Assignment 1

CO 330 (Fall 2026)

Due Friday, September 25, 2026 at 3pm Eastern time.

Instructions

To get full marks on this assignment you must do questions worth 12 marks total. Any additional questions you do beyond 12 marks-worth will not be counted. Each question has two possible point values, one if you would like it marked for correctness and one if you would like it marked for completion and feedback, you will indicate which you want when you submit.[1] Point values of each question are listed before the question.

All answers should be justified unless specifically indicated otherwise.

This assignment should be done and submitted individually. You are welcome to discuss with your classmates, but you should not use generative AI for your ideas and you must write up your solutions independently (neither writing it with other people nor with tools such as generative AI). In your write up explicitly acknowledge any discussions or non-class resources and tools that you may have used.

This assignment should be submitted in crowdmark. You should receive a link no later than Friday September 18. If not please contact `ren.yeats@uwaterloo.ca` well before the deadline.

Questions

(1) **{4 marks graded for content, 2 marks graded for completion}**
   (a) Prove that if R is an integral domain then so is $R[[x]]$.
   (b) Prove that if we instead defined multiplication of formal power series with coefficients in an integral domain R to be term-wise (keeping the rest of the definition the same):
$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} a_n b_n x^n$$
   then the resulting object would be a ring but not an integral domain.

(2) **{4 marks graded for content, 2 marks graded for completion}** Give combinatorial (bijective or counting-in-two-ways) proofs of the following identities.
   (a)
$$(d+1)^n = \sum_{k=0}^{n} \binom{n}{k} d^k \qquad \text{for } n, d \in \mathbb{Z}_{\geq 1}$$
   (b)
$$\binom{n}{k}^2 = \sum_{i=0}^{k} \binom{n}{i}\binom{n-i}{k-i}\binom{n-k}{k-i} \qquad \text{for } k, n \in \mathbb{Z}_{\geq 0}, k \leq n.$$

[1]Make good use of this feature; it has always been valuable to recognize what you don't know, but it is increasingly important with today's AI tools to be able to judge the correctness (or incorrectness) of an argument.

(3) **{4 marks graded for content, 2 marks graded for completion}** Let $n \geq 1$.

A $2 \times n$ standard Young tableau is a $2 \times n$ matrix in which the numbers $1, 2, \dots, 2n$ each occur exactly once, the rows increase from left to right, and the columns increase from bottom to top.

A Dyck path with $2n$ steps is a lattice path starting at the origin, ending on the x-axis, using steps $\nearrow$ and $\searrow$, and never going strictly below the x-axis.

Prove that the number of $2 \times n$ standard Young tableau is the same as the number of Dyck paths with $2n$ steps.

(4) **{4 marks graded for content, 2 marks graded for completion}** Prove that for $n > 0$

$$\binom{\frac{1}{2}}{n}(-4)^n = \frac{-2}{n}\binom{2(n-1)}{n-1}$$

You can use this (with the generalized binomial theorem) to do the coefficient extraction for the generating series of binary trees and see explicitly that it is the Catalan numbers.

(5) **{4 marks graded for content, 2 marks graded for completion}**

(a) Give two different combinatorial classes with an infinite number of elements and the same underlying set.

(b) Give a third weight function on the same set so that the result is not a combinatorial class