Assignment 3

CO 330 (Fall 2026)

Due Friday, October 30, 2026 at 3pm Eastern time.

Instructions

To get full marks on this assignment you must do questions worth 12 marks total. Any additional questions you do beyond 12 marks-worth will not be counted. Each question has two possible point values, one if you would like it marked for correctness and one ("engagement") if you would like it marked for feedback after you have done some work on it. You will indicate which you want when you submit.

All answers should be justified unless specifically indicated otherwise.

This assignment should be done and submitted individually. You are welcome to discuss with your classmates, but you should not use generative AI for your ideas and you must write up your solutions independently (neither writing it with other people nor with tools such as generative AI). In your write up explicitly acknowledge any discussions or non-class resources and tools that you may have used.

This assignment should be submitted in crowdmark.

Questions

(1) **{4 marks graded for content, 2 marks graded for engagement}**

(a) Suppose $T(x)$ is a formal power series satisfying $T(x) = x\exp(T(x))$ and let c be any invertible element of the ring we are working over. Show that

$$[x^n]\exp(cT(x)) = \frac{c(c+n)^{n-1}}{n!}$$

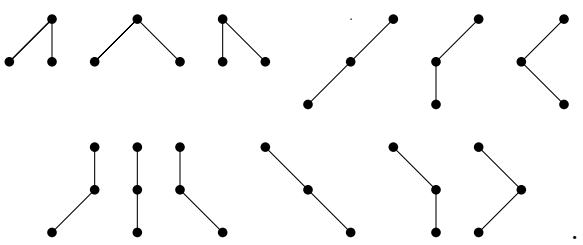
Hint: finally a case where you don't want $F(u) = u$ in the Lagrange Implicit Function Theorem.

(b) Let a, b and $a+b$ be invertible. Prove *Abel's extension of the binomial theorem*

$$(a+b)(a+b+n)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} a(a+k)^{k-1} b(b+n-k)^{n-k-1}$$

by calculating the coefficient of x^n in $\exp((a+b)T(x))$ in two different ways.

(2) **{4 marks graded for content, 2 marks graded for engagement}** Consider the class of rooted trees where the children at every vertex are designated either left, middle, or right, and some or all of them may be empty. For example there are 12 trees from this class with 3 vertices:



(a) Let $T(x,t)$ be the generating series for these trees where x counts the number of vertices and t counts the number of leaves. Explain why

$$T(x,t) = x(t + 3T(x,t) + 3T(x,t)^2 + T(x,t)^3)$$

(b) What is the average number of leaves among such trees with n vertices?

(3) **{4 marks graded for content, 2 marks graded for engagement}**

(a) Prove combinatorially

$$\binom{a+b}{b} = \sum_{j=0}^{a} \binom{c}{c-j}\binom{a+b-c}{j+b-c}$$

for integers $0 \leq a \leq c \leq b$.

(b) State and prove a q-binomial generalization to the identity in the previous part.

(4) **{4 marks graded for content, 2 marks graded for engagement}** We mentioned in class that if k and n are integers with $0 \leq k \leq n$ and $q = p^c$ is a prime power, then the number of k-dimensional subspaces of an n-dimensional vector space over the finite field with q elements is $\begin{bmatrix} n \\ k \end{bmatrix}_q$. Prove this fact.

(5) **{4 marks graded for content, 2 marks graded for engagement}** Problem 6 from `https://enumeration.ca/extensions/q-analogues/`.