

Solutions to Assignment 1

CO 330 (Fall 2026)

Question 1.

- (a) Prove that if R is an integral domain then so is $R[[x]]$.
- (b) Prove that if we instead defined multiplication of formal power series with coefficients in an integral domain R to be term-wise (keeping the rest of the definition the same):

$$\left(\sum_{n \geq 0} a_n x^n\right) \left(\sum_{n \geq 0} b_n x^n\right) = \sum_{n \geq 0} a_n b_n x^n$$

then the resulting object would be a ring but not an integral domain.

Solution. Recall that an integral domain is a commutative ring having no zero divisors.

(a) Assuming that $R[[x]]$ is a commutative ring, it suffices to show that $R[[x]]$ has no zero divisors. Let $A(x) = \sum_{n \geq 0} a_n x^n$ and $B(x) = \sum_{n \geq 0} b_n x^n$ be nonzero series in $R[[x]]$. Since $A(x)$ and $B(x)$ are nonzero series, there exist minimal indices i and j such that $a_i \neq 0$ and $b_j \neq 0$, respectively. The coefficient of x^{i+j} in $A(x)B(x)$ is

$$[x^{i+j}]A(x)B(x) = \sum_{k=0}^{i+j} a_k b_{i+j-k} = a_i b_j.$$

Since R is an integral domain and $a_i, b_j \neq 0$, we have $a_i b_j \neq 0$. Thus $A(x)B(x) \neq 0$, proving that $R[[x]]$ is an integral domain.

(b) The ring $R[[x]]$ with term-wise multiplication is not an integral domain because the nonzero series $A(x) = 1$ and $B(x) = x$ satisfy

$$A(x)B(x) = (1 + 0x + 0x^2 + \cdots)(0 + 1x + 0x^2 + \cdots) = 0 + 0x + 0x^2 + \cdots = 0.$$

Comments.

- In part (a), the proof that $R[[x]]$ is a commutative ring is not difficult, and is just a follow your nose sort of proof.
- The proof that $R[[x]]$ in part (b) is a ring is also a follow your nose proof, although defining the term-multiplicative identity as $\sum_{n \geq 0} x^n$ is potentially a bit tricky.

- For part (a), You may be tempted to use some form of your regular, run of the mill induction for this question. The proof given above is actually secretly using induction, but in a very different form, sometimes called “minimal counterexample:” If the induction is not true, then there must be some smallest value in which it is not true. So assume towards a contradiction that there is such a smallest value and then show a contradiction. Sometimes for proofs you just need to try a bunch of versions of the same thing before you find the easiest proof.

Question 2. Give combinatorial (bijective or counting-in-two-ways) proofs of the following identities.

(a)

$$(d+1)^n = \sum_{k=0}^n \binom{n}{k} d^k \quad \text{for } n, d \in \mathbb{Z}_{\geq 1}$$

(b)

$$\binom{n}{k}^2 = \sum_{i=0}^k \binom{n}{i} \binom{n-i}{k-i} \binom{n-k}{k-i} \quad \text{for } k, n \in \mathbb{Z}_{\geq 0}, k \leq n$$

Solution. The following solutions might be more detailed than necessary. Read the comment after the solutions for a discussion on how much detail is appropriate.

(a) Fix non-negative integers n and d . Consider the set

$$S = \{0, 1, \dots, d\}^n$$

of n -tuples with entries from $\{0, 1, \dots, d\}$. Then, $|S| = (d+1)^n$. For each $k \in \{0, 1, \dots, n\}$, let

$$S_k = \left\{ w \in S : \begin{array}{l} w \text{ has exactly} \\ k \text{ non-zero entries} \end{array} \right\}, \quad \text{and} \quad T_k = \left\{ (Z, x) : \begin{array}{l} Z \subseteq [n], |Z| = k, \\ x \in \{1, \dots, d\}^k \end{array} \right\}.$$

Since the number of non-zero entries of any n -tuple $w \in S$ is an integer between 0 and n , the sets $S_0, S_1, \dots, S_n$ are pairwise disjoint and cover S , giving the disjoint union

$$S = \bigsqcup_{k=0}^n S_k = S_0 \sqcup S_1 \sqcup \dots \sqcup S_n.$$

Define the map $f: S_k \rightarrow T_k$ by $f(w) = (Z, x)$, where $Z = \{i \in [n] : w_i \neq 0\}$ is the set of indices of the non-zero entries of w , and $x \in \{1, \dots, d\}^k$ is the k -tuple obtained from w by deleting all zeroes. Conversely, define $g: T_k \rightarrow S_k$ by sending (Z, x) to the n -tuple obtained by placing the entries of x , in order, into the positions specified by Z and filling all remaining positions with 0. For example,

$$f(0, 3, 0, 1, 2) = (\{2, 4, 5\}, (3, 1, 2)) \quad \text{and} \quad g(\{2, 4, 5\}, (3, 1, 2)) = (0, 3, 0, 1, 2).$$

It is necessary but straightforward to check that $f(w) \in T_k$ for all $w \in S_k$, that $g(Z, x) \in S_k$ for all $(Z, x) \in T_k$, that $g(f(w)) = w$ for all $w \in S_k$, and that $f(g(Z, x)) = (Z, x)$ for all $(Z, x) \in T_k$, proving that f is a bijection with inverse g . Therefore, we have $|S_k| = |T_k|$.

Since $T_k = \{Z \subseteq [n] : |Z| = k\} \times \{1, \dots, d\}^k$, we have $|T_k| = \binom{n}{k} d^k$ by the product rule.

Finally, since S is the disjoint union of $S_0, S_1, \dots, S_n$, we have

$$(d+1)^n = |S| = \sum_{k=0}^n |S_k| = \sum_{k=0}^n |T_k| = \sum_{k=0}^n \binom{n}{k} d^k.$$

(b) Fix non-negative integers k and n with $k \leq n$. Consider the set

$$\mathcal{S} = \{(A, B) : A, B \subseteq [n], |A| = |B| = k\}$$

of pairs of k -subsets of $[n]$. Then, $|\mathcal{S}| = \binom{n}{k}^2$. Consider also the set

$$\mathcal{T} = \left\{ (I, A', B') : \begin{array}{l} I, A', B' \subseteq [n] \text{ pairwise disjoint,} \\ |A'| = |B'| = k - |I| \end{array} \right\}.$$

The maps $f: \mathcal{S} \rightarrow \mathcal{T}$ and $g: \mathcal{T} \rightarrow \mathcal{S}$ defined by

$$f(A, B) = (A \cap B, A \setminus B, B \setminus A) \quad \text{and} \quad g(I, A', B') = (I \cup A', I \cup B')$$

are inverses of each other. *(It is necessary but straightforward to check that $f(A, B) \in \mathcal{T}$ for all $(A, B) \in \mathcal{S}$, that $g(I, A', B') \in \mathcal{S}$ for all $(I, A', B') \in \mathcal{T}$, that $g(f(A, B)) = (A, B)$ for all $(A, B) \in \mathcal{S}$, and that $f(g(I, A', B')) = (I, A', B')$ for all $(I, A', B') \in \mathcal{T}$, proving that f is a bijection with inverse g .)* Therefore, we have $|\mathcal{S}| = |\mathcal{T}|$.

For each $i \in \{0, 1, \dots, k\}$, let $\mathcal{T}_i = \{(I, A', B') \in \mathcal{T} : |I| = i\}$. Since the size $|I|$ of any triple $(I, A', B') \in \mathcal{T}$ is an integer between 0 and k , the sets $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_k$ are pairwise disjoint and cover $\mathcal{T}$, giving the disjoint union

$$\mathcal{T} = \bigsqcup_{i=0}^k \mathcal{T}_i = \mathcal{T}_0 \sqcup \mathcal{T}_1 \sqcup \dots \sqcup \mathcal{T}_k.$$

To count $|\mathcal{T}_i|$, we successively choose the subset $I \subseteq [n]$ of size i in $\binom{n}{i}$ ways, the subset $A' \subseteq [n] \setminus I$ of size $k - i$ in $\binom{n-i}{k-i}$ ways, and the subset $B' \subseteq [n] \setminus (I \cup A')$ of size $k - i$ in $\binom{n-k}{k-i}$ ways, giving

$$|\mathcal{T}_i| = \binom{n}{i} \binom{n-i}{k-i} \binom{n-k}{k-i}.$$

Finally, since $\mathcal{T}$ is the disjoint union of $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_k$, we have

$$\binom{n}{k}^2 = |\mathcal{S}| = |\mathcal{T}| = \sum_{i=0}^k |\mathcal{T}_i| = \sum_{i=0}^k \binom{n}{i} \binom{n-i}{k-i} \binom{n-k}{k-i}.$$

Comments.

- The solutions above are perhaps more detailed than necessary, but it's important that you'd be able to write them like this if you were prompted to. Some arguments can be summarized more briefly. For example, in (a), instead of defining the sets S_k, T_k and the maps f, g , you could say something like the following:

An n -tuple $w \in S$ containing k non-zero entries is uniquely determined by successively choosing the k -subset $Z \subseteq [n]$ of indices of the non-zero entries of w , which can be chosen in $\binom{n}{k}$ ways, and the values of w at the k indices in Z , each chosen from $\{1, \dots, d\}$, which can be done in d^k ways.

Therefore, the number of n -tuples $w \in S$ with exactly k non-zero entries is $\binom{n}{k}d^k$, and consequently, since the number of non-zero entries of any n -tuple $w \in S$ is an integer between 0 and n , we have $|S| = \sum_{k=0}^n \binom{n}{k}d^k$.

Similarly, in (b), instead of defining the sets $\mathcal{T}$, $\mathcal{T}_i$ and the maps f and g , you could say something like the following:

An element $(A, B) \in \mathcal{S}$ satisfying $|A \cap B| = i$ is uniquely determined by successively choosing

- the intersection $I := A \cap B \subset [n]$ of size i , which can be chosen in $\binom{n}{i}$ ways,
- the set difference $A' := A \setminus B \subset [n] \setminus I$ of size $k - i$, which can be chosen in $\binom{n-i}{k-i}$ ways, and
- the set difference $B' := B \setminus A \subset [n] \setminus (I \cup A')$ of size $k - i$, which can be chosen in $\binom{n-k}{k-i}$ ways.

Therefore, the number of pairs $(A, B) \in \mathcal{S}$ with $|A \cap B| = i$ is $\binom{n}{i}\binom{n-i}{k-i}\binom{n-k}{k-i}$, and consequently, since the size of the intersection $A \cap B$ of any pair $(A, B) \in \mathcal{S}$ is an integer between 0 and k , we have $|\mathcal{S}| = \sum_{i=0}^k \binom{n}{i}\binom{n-i}{k-i}\binom{n-k}{k-i}$.

Nevertheless, it's important that you only write this if you are confident that you could write the longer version if prompted to do so.

- Another comment on amount of details: whenever defining a maps f, g which you claim are inverses of each other, you should at least acknowledge the need to check that f and g are well-defined and $g \circ f = \text{id}$ and $f \circ g = \text{id}$. If you truly believe that these are straightforward to check, you can say so, but you should not just omit the comment entirely. In some problems (like Question 3 in this assignment), some of these checks are not straightforward and are expected to be included in the solution.

- It's a good idea to give a name to the set that you are counting, like $\mathcal{S}$.
- Never write a number by itself without explaining what it counts. Some examples:

Bad: Pick k elements from $[n]$: $\binom{n}{k}$. Now pick i elements: $\binom{k}{i}$. Therefore: $\binom{n}{k}\binom{k}{i}$.

Good: There are $\binom{n}{k}$ ways to pick k elements from $[n]$, and there are $\binom{k}{i}$ ways to pick i elements from the k elements that were chosen. Therefore, there are $\binom{n}{k}\binom{k}{i}$ ways to pick k elements from $[n]$ and then pick i elements from the k elements that were chosen.

Better: Let's count the number of subsets $B \subset A \subset [n]$ with $|A| = k$ and $|B| = i$. There are $\binom{n}{k}$ ways to choose A as a k -subset of $[n]$, and then there are $\binom{k}{i}$ ways to choose B as an i -subset of A . Therefore, there are $\binom{n}{k}\binom{k}{i}$ such pairs (A, B) .

- The product of binomial coefficients in the summand of (b) is a multinomial coefficient

$$\binom{n}{i}\binom{n-i}{k-i}\binom{n-k}{k-i} = \binom{n}{i, k-i, k-i, n-2k+i},$$

which counts the ways to partition $[n]$ into the parts I , A' , B' , and $[n] \setminus (I \cup A' \cup B')$ of sizes i , $k - i$, $k - i$, and $n - 2k + i$, respectively.

- You might recognize the identity in (a) as the binomial theorem, which actually holds for any real number d (not just integers). The combinatorial proof above only works for integers d , but this problem proves that the polynomial $(x + 1)^n - \sum_{k=0}^n \binom{n}{k} x^k$ of degree at most n has infinitely many roots (the positive integers), and therefore must be the zero polynomial, so the identity holds for all real numbers d .

Question 3. Let $n \geq 1$. A $2 \times n$ standard Young tableau is a $2 \times n$ matrix in which the numbers $1, 2, \dots, 2n$ each occur exactly once, the rows increase from left to right, and the columns increase from bottom to top.

A Dyck path with $2n$ steps is a lattice path starting at the origin, ending on the x -axis, using steps $\nearrow$ and $\searrow$, and never going strictly below the x -axis.

Prove that the number of $2 \times n$ standard Young tableau is the same as the number of Dyck paths with $2n$ steps.

Solution. Let $\mathcal{T}_n$ denote the set of $2 \times n$ standard Young tableaux, and let $\mathcal{D}_n$ denote the set of Dyck paths with $2n$ steps. We define a bijection $f: \mathcal{T}_n \rightarrow \mathcal{D}_n$ and its inverse $g: \mathcal{D}_n \rightarrow \mathcal{T}_n$.

Given a tableau $T \in \mathcal{T}_n$, define $f(T)$ to be the lattice path of length $2n$ whose i -th step is $\nearrow$ if i is in the bottom row of T , and $\searrow$ if i is in the top row of T , for each $i \in \{1, 2, \dots, 2n\}$. Conversely, given a Dyck path $P \in \mathcal{D}_n$, define $g(P)$ to be the $2 \times n$ matrix whose bottom row consists of the indices i where P takes a $\nearrow$ step, and whose top row consists of the indices i where P takes a $\searrow$ step, both written in increasing order.

For example,

$$f\left(\begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 1 & 3 & 5 \\ \hline \end{array}\right) = \text{Dyck path} \quad \text{and} \quad g\left(\text{Dyck path}\right) = \begin{array}{|c|c|c|} \hline 3 & 5 & 6 \\ \hline 1 & 2 & 4 \\ \hline \end{array}.$$

To see that f is well-defined, let $T \in \mathcal{T}_n$. Because each row of T has length n , the path $f(T)$ contains n steps $\nearrow$ and n steps $\searrow$, ending at $(2n, 0)$. Furthermore, because the j -th entry of the bottom row is strictly less than the j -th entry of the top row for each $j \in \{1, \dots, n\}$, the j -th step $\nearrow$ in $f(T)$ precedes the j -th step $\searrow$. Thus, $f(T)$ never dips below the x -axis, so $f(T) \in \mathcal{D}_n$.

Conversely, to see that g is well-defined, let $P \in \mathcal{D}_n$. Because P has n steps $\nearrow$ and n steps $\searrow$, the matrix $g(P)$ partitions $\{1, 2, \dots, 2n\}$ into two rows of size n , each increasing by construction. Since P never dips below the x -axis, the j -th step $\nearrow$ precedes the j -th step $\searrow$ for each $j \in \{1, \dots, n\}$. Consequently, the j -th entry of the bottom row is strictly less than the j -th entry of the top row, so columns increase from bottom to top, meaning $g(P) \in \mathcal{T}_n$.

Checking that $g(f(T)) = T$ for all $T \in \mathcal{T}_n$ and $f(g(P)) = P$ for all $P \in \mathcal{D}_n$ is immediate from the definitions. Thus, f and g are bijections, and therefore $|\mathcal{T}_n| = |\mathcal{D}_n|$, as desired.

Comments.

- Finding the bijection here is not too bad. The Dyck paths and Young tableaux each have $2n$ things in them, so you just have to match them up nicely. You also have two types of steps in the Dyck path and two rows in the tableau, which suggests that you should match those up in some way. From there, there is really only one canonical way to complete the bijection.

- Once you find the bijection, the hardest part of this proof is proving that these maps are well-defined. The best way to do so in this case is to notice that the Dyck path condition

is equivalent to having j up steps before having $j + 1$ down steps, and noticing what the columns represent in the Young tableau.

- Side note: formalizing proofs is hard because you need to be able to explain things well without making it too detailed. If you struggle with this, it helps to first write down your intuition, and then from there write down a formal proof, and then reread as many times as you need to make it well.

Question 4. Prove that for $n > 0$:

$$\binom{1/2}{n}(-4)^n = \frac{-2}{n} \binom{2(n-1)}{n-1}$$

(You can use this with the generalized binomial theorem to do the coefficient extraction for the generating series of binary trees and see explicitly that it is the Catalan numbers.)

Solution 1. Recall that the generalized binomial coefficient is defined for any real number r and non-negative integer n by

$$\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-(n-1))}{n!}.$$

Therefore, we have

$$\binom{1/2}{n} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-(n-1))}{n!} = \frac{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!}.$$

Multiplying by $(-4)^n = (-1)^n 2^{2n}$, we obtain

$$\binom{1/2}{n}(-4)^n = \frac{(-1)^{2n-1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot 2^{2n}}{2^n n!} = -1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot \frac{2^n}{n!}.$$

We express the product of the first $n-1$ odd positive integers using factorials:

$$1 \cdot 3 \cdot 5 \cdots (2n-3) = \frac{(2n-2)!}{2 \cdot 4 \cdot 6 \cdots (2n-2)} = \frac{(2n-2)!}{2^{n-1}(n-1)!}.$$

Substituting this identity into our expression yields

$$\binom{1/2}{n}(-4)^n = -\frac{(2n-2)!}{2^{n-1}(n-1)!} \cdot \frac{2^n}{n!} = -\frac{2(2n-2)!}{n!(n-1)!} = -\frac{2}{n} \frac{(2n-2)!}{(n-1)!(n-1)!} = -\frac{2}{n} \binom{2(n-1)}{n-1},$$

completing the proof.

Solution 2. Let $x_i = \frac{4i-2}{i+1}$. We can check that

$$x_i = \frac{(-4) \left(\frac{1}{2} - i\right)}{i+1} \text{ for all } i \geq 0 \quad \text{and} \quad x_i = \frac{i}{i+1} \frac{(2i)(2i-1)}{i^2} \text{ for all } i \geq 1.$$

On the one hand, from the first equation, we have

$$\prod_{i=0}^{n-1} x_i = \prod_{i=0}^{n-1} \frac{(-4) \left(\frac{1}{2} - i\right)}{i+1} = (-4)^n \frac{\prod_{i=0}^{n-1} \left(\frac{1}{2} - i\right)}{n!} = (-4)^n \binom{1/2}{n}.$$

On the other hand, if we extract the $i=0$ term and use the second equation, we have

$$\prod_{i=0}^{n-1} x_i = x_0 \left(\prod_{i=1}^{n-1} \frac{i}{i+1} \right) \left(\prod_{i=1}^{n-1} \frac{(2i)(2i-1)}{i^2} \right) = -2 \cdot \frac{1}{n} \cdot \frac{(2(n-1))!}{((n-1)!)^2} = -\frac{2}{n} \binom{2(n-1)}{n-1}.$$

The result follows by equating the two expressions for $\prod_{i=0}^{n-1} x_i$.

Comments.

- Two student submitted a solutions by induction, which is a valid approach; but whenever possible, not using induction tends to be nicer. Luckily, their solution can be adapted to a non-inductive solution and is how we presented it in the very slick Solution 2 above. (The inductive step is basically going from $\prod_{i=0}^{n-1} x_i$ to $\prod_{i=0}^n x_i$ by multiplying by x_n .)
- In Solution 1, while you might be worried about the “product of the first $n - 1$ odd positive integers” when $n = 1$, the product is empty in that case, and the empty product is defined to be 1. It would break if we had $n = 0$, but the problem statement explicitly assumes $n > 0$ (besides the denominator of the right-hand side being zero).

Question 5.

(a) Give two different combinatorial classes with an infinite number of elements and the same underlying set.

(b) Give a third weight function on the same set so that the result is not a combinatorial class.

Solution. Recall that a combinatorial class is a pair $(\mathcal{A}, w)$, where $\mathcal{A}$ is a set and $w: \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$ is a function such that

$$w^{-1}(n) = \{a \in \mathcal{A} : w(a) = n\}$$

is finite for all $n \geq 0$. (The set $w^{-1}(n)$ is also denoted by $\mathcal{A}_n$, but this might be confusing here since we are considering multiple weight functions.)

Because this is an open-ended question, there are many valid choices for the set $\mathcal{A}$ and the weight functions; the following construction is just one of many plausible solutions.

Let $\mathcal{A}$ be the set of all non-empty rooted unary-binary trees, that is, rooted trees in which every vertex has zero, one, or two (ordered) children.

(a) We define two distinct weight functions $w_1, w_2: \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$:

Let $w_1(T) = |V(T)|$ be the number of vertices of T . For any $n \geq 1$, the preimage $w_1^{-1}(n)$ is finite, since there are only finitely many rooted trees with n vertices. (For a very crude bound, the number of graphs on n labelled vertices equals $2^{\binom{n}{2}}$.) Therefore, $(\mathcal{A}, w_1)$ is a combinatorial class.

Let $w_2(T)$ be the number of non-leaf vertices of T . A tree with n non-leaf vertices can have at most $2n$ leaves, since each non-leaf has at most two children. Thus $|V(T)| \leq 3n$, meaning $w_2^{-1}(n) \subseteq \bigcup_{m=1}^{3n} w_1^{-1}(m)$, which is finite. Therefore, $(\mathcal{A}, w_2)$ is also a combinatorial class.

(b) Define $w_3: \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$ by setting $w_3(T)$ to be the number of leaves of T . For every integer $n \geq 1$, a path of n vertices where each non-leaf has exactly one child forms a tree with exactly one leaf. Therefore, the preimage $w_3^{-1}(1)$ is infinite, and consequently $(\mathcal{A}, w_3)$ is not a combinatorial class.

Here are some other constructions that you came up with: the set $\mathbb{Z}_{\geq 0}$ with

$$w_1(n) = n, \quad w_2(n) = 2n, \quad w_3(n) = 0;$$

the set of finite binary strings with

$$w_1(s) = \text{length}(s), \quad w_2(s) = \text{length}(s) + \#(\text{ones in } s), \quad w_3(s) = \#(\text{ones in } s);$$

the set of integer partitions with

$$w_1(\lambda) = |\lambda|, \quad w_2(\lambda) = \text{length}(\lambda) \cdot \text{width}(\lambda), \quad w_3(\lambda) = \text{length}(\lambda).$$

Comments.

- While not necessary, it's a good idea to recall the definition of a combinatorial class in your solution.
- In (a), you need to acknowledge the need to check that the preimage of each weight is finite, and give an argument (possibly very brief) for why this is the case.
- In (b), you need to give an argument for why some preimage $w^{-1}(n)$ is infinite. The simplest way is to give an infinite family of objects of same weight.