# Solutions to Assignment 3
# CO 330 (Fall 2026)

**Question 1.**

**(a)** Suppose $T(x)$ is a formal power series satisfying $T(x) = x\exp(T(x))$ and let $c$ be any invertible element of the ring we are working over. Show that

$$[x^n]\exp(cT(x)) = \frac{c(c+n)^{n-1}}{n!}.$$

*Hint: finally a case where you don't want $F(u) = u$ in the Lagrange Implicit Function Theorem.*

**(b)** Let $a, b$ and $a+b$ be invertible. Prove *Abel's extension of the binomial theorem*

$$(a+b)(a+b+n)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} a(a+k)^{k-1} b(b+n-k)^{n-k-1}$$

by calculating the coefficient of $x^n$ in $\exp((a+b)T(x))$ in two different ways.

**Solution.**

**(a)** We apply the Lagrange Implicit Function Theorem. Let $G(u) = \exp(u)$ and $F(u) = \exp(cu)$, so that $T(x) = xG(T(x))$, the constant term $[u^0]G(u) = 1$ is invertible, and $F'(u) = c\exp(cu)$. For $n = 0$, we have $T(0) = 0$, which yields

$$[x^0]\exp(cT(x)) = F(0) = 1 = \frac{c(c+0)^{-1}}{0!}.$$

For $n \geq 1$, the Lagrange Implicit Function Theorem gives

$$\begin{aligned}
[x^n]\exp(cT(x)) &= \frac{1}{n}[u^{n-1}]F'(u)G(u)^n \\
&= \frac{c}{n}[u^{n-1}]\exp((c+n)u) \\
&= \frac{c}{n}\frac{(c+n)^{n-1}}{(n-1)!} \\
&= \frac{c(c+n)^{n-1}}{n!},
\end{aligned}$$

which completes the proof of part (a).

**(b)** We compute $[x^n]\exp((a+b)T(x))$ in two ways. First, applying part (a) directly with $c = a + b$ gives

$$[x^n]\exp((a+b)T(x)) = \frac{(a+b)(a+b+n)^{n-1}}{n!}.$$

Second, using $\exp((a+b)T(x)) = \exp(aT(x))\exp(bT(x))$ and applying part (a) with $c = a$ and $c = b$, we obtain

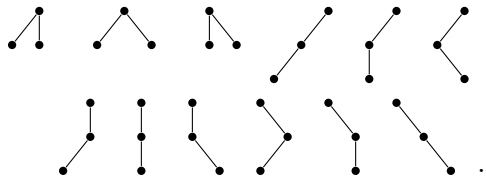
$$\begin{aligned}[x^n]\exp((a+b)T(x)) &= \sum_{k=0}^{n} \left([x^k]\exp(aT(x))\right)\left([x^{n-k}]\exp(bT(x))\right) \\ &= \sum_{k=0}^{n} \frac{a(a+k)^{k-1}}{k!}\frac{b(b+n-k)^{n-k-1}}{(n-k)!}.\end{aligned}$$

Equating the two expressions and multiplying both sides by $n!$ completes the proof.

**Comments.**

• It is important to verify the $n = 0$ case separately in part (a), since the formula $\frac{1}{n}[u^{n-1}]F'(u)G(u)^n$ in the Lagrange Implicit Function Theorem applies only for $n \geq 1$, and the $k = 0$ and $k = n$ terms in part (b) rely on the $n = 0$ case of part (a). Notice that the $n = 0$ case is also where we use the assumption that $c$ (and thus $a$, $b$, and $a + b$ in part (b)) is invertible, so that $c^{-1}$ is well-defined.

**Question 2.** Consider the class of rooted trees where the children at every vertex are designated either left, middle, or right, and some or all of them may be empty. For example there are 12 trees from this class with 3 vertices:



**(a)** Let $T(x,t)$ be the generating series for these trees where $x$ counts the number of vertices and $t$ counts the number of leaves. Explain why

$$T(x,t) = x(t + 3T(x,t) + 3T(x,t)^2 + T(x,t)^3).$$

**(b)** What is the average number of leaves among such trees with $n$ vertices?

**Solution.**

**(a)** Let $\mathcal{T}$ be the class of such rooted trees. For each $k \in \{0,1,2,3\}$, let $\mathcal{T}_k$ be the subclass of trees in $\mathcal{T}$ whose root has $k$ nonempty children, and let $T_k(x,t)$ be its generating series. Since the root has at most three children (left, middle, and right), we have the disjoint union

$$\mathcal{T} = \bigsqcup_{k=0}^{3} \mathcal{T}_k = \mathcal{T}_0 \sqcup \mathcal{T}_1 \sqcup \mathcal{T}_2 \sqcup \mathcal{T}_3.$$

The class $\mathcal{T}_0$ consists only of the single-vertex tree, whose root is a leaf, so $T_0(x,t) = xt$. For $k \in \{1,2,3\}$, a tree in $\mathcal{T}_k$ consists of a non-leaf root (contributing $x$) and $k$ subtrees from $\mathcal{T}$ placed in one of $\binom{3}{k}$ choices of positions. Aside from the root, the numbers of vertices and leaves of the tree equal the sums of the numbers of vertices and leaves of its $k$ subtrees, so each choice of positions contributes $xT(x,t)^k$ by the product lemma, giving $T_k(x,t) = \binom{3}{k} xT(x,t)^k$. Applying the sum lemma to the disjoint union above yields

$$\begin{aligned} T(x,t) &= T_0(x,t) + T_1(x,t) + T_2(x,t) + T_3(x,t) \\ &= xt + 3xT(x,t) + 3xT(x,t)^2 + xT(x,t)^3 \\ &= x\left(t + 3T(x,t) + 3T(x,t)^2 + T(x,t)^3\right). \end{aligned}$$

**(b)** Following Lecture 10, for $n \geq 1$, the total number of leaves across all trees in $\mathcal{T}$ with $n$ vertices is $[x^n]\left(\frac{\partial}{\partial t}T(x,t)\right)_{t=1}$, and the number of such trees is $[x^n]T(x,1)$, so the average number of leaves among trees in $\mathcal{T}$ with $n$ vertices is

$$\frac{[x^n]\left(\frac{\partial}{\partial t}T(x,t)\right)_{t=1}}{[x^n]T(x,1)}.$$

Rewriting the equation from part (a) as $T(x,t) = xG(T(x,t))$ with $G(u) = t - 1 + (1+u)^3$, the Lagrange Implicit Function Theorem with $F(u) = u$ gives

$$[x^n]T(x,t) = \frac{1}{n}[u^{n-1}]\left(t - 1 + (1+u)^3\right)^n.$$

For the denominator, evaluating at $t = 1$ and using the binomial theorem gives

$$[x^n]T(x,1) = \frac{1}{n}[u^{n-1}](1+u)^{3n} = \frac{1}{n}\binom{3n}{n-1}.$$

For the numerator, because differentiation with respect to $t$ and evaluation at $t = 1$ act coefficient-wise on powers of $x$ and powers of $u$, both operators commute with the coefficient extractions $[x^n]$ and $[u^{n-1}]$. Computing slowly, we obtain

$$\begin{aligned}
[x^n]\left(\frac{\partial}{\partial t}T(x,t)\right)_{t=1} &= \left(\frac{\partial}{\partial t}[x^n]T(x,t)\right)_{t=1} && \text{(commuting } [x^n]\text{)}\\
&= \left(\frac{\partial}{\partial t}\left(\frac{1}{n}[u^{n-1}]\left(t-1+(1+u)^3\right)^n\right)\right)_{t=1} && \text{(by LIFT)}\\
&= \frac{1}{n}[u^{n-1}]\left(\frac{\partial}{\partial t}\left(t-1+(1+u)^3\right)^n\right)_{t=1} && \text{(commuting } [u^{n-1}]\text{)}\\
&= \frac{1}{n}[u^{n-1}]\left(n\left(t-1+(1+u)^3\right)^{n-1}\right)_{t=1} && \text{(differentiating)}\\
&= \frac{1}{n}[u^{n-1}]n(1+u)^{3(n-1)} && \text{(evaluating at } t=1\text{)}\\
&= [u^{n-1}](1+u)^{3(n-1)}\\
&= \binom{3n-3}{n-1} && \text{(by the binomial theorem).}
\end{aligned}$$

Therefore, the average number of leaves among trees with $n$ vertices is

$$\frac{\binom{3n-3}{n-1}}{\frac{1}{n}\binom{3n}{n-1}} = \frac{n(2n+1)(2n)(2n-1)}{3n(3n-1)(3n-2)} = \frac{2n(2n+1)(2n-1)}{3(3n-1)(3n-2)}.$$

**Comments.**

• In part (a), one detail that almost everyone overlooked (perhaps because it feels like an excess of caution) is justifying why the product lemma applies. For instance, when $k = 3$, decomposing each tree $T \in \mathcal{T}_3$ into its root $r$ and its left, middle, and right subtrees $T_1, T_2, T_3 \in \mathcal{T}$ gives a bijection

$$\mathcal{T}_3 \leftrightharpoons \{r\} \times \mathcal{T} \times \mathcal{T} \times \mathcal{T}$$

sending $T$ to $(r, T_1, T_2, T_3)$. One should acknowledge that this bijection is weight preserving, that is, both parameters are additive, namely

$$|T| = |r| + |T_1| + |T_2| + |T_3|, \qquad \ell(T) = \ell(r) + \ell(T_1) + \ell(T_2) + \ell(T_3),$$

where $|\cdot|$ counts vertices and $\ell(\cdot)$ counts leaves, and we take $|r| = 1$ and $\ell(r) = 0$.

• In part (b), to apply LIFT when computing the numerator (with $t$ kept symbolic prior to differentiating and setting $t = 1$), we take $\mathbb{Q}[[t]]$ as the coefficient ring, viewing $T(x,t) \in \mathbb{Q}[[t]][[x]]$ and $G(u) = t - 1 + (1+u)^3 \in \mathbb{Q}[[t]][[u]]$ as univariate formal power series in $x$ and $u$, respectively, over $\mathbb{Q}[[t]]$.

• Before applying LIFT in part (b), make sure to explicitly state what series $G(u)$ is being used (for instance, write $T(x,t) = xG(T(x,t))$ with $G(u) = t - 1 + (1+u)^3$).

**Question 3.**

**(a)** Prove combinatorially

$$\binom{a+b}{b} = \sum_{j=0}^{a} \binom{c}{c-j}\binom{a+b-c}{j+b-c}$$

for integers $0 \le a \le c \le b$.

**(b)** State and prove a $q$-binomial generalization to the identity in the previous part.

**Solution.**

**(a)** For integers $m, n \ge 0$, let $\mathcal{L}(m,n)$ be the set of lattice paths from $(0,0)$ to $(m,n)$ taking unit east and north steps, so that $|\mathcal{L}(m,n)| = \binom{m+n}{n}$, and set $\mathcal{L} = \mathcal{L}(a,b)$. Every path $P \in \mathcal{L}$ has $a+b \ge c$ steps, so after its first $c$ steps it reaches a unique point $(j, c-j)$, where $j$ is the number of east steps among those first $c$ steps. Since $P$ takes $a$ east steps in total and $0 \le a \le c \le b$, we have $0 \le j \le a$ and $0 \le c-j \le b$. Letting $\mathcal{L}_j \subseteq \mathcal{L}$ denote the subset of paths reaching $(j, c-j)$ after $c$ steps gives the disjoint union

$$\mathcal{L} = \bigsqcup_{j=0}^{a} \mathcal{L}_j.$$

Splitting each path $P \in \mathcal{L}_j$ after $c$ steps gives a bijection

$$\mathcal{L}_j \leftrightharpoons \mathcal{L}(j, c-j) \times \mathcal{L}(a-j, j+b-c)$$

sending $P$ to $(P_1, P_2)$, where $P_1$ consists of the first $c$ steps of $P$ (taking $j$ east and $c-j$ north steps) and $P_2$ consists of the remaining $a+b-c$ steps translated to start at $(0,0)$ (taking $a-j$ east and $b-(c-j) = j+b-c$ north steps). Because $|\mathcal{L}(j,c-j)| = \binom{c}{c-j}$ and $|\mathcal{L}(a-j, j+b-c)| = \binom{a+b-c}{j+b-c}$, taking cardinalities in the disjoint union yields

$$\binom{a+b}{b} = |\mathcal{L}| = \sum_{j=0}^{a} |\mathcal{L}_j| = \sum_{j=0}^{a} \binom{c}{c-j}\binom{a+b-c}{j+b-c}.$$

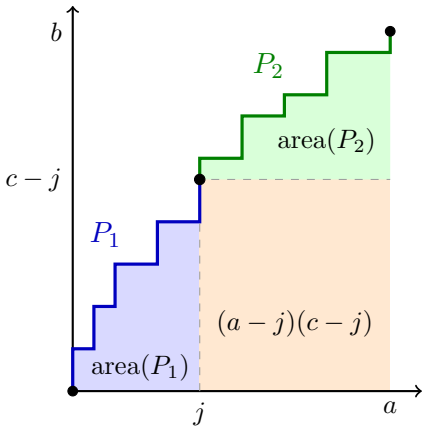
**(b)** For integers $0 \le a \le c \le b$, a $q$-analogue of the identity in part (a) is

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \sum_{j=0}^{a} q^{(a-j)(c-j)} \begin{bmatrix} c \\ c-j \end{bmatrix}_q \begin{bmatrix} a+b-c \\ j+b-c \end{bmatrix}_q.$$

By Lecture 14, for any integers $m, n \ge 0$, we have $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \sum_{P \in \mathcal{L}(m,n)} q^{\mathrm{area}(P)}$, where $\mathrm{area}(P)$ is the area between $P$ and the $x$-axis on $[0,m]$. For each $j \in \{0, 1, \ldots, a\}$, decomposing any path $P \in \mathcal{L}_j$ into $(P_1, P_2) \in \mathcal{L}(j, c-j) \times \mathcal{L}(a-j, j+b-c)$ as in part (a) partitions the region beneath $P$ into three parts:

- the region under $P_1$ on $[0, j]$, with area $\mathrm{area}(P_1)$;

- the rectangle $[j, a] \times [0, c-j]$, with area $(a-j)(c-j)$; and
- the region between the second subpath and $y = c - j$ on $[j, a]$, with area area$(P_2)$ after translation.




Consequently, we have area$(P) = (a-j)(c-j) + \text{area}(P_1) + \text{area}(P_2)$. Summing $q^{\text{area}(P)}$ over the disjoint union $\mathcal{L} = \bigsqcup_{j=0}^{a} \mathcal{L}_j$ and using the bijection for each $\mathcal{L}_j$ yields

$$\begin{aligned}
\begin{bmatrix} a+b \\ b \end{bmatrix}_q &= \sum_{P \in \mathcal{L}} q^{\text{area}(P)} = \sum_{j=0}^{a} \sum_{P \in \mathcal{L}_j} q^{\text{area}(P)} \\
&= \sum_{j=0}^{a} \sum_{(P_1,P_2) \in \mathcal{L}(j,c-j) \times \mathcal{L}(a-j,j+b-c)} q^{(a-j)(c-j)+\text{area}(P_1)+\text{area}(P_2)} \\
&= \sum_{j=0}^{a} q^{(a-j)(c-j)} \left( \sum_{P_1 \in \mathcal{L}(j,c-j)} q^{\text{area}(P_1)} \right) \left( \sum_{P_2 \in \mathcal{L}(a-j,j+b-c)} q^{\text{area}(P_2)} \right) \\
&= \sum_{j=0}^{a} q^{(a-j)(c-j)} \begin{bmatrix} c \\ c-j \end{bmatrix}_q \begin{bmatrix} a+b-c \\ j+b-c \end{bmatrix}_q,
\end{aligned}$$

completing the proof.

**Comments.**

- There is an analogous solution to both parts (a) and (b) using the formula for $\begin{bmatrix} n \\ k \end{bmatrix}_q$ from Lecture 13, though the resulting $q$-exponent calculation gets nastier.
- In part (b), notice how the diagram improves the clarity of the solution. While your reasoning should not rely on pictures, they can be good complements to written arguments.
- Another $q$-generalization that you can obtain is

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \sum_{j \in \mathbb{Z}} q^{j(j+b-c)} \begin{bmatrix} c \\ c-j \end{bmatrix}_q \begin{bmatrix} a+b-c \\ j+b-c \end{bmatrix}_q.$$

- A student came up with the following alternative solution, using the $q$-Binomial Theorem. It states that $\prod_{k=1}^{n}(1+q^k z) = \sum_{k \in \mathbb{Z}} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k$. We compute $[z^b] \prod_{k=1}^{a+b}(1+q^k z)$ in two

ways. Directly, the coefficient is

$$[z^b]\prod_{k=1}^{a+b}(1+q^k z) = q^{\binom{b+1}{2}}\begin{bmatrix} a+b \\ b \end{bmatrix}_q.$$

On the other hand, splitting the product after $c$ terms gives

$$\prod_{k=1}^{a+b}(1+q^k z) = \left(\prod_{k=1}^{c}(1+q^k z)\right)\left(\prod_{i=1}^{a+b-c}(1+q^i(q^c z))\right),$$

and applying the $q$-Binomial Theorem to each factor and extracting $[z^b]$ gives

$$\begin{aligned}
[z^b]\prod_{k=1}^{a+b}(1+q^k z) &= \sum_{u+v=b}\left([z^u]\prod_{k=1}^{c}(1+q^k z)\right)\left([z^v]\prod_{i=1}^{a+b-c}(1+q^i(q^c z))\right) \\
&= \sum_{u+v=b} q^{\binom{u+1}{2}}\begin{bmatrix} c \\ u \end{bmatrix}_q q^{\binom{v+1}{2}+cv}\begin{bmatrix} a+b-c \\ v \end{bmatrix}_q \\
&= \sum_{j} q^{\binom{c-j+1}{2}+\binom{j+b-c+1}{2}+c(j+b-c)}\begin{bmatrix} c \\ c-j \end{bmatrix}_q\begin{bmatrix} a+b-c \\ j+b-c \end{bmatrix}_q.
\end{aligned}$$

After some algebra, the exponent of $q$ simplifies to $\binom{b+1}{2} + j(j+b-c)$, and canceling $q^{\binom{b+1}{2}}$ from both expressions gives the desired identity.

**Question 4.** We mentioned in class that if $k$ and $n$ are integers with $0 \leq k \leq n$ and $q = p^c$ is a prime power, then the number of $k$-dimensional subspaces of an $n$-dimensional vector space over the finite field with $q$ elements is $\begin{bmatrix} n \\ k \end{bmatrix}_q$. Prove this fact.

**Solution.** Let $V$ be an $n$-dimensional vector space over the finite field $\mathbb{F}_q$, and let $\mathcal{I}_k$ denote the set of ordered linearly independent $k$-tuples $(v_1, \ldots, v_k)$ of vectors in $V$. We count $|\mathcal{I}_k|$ in two ways.

First, we choose the vectors $v_1, \ldots, v_k$ successively. For each $i \in \{1, \ldots, k\}$, once the linearly independent vectors $v_1, \ldots, v_{i-1}$ have been chosen, their span is an $(i-1)$-dimensional subspace of $V$ and therefore contains $q^{i-1}$ vectors. Because $|V| = q^n$, the vector $v_i$ can be any of the $q^n - q^{i-1}$ vectors outside $\operatorname{span}(v_1, \ldots, v_{i-1})$. Thus, the total number of ordered linearly independent $k$-tuples in $V$ is

$$|\mathcal{I}_k| = \prod_{i=0}^{k-1} (q^n - q^i).$$

Second, every ordered linearly independent $k$-tuple $(v_1, \ldots, v_k) \in \mathcal{I}_k$ spans a unique $k$-dimensional subspace $W \subseteq V$ and forms an ordered basis of $W$. Applying the previous count to $W$ (replacing $n$ with $k$), we see that each $k$-dimensional subspace $W$ has $\prod_{i=0}^{k-1}(q^k - q^i)$ ordered bases. Letting $N(n,k)$ denote the number of $k$-dimensional subspaces of $V$, we obtain

$$|\mathcal{I}_k| = N(n,k) \prod_{i=0}^{k-1} (q^k - q^i).$$

Equating the two expressions for $|\mathcal{I}_k|$ and solving for $N(n,k)$ yields

$$\begin{aligned} N(n,k) &= \frac{\prod_{i=0}^{k-1}(q^n - q^i)}{\prod_{i=0}^{k-1}(q^k - q^i)} = \frac{\prod_{i=0}^{k-1} q^i(q^{n-i} - 1)}{\prod_{i=0}^{k-1} q^i(q^{k-i} - 1)} \\ &= \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1} = \prod_{i=0}^{k-1} \frac{[n-i]_q}{[k-i]_q} = \frac{[n]_q!}{[k]_q![n-k]_q!} = \begin{bmatrix} n \\ k \end{bmatrix}_q, \end{aligned}$$

which completes the proof.

**Comments.**

• One thing to watch out for is explicitly explaining that the counting method used for the number of ordered linearly independent tuples in $\mathbb{F}_q^n$ is the same as the number of ordered linearly independent tuples in $\mathbb{F}_q^k$.

**Question 5.** Problem 6 from `https://enumeration.ca/extensions/q-analogues/`:

Fix a positive integer $t$ and recall that a multiset of size $n$ with $t$ types is a set containing $n$ of the numbers $\{1,\dots,t\}$ where numbers can appear multiple times. Define a parameter on the class of multisets with $t$ types and use it to prove the **Negative $q$-Binomial Theorem**

$$\frac{1}{(1-z)(1-qz)\cdots(1-q^{t-1}z)} = \sum_{n\geq 0} \begin{bmatrix} n+t-1 \\ t-1 \end{bmatrix}_q z^n.$$

**Solution.** Let $\mathcal{M}$ be the class of multisets with elements from $\{1,\dots,t\}$. For each $i \in \{1,\dots,t\}$, let $m_i \geq 0$ denote the multiplicity of $i$ in a multiset $M \in \mathcal{M}$, so that $|M| = \sum_{i=1}^{t} m_i$. We define the parameter $p\colon \mathcal{M} \to \mathbb{Z}_{\geq 0}$ by $p(M) = \sum_{x\in M}(x-1) = \sum_{i=1}^{t}(i-1)m_i$, and we compute the bivariate generating series $F(z,q) = \sum_{M\in\mathcal{M}} q^{p(M)} z^{|M|}$ in two ways.

First, every multiset $M \in \mathcal{M}$ corresponds uniquely to an element of $\mathsf{Seq}(\{1\}) \times \cdots \times \mathsf{Seq}(\{t\})$. Because each copy of type $i$ contributes 1 to $|M|$ and $i-1$ to $p(M)$, it carries the weight $q^{i-1}z$. By the product and sequence rules, we obtain

$$F(z,q) = \prod_{i=1}^{t} \sum_{m_i \geq 0} (q^{i-1}z)^{m_i} = \frac{1}{(1-z)(1-qz)\cdots(1-q^{t-1}z)}.$$

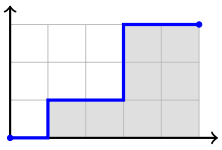
Second, for each $n \geq 0$, let $\mathcal{M}_n$ be the set of multisets in $\mathcal{M}$ of size $n$, and let $\mathcal{L}_n$ be the set of lattice paths from $(0,0)$ to $(n,t-1)$ taking unit east and north steps. We define a bijection $\phi\colon \mathcal{M}_n \to \mathcal{L}_n$ by sending each multiset $M$ with multiplicities $(m_1,\dots,m_t)$ to the path that takes $m_i$ east steps at height $i-1$ for each $i \in \{1,\dots,t\}$, separated by north steps. Equivalently, viewing a path as sequence of elements of $\{\rightarrow,\uparrow\}$, we have

$$\phi(M) = {\rightarrow}^{m_1}\uparrow{\rightarrow}^{m_2}\uparrow\cdots\uparrow{\rightarrow}^{m_t} = \underbrace{\rightarrow\cdots\rightarrow}_{m_1}\uparrow\underbrace{\rightarrow\cdots\rightarrow}_{m_2}\uparrow\cdots\uparrow\underbrace{\rightarrow\cdots\rightarrow}_{m_t}.$$

For example, for $t=4$, the multiset $M=\{1,2,2,4,4\}$ with multiplicities $(m_1,m_2,m_3,m_4) = (1,2,0,2)$ maps to

$$\phi(M) = \underbrace{\rightarrow}_{m_1=1}\uparrow\underbrace{\rightarrow\rightarrow}_{m_2=2}\uparrow\underbrace{\ }_{m_3=0}\uparrow\underbrace{\rightarrow\rightarrow}_{m_4=2} = {\rightarrow}\uparrow{\rightarrow}{\rightarrow}\uparrow\uparrow{\rightarrow}{\rightarrow}\ .$$

which is the lattice path



The $m_i$ east steps at height $i-1$ lie above a rectangle of area $(i-1)m_i$, so $\text{area}(\phi(M)) = \sum_{i=1}^{t}(i-1)m_i = p(M)$. Summing over all paths in $\mathcal{L}_n$ yields

$$[z^n]F(z,q) = \sum_{M\in\mathcal{M}_n} q^{p(M)} = \sum_{P\in\mathcal{L}_n} q^{\text{area}(P)} = \begin{bmatrix} n+t-1 \\ t-1 \end{bmatrix}_q.$$

Equating the coefficient of $z^n$ across both expressions for $F(z,q)$ completes the proof.

**Comments.**

• Although $\mathcal{L}_n$ and $\phi$ may seem unmotivated at first, once $p(M)$ reduces the problem to showing $\sum_{M\in\mathcal{M}_n} q^{p(M)} = \begin{bmatrix} n+t-1 \\ t-1 \end{bmatrix}_q$, the natural strategy is to pick a known combinatorial model for $\begin{bmatrix} n+t-1 \\ t-1 \end{bmatrix}_q$ (such as lattice paths $\mathcal{L}_n$ by area) and seek a weight-preserving bijection between $\mathcal{M}_n$ and that model.