Solutions to Assignment 2
CO 330 (Fall 2026)

Question 1.

(a) Let $A_k(x)$ be formal power series for which the partial sums $\sum_{k=0}^{n} A_k(x)$ converge in $R[[x]]$. Prove that

$$\left(\sum_{n\geq 0} A_n(x)\right)' = \sum_{n\geq 0} A_n'(x).$$

(b) Prove the chain rule of formal derivatives of formal power series.

Solution to (a). Let $S(x) = \sum_{n\geq 0} A_n(x)$, which converges in $R[[x]]$, and denote the partial sums by $S_N(x) = \sum_{n=0}^{N} A_n(x)$. By finite linearity, $S_N'(x) = \sum_{n=0}^{N} A_n'(x)$.

Because $S_N(x)$ converges to $S(x)$, there exists an integer N_0 such that $[x^{m+1}]S_N(x) = [x^{m+1}]S(x)$ for all $N \geq N_0$. By the definition of the formal derivative, for all $N \geq N_0$,

$$[x^m]S_N'(x) = (m+1)[x^{m+1}]S_N(x) = (m+1)[x^{m+1}]S(x) = [x^m]S'(x).$$

Therefore, the coefficients of $S_N'(x)$ stabilize to those of $S'(x)$ as $N \to \infty$, so $S_N'(x) \to S(x)$ and we conclude that

$$\sum_{n\geq 0} A_n'(x) = \lim_{N\to\infty} \sum_{n=0}^{N} A_n'(x) = \lim_{N\to\infty} \left(\sum_{n=0}^{N} A_n(x)\right)' = \lim_{N\to\infty} S_N'(x) = S'(x) = \left(\sum_{n\geq 0} A_n(x)\right)'.$$

Alternative proof of $S_N(x) \to S(x)$ in (a). We use the equivalent definition of convergence in $R[[x]]$ via the order of formal power series (the minimum degree of a non-zero term, also called valuation): a sequence $F_N(x)$ converges to $F(x)$ in $R[[x]]$ if and only if $\operatorname{ord}(F_N(x) - F(x)) \to \infty$ as $N \to \infty$.

Because formal differentiation drops the order of any power series by at most 1,

$$\operatorname{ord}\left(S_N'(x) - S'(x)\right) = \operatorname{ord}\left((S_N(x) - S(x))'\right) \geq \operatorname{ord}(S_N(x) - S(x)) - 1.$$

Since $S_N(x) \to S(x)$, we have $\operatorname{ord}(S_N(x) - S(x)) \to \infty$ as $N \to \infty$. This directly implies

$$\operatorname{ord}\left(S_N'(x) - S'(x)\right) \to \infty,$$

which proves $S_N'(x) \to S'(x)$.

Solution to (b). Write $A(x) = \sum_{n\geq 0} a_n x^n$ and assume that the composition $A(B(x)) = \sum_{n\geq 0} a_n B(x)^n$ converges in $R[[x]]$. *(This happens if and only if $B(0) = 0$ or $A(x)$ is a polynomial.)* Applying the infinite sum rule from part (a) and the product rule,

$$(A(B(x)))' = \left(\sum_{n\geq 0} a_n B(x)^n\right)' \overset{\text{(a)}}{=\!=} \sum_{n\geq 0} a_n (B(x)^n)'$$
$$\overset{\text{p.r.}}{=\!=} \sum_{n\geq 1} a_n n B'(x) B(x)^{n-1} = B'(x) \sum_{n\geq 1} n a_n B(x)^{n-1} = B'(x) A'(B(x)).$$

(For completeness, the product rule is true since, for all $n \geq 0$,

$$[x^n](F(x)G(x))' = (n+1)[x^{n+1}](F(x)G(x)) = \sum_{k=0}^{n+1} k f_k g_{n+1-k} + \sum_{k=0}^{n+1} (n+1-k) f_k g_{n+1-k}$$
$$= \sum_{j=0}^{n} (j+1) f_{j+1} g_{n-j} + \sum_{k=0}^{n} f_k (n-k+1) g_{n-k+1} = [x^n](F'(x)G(x) + F(x)G'(x)).$$

Then, by induction, $(B(x)^n)' = nB'(x)B(x)^{n-1}$ for all $n \geq 1$.)

Comments.

- In part (a), you cannot assume that $\sum_{n\geq 0} A_n'(x)$ exists, that is, you cannot assume that the sequence of partial sums $\sum_{k=0}^{n} A_k'(x)$ converges in $R[[x]]$.
- You cannot freely swap the order of a double sum. For example, define

$$A_{m,n} = \begin{cases} 1 & \text{if } n = m, \\ -1 & \text{if } n = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

You can compute that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n} = 0 \neq 1 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{m,n}.$$

One order can fail to converge entirely. For example, define

$$A_{m,n} = \begin{cases} x & \text{if } m = 0, \\ -x & \text{if } m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{m,n} = \sum_{n=0}^{\infty} (x - x) = 0,$$

while $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}$ does not converge because the inner sum $\sum_{n=0}^{\infty} x$ diverges in $R[[x]]$.

Question 2. Determine whether or not the sequence $A_k(x) = \left(1 + \frac{x}{k}\right)^k$ converges in $\mathbb{Q}[[x]]$ as $k \to \infty$. If it does, determine the limit.

Solution. The sequence does not converge in $\mathbb{Q}[[x]]$. By the binomial theorem, the coefficient of x^2 in $A_k(x)$ is

$$[x^2]A_k(x) = \binom{k}{2}\frac{1}{k^2} = \frac{k-1}{2k} = \frac{1}{2} - \frac{1}{2k} \quad \text{for all } k \geq 1.$$

Because convergence in $\mathbb{Q}[[x]]$ requires each coefficient to eventually stabilize, and $[x^2]A_k(x)$ is strictly increasing with k, the sequence $A_k(x)$ does not converge in $\mathbb{Q}[[x]]$.

Comments.

• You might be confused by the fact that, in calculus, you learn that for any real number α we have $\lim_{k\to\infty}\left(1 + \frac{\alpha}{k}\right)^k = \exp(\alpha)$. Since we have defined the exponential power series $\exp(x) = \sum_{n\geq 0}\frac{x^n}{n!} \in \mathbb{Q}[[x]]$, you could be tempted to think that $A_k(x)$ converges to $\exp(x)$ in $\mathbb{Q}[[x]]$. The solution above shows that this is not the case: formal power series convergence requires each coefficient sequence to eventually stabilize to a constant value, rather than merely converging as a sequence of real numbers.

• If you're curious about what this means in terms of topology, it means that the we are requiring the sequence $([x^n]A_k(x))_{k\geq 0}$ to converge in the *discrete topology* on $\mathbb{Q}$ for each $k \geq 0$, rather than in the usual topology on $\mathbb{Q}$ (or $\mathbb{R}$).

Question 3. Let $\mathcal{A}$ be a combinatorial class with no objects of weight 0. Then $\mathsf{MSet}(\mathcal{A})$ is defined to be the combinatorial class of all finite multisets (i.e., repetition is allowed) of objects of $\mathcal{A}$.

(a) Justify that $\mathsf{MSet}(\mathcal{A})$ is a combinatorial class with the expected weight function.

(b) Let $\mathcal{B} = \mathsf{MSet}(\mathcal{A})$. Explain why

$$B(x) = \prod_{a \in \mathcal{A}} \left(\frac{1}{1 - x^{w(a)}} \right)$$

and conclude that $B(x) = \prod_{n \geq 1} (1 - x^n)^{-a_n}$.

(c) Use properties of formal exp and log to prove that

$$B(x) = \exp\left(\sum_{n \geq 1} \frac{A(x^n)}{n} \right).$$

This is a good example of how useful combinatorial constructions can be admissible while going beyond sum, product, and composition of series.

Solution. The following is adapted from Flajolet and Sedgewick, *Analytic Combinatorics*, p. 29.

(a) Let $\mathcal{B} = \mathsf{MSet}(\mathcal{A})$, with $w(M) = \sum_{a \in M} w(a)$, and let $\mathcal{A}^{(\leq m)} = \bigcup_{j=1}^{m} \mathcal{A}_j$, which is finite since each $\mathcal{A}_j$ is finite. Since every element in $\mathcal{A}$ has weight at least 1, any $M \in \mathcal{B}_n$ contains only objects from $\mathcal{A}^{(\leq n)}$, each with multiplicity at most n. Choosing a multiplicity in $\{0, 1, \ldots, n\}$ for each such object yields the bound

$$|\mathcal{B}_n| \leq (n+1)^{|\mathcal{A}^{(\leq n)}|} < \infty.$$

Thus, $\mathcal{B}_n$ is finite for all $n \geq 0$, so $\mathcal{B} = \mathsf{MSet}(\mathcal{A})$ is a combinatorial class.

(b) First assume that $\mathcal{A}$ is finite. Any multiset of objects from $\mathcal{A}$ can be sorted according to a canonical ordering $\alpha_1, \alpha_2, \ldots, \alpha_k$ of the elements of $\mathcal{A}$. Thus, every multiset can be viewed uniquely as a sequence of repeated elements α_1, followed by a sequence of repeated elements α_2, and so forth. The multiset construction is therefore definable by

$$\mathsf{MSet}(\mathcal{A}) \leftrightharpoons \mathsf{Seq}(\{\alpha_1\}) \times \mathsf{Seq}(\{\alpha_2\}) \times \cdots \times \mathsf{Seq}(\{\alpha_k\}).$$

By the product and sequence rules, this relation translates into generating functions as

$$B(x) = \prod_{a \in \mathcal{A}} (1 - x^{w(a)})^{-1} = \prod_{n \geq 1} \prod_{\substack{a \in \mathcal{A} \\ w(a) = n}} (1 - x^n)^{-1} = \prod_{n \geq 1} (1 - x^n)^{-a_n}.$$

The argument extends to an infinite class $\mathcal{A}$ by extracting the coefficient of x^m for each integer $m \geq 0$. For each $m \geq 0$, let $\mathcal{A}^{(\leq m)} = \bigcup_{j=1}^{m} \mathcal{A}_j$ and $\mathcal{B}^{(\leq m)} = \mathsf{MSet}(\mathcal{A}^{(\leq m)})$. Because

every object in $\mathcal{A}$ has weight at least 1, any multiset in $\mathcal{B}$ of total weight m can contain only objects of weight at most m. Therefore, the multisets of weight m in $\mathcal{B}$ and in $\mathcal{B}^{(\leq m)}$ coincide, so

$$[x^m]B(x) = [x^m]B^{(\leq m)}(x).$$

Since $\mathcal{A}^{(\leq m)}$ is finite, the finite case gives

$$B^{(\leq m)}(x) = \prod_{a \in \mathcal{A}^{(\leq m)}} (1 - x^{w(a)})^{-1} = \prod_{n=1}^{m} (1 - x^n)^{-a_n}.$$

Furthermore, every object $a \in \mathcal{A}$ with $w(a) > m$ satisfies $(1-x^{w(a)})^{-1} = 1+x^{w(a)}+\cdots$, which contributes no non-constant terms of degree at most m. Similarly, every factor $(1 - x^n)^{-a_n}$ with $n > m$ has constant term 1 and contributes no further terms of degree at most m. Consequently,

$$[x^m] \prod_{a \in \mathcal{A}} (1 - x^{w(a)})^{-1} = [x^m] \prod_{a \in \mathcal{A}^{(\leq m)}} (1 - x^{w(a)})^{-1} = [x^m]B^{(\leq m)}(x) = [x^m]B(x).$$

and

$$[x^m] \prod_{n \geq 1} (1 - x^n)^{-a_n} = [x^m] \prod_{n=1}^{m} (1 - x^n)^{-a_n} = [x^m]B^{(\leq m)}(x) = [x^m]B(x).$$

Because their coefficients agree for all $m \geq 0$, we conclude that

$$B(x) = \prod_{a \in \mathcal{A}} (1 - x^{w(a)})^{-1} = \prod_{n \geq 1} (1 - x^n)^{-a_n}.$$

(c) We compute the logarithm of $B(x)$ using the formula from part (b) as

$$\log B(x) = \sum_{n=1}^{\infty} -a_n \log(1 - x^n) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_n x^{nk}}{k} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n x^{nk}}{k} = \sum_{k=1}^{\infty} \frac{A(x^k)}{k},$$

where swapping the summations is valid in $\mathbb{R}[[x]]$ because each power x^m receives contributions from only finitely many pairs (n, k) with $nk = m$. The result follows from $B(x) = \exp(\log B(x))$.

Comments.

• In part (b), assuming $\mathcal{A}$ is infinite, you can prove that $\mathcal{A}$ is countably infinite and therefore write it as $\mathcal{A} = \{\alpha_1, \alpha_2, \dots\}$. However, you still cannot write

$$\mathsf{MSet}(\mathcal{A}) \leftrightharpoons \mathsf{Seq}(\{\alpha_1\}) \times \mathsf{Seq}(\{\alpha_2\}) \times \cdots,$$

because the right-hand side includes, for example, the infinite multiset where each element α_i has multiplicity 1. Perhaps a more accurate caveat would be to say that the right-hand is not even a combinatorial class, since there are elements without a well-defined weight value.

Question 4. Prove that the generating series of the class of rooted non-plane (where left and right child are not distinguished) binary trees (still with every vertex having 0 or 2 children) is

$$A(x) = x + x\frac{A(x)^2 + A(x^2)}{2}.$$

Solution. Let $\mathcal{A}$ be the class of rooted non-plane binary trees where each vertex has zero or two children, where the weight of a tree $T \in \mathcal{A}$ is its number of vertices, denoted $|T|$. Every such tree is either a single root vertex with no children (which has weight 1), or a root vertex attached to an unordered pair $\{T_1, T_2\}$ of trees from $\mathcal{A}$ (which has weight $1 + |T_1| + |T_2|$). Decomposing the set of trees according to these two possibilities, the generating series satisfies

$$A(x) = \sum_{T\in\mathcal{A}} x^{|T|} = x + x \sum_{\{T_1,T_2\}\subseteq\mathcal{A}} x^{|T_1|+|T_2|} = x + x\left(\sum_{\substack{\{T_1,T_2\}\subseteq\mathcal{A}\\ T_1=T_2}} x^{|T_1|+|T_2|} + \sum_{\substack{\{T_1,T_2\}\subseteq\mathcal{A}\\ T_1\neq T_2}} x^{|T_1|+|T_2|}\right).$$

We now compute the two sums inside the parentheses separately. For the first sum, where the two subtrees are isomorphic ($T_1 = T_2$), the pair is uniquely determined by a single tree $T_1 \in \mathcal{A}$, giving

$$\sum_{\substack{\{T_1,T_2\}\subseteq\mathcal{A}\\ T_1=T_2}} x^{|T_1|+|T_2|} = \sum_{T_1\in\mathcal{A}} x^{2|T_1|} = \sum_{T\in\mathcal{A}} \left(x^2\right)^{|T|} = A(x^2).$$

For the second sum, where the two subtrees are distinct ($T_1 \neq T_2$), we subtract the pairs with $T_1 = T_2$ from all ordered pairs in $\mathcal{A}^2$, and divide by 2 since each unordered pair appears twice:

$$\sum_{\substack{\{T_1,T_2\}\subseteq\mathcal{A}\\ T_1\neq T_2}} x^{|T_1|+|T_2|} = \frac{1}{2}\left(\sum_{(T_1,T_2)\in\mathcal{A}^2} x^{|T_1|+|T_2|} - \sum_{T\in\mathcal{A}} x^{2|T|}\right) = \frac{A(x)^2 - A(x^2)}{2}.$$

Substituting these two evaluations back into our initial expression for $A(x)$, we conclude that

$$A(x) = x + x\left(A(x^2) + \frac{A(x)^2 - A(x^2)}{2}\right) = x + x\frac{A(x)^2 + A(x^2)}{2}.$$

Comments.

• An alternative way to view the pair of children is via a restricted version of the multiset construction from Question 3. The class $\mathcal{A}$ has specification

$$\mathcal{A} \leftrightharpoons \mathcal{Z} \cup (\mathcal{Z} \times \mathsf{MSet}_2(\mathcal{A})),$$

where the generating series for $\mathsf{MSet}_2(\mathcal{A})$ is $\frac{A(x)^2+A(x^2)}{2}$. It's possible to extract this formula from the general multiset construction from Question 3 using that

$$[y^2] \exp\left(\sum_{n\geq 1} \frac{A(x^n)y^n}{n}\right) = \frac{A(x)^2 + A(x^2)}{2}.$$

Question 5. Express each of these combinatorial classes using operations we know ($\mathcal{E}$, $\mathcal{Z}$, other explicit finite sets, $\cup$, $\times$, Seq, etc. You may also use MSet from earlier in this assignment even if you didn't do that question.)

(a) Ternary strings where every block of 2s is followed by at least one 0.

(b) Compositions with an odd number of parts and where each part is even.

(c) Rooted plane trees where each vertex has an even number of children.

(d) Rooted non-plane (order of the children is not distinguished) trees.

Solution.

(a) Let the alphabet be $\{0, 1, 2\}$, with each letter having weight 1. In any valid string, every maximal contiguous block of 2s must be followed by at least one 0. Associating each block of 2s with the first 0 that follows it, the string factors uniquely into a sequence of units: each unit is either a single 1, or a block of zero or more 2s followed by a 0 (representing a single 0 when the block of 2s is empty, and a block of 2s followed by a 0 otherwise). Thus, the class $\mathcal{S}$ of such strings satisfies

$$\mathcal{S} \leftrightharpoons \mathsf{Seq}\big(\{1\} \cup (\mathsf{Seq}(\{2\}) \times \{0\})\big).$$

(b) Let the weight of a composition be the sum of its parts, where each atomic unit has weight 1 represented by $\mathcal{Z}$. An even positive integer is a sequence of at least one pair of atoms, so the class of parts satisfies

$$\mathcal{P}_{\text{even}} \leftrightharpoons \mathcal{Z}^2 \times \mathsf{Seq}(\mathcal{Z}^2).$$

A composition with an odd number of parts consists of one part followed by an arbitrary sequence of pairs of parts, which gives

$$\mathcal{C} \leftrightharpoons \mathcal{P}_{\text{even}} \times \mathsf{Seq}(\mathcal{P}_{\text{even}} \times \mathcal{P}_{\text{even}}).$$

(c) Let the weight be the number of vertices, with each vertex represented by $\mathcal{Z}$. A rooted plane tree consists of a root vertex followed by an ordered sequence of child subtrees. Since each vertex must have an even number of children, the children form a sequence of pairs of trees, yielding

$$\mathcal{T} \leftrightharpoons \mathcal{Z} \times \mathsf{Seq}(\mathcal{T} \times \mathcal{T}).$$

(d) Let the weight be the number of vertices, with the root vertex represented by $\mathcal{Z}$. A rooted non-plane tree consists of a root vertex attached to an unordered collection of children subtrees. Because the order of children is not distinguished and repetition of isomorphic subtrees is allowed, the children form a finite multiset of trees, giving

$$\mathcal{T} \leftrightharpoons \mathcal{Z} \times \mathsf{MSet}(\mathcal{T}).$$

Comments.

• In part (b), one could also write $\mathcal{P}_{\text{even}} \leftrightharpoons \mathsf{Seq}_{\geq 1}(\mathcal{Z}^2)$ and $\mathcal{C} \leftrightharpoons \mathsf{Seq}_{\text{odd}}(\mathcal{P}_{\text{even}})$, using the notation $\mathsf{Seq}_{\text{condition}}$ introduced in Lecture 7.