,d}$, and call this vertex x (if $x \notin V(P_{u,d})$, then either $((A_1', B_1'), \dots, (A_n', B_n'))$ is not a dissection, or not a terminal separating chain). If $x = v$, then $(A, B), (A_1', B_1'), (A_2', B_2')$ plus $((A_1, B_1), \dots (A_n, B_n))$ satisfies obstruction 3 in in G, a contradiction. Therefore $x \neq v$. But notice that $uv \in E(C_B)$, which implies that x lies on the (v, d)-subpath of $P_{u,d}$. But then the vertex in $A_1' \cap B_1' \setminus \{u\}$ and v are a 2-vertex cut. Let (A', B') be the 2-separation such that $A' \cap B' = \{u, (A_1' \cap B_1' \setminus \{u\})\}$. Then $(A, B), (A', B'), (A_1', B_1')$ plus $((A_1, B_1), \dots, (A_n, B_n))$ satisfy obstruction 3 in G, a contradiction.

Therefore we can assume that the vertex in $A_1' \cap B_1' \setminus \{u\}$ lies in B_1. Let x be the vertex in $A_1 \cap B_1 \setminus \{v\}$. Notice that $x \neq c$ as the vertex in $A_1' \cap B_1' \setminus \{u\}$ lies in B_1. Since the vertex in $A_1' \cap B_1' \setminus \{u\}$ lies in B_1, we have that $\{x, u\}$ is a 2-vertex cut. Let (A', B') be this 2-separation. Then $(A, B), (A', B'), (A_1, B_1)$ plus $((A_2, B_2), \dots (A_n, B_n))$ is a terminal separating triangle plus terminal separating 2-chain satisfying obstruction 3 a contradiction.

Therefore we can assume that the vertex in $A_1' \cap B_1' \setminus \{u\}$ and the vertex in $A_1 \cap B_1 \setminus \{v\}$ both do not lie on $P_{b,c} \setminus \{b\}$. By essentially the same argument, we can assume that the vertex in $A_1' \cap B_1' \setminus \{u\}$ and the vertex in $A_1 \cap B_1 \setminus \{v\}$ both do not lie on $P_{c,d} \setminus \{d\}$.

Now consider the case where the vertex in $A_1 \cap B_1 \setminus \{v\}$ lies in $P_{b,c} \setminus \{b\}$ and the vertex in $A_1' \cap B_1' \setminus \{u\}$ lies in $P_{c,d} \setminus \{d\}$. Notice that at c is not both in $A_1 \cap B_1 \setminus \{v\}$ and $A_1' \cap B_1' \setminus \{u\}$. Let x be the vertex in $A_1 \cap B_1 \setminus \{v\}$ and suppose $x \neq c$. Then (A_2, B_2) exists and $A_2 \cap B_2 \setminus \{x\}$ lies on $P_{v,d}$. Then notice that $\{u, x\}$ is a 2-vertex cut in G. Let (A', B') be the separation such that $A' \cap B' = \{u, x\}$. Then $(A, B), (A_1, B_1), (A', B')$ plus $((A_2, B_2), \dots, (A_n, B_n))$ is a terminal separating triangle and terminal separating chain as in obstruction 3. The case where c is not in $A_1' \cap B_1' \setminus \{u\}$ follows similarly.

Now consider the case where the vertex in $A_1 \cap B_1 \setminus \{v\}$ lies in $P_{c,d} \setminus \{d\}$ and the vertex in $A_1' \cap B_1' \setminus \{u\}$ lies in $P_{b,c} \setminus \{b\}$. Notice that at least one of the vertices in $A_1' \cap B_1' \setminus \{u\}$ and $A_1 \cap B_1 \setminus \{v\}$ is not c. Let x be the vertex in $A_1 \cap B_1 \setminus \{v\}$ and suppose that $x \neq c$. Then since $x \neq c$, we have that (A_2, B_2) exists and lies on $P_{v,b}$. Then since $uv \in E(C_B)$, notice that $\{u, x\}$ is a 2-vertex cut. Let (A', B') be the separation such that $A' \cap B' = \{u, x\}$. Then $(A, B), (A', B'), (A_1, B_1)$ plus $((A_2, B_2), \dots, (A_n, B_n))$ is a terminal separating 2-chain plus terminal separating triangle satisfying obstruction 3, a contradiction. The case where c is not in $A_1' \cap B_1' \setminus \{u\}$ follows similarly.

Subcase 2: Suppose that under X_1, we get a terminal separating triangle (A_1', B_1'), $(A_2', B_2'), (A_3', B_3')$ satisfying obstruction 2.

Suppose that the vertex in $A_1' \cap B_1' \setminus u$ lies in A_1. Call this vertex x. First suppose $x \in A_1 \cap B_1 \setminus \{v\}$. Notice that $x \neq c$, as if $x = c$ then $(A_1', B_1'), (A_2', B_2'), (A_3', B_3')$ would not satisfy the definition of a terminal separating triangle, a contradiction. Therefore $x \neq c$, but then $(A, B), (A_1', B_1'), (A_1, B_1)$ and $((A_2, B_2), \dots, (A_n, B_n))$ is a terminal separating 2-chain satisfying obstruction 3, a contradiction.

Now consider when $x \in A_1 \setminus (A_1 \cap B_1)$ and suppose that x lies in $P_{b,c} \setminus \{b\}$. Then the vertex in $A_2' \cap B_2' \setminus \{x\}$ lies in either $P_{u,d} \setminus \{u, v\}$ or $P_{c,d}$. In either case, notice that $\{x, v\}$ is a 2-vertex cut. Let (A', B') be the separation such that $A' \cap B' = \{x, v\}$. But then $(A, B), (A', B'), (A_1', B_1')$ plus $((A_1, B_1), \ldots, (A_n, B_n))$ satisfies obstruction 3 in G, a contradiction. Now suppose that x lies in $P_{c,d} \setminus \{d\}$. Then since $x \in A_1 \setminus (A_1 \cap B_1)$, we have that u and the vertex in $A_1 \cap B_1 \setminus \{v\}$ form a 2-vertex cut. Let (A', B') be the separation induced by this 2-vertex cut. Then if $n = 1$, we have $(A, B), (A', B'), (A_1, B_1)$ form a terminal separating triangle in G, a contradiction. Otherwise $n > 1$ and $(A, B), (A', B'), (A_1, B_1)$ plus $((A_2, B_2), \ldots, (A_n, B_n))$ form a terminal separating triangle plus a terminal separating 2-chain in G, a contradiction.

Therefore we can assume x lies in $B_1 \setminus (A_1 \cap B_1)$. Let y be the vertex in $A_1 \cap B_1 \setminus \{v\}$. Then since $x \in B_1 \setminus (A_1 \cap B_1)$, notice that u, y induces a 2-separation. Let (A', B') be the 2-separation such that $A' \cap B' = \{u, y\}$. If $n = 1$, then $(A, B), (A', B'), (A_1, B_1)$ is a terminal separating triangle satisfying obstruction 2 in G, a contradiction. Otherwise, $(A, B), (A', B'), (A_1, B_1)$ plus $((A_2, B_2), \ldots, (A_n, B_n))$ satisfy obstruction 3 in G, a contradiction.

Subcase 3: Suppose that under X_1, we get a terminal separating triangle (A_1', B_1'), $(A_2', B_2'), (A_3', B_3')$ and a terminal separating 2-chain $((A_1'', B_1''), \ldots, (A_n'', B_n''))$ satisfying obstruction 3.

Notice that if $u \notin A_1'' \cap B_1''$ or $u \notin A_n'' \cap B_n''$, then the obstruction exists already in G, a contradiction. Therefore without loss of generality, $u \in A_1'' \cap B_1''$. Now by the same arguments as in subcase 1, we can look at the position of the vertex in $A_1'' \cap B_1'' \setminus \{u\}$ and show that some obstruction 3 or 2 always exists in G.

Case 2: Under X_2, we get a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfying obstruction 2.

Notice that $A_1 \cap B_1$ contains v as otherwise the obstruction is an obstruction of G, a contradiction. Then since $v \in A_1 \cap B_1$, we may assume that no other vertex of X_2 is in $A_1 \cap B_1$, $A_2 \cap B_2$, or $A_3 \cap B_3$. Also note that u is not in any of $A_1 \cap B_1$, $A_2 \cap B_2$, $A_3 \cap B_3$ as otherwise $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ would not be a terminal separating triangle. Then we may assume that $u \in A_1 \setminus (A_1 \cap B_1)$. Also note that we may assume the vertex in $A_1 \cap B_1 \setminus \{v\}$ lies in $P_{b,c}$. Notice by symmetry and the above cases, we do not need to consider the case where under X_1 we get a terminal separating 2-chain.

Subcase 1: Under X_1, we get a terminal separating triangle satisfying obstruction 2, $(A_1', B_1'), (A_2', B_2'), (A_3', B_3')$.

By a similar argument as above, we may assume that $u \in A_1' \cap B_1'$ as otherwise the obstructions exists in G. Without loss of generality we may assume that the vertex in $A_1' \cap B_1' \setminus \{u\}$ lies in $P_{b,c} \setminus \{c\}$. Let x be the vertex in $A_1' \cap B_1' \setminus \{u\}$. If $x \in A_1 \cap B_1 \setminus \{v\}$, then $(A, B), (A_1', B_1'), (A_1, (B_1)$ and $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfy obstruction 5 in G, a contradiction.

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Now suppose that x is in $A_1 \setminus (A_1 \cap B_1)$. Notice that $A_2' \cap B_2' \setminus \{x\}$ lies on $P_{u,d}$. But then since $uv \in E(C_B)$, $\{x, v\}$ is a 2-vertex cut. Let (A', B') be the separation where $A' \cap B' = \{x, v\}$. But then $(A, B), (A', B'), (A_1', B_1')$ and $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfy obstruction 5 in G, a contradiction.

Now suppose that x lies in $B_1 \setminus (A_1 \cap B_1)$. Let $y \in A_1 \cap B_1 \setminus \{v\}$. Then $\{u, y\}$ is a 2 vertex cut. Let (A', B') be the separation such that $A' \cap B' = \{u, y\}$. Then $(A, B), (A', B'), (A_1, B_1)$ and $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ are terminal separating triangles satisfying obstruction 5 in G, a contradiction.

Now suppose that x is in $A_1 \cap B_1 \setminus \{v\}$. Then easily $(A, B), (A_1, B_1), (A_1', B_1')$ and $(A_1', B_1'), (A_2', B_2'), (A_3', B_3')$ satisfies obstruction 5 in G a contradiction.

Subcase 2: Suppose under X_1, we get a terminal separating triangle $(A_1', B_1'), (A_2', B_2')$, (A_3', B_3') and terminal separating 2-chain $((A_1'', B_1'') \ldots (A_{n'}'', B_{n'}''))$ as in obstruction 3.

Notice that if $u \notin A_1'' \cap B_1''$ and $u \notin A_n'' \cap B_n''$ then the obstruction exists in G, a contradiction. Thus without loss of generality, we may assume that $u \in A_1'' \cap B_1''$. Then notice that the vertex in $A_1'' \cap B_1'' \setminus \{u\}$ lies on $P_{b,c} \setminus \{b\}$ or $P_{c,d} \setminus \{d\}$. Let x be the vertex in $A_1'' \cap B_1''$.

Now suppose that x is in $A_1 \setminus (A_1 \cap B_1)$. Since we assumed that the other vertex in $A_1 \cap B_1 \setminus \{v\}$ was in $P_{b,c}$, this implies that $x \in P_{b,c}$. Then notice that since $c \notin A_1 \cap B_1 \setminus \{v\}$, then either (A_2'', B_2'') exists and the vertex in $A_2'' \cap B_2'' \setminus \{x\}$ lies on $P_{u,d}$ or (A_2'', B_2'') does not exist and the vertex in $A_1' \cap B_1' \setminus \{x\}$ lies in $P_{u,d}$. First suppose the vertex in $A_2'' \cap B_2'' \setminus \{x\}$ lies on $P_{u,d}$. But then since $uv \in E(C_B)$, $\{x, v\}$ is a 2-vertex cut. Let (A', B') be the separation where $A' \cap B' = \{x, v\}$. But then $(A, B), (A', B'), (A_1', B_1')$ and $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfy obstruction 5 in G, a contradiction. A similar analysis holds for the case when (A_2'', B_2'') does not exist and the vertex in $A_1' \cap B_1' \setminus \{x\}$ lies in $P_{u,d}$.

Now suppose that x lies in $B_1 \setminus (A_1 \cap B_1)$. Let $y \in A_1 \cap B_1 \setminus \{v\}$. Then $\{u, y\}$ is a 2 vertex cut. Let (A', B') be the separation such that $A' \cap B' = \{u, y\}$. Then $(A, B), (A', B'), (A_1, B_1)$ and $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ are terminal separating triangles satisfying obstruction 5, a contradiction.

Case 3: Under X_2, we have a terminal separating 2-chain $((A_1', B_1'), \ldots, (A_n', B_n'))$ and a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$.

Notice that if $v \notin A_1' \cap B_1'$ and $v \notin A_n' \cap B_n'$ then the obstruction exists in G, a contradiction. Without loss of generality, assume that $v \in A_1' \cap B_1'$. Notice that the vertex in $v \in A_1' \cap B_1' \setminus \{v\}$ lies in either $P_{b,c}$ or $P_{c,d}$. By symmetry and the previous cases, we only need to consider the case when under X_1 we get obstruction 3.

Subcase 1: Under X_1 we get a terminal separating 2 chain $((A_1'', B_1''), \ldots (A_n'', B_n''))$ and a terminal separating triangle $(A_1''', B_1'''), (A_2''', B_2'''), (A_3''', B_3''')$ satisfying obstruction 3.

By the same arguments as above, without loss of generality we may assume that $u \in A_1'' \cap B_1''$. Then let x be the vertex in $A_1'' \cap B_1'' \setminus \{u\}$. Notice that x lies in $P_{b,c} \setminus \{b\}$ or

$P_{c,d} \setminus \{d\}$. Consider the case where x lies on $P_{b,c} \setminus \{b\}$ and the vertex in $A'_1 \cap B'_1 \setminus \{v\}$ lies on $P_{b,c} \setminus \{b\}$.

First suppose that $x \in A_1 \cap B_1 \setminus \{v\}$. Then if $n = 1$, $(A, B), (A''_1, B''_1), (A'_1, B'_1)$ and $(A_1, B_1), (A_2, B_2), (A_3, B_3))$ satisfies obstruction 5. Otherwise $n \geq 1$ and $(A, B), (A''_1, B''_1)$, (A'_1, B'_1) plus $((A'_2, B'_2) \ldots (A'_n, B'_n))$ plus $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfy obstruction 4 in G, a contradiction.

Now suppose that $x \in A'_1 \setminus (A'_1 \cap B'_1)$. Then either the vertex in $A''_2 \cap B''_2 \setminus \{x\}$ lies on $P_{u,d}$ or (A''_2, B''_2) does not exist and the vertex in $A'''_1 \cap B'''_1 \setminus \{x\}$ lies on $P_{u,d}$. Consider the case where $A''_2 \cap B''_2 \setminus \{x\}$ lies on $P_{u,d}$. Then since $uv \in E(C_B)$, we have that $\{x, v\}$ is a 2-vertex cut. Let (A', B') be the separation where $A' \cap B' = \{x, v\}$. But then $(A, B), (A', B'), (A''_1, B''_1)$ plus $((A_1, B_1), \ldots, (A_n, B_n))$ plus $(A'_1, B'_1), (A'_2, B'_2), (A'_3, B'_3)$ satisfies obstruction 4 in G, a contradiction.

Now suppose that $x \in B'_1 \setminus (A'_1 \cap B'_1)$. Let $y \in A'_1 \cap B'_1 \setminus \{v\}$. Then notice that $\{y, u\}$ is a 2-vertex cut in G_B. Let (A', B') be the separation such that $A' \cap B' = \{y, u\}$. Then if $n = 1$, $(A, B), (A', B'), (A'_1, B'_1)$ plus $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfies obstruction 5 in G, a contradiction. Otherwise, $n \geq 2$, and $(A, B), (A', B'), (A'_1, B'_1)$ plus $((A'_2, B'_2), \ldots, (A'_n, B'_n))$ plus $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfies obstruction 4 in G, a contradiction.

Notice the case where x lies on $P_{c,d} \setminus \{d\}$ and the vertex in $A'_1 \cap B'_1 \setminus \{v\}$ lies on $P_{b,d}$ follows in a similar fashion.

Now suppose that x lies on $P_{b,c} \setminus \{b\}$ and the vertex in $A'_1 \cap B'_1 \setminus \{v\}$ lies on $P_{c,d} \setminus \{d\}$. Let y be the vertex in $A'_1 \cap B'_1 \setminus \{v\}$. Then $\{u, y\}$ is a 2-vertex cut. Let (A', B') be the separation induced by this 2-vertex cut. Then if $n = 1$, we have $(A, B), (A', B'), (A'_1, B'_1)$ plus $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ form obstruction 5, a contradiction. Otherwise, $n > 1$ and $(A, B), (A', B'), (A'_1, B'_1)$ plus $((A'_2, B'_2) \ldots (A'_n, B'_n))$ and $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfies obstruction 4, a contradiction.

Now suppose that x lies on $P_{c,d} \setminus \{d\}$ and the vertex in $A'_1 \cap B'_1 \setminus \{v\}$ lies in $P_{b,c} \setminus \{c\}$. We note that a similar argument to when both x and the vertex in $A'_1 \cap B'_1 \setminus \{v\}$ were in $P_{b,c}$ and $x \in B'_1 \setminus (A'_1 \cap B'_1)$ works in this case.

Therefore, there are no 2-separations of the form (A, B) such that $A \cap B = \{u, v\}$, $a \in A \setminus \{u, v\}$, and $b, c, d \in B \setminus \{u, v\}$.

Claim 5: There are no 2-separations (A, B) such that $A \cap B$ contain two vertices from X, and $A \setminus (A \cap B)$ contains a vertex from X and $B \setminus (A \cap B)$ contains a vertex from X.

If such separation existed, it would be a terminal separating 2-chain, a contradiction.

Therefore, our graph G has no 2-separations. Therefore G is 3-connected. But every 3-connected graph which does not have a $K_4(X)$-minor has a $W_4(X)$-minor by Theorem 74. But this contradicts our choice of G, completing the claim. $\qquad\square$
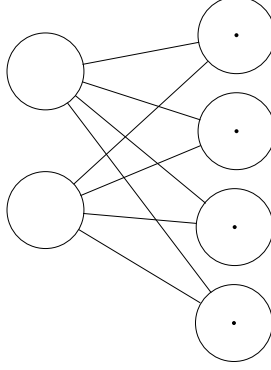
Figure 3.8: A graph G with a model of a $K_{2,4}(X)$-minor. The circles represent connected subgraphs, and the vertices in the circles represent vertices from X.

3.3 A characterization of graphs without a $K_4(X)$, $W_4(X)$, or a $K_{2,4}(X)$-minor

First we define what we mean by $K_{2,4}(X)$-minor. Let $V(K_{2,4}) = \{t_1, t_2, t_3, t_4, s_1, s_2\}$ where $E(K_{2,4}) = \{t_i s_j \mid \forall i \in \{1,2,3,4\}, j \in \{1,2\}\}$. Let G be a graph and $X = \{a, b, c, d\} \subseteq V(G)$. Let $\mathcal{F}$ be the family of maps from X to $V(K_{2,4})$ such that each vertex of X goes to a distinct vertex in $\{t_1, t_2, t_3, t_4\}$. For the purposes of this thesis, a $K_{2,4}(X)$ minor refers to the X and family of maps given above.

Graphs not containing $K_{2,4}(X)$-minors was the subject of Demasi's PhD thesis ([17]). He managed to give a complete chracterization of planar graphs without a $K_{2,4}(X)$-minor. We do not require the full strength of the planar $K_{2,4}(X)$ characterization, but in obtaining a full characterization of graphs without a $K_4(X)$, $W_4(X)$ or $K_{2,4}(X)$-minor we will make use of the reduction lemmas given in [17] to throw out most of the non-web classes of $K_4(X)$ and $W_4(X)$ free graphs. Then by appealing to Theorem 82, we will show that if an $\{a, b, c, d\}$-web has a $K_{2,4}(X)$-minor, then it also has a $W_4(X)$-minor (see Theorem 97).

Of particular interest will be $K_{2,2}(X)$-minors due to a useful reduction lemma. Let $V(K_{2,2}) = \{t_1, t_2, s_1, s_2\}$ where $E(K_{2,2}) = \{t_i s_j \mid \forall i, j \in \{1,2\}\}$. Let G be a graph and $X = \{a, b, c, d\} \subseteq V(G)$. Define $\mathcal{F}$ to be the family of maps where a and b are mapped to $\{s_1, s_2\}$ and c and d are mapped to $\{t_1, t_2\}$. For the purpose of this thesis, a $K_{2,2}(X)$-minor refers to the X and $\mathcal{F}$ above.

Theorem 86 ([17]). *Let a, b, c, d be distinct vertices in a graph G. Then G contains a $K_{2,2}(X)$-minor if and only if there exists an (a, c)-path $P_{a,c}$ and a (b, d)-path $P_{b,d}$ such that $P_{a,c} \cap P_{b,d} = \emptyset$ and there exists an (a, d)-path $P_{a,d}$ and a (b, c)-path $P_{b,c}$ such that $P_{b,c} \cap P_{a,d} = \emptyset$.*

Since $K_{2,4}$ is 2-connected, by previous discussion, we may assume we are dealing with 2-connected graphs (We note that [17] has the same cut vertex reductions specialized to
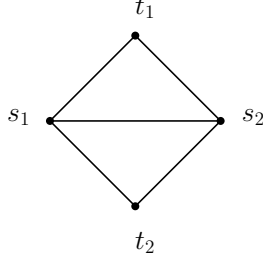
Figure 3.9: The small graph H, a diamond from Lemma 91.

$K_{2,4}(X)$-minors, though we note the proofs are in essence the same as those given in the cut vertex section).

Now we record some of the 2-connected reductions from [17]. For the next three lemmas, suppose G is a 2-connected graph, $X = \{a, b, c, d\} \subseteq V(G)$ and (A, B) a 2-separation such that $A \cap B = \{u, v\}$. Also let $G_A = G[A] \cup \{uv\}$ and $G_B = G[B] \cup \{uv\}$, as before.

Lemma 87. *Suppose that $X \subseteq B$. Then G has a $K_{2,4}(X)$ minor if and only if G_B has a $K_{2,4}(X)$-minor.*

Lemma 88. *If $u, v \in X$, then G has a $K_{2,4}(X)$-minor if and only if $X \subseteq A$ and G_A has a $K_{2,4}(X)$-minor or $X \subseteq B$ and G_B has a $K_{2,4}(X)$-minor. Suppose $u \in X$ and $v \notin X$. Furthermore, suppose $a \in A \setminus (A \cap B)$ and $b, c, d \in B$. For each $\pi \in \mathcal{F}$, let $\pi_B : \{u, b, c, d\} \to V(K_{2,4})$ be such that $\pi_B(u) = \pi(a)$ and on $\{b, c, d\}$, $\pi_B = \pi$. Then G has a $K_{2,4}(X)$-minor if and only if G_B has a $K_{2,4}(X)$-minor.*

Lemma 89. *Suppose $a, b \in A \setminus (A \cap B)$ and $c, d \in B \setminus (A \cap B)$. For each $\pi \in \mathcal{F}$, let $\pi_A : \{a, b, u, v\} \to V(K_{2,4})$ be such that $\pi_A = \pi$ on $\{a, b\}$ and $\pi_A(u) = \pi(c)$ and $\pi_A(v) = \pi(d)$. Similarly for each $\pi \in \mathcal{F}$, define $\pi_B : \{c, d, u, v\} \to V(K_{2,4})$. Additionally for each $\pi \in \mathcal{F}$, let $\pi'_A : \{a, b, u, v\} \to V(K_{2,2})$ be such that $\pi'_A = \pi$ on $\{a, b\}$ and $\pi'_A(u) = \pi(c)$ and $\pi'_A(v) = \pi(d)$. Also, for each $\pi \in \mathcal{F}$, let $\pi'_B : \{c, d, u, v\} \to V(K_{2,2})$ be such that $\pi'_B = \pi$ on $\{c, d\}$ and, $\pi'_A(u) = \pi(a)$ and $\pi'_A(v) = \pi(b)$. Then G has a $K_{2,4}(X)$-minor if and only if either G_A or G_B has a $K_{2,4}(X)$-minor, or both of G_A and G_B have a $K_{2,2}(X)$ minor.*

That completes the 2-connected reductions from [17] that will be needed. There is one 3-connected reduction from [17] which is useful.

Lemma 90. *Let G be a graph and let (A, B) be a tight 3-separation where $A \cap B = \{v_1, v_2, v_3\}$. Suppose that $X \subseteq A$. Let $G' = G[A] \cup \{v_1 v_2, v_1 v_3, v_2 v_3\}$. Then G has a $K_{2,4}(X)$-minor if and only if G' has a $K_{2,4}(X)$-minor.*

Additional reduction lemmas are proven in [17], but these suffice for our needs. To avoid repeating the same statements in the next lemmas, we make the following observation.

Lemma 91. *Let H be the graph where $V(H) = V(K_{2,2})$ and $E(H) = E(K_{2,2}) \cup \{s_1 s_2\}$. Let G be a 2-connected spanning subgraph of H^+. From our definition of $K_{2,2}(X)$-minors, let t_1, t_2 take the place of a, b and s_1, s_2 take the place of c and d. Then G has a $K_{2,2}(X)$-minor, and G does not have a $K_{2,4}(X)$-minor.*

Proof. Notice that $\{s_1, s_2\}$ are the vertex boundary for a 2-separation (A, B) in H^+ such that $t_1 \in A \setminus \{s_1, s_2\}$ and $t_2 \in B \setminus \{s_1, s_2\}$. Then by Lemma 88, H^+ has no $K_{2,4}(X)$-minor and thus G has no $K_{2,4}(X)$ minor.

As G is 2-connected, we can find a (s_1, t_1)-path in $G[A]$ which does not contain s_2 and a (s_2, t_2)-path in $G[B]$ which does not use s_1. Similarly there exists a (s_1, t_2)-path in $G[B]$ which does not use s_2 and a (s_2, t_1)-path in $G[A]$ which does not use $G[B]$. Therefore by Theorem 86, G has a $K_{2,2}(X)$-minor. $\square$

Corollary 92. *Let G be a 2-connected spanning subgraph of a class $\mathcal{A}, \mathcal{B}$, or $\mathcal{C}$. If G is the spanning subgraph of a class $\mathcal{A}$ graph, G does not have a $K_{2,4}(X)$-minor. If G is the spanning subgraph of a class $\mathcal{B}$, or $\mathcal{C}$, then G has a $K_{2,4}(X)$-minor.*

Proof. If G is the spanning subgraph of a class $\mathcal{A}$ graph, then $\{d, e\}$ is the vertex boundary of a 2-separation (A, B) where two vertices of X lie in $A \setminus \{d, e\}$ and one lies in $B \setminus \{d, e\}$. By applying Lemma 88 we see that G has a $K_{2,4}(X)$-minor if and only if the graph in Lemma 91 has a $K_{2,4}(X)$-minor. By Lemma 91, it does not. Therefore G is $K_{2,4}(X)$-minor free.

If G is the spanning subgraph of a class $\mathcal{B}$ or $\mathcal{C}$ graph, then $\{e, f\}$ is the vertex boundary of a 2-separation (A, B) where two vertices of X lie in $A \setminus \{e, f\}$ and two vertices of X lies in $B \setminus \{e, f\}$. Then applying Lemma 89, we see that G has a $K_{2,4}(X)$-minor if the graph in Lemma 91 has a $K_{2,2}(X)$-minor. By Lemma 91, the graph in question has a $K_{2,2}(X)$-minor and therefore G has a $K_{2,4}(X)$-minor. $\square$

Now we deal with class $\mathcal{E}$ and $\mathcal{F}$ graphs.

Lemma 93. *Let G be a 2-connected spanning subgraph of a $\{t_1, t_2, s_1, s_2\}$-web, $H^+ = (H, F)$. Then there is a planar 2-connected graph G' such that G has a $K_{2,2}(X)$-minor if and only if G' has a $K_{2,2}(X)$-minor.*

Proof. We construct G' in the same way that we construct G' in Corollary 78. We refer the reader to the proof of Corollary 78 for the verification that G' is 2-connected and planar. Notice that if G' has a $K_{2,2}(X)$-minor, then immediately G has a $K_{2,2}(X)$-minor.

Therefore we assume that G has a $K_{2,2}(X)$-minor. Then by Theorem 86, there is an (t_1, s_1)-path P_{t_1, s_1} and a (t_2, s_2)-path, P_{t_2, s_2} such that $P_{t_1, s_1} \cap P_{t_2, s_2} \neq \emptyset$ and there is an (t_1, s_2)-path, P_{t_1, s_2} and a (t_2, s_1)-path P_{t_2, s_1} such that $P_{t_1, s_2} \cap P_{t_2, s_1} \neq \emptyset$.

Suppose P_{t_1, s_1} contains a vertex from F_T for some T. Then at least two vertices from T are in P_{t_1, s_1}. Then since $|V(T)| = 3$, P_{t_2, s_2} does not contain any vertex from F_T. Therefore

if we contract F_T down to a vertex, after contracting appropriately P_{t_1,s_1} is still a (t_1, s_1)-path, and $P_{t_1,s_1} \cap P_{t_2,s_2} = \emptyset$. A similar statement holds for P_{t_2,s_1} and P_{t_1,s_2}. Applying that argument to each triangle T in H and appealing to Theorem 86, G' has a $K_{2,2}(X)$-minor. □

Lemma 94. *Let G be a 2-connected spanning subgraph of an $\{t_1, t_2, s_1, s_2\}$-web, $H^+ = (H, F)$. Assume that t_1, t_2, s_1 and s_2 appear in that order in the outerface of H. Then G does not contain a $K_{2,2}(X)$-minor.*

Proof. We note it suffices to show that H^+ does not have a $K_{2,2}(X)$-minor. By Lemma 93, we may assume that H^+ is planar. Then by observation 76 the cycle with edge set $t_1t_2, t_2s_1, s_1s_2, s_2t_1$ is a face. Consider any (t_1, s_1)-path P_{t_1,s_1} and any (t_2, s_2)-path P_{t_2,s_2}. We claim that $P_{t_1,s_1} \cap P_{t_2,s_2} \neq \emptyset$. If $t_2, s_2 \in P_{t_1,s_1}$ or $t_1, s_1 \in P_{t_2,s_2}$ then we are done. Therefore we assume that $t_2, s_2 \notin P_{t_1,s_1}$ and $t_1, s_1 \notin P_{t_2,s_2}$. But then by the Jordan Curve Theorem, $P_{t_1,s_1} \cap P_{t_2,s_2}$ is non-empty, completing the claim. □

Lemma 95. *Let G be a 2-connected spanning subgraph of a class $\mathcal{E}$ or a $\mathcal{F}$ graph. If G is a spanning subgraph of a class $\mathcal{E}$ graph, then G has a $K_{2,4}(X)$-minor if and only if the $\{e, f, c, d\}$-web has a $K_{2,4}(X)$-minor. If G is a spanning subgraph of a class $\mathcal{F}$ graph, then G has a $K_{2,4}(X)$-minor if and only if the $\{e, f, g, h\}$-web has a $K_{2,4}(X)$-minor.*

Proof. First suppose that G is a 2-connected spanning subgraph of a class $\mathcal{E}$ graph. If the $\{e, f, c, d\}$-web has a $K_{2,4}(X)$-minor, then immediately G has a $K_{2,4}(X)$-minor.

Therefore assume that G has a $K_{2,4}(X)$-minor. Apply Lemma 89 to the two separation with vertex boundary $\{e, f\}$. Then by appealing to Lemma 94 and Lemma 91, we get that G having a $K_{2,4}(X)$-minor implies the $\{e, f, c, d\}$-web has a $K_{2,4}(X)$-minor, completing the claim. Essentially the same argument gives the claim for the class $\mathcal{F}$ graphs. □

Now it suffices to deal with webs to complete the characterization. Note we can reduce the problem of finding $K_{2,4}(X)$-minors down to the planar case.

Lemma 96. *Let $H^+ = (H, F)$ be an $\{a, b, c, d\}$-web. Let G be 2-connected spanning subgraph of H^+. Then there is a planar graph K such that G has a $K_{2,4}(X)$-minor if and only if K has a $K_{2,4}(X)$-minor.*

Proof. For each triangle T in H, and every two element subset of $V(T)$ which induces a 2-separation (A, B) such that $B \setminus (A \cap B) = V(F_T)$, apply Lemma 68. After doing this to every triangle, notice that for every triangle $T \in H$, T induces a tight 3-separation (A, B) such that $B = V(F_T) \cup V(T)$. Then we may apply Lemma 90 to (A, B). Call the resulting graph K. By construction, K has a $K_{2,4}(X)$-minor if and only if G has a $K_{2,4}(X)$-minor. Additionally, notice that K is a subgraph of H, and H is planar, so thus K is planar. □

Since Demasi characterized exactly when planar graphs have $K_{2,4}(X)$ minors ([17]), a full characterization of graphs without $K_{2,4}(X)$ and $K_4(X)$ is known. It is exactly when

the the characterizations given in [45] and [17] agree. We do not explore this to try and give a cleaner characterization, we just note that for webs, $K_{2,4}(X)$ minors only appear when $W_4(X)$-minors appear for webs.

Theorem 97. *Let G be a 2-connected graph which is the spanning subgraph of an $\{a, b, c, d\}$-web. Let $X = \{a, b, c, d\} \subseteq V(G)$. If G does not have a $W_4(X)$-minor, then G does not have a $K_{2,4}(X)$-minor.*

Proof. Since G does not have a $W_4(X)$-minor, by Theorem 82 for every cycle C such that $X \subseteq V(C)$, we have one of five obstructions. By Corollary 78 we know at least one such cycle exists. Suppose for sake of contradiction, that G is a minimal counterexample with respect to vertices. We proceed by checking all the cases.

Case 1: Suppose that there is a terminal separating 2-chain $((A_1, B_1), (A_2, B_2), \ldots, (A_n, B_n))$ such that $A_i \cap B_i \subseteq V(C)$ for all $i \in \{1, \ldots, n\}$. If $A_1 \cap B_1$ contains two vertices of X, then by Lemma 88 there is no $K_{2,4}(X)$-minor. Therefore we can assume that $A_1 \cap B_1$ contains exactly one vertex from X. Then by Lemma 88, the graph G has $K_{2,4}(X)$-minor if and only if the graph G_{B_1} has a $K_{2,4}(X_1)$-minor, where X_1 is defined from Lemma 88. Since G did not have a $W_4(X)$-minor, G_{B_1} does not have a $W_4(X_1)$-minor, and thus by minimality G_{B_1} does not have a $K_{2,4}(X_1)$-minor.

Case 2: Suppose that there is a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ such that $A_i \cap B_i \subseteq V(C)$ and $A_1 \cap B_1$ contains a vertex of X. Then applying Lemma 88 to the separation (A_1, B_1) we that G has a $K_{2,4}(X)$-minor if and only if the graph G_{B_1} has a $K_{2,4}(X_1)$-minor, where X_1 is defined from Lemma 88. Since G does not have a $W_4(X)$-minor, G_{B_1} does not have a $W_4(X_1)$-minor, and thus by minimality, G_{B_1} does not have a $K_{2,4}(X_1)$-minor, and therefore G does not have a $K_{2,4}(X)$-minor.

Case 3: Suppose that there is a terminal separating triangle $(A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1)$ and a terminal separating 2-chain $((A_1, B_1), \ldots (A_n, B_n))$ in $G_{A_1^1}$. Then suppose that $A_n \cap B_n$ contains a vertex of X. Then applying Lemma 88 to the separation (A_n, B_n), we have that G has a $K_{2,4}(X)$-minor if and only if G_{A_n} has a $K_{2,4}(X_1)$-minor, where X_1 is defined from Lemma 88. Since G does not have a $W_4(X)$-minor, G_{A_n} does not have a $W_4(X_1)$-minor, and thus by minimality, G_{A_n} does not have a $K_{2,4}(X_1)$-minor so G does not have a $K_{2,4}(X)$-minor.

Case 4: Suppose we have terminal separating triangles $((A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1))$, $((A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2))$ where for all $i \in \{1, 2, 3\}$, $A_i^1 \cap B_i^1 \subseteq A_1^2$ and $A_i^2 \cap B_i^2 \subseteq A_3^1$, and if we consider the graph $G[A_1^2 \cap A_1^1]$, and the cycle $C' = G[V(C) \cap A_1^2 \cap A_1^1] \cup \{xy | x, y \in A_1^i \cap B_1^i, i \in \{1, 2\}\}$ and we let X' be defined to be the vertices $A_1^2 \cap B_1^2$ and $A_3^2 \cap B_3^2$, then there is a terminal separating 2-chain in $G[A_3^2 \cap A_1^1]$ with respect to X'.

Consider the two separation (A_1^1, B_1^1). From previous analysis, there are two vertices of $X \in A_1^1 \setminus (A_1^1 \cap B_1^1)$ and two vertices of X are in $B_1^1 \setminus (A_1^1 \cap B_1^1)$. Without loss of generality, let a, b be the two vertices in $X \in A_1^1 \setminus (A_1^1 \cap B_1^1)$. Therefore we can apply Lemma 89 to

(A_1^1, B_1^1) and get a two graphs $G_{A_1^1}$, $G_{B_1^1}$ such that G has a $K_{2,4}(X)$-minor if and only if either $G_{A_1^1}$ has a $K_{2,4}(X_1)$ or both have a $K_{2,2}(X_1)$-minor, where X_1 is defined from Lemma 89. Since G did not have a $W_4(X)$-minor both $G_{A_1^1}$ and $G_{B_1^1}$ do not have a $W_4(X_1)$-minor, and thus by minimality, both $G_{A_1^1}$ and $G_{B_1^1}$ do not have $K_{2,4}(X_1)$-minors.

Therefore it suffices to show that $G_{B_1^1}$ does not have a $K_{2,2}(X_1)$-minor (see Figure 3.10 for a picture). Let $A_1^1 \cap B_1^1 = \{t_1, t_2\}$. Without loss of generality, let $A_2^1 \cap B_2^1 = \{v, t_1\}$ and $A_3^1 \cap B_3^1 = \{v, t_2\}$. Now since we had a terminal separating triangle, we have that $v \neq a, b$. So without loss of generality, we may assume $a \in A_2^1 \setminus (A_2^1 \cap B_2^1)$ and $b \in A_3^1 \setminus (A_3^1 \cap B_3^1)$. Now by Theorem 86, it suffices to show that for every (a, t_2)-path, P_{a,t_2} and every (b, t_1)-path, P_{b,t_1} in $G_{B_1^1}$ we have $P_{a,t_2} \cap P_{b,t_1} \neq \emptyset$. Let P_{a,t_2} be any (a, t_2)-path. Since $a \in A_2^1 \setminus \{v, t_1\}$ and $t_2 \in B_2^1 \setminus \{v, t_1\}$, either v or $t_1 \in V(P_{a,t_2})$. If $t_1 \in V(P_{a,t_2})$, then any (b, t_2)-path P_{b,t_2} contains t_1 by definition, so $P_{a,t_2} \cap P_{b,t_1} \neq \emptyset$. Therefore we only have to consider when $v \in P_{a,t_2}$. Now since $b \in A_3^1 \setminus \{v, t_2\}$ and $t_1 \in B_3^1 \setminus \{v, t_2\}$ every (b, t_1)-path P_{b,t_1} contains either v or t_2. By similar reasoning as above, we may assume that $t_2 \notin P_{b,t_1}$. Therefore $v \in P_{b,t_1}$. But then $P_{b,t_1} \cap P_{a,t_2} \neq \emptyset$, which implies $G_{B_1^1}$ does not have a $K_{2,2}(X_1)$-minor. Combining this with what we already showed, this implies that G does not have a $K_{2,4}(X)$-minor.

Case 5: Now suppose there are 2 distinct terminal separating triangles $((A_1^1, B_1^1),$ $(A_2^1, B_2^1), (A_3^1, B_3^1)), ((A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2))$ where for all $i \in \{1, 2, 3\}$, $A_i^1 \cap B_i^1 \subseteq A_i^2$ and $A_i^2 \cap B_i^2 \subseteq A_i^1$, $A_1^1 \cap B_1^1 \cap A_1^1 \cap B_1^1$ is not empty and $A_i^j \cap B_i^j \subseteq V(C)$ for all $i \in \{1, 2, 3\}$, $j \in \{1, 2\}$. Now notice if we apply Lemma 89 to (A_1^1, B_1^1), we get two graphs $G_{A_1^1}$ and $G_{B_1^1}$ such that G has a $K_{2,4}(X)$-minor if and only if either one of $G_{A_1^1}$ or $G_{B_1^1}$ has a $K_{2,4}(X_1)$-minor or both have $K_{2,2}(X_1)$-minors, where X_1 is defined from Lemma 89. Now since G does not have a $W_4(X)$-minor both of $G_{B_1^1}$ and $G_{A_1^1}$ do not have $W_4(X_1)$-minors and thus by minimality both do not have $K_{2,4}(X_1)$-minors. Observe that we can apply the same argument as case four to $G_{B_1^1}$ to obtain that $G_{B_1^1}$ does not have a $K_{2,2}(X_1)$-minor. Therefore G does not have a $K_{2,4}(X)$-minor, completing the claim. $\qquad \square$

As a recap of what we have so far.

Corollary 98. *Let G be a 2-connected graph and $X = \{a, b, c, d\} \subseteq V(G)$. The graph G has no $K_4(X)$, $K_{2,4}(X)$ or $W_4(X)$-minor if and only if G belongs to class $\mathcal{A}$ (see Theorem 56) or G is the spanning subgraph of a class $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ graph and the corresponding web does not have a $W_4(X)$-minor (see Theorem 82).*

3.4 A characterization of graphs without a $K_4(X)$, $W_4(X)$, $K_{2,4}(X)$ or an $L(X)$-minor

We define the graph L to have vertex set $V(L) = \{v_1, \ldots, v_8\}$ and $E(L) = \{v_1 v_2, v_1 v_5, v_2 v_7,$ $v_2 v_8, v_2 v_3, v_3 v_4, v_4 v_5, v_4 v_7, v_5 v_6, v_6 v_7, v_6 v_8, v_7 v_8\}$ (see Figure 3.11). Let G be a graph and
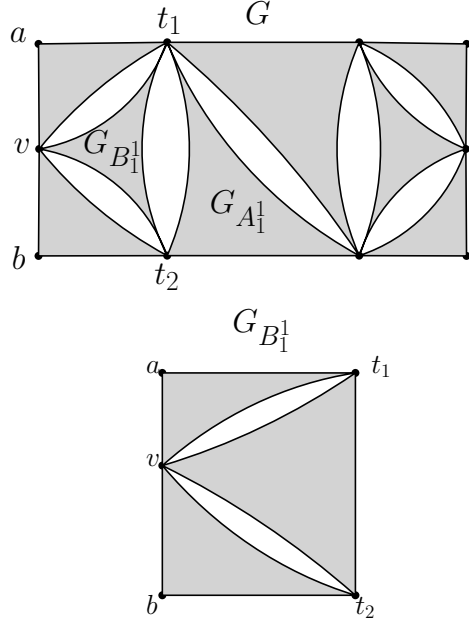
Figure 3.10: The graph $G_{B_1^1}$ in case four of Theorem 97. The shaded sections are spanning subgraphs of webs.
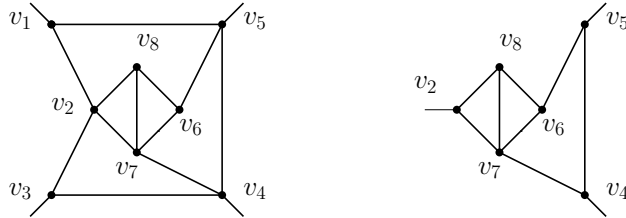

Figure 3.11: The graph L and the graph L'. Vertices with lines with only one endpoint indicate the vertices where the roots are being mapped to.

$X = \{a, b, c, d\} \subseteq V(G)$. Let $\mathcal{F}$ be the family of maps from X to $V(L)$ where each vertex of X goes to a distinct vertex in $\{v_1, v_3, v_4, v_5\}$. For the purposes of this thesis, an $L(X)$-minor refers to the X and family of maps defined above. It is easy to see that the graph L is 2-connected, so the cut vertex section applies. Therefore we may assume that all graphs are at least 2-connected.

We let L' denote the graph induced by $\{v_2, v_8, v_7, v_6, v_5, v_4\}$ in L. Let G be a graph and $X = \{a, b, c\} \subseteq V(G)$. Let $\mathcal{F}$ be the family of surjective maps from X to $\{v_2, v_4, v_5\}$. An $L'(X)$-minor will refer to the $\mathcal{F}$ and X above. It is easy to see that L' is 2-connected and thus we may assume all graphs are 2-connected.

Lemma 99. *Let G be a 2-connected graph and let $X = \{a, b, c\}$. Then G has an $L'(X)$-minor if and only if there are three distinct cycles C_1, C_2, C_3 and three distinct paths P_1, P_2, P_3 satisfying the following properties:*

1. $|V(C_1) \cap V(C_2)| \geq 2$, $|V(C_2) \cap V(C_3)| \geq 2$, $|C_3| \geq 4$, *there is at least one edge in* $E(C_2)$ *which is not contained in either of* $E(C_1)$ *and* $E(C_2)$, *and there exists a vertex* $v_1 \in (V(C_2) \cap V(C_3)) \setminus V(C_1)$ *and a vertex* $v_2 \in (V(C_1) \cap V(C_2)) \setminus V(C_3)$.

2. *The vertices* a, b *and* c *are endpoints of* P_1, P_2 *and* P_3 *respectively. Additionally, the other endpoint of* P_1 *is in* $V(C_1)$, *and the other endpoint of point* P_2 *and* P_3 *is in* $V(C_3)$. *Furthermore,* $P_1 \cap C_j = \emptyset$ *for all* $j \in \{2, 3\}$, *and* $P_i \cap C_j = \emptyset$ *for all* $i \in \{2, 3\}, j \in \{1, 2\}$.

Proof. Let $\{G_x | x \in V(L')\}$ be a model of an $L'(X)$-minor in G. Let x_1 be a vertex in G_{v_2} which is adjacent to a vertex $x_2 \in G_{v_7}$. Let x_3 be a vertex in G_{v_7} which is adjacent to a vertex x_4 in G_{v_8}. Let x_5 be a vertex in G_{v_8} which is adjacent to a vertex x_6 in G_{v_2}. Then since G_{v_2} is connected, there is a (x_1, x_6)-path, P_{x_1, x_6}, contained in G_{v_2}. Similarly, there is a (x_2, x_3)-path, P_{x_2, x_3}, contained in G_{v_7} and a (x_4, x_5)-path, P_{x_4, x_5}, contained in G_{v_8}. Then $C_1 = P_{x_1, x_6} \cup P_{x_2, x_3} \cup P_{x_4, x_5} \cup \{x_1 x_2, x_3 x_4, x_5 x_6\}$ is a cycle.

Now there is a vertex $v_1 \in G_{v_7}$ which is adjacent to a vertex $v_2 \in G_{v_6}$. Additionally there is a vertex $v_3 \in G_{v_6}$ which is adjacent to a vertex, $v_4 \in G_{v_8}$. As G_{v_6} is connected, there is a (v_2, v_3)-path, P_{v_2, v_3}, contained in G_{v_6}. As v_4 and x_4 are in G_{v_8}, there is a (v_4, x_4)-path, P_{v_4, x_4}, contained in G_{v_8}. Similarly, as v_1 and x_3 are in G_{v_7}, there is a (v_1, x_3)-path, P_{v_1, x_3}, contained in G_{v_7}. Then $C_2 = P_{v_2, v_3} \cup P_{v_1, x_3} \cup P_{v_4, x_4} \cup \{v_3 v_4, x_3 x_4, v_1 v_2\}$ is a cycle.

By definition of an $L(X)$-model, there is a vertex $y_1 \in G_{v_5}$ which is adjacent to a vertex $y_2 \in G_{v_4}$. There is a vertex $y_3 \in G_{v_4}$ which is adjacent to a vertex $y_4 \in G_{v_7}$. There is a vertex y_5 in G_{v_5} which is adjacent to a vertex y_6 in G_{v_6}. As G_{v_7} is connected, this is a (v_1, y_4)-path, P_{v_1, y_4}, contained in G_{v_7}. As G_{v_6} is connected, there is a (v_2, y_6)-path, P_{v_2, y_6}, which is contained in G_{v_6}. As G_{v_5} is connected, there is a (y_1, y_5)-path, P_{y_1, y_5} contained in G_{v_5}. As G_{v_4} is connected there is a (y_3, y_2)-path, P_{y_3, y_2} which is contained inside G_{v_4}. Then let $C_3 = P_{y_3, y_2} \cup P_{v_1, y_4} \cup P_{y_3, y_2} \cup P_{y_1, y_5} \cup \{y_3 y_4, v_1 v_2, y_6 y_5, y_1 y_2\}$ is a cycle. Notice that $|C_3| \geq 4$ and $|C_3 \cap C_2| \geq 2$. Now without loss of generality let $a \in G_{v_2}$. There is a (a, x_1)-path P_{a, x_1} contained in G_{v_2} as G_{v_2} is connected. Without loss of generality let $b \in G_{v_5}$. There is a (b, y_1)-path, P_{b, y_1}, contained in G_{v_5} since G_{v_5} is connected. Then $c \in G_{v_4}$ and there is a (c, y_2)-path P_{c, y_2} contained in G_{v_4}. Then notice it is easy to see that that $P_1, P_2, P_3, C_1, C_2, C_3$ satisfy the claim.

Conversely, if given P_1, P_2, P_3, C_1, C_2 and C_3 satisfying the lemma statement, we simply contract P_1, P_2, P_3 down to a single vertex, contract C_1 and C_2 to a diamond, and then C_3 to a 4-cycle. $\square$

As with the other rooted minor sections, we start off with some lemmas about how the $L(X)$-minor behaves across 2-separations.

Lemma 100. *Let G be a graph and (A, B) be a 2-separation with vertex boundary $\{x, y\}$. If $X = \{a, b, c, d\} \subseteq A$, then G has an $L(X)$-minor if and only if $G_A = G[A] \cup \{xy\}$ has an $L(X)$-minor.*

Proof. If G_A has an $L(X)$-minor then since G_A is a minor of G by contracting all of $G[B]$ onto $\{x, y\}$, we get that G has an $L(X)$-minor.

Conversely, let $\{G_x | x \in V(L)\}$ be a model for an $L(X)$-minor. We claim that $\{G_A[V(G_x) \cap A] | x \in V(L)\}$ is a model of an $L(X)$-minor in G_A. If there is only one branch set containing vertices of B then the result trivially holds. Therefore we may assume there are at least two distinct branch sets containing vertices of B. If the branch sets containing vertices of B are two of $G_{v_1}, G_{v_3}, G_{v_4}, G_{v_5}$, then since $X \subseteq A$, all other branch sets are contained inside $G[A]$. Then since $xy \in E(G_A)$, $\{G_A[V(G_x) \cap A] | x \in V(L)\}$ is a model of an $L(X)$-minor.

Now suppose that exactly one of $G_{v_1}, G_{v_3}, G_{v_4}$, and G_{v_5} contains a vertex from B. Suppose that G_{v_1} is the branch set which contains the vertex. Then suppose that G_{v_2} is the other branch set which contains a vertex from B. Since $\deg(v_2) > 2$, we may assume that $x \in G_{v_1}$ and $y \in G_{v_2}$ and all other branch sets are contained in $G[A \setminus \{x, y\}]$. Then since $xy \in E(G_A)$, $\{G_A[V(G_x) \cap A] \mid x \in V(L)\}$ is a model for an $L(X)$-minor. A similar analysis works for the other cases.

Therefore we can assume that two of G_{v_2}, G_{v_8}, G_{v_7}, and G_{v_6} contain vertices from B. Suppose G_{v_2} and G_{v_8} are two branch sets containing vertices from B. Then since $\deg(v_2) > 2$ and $\deg(v_8) > 2$ in L, we may assume that $u \in G_{v_2}$ and $x \in G_{v_8}$ and that all other branch sets are contained in $A \setminus \{x, y\}$. But then since $xy \in E(G_A)$, $\{G_A[V(G_x) \cap A] | x \in V(L)\}$ is a model for an $L(X)$-minor. The other cases follow by essentially the same argument. $\square$

Lemma 101. *Let G be a graph and (A, B) a 2-separation with vertex boundary $\{x, y\}$. Let $X = \{a, b, c, d\} \subseteq V(G)$. If $x, y \in X$ and one vertex of X lies in $A \setminus \{x, y\}$, and the other lies in $B \setminus \{x, y\}$, then G does not have an $L(X)$-minor.*

Proof. Suppose towards a contradiction that $\{G_x | x \in V(L)\}$ was a model of an $L(X)$-minor in G. If $x = v_1$ and $y = v_3$ then without loss of generality we have that $G_{v_4} \subseteq G[A \setminus \{x, y\}]$ and $G_{v_5} \subseteq G[B \setminus \{x, y\}]$. But this contradicts that there is a vertex in G_{v_4} which is adjacent to a vertex in G_{v_5}. The same argument works if $x = v_1$, $y = v_5$ or if $x = v_3$ or $y = v_4$.

Now assume that $x = v_1$ and $y = v_4$. Then without loss of generality we may assume that $G_{v_3} \subseteq G[A \setminus \{x, y\}]$ and $G_{v_5} \subseteq G[B \setminus \{x, y\}]$. Then $G_{v_2} \subseteq G[A \setminus \{x, y\}]$ as G_{v_2} has a vertex which is adjacent to a vertex in G_{v_3}. By similar reasoning, G_{v_8}, G_{v_7} and G_{v_6} are all contained in $G[A \setminus \{x, y\}]$. But then there is no vertex in G_{v_6} which is adjacent to a vertex in G_{v_5}, a contradiction. The case where $x = v_3$ and $y = v_5$ follows similarly.

Lastly, assume that $x = v_4$ and $y = v_5$. Without loss of generality, we may assume that G_{v_3} is contained in $G[B \setminus \{x, y\}]$ and G_{v_1} is contained in $G[A \setminus \{x, y\}]$. Then G_{v_2} is contained in either $G[A \setminus \{x, y\}]$ or $G[B \setminus \{x, y\}]$. Suppose that G_{v_2} is contained in $G[A \setminus \{x, y\}]$. But then there is no vertex in G_{v_3} which is adjacent to a vertex in G_{v_2}. Then G_{v_2} is contained in $G[B \setminus \{x, y\}]$. But then there is no vertex in G_{v_1} which is adjacent to a vertex in G_{v_2}, a contradiction. $\square$

Lemma 102. *Let G be a graph and (A, B) be a 2-separation with vertex boundary $\{u, v\}$. Let $X = \{a, b, c, d\} \subseteq V(G)$. Suppose that there is exactly one vertex, z, such that $z \in X \cap (A \backslash \{u, v\})$, and exactly two vertices from X in $B \backslash \{x, y\}$ and $u \in X$. Let $X_1 = X \backslash \{z\} \cup \{v\}$. For each $\pi \in \mathcal{F}$, define $\pi' : X_1 \to V(L)$ such that $\pi' = \pi$ on $X \backslash \{z\}$ and $\pi'(v) = \pi(z)$. If there is a model of an $L(X)$-minor, $\{G_x | x \in L(X)\}$, then either $\{G_B[V(G_x) \cap B)] | x \in V(L)\}$ is a model of an $L(X_1)$-minor in G_B, or the vertex from X in $A \backslash \{u, v\}$ is not in branch sets G_{v_4} or G_{v_5}.*

Proof. Suppose $\{G_x | x \in L(X)\}$ is a model of an $L(X)$-minor, and suppose that $\{G_B[V(G_x) \cap B)] \mid x \in V(L)\}$ is not a model of an $L(X_1)$-minor. Furthermore, suppose the vertex from X in $A \backslash \{u, v\}$ is in G_{v_4}.

First consider the case when $v \in G_{v_4}$. Suppose any of G_{v_2}, G_{v_7}, G_{v_6}, or G_{v_8} is contained in $G[A \backslash \{u, v\}]$. Since v_2, v_7, v_6 and v_8 induce a diamond in L, and $v \in G_{v_4}$, and $u \in X$, each of G_{v_2}, G_{v_7}, G_{v_6} and G_{v_8} are contained in $G[A \backslash \{u, v\}]$. But then at least two of G_{v_1}, G_{v_3}, and G_{v_5} are contained in $G[B \backslash \{u, v\}]$. But this is a contradiction, since in L, all of v_1, v_5 and v_4 are adjacent to at least one of v_2, v_7 and v_6. Therefore we can assume that none of G_{v_2}, G_{v_7}, G_{v_6}, or G_{v_8} are in $G[A \backslash \{u, v\}]$. But then since two vertices of X lie in $B \backslash \{u, v\}$, there are at most two branch sets containing vertices from A. But then $\{G_B[V(G_x) \cap B)] | x \in V(L)\}$ is a model of an $L(X_1)$-minor in G_B, a contradiction.

Therefore we can assume that $v \notin V(G_{v_4})$, and thus $G_{v_4} \subseteq G[A \backslash \{u, v\}]$. Now at least one of G_{v_3} and G_{v_5} contains a vertex from B which is not u, and thus either $v \in G_{v_3}$ or $v \in G_{v_5}$. In either case, this implies that G_{v_7} is contained in $G[A \backslash \{u, v\}]$. By the same reasoning as before, this implies that all of G_{v_2}, G_{v_8} and G_{v_6} are contained in $G[A \backslash \{u, v\}]$. But then since at least one of G_{v_1}, G_{v_3} and G_{v_5} are contained in $G[B \backslash \{u, v\}]$, which contradicts that $\{G_x | x \in L(X)\}$ is an $L(X)$-model. The case where the vertex from X in $A \backslash \{u, v\}$ is in G_{v_5} follows similarly. $\square$

Lemma 103. *Let G be a 2-connected graph and (A, B) be a 2-separation with vertex boundary $\{x, y\}$. Let $X = \{a, b, c, d\} \subseteq V(G)$. Suppose $a, b \in A \backslash (A \cap B)$ and $c, d \in B \backslash (B \cap A)$. Let $X_1 = (X \cap A) \cup \{x, y\}$. For each $\pi \in \mathcal{F}$, define $\pi_1 : X_1 \to V(L)$ such that $\pi_1 = \pi$ on a, b and $\pi_1(x) = \pi(c)$ and $\pi_1(d) = \pi$. Let $X_2 = (X \cap B) \cup \{x, y\}$. For each $\pi \in \mathcal{F}$, define $\pi_2 : X \to V(L)$ such that $\pi_2 = \pi$ on $\{c, d\}$ and $\pi_2(x) = \pi(a)$ and $\pi_2(y) = \pi(b)$. Then G has an $L(X)$-minor if and only if $G_A = G[A] \cup \{xy\}$ has an $L(X_1)$-minor or $G_B = G[B] \cup \{xy\}$ has an $L(X_2)$-minor.*

Proof. If G_A has an $L(X_1)$-minor, then we can contract B onto $\{x, y\}$ such that the vertices of X do not get contracted together. This is possible since G is 2-connected. Then G has an $L(X)$-minor. A similar argument works when G_B has an $L(X_2)$-minor. Now assume that $\{G_x | x \in V(L)\}$ is a model of an $L(X)$-minor in G.

Suppose x is in one of G_{v_2}, G_{v_7}, G_{v_6}, or G_{v_8}. We consider cases based on which branch sets contain the vertices of X.

First, suppose that the vertices from X in $A \setminus \{x, y\}$ are in branch sets G_{v_1} and G_{v_3}. Then the vertices in X in $B \setminus \{x, y\}$ are in branch sets G_{v_5} and G_{v_4}. But then since at most one branch set contains y, either there is no vertex in G_{v_3} adjacent to a vertex in G_{v_4} or there is no vertex in G_{v_1} which is adjacent to a vertex in G_{v_5}.

Now suppose the vertices from X in $A \setminus \{x, y\}$ are in branch sets G_{v_1} and G_{v_4}. Then the vertices in X in $B \setminus \{x, y\}$ are in branch sets G_{v_5} and G_{v_3}. But then since at most one branch set contains y, either there is no vertex in G_{v_3} adjacent to a vertex in G_{v_4} or there is no vertex in G_{v_1} which is adjacent to a vertex in G_{v_5}. In either case, this is a contradiction.

Now suppose the vertices from X in $A \setminus \{x, y\}$ are in branch sets G_{v_1} and G_{v_5}. Then the vertices from X in $B \setminus \{x, y\}$ are in branch sets G_{v_3} and G_{v_4}. First suppose that $x \notin G_{v_2}$. Then since v_1 and v_3 are adjacent to v_2 in L, we have that $y \in G_{v_2}$. But then there is no vertex in G_{v_4} which is adjacent to a vertex in G_{v_5}, a contradiction. Therefore $x \in G_{v_2}$. Then $y \in G_{v_4}$ or G_{v_5}. Then G_{v_8}, G_{v_7} and G_{v_6} all have vertex sets in either $B \setminus \{x, y\}$ or $A \setminus \{x, y\}$. But then either G_{v_6} does not have a vertex adjacent to a vertex in G_{v_5} or G_{v_4} does not have a vertex adjacent to a vertex in G_{v_7}. In either case, this is a contradiction.

Therefore x is not in $V(G_{v_2}), V(G_{v_6}), V(G_{v_7})$, or $V(G_{v_8})$. Similarly, y is not in $V(G_{v_2}), V(G_{v_6}), V(G_{v_7})$, or $V(G_{v_8})$. Then x and y belong to two of $G_{v_1}, G_{v_3}, G_{v_4}$ and G_{v_5}. Now suppose that G_{v_1} is contained in $G[A \setminus \{x, y\}]$ and G_{v_3} is contained in $G[B \setminus \{x, y\}]$. Then since x, y belong to two of $G_{v_1}, G_{v_3}, G_{v_4}$ and G_{v_5}, we have that G_{v_2} is contained in one of $G[B \setminus \{x, y\}]$ or $G[A \setminus \{x, y\}]$. But then without loss of generality there is no vertex in G_{v_2} which is adjacent to a vertex in G_{v_1}, a contradiction. A similar analysis shows that for any two of $G_{v_1}, G_{v_3}, G_{v_4}$ and G_{v_5}, if one of the branch sets is contained in $G[B \setminus \{x, y\}]$ and the other in $G[A \setminus \{x, y\}]$ we get a contradiction. Therefore the two branch sets from $G_{v_1}, G_{v_3}, G_{v_4}$ and G_{v_5} which do not contain x, y are contained on the same side of the 2-separation. Suppose the two branch sets from $G_{v_1}, G_{v_3}, G_{v_4}$ and G_{v_5} which do not contain x, y are contained in $G[A \setminus \{x, y\}]$. Then by the same reasoning as above, all of $G_{v_2}, G_{v_8}, G_{v_6}$, and G_{v_7} are contained in $G[A \setminus \{x, y\}]$. But then $\{G_A[V(G_x) \cap A] | x \in V(L)\}$ is a model of an $L(X_1)$-minor in G_A. In the other case, by the same argument we get an $L(X_2)$-minor in G_B. $\square$

Now we show that class $\mathcal{A}$ graphs do not have an $L(X)$-minors.

Lemma 104. *Let G be 2-connected spanning subgraph of a class $\mathcal{A}$ graph. Then G does not have an $L(X)$-minor.*

Proof. Suppose towards a contradiction that we have a model of an $L(X)$-minor in G, $\{G_x | x \in V(L)\}$. Then observe that $\{e, d\}$ is the vertex boundary of a separation (A, B) such that $a \in A \setminus \{d, e\}$ and $b, c \in B \setminus \{d, e\}$ so the hypotheses of Lemma 102 are satisfied. Then we consider two cases. If $\{G_B[V(G_x \cap B)] | x \in V(L)\}$ is a model of an $L(X_1)$-minor, then notice that this means that a spanning subgraph of a graph H^+, where H is the graph defined in Lemma 91 has an $L(X_1)$-minor. But in this graph H^+, $\{d, e\}$ is the
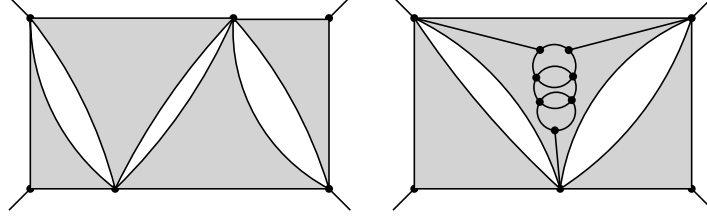
Figure 3.12: Graph on the left has no $L(X)$-minor by Lemma 106 and the graph on the right has an $L(X)$-minor and satisfies the conditions of Lemma 106. Curved lines indicate 2-separations. The shaded sections are spanning subgraphs of webs.

vertex boundary of a 2-separation satisfying Lemma 101, and thus H^+ does not have an $L(X_1)$-minor, a contradiction.

Therefore we must be in the case where $a \in G_{v_1}$ or $a \in G_{v_3}$. Notice that $\{e, d\}$ induces a 2-separation in G, (A', B') such that $c \in A' \setminus \{e, d\}$ and $a, b \in B' \setminus \{e, d\}$. Additionally, $\{e, d\}$ is the vertex boundary of a 2-separation (A'', B'') such that $b \in A'' \setminus \{e, d\}$, and $a, c \in B'' \setminus \{e, d\}$. Therefore by applying the exact same analysis as the the (A, B) separation, we get that all of a, b, and c are in G_{v_1} or G_{v_3}, which is a contradiction. $\square$

Now we reduce class $\mathcal{E}$ and $\mathcal{F}$ down to looking at webs.

Lemma 105. *Let G be a 2-connected spanning subgraph of a class $\mathcal{E}$ or a $\mathcal{F}$ graph. If G is a spanning subgraph of a class $\mathcal{E}$ graph, then G has an $L(X)$-minor if and only if the $\{e, f, c, d\}$-web has an $L(X)$-minor. If G is a spanning subgraph of a class $\mathcal{F}$ graph, then G has an $L(X)$-minor if and only if the $\{e, f, g, h\}$-web has an $L(X)$-minor.*

Proof. First suppose that G is a 2-connected spanning subgraph of a class $\mathcal{E}$ graph. Then apply Lemma 103 to the separation whose vertex boundary is $\{e, f\}$. Notice that one of the graphs we get from Lemma 103 is the graph H^+ where H is the graph from Lemma 91. Notice that H^+ does not have an $L(X)$-minor since $\{e, f\}$ forms a separation satisfying Lemma 101. Notice that the other graph we obtain from Lemma 103 is the $\{e, f, c, d\}$-web, which completes the claim. The argument for the class $\mathcal{F}$ graphs is essentially the same. $\square$

Therefore it suffices to look at graphs which are spanning subgraphs of webs satisfying the obstructions given in Theorem 82. We will look at each case separately, but it turns out that essentially all the obstructions reduce to instances of terminal separating 2-chains having $L(X)$-minors. First we look at the terminal separating 2-chain obstruction.

Lemma 106. *Let G be a 2-connected spanning subgraph of an $\{a, b, c, d\}$-web. Let $X = \{a, b, c, d\} \subseteq V(G)$ and suppose that G does not have a $W_4(X)$-minor. Consider any cycle C for which $X \subseteq V(C)$. Suppose a, b, c, d appears in that order on C. Suppose there is a terminal separating 2-chain $((A_1, B_1), (A_2, B_2), \ldots, (A_n, B_n))$ satisfying obstruction 1 from Theorem 82. Furthermore, suppose that $A_1 \cap B_1 \cap X = \{a\}$, and $b = A_1 \cap X \setminus$*

$(A_1 \cap B_1)$. *If $n \equiv 1$ (mod 2), then G has no $L(X)$-minor. Now consider when $n \equiv 0$ (mod 2). Then $c \in A_n \cap X \setminus (A_n \cap B_n)$. Then G has an $L(X)$-minor if and only if the following occurs. There exists an $i \in \{1, \ldots, n-1\}$, $i \equiv 1$ (mod 2), such that the graph $G_{B_i \cap A_{i+1}} = G[B_i \cap A_{i+1}] \cup \{xy | x, y \in A_i \cap B_i\} \cup \{xy | x, y \in A_{i+1} \cap B_{i+1}\}$ has an $L'(X')$-minor, where $X' = \{(A_1 \cap B_1), (A_2 \cap B_2)\}$. Furthermore, the vertex from X' which is contained in the branch set G_{v_2} lies on the (b, c)-path $P_{b,c}$ where $a, d \notin V(P_{b,c})$ and $V(P_{b,c}) \subseteq V(C)$.*

Proof. First suppose that $n \equiv 1$ (mod 2). If $n = 1$, then (A_1, B_1) satisfies Lemma 101 and thus G does not have an $L(X)$-minor. Now suppose $n \geq 3$ and G is a vertex minimal counterexample to the claim. Then apply Lemma 103 to (A_2, B_2) gives two graphs G_{A_2} and G_{B_2} such that G has an $L(X)$-minor if and only if G_{A_2} has an $L(X_1)$-minor or G_{B_2} has an $L(X_2)$-minor. Now both G_{A_2} and G_{B_2} have terminal separating 2-chains of odd length. Namely, (A_1, B_1) is a terminal separating 2-chain in G_{A_2} and $(A_3, B_3), \ldots, (A_n, B_n)$ is a terminal separating 2-chain in G_{B_2} which has length $n - 2$, which is odd. Then since G is a vertex minimal counterexample, both G_{A_2} and G_{B_2} do not have $L(X)$-minors and therefore G does not have an $L(X)$-minor.

Now suppose that $n \equiv 0$ (mod 2). First suppose that $n \geq 4$ and suppose that G is a vertex minimal counterexample. Then apply Lemma 103 to (A_2, B_2). Then we get two graphs G_{A_2} and G_{B_2} such that G has an $L(X)$-minor if and only if either G_{A_2} has an $L(X_1)$-minor or G_{B_2} has an $L(X_2)$-minor. We note that G_{A_2} does not have an $L(X_1)$-minor since (A_1, B_1) satisfies Lemma 101. Therefore since G has an $L(X)$-minor, G_{B_2} has an $L(X_2)$-minor. Furthermore, $((A_3, B_3), \ldots, (A_n, B_n))$ is an even length terminal separating 2-chain in G_{B_2}. Then as G is a vertex minimal counterexample, G_{B_2} satisfies the claim. But then notice that the vertex in X_2 which takes the place of b in the claim lies on $P_{b,c}$ since $b \in A_1 \setminus (A_1 \cap B_1)$ implies that the vertex in $(A_1 \cap B_1) \setminus \{a\}$ lies on $P_{b,c}$. Therefore G satisfies the claim, a contradiction.

Therefore we can assume that $n = 2$, and suppose $\{G_x \mid x \in V(L)\}$ is a model of an $L(X)$-minor in G. Apply Lemma 102 to (A_1, B_1). Then we have two cases. First suppose we have an $L(X_1)$-minor in G_{B_1} where $X_1 = X \setminus \{b\} \cup (A_1 \cap B_1)$. Then in G_{B_1} under X_1, the 2-separation (A_2, B_2) satisfies Lemma 101, implying that G_{B_1} does not have an $L(X_1)$-minor, a contradiction. Therefore $b \in G_{v_1}$ or $b \in G_{v_3}$. First suppose that $b \in G_{v_1}$. Notice that (A_2, B_2) satisfies Lemma 102, and that by symmetry we may assume that $c \in G_{v_1}$ or $c \in G_{v_3}$. As we assumed $b \in G_{v_1}$, we assume that $c \in G_{v_3}$. Suppose that G_{v_1} is not contained in $G[A_1 \setminus (A_1 \cap B_1)]$. Then the vertex in $A_1 \cap B_1 \setminus \{a\}$ is in G_{v_1}. Then G_{v_3} is contained in $G[B_2]$. Then since G_{v_3} has a vertex adjacent to a vertex in G_{v_2}, we have that G_{v_2} is contained in $G[B_1]$. Then this implies that G_{v_8}, G_{v_6}, and G_{v_7} are contained in $G[B_2 \setminus (A_2 \cap B_2)]$. But $a \in G_{v_5}$ or $a \in G_{v_4}$, so either G_{v_4} is contained in $G[A_2 \setminus (A_2 \cap B_2)]$ or G_{v_5} is contained in $G[A_2 \setminus (A_2 \cap B_2)]$. But that contradicts that $\{G_x \mid x \in L(X)\}$ is an $L(X)$-minor. Therefore G_{v_1} is contained in $G[A_1 \setminus (A_1 \cap B_1)]$. By symmetry, G_{v_3} is contained in $G[B_2 \setminus (A_2 \cap B_2)]$. That implies that G_{v_2} contains the vertex in $A_1 \cap B_1 \setminus \{a\}$.

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Suppose that G_{v_8} is contained in $G[A_1 \setminus (A_1 \cap B_1)]$. Then since G_{v_2} contains the vertex in $A_1 \cap B_1 \setminus \{a\}$, the branch sets G_{v_7} and G_{v_6} are contained in $G[A_1 \setminus (A_1 \cap B_1)]$. But then since $d \in G_{v_5}$ or $d \in G_{v_4}$, this implies that either G_{v_5} is contained in $G[B_1 \setminus (A_1 \cap B_1)]$ or G_{v_4} is contained in $G[B_1 \setminus (A_1 \cap B_1)]$. But this contradicts that $\{G_x \mid x \in V(L)\}$ is a model of an $L(X)$-minor. By essentially the same argument, none of G_{v_8}, G_{v_7} or G_{v_6} are contained in $G[B_2 \setminus (A_2 \cap B_2)]$. Therefore all of G_{v_8}, G_{v_7} and G_{v_6} are contained in $G[B_1 \cap A_2]$. But then the set $\{G_{B_1 \cap A_2}[G_x \cap B_1 \cap A_2] \mid x \in V(L')\}$ is a model of $L'(X')$ minor satisfying the properties of the lemma.

Now we prove the converse. Suppose there exists an $i \in \{1, \ldots, n-1\}$ and $i \equiv 1 \pmod 2$ such that the graph $G_{B_i \cap A_{i+1}}$ has an $L'(X')$-minor satisfying the properties in the lemma. Let $\{G_x | x \in V(L')\}$ be a model of an $L'(X')$-minor satisfying the above properties. Let v be the vertex in X' which is also in G_{v_2} in the $L'(X)$ model. Observe that since $i \equiv 1$ $\pmod 2$, there is exactly one vertex in X' which lies on $P_{b,c}$, so therefore $v \in P_{b,c}$ and both other vertices of X' lie on $P_{a,d}$, the (a, d)-path such that $V(P_{a,d}) \subseteq V(C)$ and $b, c \notin V(P_{a,d})$. Then to get an $L(X)$-minor in G, first let x be the vertex in X' such that the (a, x)-subpath on $P_{a,d}$ does not contain the other vertex from X' on $P_{a,d}$, and without loss of generality, let $x \in G_{v_5}$. Then extend G_{v_5} to include the subpath from (a, x) on $P_{a,d}$. Now let y be the vertex in X such that the (y, d)-subpath on $P_{a,d}$ does not contain x. Since $x \in G_{v_5}$, we have $y \in G_{v_4}$. Extend G_{v_4} along the (y, d)-subpath on $P_{a,d}$. Now we create G_{v_1} by letting it be the (a, v)-path, $P_{a,v}$ such that $V(P_{a,v}) \subseteq V(C)$ and $d, c \notin P_{a,v}$ and do not include either a or v. Note that $b \in V(P_{a,v})$. Similarly, let G_{v_3} be the (d, v)-path, $P_{d,v}$ such that $V(P_{d,v}) \subseteq V(C)$ and $a, b \notin P_{d,v}$, not including either d or v. we G_{v_1} contain the path from v to b on $P_{b,c}$, not including v, we let G_{v_3} contain the path from v to c on $P_{b,c}$ not including v. Then by construction and since we already had an $L'(X')$-minor, we have an $L(X)$-minor in G. □

Now we look at when our graph has a terminal separating triangle as in obstruction 2.

Lemma 107. *Let G be a 2-connected spanning subgraph of an $\{a, b, c, d\}$-web. Let $X = \{a, b, c, d\} \subseteq V(G)$ and suppose that G does not have a $W_4(X)$-minor. Consider any cycle C for which $X \subseteq V(C)$, and suppose that there is a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ satisfying obstruction 2 of Theorem 82. Then G has an $L(X)$-minor if and only if either the graph G_{B_1} has an $L(X_1)$-minor where $X_1 = (X \cap B_1) \cup (A_1 \cap B_1)$ or the graph G_{B_3} has an $L(X_2)$-minor where $X_2 = (X \cap B_3) \cup (A_3 \cap B_3)$.*

Proof. If G_{B_1} has an $L(X_1)$-minor then since G is 2-connected we can contract the vertex in X in A_1 to $A_1 \cap B_1$ such that it does not get contracted together with another vertex of X. But then G has an $L(X)$-minor. A similar argument holds for G_{B_3}.

Now suppose that $\{G_x | x \in V(L)\}$ is a model of an $L(X)$-minor in G, and furthermore suppose we have a vertex minimal counterexample to the claim. Without loss of generality
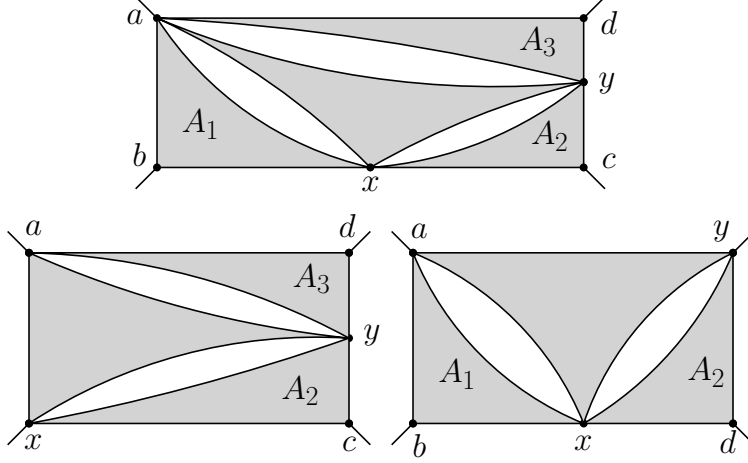
Figure 3.13: A terminal separating triangle and the graphs G_{B_1} and G_{B_3} as in Lemma 107. Note both G_{B_1} and G_{B_3} have terminal separating 2-chain obstructions. Shaded sections are spanning subgraphs of webs.

let a, b, c, d appear in that order on C, and without loss of generality suppose that a is the vertex from X in $A_1 \cap B_1$, and that $b \in A_1 \setminus (A_1 \cap B_1)$. Similarly, without loss of generality we may assume that $c \in A_2 \setminus (A_2 \cap B_2)$ and $d \in A_3 \setminus (A_3 \cap B_3)$. Then notice that (A_1, B_1) satisfies Lemma 102. We consider two cases.

First consider the case where $b \in G_{v_1}$ or $b \in G_{v_3}$. We assume that $b \in G_{v_1}$. Now notice that since we assumed $d \in A_3$, (A_3, B_3) is a 2-separation satisfying Lemma 102. First suppose that $d \in G_{v_3}$ (note this is without loss of generality, since $b \in G_{v_1}$). First consider the case where G_{v_1} is contained in $G[A_1 \setminus (A_1 \cap B_1)]$. Then G_{v_2} contains the vertex in $(A_1 \cap B_1)$ which is not a. Similarly, if G_{v_3} is contained in $G[A_3 \setminus (A_3 \cap B_3)]$, then G_{v_2} contains the vertex in $A_3 \cap B_3 \setminus \{a\}$. But then $a \in B_2 \setminus (A_2 \cap B_2)$ and $c \in A_2 \setminus (A_2 \cap B_2)$, which implies that there is no vertex in G_{v_5} which is adjacent to a vertex in G_{v_4}, a contradiction. Therefore G_{v_3} is not contained in $G[A_3 \setminus (A_3 \cap B_3)]$ and the vertex in $A_3 \cap B_3 \setminus \{a\}$ is in G_{v_3}. But then as before, $a \in B_2 \setminus (A_2 \cap B_2)$ and $c \in A_2 \setminus (A_2 \cap B_2)$, contradicting that there is a vertex in G_{v_5} which is adjacent to a vertex in G_{v_4}. Therefore we can assume that $d \notin G_{v_3}$. But then by Lemma 102, G_{B_3} has an $L(X_2)$-minor, which satisfies the claim.

Therefore we can assume that G_{v_1} is not contained in $G[A_1 \setminus (A_1 \cap B_1)]$. Since the cases are symmetric, the only situation left to consider is when $d \in G_{v_3}$ and not contained in $G[A_3 \setminus (A_1 \cap B_1)]$. Then the vertex in $(A_1 \cap B_1) \setminus \{a\}$ and the vertex in $(A_3 \cap B_3) \setminus \{a\}$ are in G_{v_1} and G_{v_3} respectively. But then there is no vertex in G_{v_4} which is adjacent to G_{v_5}, a contradiction. The same situation happens when $b \in G_{v_3}$. Therefore we assume that $b \notin G_{v_3}$ and $b \notin G_{v_1}$. But then by Lemma 102, G_{B_1} has an $L(X_1)$-minor, which completes the claim. $\qquad\square$

Observe that both the graphs G_{B_1} and G_{B_2} have terminal separating 2-chains as obstructions, and thus when we have a terminal separating triangle as an obstruction, the problem reduces to appealing to Lemma 106. Now we look at what happens when the graph has a terminal separating 2-chain and a terminal separating triangle as in obstruction 3.

Lemma 108. *Let G be a 2-connected spanning subgraph of an $\{a, b, c, d\}$-web. Let $X = \{a, b, c, d\} \subseteq V(G)$ and suppose that G does not have a $W_4(X)$-minor. Consider any cycle C for which $X \subseteq V(C)$, and suppose there is a terminal separating triangle $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ such that $A_1 \cap B_1 = \{x, y\}$, where $x, y \notin X$. Furthermore, the graph $G_{A_1} = G[A_1] \cup \{xy\}$ and $C_A = G[V(C) \cap A] \cup \{x, y\}$ has a terminal separating 2-chain $(A'_1, B'_1), \ldots, (A'_n, B'_n)$ where we let x and y replace the two vertices in X from G not in G_{A_1} and $A_i \cap B_i \subseteq V(C)$, $i \in \{1, 2, 3\}$, and $A'_i \cap B'_i \subseteq V(C_A)$ for all $i \in \{1, \ldots, n\}$. Then G has an $L(X)$-minor if and only if the graph $G_{B_1} = G[B_1] \cup \{xy | x, y \in A_1 \cap B_1\}$ has an $L(X_1)$-minor where $L(X_1) = X \cap B_1 \cup (A_1 \cap B_1)$ or $G_{A_1} = G[B_1] \cup \{xy | x, y \in A_1 \cap B_1\}$ has an $L(X_2)$-minor where $X_2 = X \cap A_1 \cup (A_1 \cap B_1)$.*

Proof. This follows immediately from applying Lemma 103 to (A_1, B_1). □

The point of the above observation is that the graphs G_{A_1} and G_{B_1} have terminal separating 2-chain obstructions, and thus the problem of finding $L(X)$-minors in that case reduces to the problem of finding $L(X)$-minors in terminal separating 2-chains case, which is done in Lemma 101. Now we look at what happens when we have two terminal separating triangles and a terminal separating 2-chain as in obstruction 4.

Lemma 109. *Let G be a 2-connected spanning subgraph of an $\{a, b, c, d\}$-web. Let $X = \{a, b, c, d\} \subseteq V(G)$ and suppose that G does not have a $W_4(X)$-minor. Consider any cycle C for which $X \subseteq V(C)$, and suppose there are two distinct terminal separating triangles $((A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1))$, $((A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2))$ where for all $i \in \{1, 2, 3\}$, $A_i^1 \cap B_i^1 \subseteq A_1^2$ and $A_i^2 \cap B_i^2 \subseteq A_3^1$. Furthermore, if we consider the graph $G[A_i^2 \cap A_1^1]$ and the cycle $C' = G[V(C) \cap A_1^2 \cap A_1^1] \cup \{xy | x, y \in A_1^i \cap B_1^i, i \in \{1, 2\}\}$ and we let X' be defined to be the vertices $A_1^2 \cap B_1^2$ and $A_3^2 \cap B_3^2$, then there is a terminal separating 2-chain in $G[A_3^2 \cap A_1^1]$ with respect to X'. Then G has an $L(X)$-minor if and only if either $G_{B_1^1}$ has an $L(X_1)$-minor where $X_1 = X \cap B_1^1 \cup (A_1^1 \cap B_1^1)$ or $G_{B_1^2}$ has an $L(X_2)$-minor where $X_2 = X \cap B_1^2 \cup (A_1^1 \cap B_1^1)$ or $G[A_3^2 \cap A_1^1]$ has an $L(X')$-minor.*

Proof. Apply Lemma 103 to (A_1^1, B_1^1). Then G has an $L(X)$-minor if and only if $G_{B_1^1}$ has an $L(X_1)$ minor where $X_1 = X \cap B_1^1 \cup (A_1^1 \cap B_1^1)$ or $G_{A_1^1}$ has an $L(X'_1)$-minor where $X'_1 = X \cap A_1^1 \cup (A_1^1 \cap B_1^1)$. Notice that in $G_{A_1^1}$, we have a terminal separating 2-chain and terminal separating triangle as in obstruction 3. But then we can apply Lemma 108 which gives us exactly the claim. □

Lemma 110. *Let G be a 2-connected spanning subgraph of an $\{a, b, c, d\}$-web. Let $X = \{a, b, c, d\} \subseteq V(G)$ and suppose that G does not have a $W_4(X)$-minor. Consider any cycle C for which $X \subseteq V(C)$, and suppose that there two distinct terminal separating triangles $((A_1^1, B_1^1), (A_2^1, B_2^1), (A_3^1, B_3^1)), ((A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2))$ which satisfy obstruction 5 in Theorem 82. Then G has an $L(X)$-minor if and only if either the graph $G_{B_1^1} = G[B_1^1] \cup \{xy | x, y \in A_1^1 \cap B_1^1\}$ has an $L(X_1)$-minor where $X_1 = (X \cap B_1^1) \cup (A_1^1 \cap B_1^1)$ or the graph $G_{B_1^2} = G[B_1^2] \cup \{xy | x, y \in A_1^2 \cap B_1^2\}$ has an $L(X_2)$-minor where $X_2 = (X \cap B_1^2) \cup (A_1^2 \cap B_1^2)$.*

Proof. If $G_{B_1^1}$ has an $L(X_1)$-minor, then we simply contract all of A_1^1 onto $B_1^1 \cap A_1^1$ such that we do not contract two vertices of X together. This is possible as G is 2-connected. The same strategy applies to $G_{B_1^2}$.

Now suppose that G has an $L(X)$-minor. Then (A_1^1, B_1^1) satisfies Lemma 103. Therefore G has an $L(X)$-minor if and only if $G_{B_1^1}$ has an $L(X_1)$-minor where $X_1 = X \cap B_1^1 \cup (B_1^1 \cap A_1^1)$ or $G_{A_1^1}$ has an $L(X_1')$-minor where $X_1' = A_1^1 \cap X \cup (A_1^1 \cap B_1^1)$. If $(A_1^1, B_1^1) = (A_1^2, B_1^2)$, then the claim follows immediately. Thus we assume that $(A_1^1, B_1^1) \neq (A_1^2, B_1^2)$. Notice that in $G_{A_1^1}$, the triangle $(A_1^2, B_1^2), (A_2^2, B_2^2), (A_3^2, B_3^2)$ is a terminal separating triangle satisfying obstruction 2. Then by Lemma 107, $G_{A_1^1}$ has an $L(X_1')$-minor if and only if the graph $G_{B_1^2}$ has an $L(X_2)$-minor, which completes the claim. $\square$

With that, one can determine exactly which graphs do not have an $K_4(X)$-minor, $W_4(X)$-minor, $K_{2,4}(X)$-minor and an $L(X)$-minor. To sum up what we have, a graph does not have one of the four above minors if and only if G is a class $\mathcal{A}$ graph, or it is the spanning subgraph of a class $\mathcal{D}$, $\mathcal{E}$, or $\mathcal{F}$ graph where the corresponding web has one of the obstructions from Theorem 82, and then after the reductions given above, we do not end up with an $L'(X')$-minor (Lemma 99) satisfying the properties in Lemma 101. We note it would be nice to obtain a cleaner structure theorem for when a graph has an $L'(X')$-minor.

3.5 $K_{3,4}$ and K_6-minors

As we noted before, $K_{3,4}$ is a forbidden minor for reduciblity of the first Symanzik polynomial. Therefore it is interesting to look at the structure of graphs not containing a $K_{3,4}$-minor. A full characterization of graphs without $K_{3,4}$-minors is not known, although partial progress has been made. Namely, when G is embeddable in the projective plane, two different but equivalent characterizations of graphs not containing $K_{3,4}$-minors are known ([30], [31]). We can use the fact that $K_{3,4}$ is not reducible with respect to the first Symanzik polynomial to gain some insight on the structure of reducible graphs.

It will be useful to consider some rooted minor classes. First, let $V(K_{2,3}) = \{s_1, s_2, t_1, t_2, t_3\}$, and $E(K_{2,3}) = \{s_i t_j \mid i \in \{1, 2\}, j \in \{1, 2, 3\}\}$. Let G be a graph and $X = \{a, b, c\} \subseteq V(G)$. Let $\mathcal{F}$ be the family of maps such that each vertex of X gets mapped to a distinct vertex

of $\{t_1, t_2, t_3\}$. For the purposes of this thesis, a $K_{2,3}(X)$-minor will refer to the X and $\mathcal{F}$ above.

Let $V(K_{3,3}) = \{s_1, s_2, s_3, t_1, t_2, t_3\}$ and $E(K_{3,3}) = \{s_i t_j \mid i, j \in \{1, 2, 3\}\}$. Let G be a graph and $X = \{a, b, c\} \subseteq V(G)$. Let $\mathcal{F}$ be the family of maps from X to $\{t_1, t_2, t_3\}$. For the purposes of this thesis, a $K_{3,3}(X)$-minor will refer to the X and $\mathcal{F}$ above.

Let $V(K_{3,1}) = \{s_1, t_1, t_2, t_3\}$ and $E(K_{3,1}) = \{s_1 t_i \mid i \in \{1, 2, 3\}\}$. Let G be a graph and $X = \{a, b, c\} \subseteq V(G)$. Let $\mathcal{F}$ be the family of maps from X to $\{t_1, t_2, t_3\}$. For the purposes of this thesis, $K_{3,1}(X)$-minor will refer to the X and $\mathcal{F}$ above.

Suppose we know a graph G does not have a $K_4(X)$-minor. Then we know from Theorem 56 that G belongs to one of six classes of graphs. We can ask the question, when do one of these graphs have a $K_{3,4}$-minor? Clearly, if one of these graphs has a $K_{3,4}$-minor, then the graph must be non-planar. By how each of the classes of graphs are constructed, that implies that one of the F_T sections must be causing the non-planarity. Thus there is a section of the graph contained inside a 3-separation which is causing the non-planarity. It is reasonable to suspect if there are a few distinct triangles where the graph induced by F_T is non-planar, and there is enough connectivity between the triangles, that you could obtain a $K_{3,4}$-minor. We show this now, using the following theorem from Robertson and Seymour.

Theorem 111 ([35]). *Let G be a 3-connected graph and $X = \{a, b, c\} \subseteq V(G)$. Then either G has a $K_{2,3}(X)$-minor on a, b, c or G is planar and a, b, c lie on a face.*

Corollary 112. *Let G be a graph. Let $T_1 = \{x_1, x_2, x_3\}$ be the vertex boundary of a 3-separation (A_1, B_1). Let $T_2 = \{y_1, y_2, y_3\}$ be the vertex boundary of a 3-separation (A_2, B_2). Suppose that $A_1 \cap A_2 = \emptyset$, $G[A_i]$ is non-planar for $i \in \{1, 2\}$, and that in $G[B_1 \cap B_2]$ there are 3 disjoint (T_1, T_2)-paths. Then G has a $K_{3,4}$-minor.*

Proof. Let $X_1 = T_1$ and $X_2 = T_2$. By Theorem 111, $G[A_i]$ has a $K_{2,3}(X_i)$ minor for $i \in \{1, 2\}$. Then since $A_1 \cap A_2 = \emptyset$ and there are three disjoint (T_1, T_2)-paths, we can contract $G[A_i]$ to a $K_{2,3}(X_i)$-minor and then contract along the three disjoint (T_1, T_2)-paths to obtain a $K_{3,4}$-minor. $\qquad\square$

Recall that for graphs without a $K_4(X)$-minor, they fall into one of six classes that are all formed based off of taking a graph H and then constructing the graph $H^+ = (H, F)$. Then for that scenario, the above corollary says to not have a $K_{3,4}$-minor, for every pair of triangles T_1 and T_2 which are not separated by a 2-separation, at most one of $T_1 \cup F_{T_1}$, and $T_2 \cup F_{T_2}$ is non-planar.

We could play the exact same game with $K_{3,3}(X)$-minors and rooted $K_{3,1}(X)$-minors. We have the following immediate proposition.

Proposition 113. *Let G be a graph and let (A, B) be a 3-separation where $A \cap B = X$. If $G[A]$ has a $K_{3,3}(X)$-minor and $G[B]$ has a $K_{3,1}(X)$ minor, then G has a $K_{3,4}$-minor.*

Proof. Contract $G[B]$ down to the $K_{3,1}(X)$-minor and $G[A]$ down to the $K_{3,3}(X)$-minor. Since $A \cap B = X$, there is a vertex from the $K_{3,1}(X)$-minor which is adjacent to all of X, and there are 3 vertices from the $K_{3,3}(X)$-minor adjacent to all of X, and all of these vertices are distinct. Therefore G has a $K_{3,4}$-minor. $\qquad\square$

Deciding if a graph has a $K_{3,1}(X)$-minor is easy.

Observation 114. *Let G be a graph and $X = \{a, b, c\} \subseteq V(G)$. Then G has a $K_{3,1}(X)$-minor if and only if there is no $X' \subset X$ such that X' is the vertex boundary of a separation (A, B) such that $B = X \setminus X'$.*

Proof. It is immediate to see if we have such a separation X' that there is no $K_{3,1}(X)$-minor. For the converse if $\{G_x | x \in V(K_{3,1})\}$ is a model of a $K_{3,1}(X)$-minor in G. Then we can delete the branch sets containing any strict subset of X and remain connected by definition of a $K_{3,1}(X)$-minor. $\qquad\square$

However it appears to be of great difficulty to give a nice characterization of $K_{3,3}(X)$-minors. The best result known appears to be the following bound.

Theorem 115 ([22]). *Let G be a 3-connected graph and let X be a set of three vertices. If $|E(G)| \geq 4|V(G)| - 9$, then G has a $K_{3,3}(X)$-minor.*

It is left as an open problem in [22] if this bound is tight. While the problem of determining when a graph has $K_{3,4}$-minor appears to be quite difficult, Bohme et al. have a result on $K_{3,k}$-minors, $k \in \mathbb{N}$, for large highly connected graphs.

Theorem 116 ([9]). *For any positive integer k, there exists a constant $N(k)$ such that every 7-connected graph G on at least $N(k)$ vertices contains $K_{3,k}$ as a minor.*

While this is of no use when looking at reduciblity of both Symanzik polynomials and 4-external momenta, it does apply to reducibility with respect to the first Symanzik polynomial.

Shifting gears, we left as an open problem whether or not K_6 was reducible with respect to the first Symanzik polynomial. We show here that even if K_6 is not reducible, we do not gain that many additional graphs which we know are not reducible. To do this we need to introduce some definitions.

Let G be a graph. If u and v are non-adjacent vertices in G, then adding an edge is simply creating the graph $G \cup \{uv\}$. Now let $v \in V(G)$ such that $\deg(v) \geq 4$. To split v is to first delete v from G, and then add two vertices to G, v_1 and v_2 such that $v_1 v_2 \in E(G)$, each neighbour of v in G is a neighbour of exactly one of v_1 or v_2, and $\deg(v_i) \geq 3$ for $i \in \{1, 2\}$. Now we give a well-known result of Tutte.

Theorem 117 ([41]). *A graph is 3-connected if and only if it is obtained from a wheel by repeatedly adding edges and splitting vertices.*
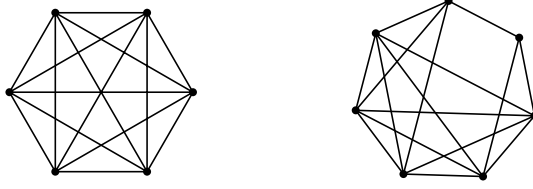
Figure 3.14: The graph K_6 and K_6 after splitting any vertex.

A partial extension is given by Seymour's splitter theorem.

Theorem 118 ([40]). *Suppose a 3-connected graph $H \neq W_3$ is a proper minor of a 3-connected graph $G \neq W_n$. Then G has a minor J, which is obtained from H by either adding an edge or splitting a vertex.*

Now we have the following corollary:

Corollary 119. *Let G be a 3-connected graph. Suppose G has either $K_{3,4}$-minor or a K_6-minor. Then either G has a $K_{3,4}$-minor, or G is isomorphic to K_6.*

Proof. It suffices to consider the case where we have a graph G which has a K_6-minor and no $K_{3,4}$-minor. We will show that G is isomorphic to K_6. Suppose the graph G is not isomorphic K_6. Observe that G non-planar and thus is not a wheel. Then since G has a K_6-minor, by the splitter theorem, G has a minor J which is obtained by either adding an edge or splitting a vertex of K_6. As K_6 is complete, we can only split a vertex, and up to isomorphism splitting any vertex results in the same graph. It is easy to check that after splitting, the resulting graph has $K_{3,4}$ as a subgraph (see Figure 3.14). But this contradicts that G is not K_6. $\square$

Since both $K_{3,4}$ and K_6 are 3-connected, Lemma 61 applies so the 3-connected assumption in Corollary 119 does not really matter. Therefore from the point of view of understanding reducibility with respect to the first Symanzik polynomial, if K_6 is not reducible then we do not add that many more non-reducible graphs. Then if K_6 is not reducible, in some sense one can consider K_6 an exception since almost all graphs are not reducible for different reasons, in the same sense that K_5 is an exception for planarity.

3.6 V_8 plus an edge minors

One of the other graphs which we know is not reducible for the first Symanzik polynomial is $V_8 \cup \{02\}$. Therefore it is of interest to understand graphs which do not contain $V_8 \cup \{02\}$ as a minor. We do not make any progress towards this, we just outline some known results and a possible approach.

One approach is to start with Robertson's characterization of V_8-topological minor free graphs. Before stating the theorem we have some definitions.

Let G be a 3-connected graph and let (G_1, G_2) be a 3-separation and suppose $|V(G)| \geq 5$. Suppose for every partition (A, B) of $E(G)$ either $|A| \leq 3$, or $|B| \leq 3$, or at least four vertices of G are incident with an edge in A and an edge in B. Any graph satisfying these conditions is called *internally 4-connected.* Now we can state Robertson's Theorem.

Theorem 120 ([29])**.** *If G is an internally-4-connected graph then either G contains a subgraph S which is isomorphic to a subdivision of V_8 or one of the following structural conditions hold:*

1. *G is planar,*

2. *$G - \{x, y\}$ is a cycle, for some adjacent $x, y \in V(G)$,*

3. *$G - \{w, x, y, z\}$ is edgeless, for some $w, x, y, z \in V(G)$,*

4. *G is isomorphic to $L(K_{3,3})$*

5. *or $|V(G)| \leq 7$.*

While the above characterization is for topological minors, as V_8 has maximum degree 3, topological minors and minors coincide. In general we have the following well known fact (see, for example, [10])

Observation 121. *Let H be a graph with maximum degree 3. Then a graph G has an H-minor if and only if G has an H-topological minor.*

Therefore the above characterization applies to minors. Unfortunately $V_8 \cup \{02\}$ is not internally 4-connected, and thus we cannot reduce the problem of determining if a graph has a $V_8 \cup \{02\}$-minor to internally 4-connected graphs. However, V_8 is internally 4-connected so we can remove the internally 4-connected condition via standard reduction lemmas.

Lemma 122 ([29])**.** *Let G be a 3-connected graph and let (G_1, G_2) be a 3-separation. Suppose $|E(G_1)| \geq 4$ and $4 \leq |E(G_2)|$. Define G_1^+, G_2^+ to be G_1 and G_2, with new vertices x_1 and x_2 respectively. Make x_1 and x_2 adjacent to each of the three vertices of $G_1 \cap G_2$. Then G has G_1^+, G_2^+ as topological minors, and G has V_8 as a topological minor if and only if one of G_1^+ or G_2^+ have V_8 as a topological minor.*

Then from the splitter theorem, to characterize graphs with no $V_8 \cup \{02\}$-minor, it suffices to show exactly what sequences of splits and adding edges you can do when you start with V_8.

We have looked at this, however it is not obvious what the exactly class ends up being. Even if we exclude both $V_8 \cup \{02\}$ and $K_{3,4}$-minors it is not clear what the class should be.

From experimenting we appear to get at infinite class of graphs with a V_8-minor but no $V_8 \cup \{02\}$ or $K_{3,4}$-minors. We leave it as an open problem to determine what the class of graphs with no $K_{3,4}$ or $V_8 \cup \{02\}$-minor is.

Chapter 4

Conclusion

The main problem of this thesis was to give a forbidden minor characterization for reducibility when the graph has no massive edges and four on-shell momenta. We did not manage to do this, and instead looked at giving a structural characterization of what graphs which might be reducible look like. Since K_4, W_4, $K_{2,4}$ and L are all forbidden minors when we have 4 on-shell momenta, we characterized the graphs not containing any of those minors. In the end, what we obtained was that if a graph is reducible with respect to both Symanzik polynomials, then it has to satisfy a few properties. It is either a class $\mathcal{A}$ graph, or it is the spanning subgraph of a class $\mathcal{D}$, $\mathcal{E}$, or $\mathcal{F}$ graph, such that on the web in each of these classes, for every cycle containing a, b, c, d one of the five obstructions from Theorem 82 occurs. Then along certain terminal separating 2-chains, we must avoid having an $L'(X')$-minor, which occur according to Lemma 99. Additionally, from looking at $K_{3,4}$-minors in $K_4(X)$-minor free graphs, we learned that in "3-connected pieces" of a reducible graph, there is at most one triangle T for which F_T is non-planar. So we gained some understanding of what it means for a graph to be reducible, but we do not have the full picture as of yet. We leave the following as a question:

Question 123. *What is the full forbidden minor set for reducibility with respect to both Symanzik polynomials when we have four on-shell momenta and no edges are massive?*

In a similar vein we have the likely much harder question is still open:

Question 124. *What is the full forbidden minor set for reducibility with respect to the first Symanzik polynomial?*

We leave the following as conjectures and questions as possible starting points for these questions.

Conjecture 125. *The graph K_6 is not reducible with respect to the first Symanzik polynomial.*

Question 126. *What is an excluded minor theorem for graphs not containing $K_{3,4}$? What is the excluded minor theorem for graphs not containing $K_{3,4}$ and $V_8 \cup \{02\}$? Or just $V_8 \cup \{02\}$?*

The next conjecture is due to Jørgensen and Kawarabayshi.

Conjecture 127 ([22]). *The bound in Theorem 115 is tight.*

Additionally, all the structural results in this thesis apply to 2-connected graphs. However reducibility with 4-external momenta does not reduce to 2-connected graphs in all cases. We ask the following question:

Question 128. *Let G be a graph with a 1-separation (A, B) such that $A \cap B = \{v\}$. Suppose G has four external momenta and no massive edges. Furthermore suppose that at least two of the momenta lie in A and two of the momenta lie in B. When is G reducible with respect to both Symanzik polynomials?*

An additional problem which we did not tackle but is interesting is the algorithmic complexity of determining when a graph has a $K_4(X)$-minor, $W_4(X)$-minor, $K_{2,4}(X)$-minor or an $L(X)$-minor. Monroy and Wood ([45]) conjecture that there is a linear time algorithm in the number of vertices for determining if a graph has a $K_4(X)$-minor. This is plausible due to the heavy relationship between $K_4(X)$-minors and the two disjoint path problem, which has a practical and efficient algorithm. Given that if we know a graph is $K_4(X)$-free, that to determine if the graph is $W_4(X)$-free can be reduced to looking a two separations, we conjecture the following:

Conjecture 129. *There is a linear time algorithm for determining if a graph G has a $K_4(X)$-minor or $W_4(X)$-minor.*

Additionally we pose the following question,

Question 130. *Is there a linear time algorithm for determining if a graph G has a $K_4(X)$-minor, $W_4(X)$-minor, $K_{2,4}(X)$-minor, or a $L(X)$-minor?*

We note there is a cubic time algorithm by appealing to the algorithm of Robertson and Seymour in [36]. However one should note that this algorithm is very impractical in general.

Lastly, in general one wants to use reducibility to actually compute integrals, and for that one needs to know a reducible order. Forbidden minors give a method for computing if a graph is reducible, but does not find a reducible order, therefore a natural question is

Question 131. *Suppose G is reducible. Is there a polynomial time algorithm to find a permutation σ such that G is reducible with respect to σ?*

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