Solutions to Assignment 4
CO 330 (Fall 2026)

Question 1. Write down the generating series for each of the following sets of partitions with each partition λ contributing $x^{n(\lambda)}y^{k(\lambda)}$ to the generating series.

(a) Partitions in which only parts divisible by 5 occur.

(b) Partitions in which odd parts occur at most twice.

(c) Partitions in which every even part occurs 0 times or an odd number of times and every odd part occurs either 0 or 4 times.

(d) Partitions for which parts greater than 6 must be distinct.

Solution. By the Template Theorem (Lecture 17), if $M_j \subseteq \mathbb{Z}_{\geq 0}$ is the set of allowed multiplicities for parts of size $j \geq 1$, the bivariate generating series is given by

$$\prod_{j=1}^{\infty} \sum_{m \in M_j} (yx^j)^m.$$

Applying this theorem to each condition gives the following:

(a) Only parts divisible by 5 occur, with no restriction on multiplicity, so $M_{5i} = \mathbb{Z}_{\geq 0}$ for all $i \geq 1$, and $M_j = \{0\}$ whenever $5 \nmid j$. By the Template Theorem, the generating series is

$$\prod_{i=1}^{\infty} \sum_{m=0}^{\infty} (yx^{5i})^m = \prod_{i=1}^{\infty} \frac{1}{1 - yx^{5i}}.$$

(b) Odd parts occur at most twice, so $M_{2i-1} = \{0, 1, 2\}$ for each $i \geq 1$. Even parts have unrestricted multiplicity, so $M_{2i} = \mathbb{Z}_{\geq 0}$ for each $i \geq 1$. By the Template Theorem, the generating series is

$$\prod_{i=1}^{\infty} (1 + yx^{2i-1} + y^2x^{4i-2}) \prod_{i=1}^{\infty} \frac{1}{1 - yx^{2i}} = \prod_{i=1}^{\infty} \frac{1 + yx^{2i-1} + y^2x^{4i-2}}{1 - yx^{2i}}.$$

(c) Every even part occurs zero times or an odd number of times, so $M_{2i} = \{0\} \cup \{2k + 1 : k \geq 0\}$ for each $i \geq 1$. The factor for parts of size $2i$ is

$$1 + \sum_{k=0}^{\infty} (yx^{2i})^{2k+1} = 1 + \frac{yx^{2i}}{1 - y^2x^{4i}} = \frac{1 + yx^{2i} - y^2x^{4i}}{1 - y^2x^{4i}}.$$

Every odd part occurs zero or four times, so $M_{2i-1} = \{0, 4\}$ for each $i \geq 1$, giving the factor $1 + (yx^{2i-1})^4 = 1 + y^4x^{8i-4}$. By the Template Theorem, the generating series is

$$\prod_{i=1}^{\infty} \left(1 + \frac{yx^{2i}}{1 - y^2x^{4i}}\right)(1 + y^4x^{8i-4}) = \prod_{i=1}^{\infty} \frac{1 + yx^{2i} - y^2x^{4i}}{1 - y^2x^{4i}}(1 + y^4x^{8i-4}).$$

(d) Parts up to size 6 have unrestricted multiplicity, so $M_j = \mathbb{Z}_{\geq 0}$ for $1 \leq j \leq 6$. Parts greater than 6 must be distinct, so $M_j = \{0, 1\}$ for $j \geq 7$. By the Template Theorem, the generating series is

$$\left(\prod_{j=1}^{6} \frac{1}{1 - yx^j}\right) \prod_{j=7}^{\infty}(1 + yx^j).$$

Comments.

- Solutions can be written either by indexing over all parts $j \geq 1$ with parity/divisibility conditions specified on the product (such as $\prod_{j \text{ odd}}$ or $\prod_{5|j}$), or by re-indexing with $j = 2i, 2i - 1, 5i$ for $i \geq 1$. Both formulations are fully correct.
- In part **(b)**, expressing the polynomial factor as $1 + yx^{2i-1} + y^2x^{4i-2}$ or in the closed geometric sum form $\frac{1-y^3x^{6i-3}}{1-yx^{2i-1}}$ are both completely standard and acceptable.
- In part **(c)**, you might write the even-part factor as $1 + \frac{yx^{2i}}{1-y^2x^{4i}}$ or combine it into the single fraction $\frac{1+yx^{2i}-y^2x^{4i}}{1-y^2x^{4i}}$; both forms should receive full credit.

Question 2. Let $\mathcal{Y}_{a,b}$ be the set of partitions with first part at most a and with at most b parts. Prove that

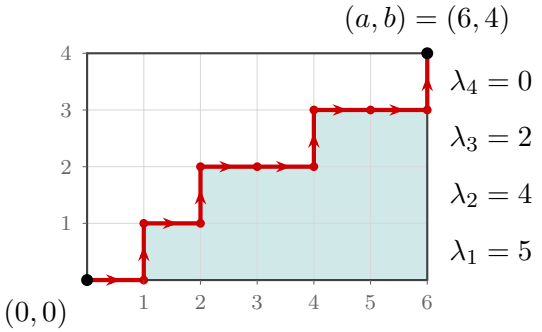
$$\sum_{\lambda \in \mathcal{Y}_{a,b}} q^{n(\lambda)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q.$$

Solution via bijection. Recall from Lecture 14 that

$$\sum_{P \in L(a,b)} q^{\text{area}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q,$$

where $L(a,b)$ is the set of lattice paths from $(0,0)$ to (a,b) consisting of a steps $\rightarrow$ and b steps $\uparrow$, and area(P) is the area bounded between P and the x-axis. We prove the identity by constructing a weight-preserving bijection $\phi\colon \mathcal{Y}_{a,b} \to L(a,b)$ satisfying area$(\phi(\lambda)) = n(\lambda)$. Formally, integer partitions do not contain trailing zeros; by convention, for any partition $\lambda \in \mathcal{Y}_{a,b}$ having $k = \ell(\lambda) \leq b$ parts, we pad with $b-k$ trailing zeros to write $\lambda = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) = (\lambda_1, \ldots, \lambda_b)$ with $a \geq \lambda_1 \geq \cdots \geq \lambda_b \geq 0$.

Pictorially, inside an $a \times b$ bounding box $[0,a] \times [0,b]$, draw the Ferrers diagram of λ with rows of lengths $\lambda_1, \ldots, \lambda_b$ placed from bottom to top and justified against the right boundary $x = a$ (with empty rows of length zero at the top whenever $\ell(\lambda) < b$). Tracing the left and top boundary of these rows from $(0,0)$ to (a,b) produces a lattice path in $L(a,b)$. The unit squares of the Ferrers diagram are precisely the squares lying beneath the path, giving area$(P) = n(\lambda)$. For example, for $(a,b) = (6,4)$ and $\lambda = (5,4,2) \in \mathcal{Y}_{6,4}$, we write $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (5,4,2,0)$, yielding the path shown below with area area$(P) = n(\lambda) = 11$:



Now we describe it formally. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_b) \in \mathcal{Y}_{a,b}$ with $a \geq \lambda_1 \geq \cdots \geq \lambda_b \geq 0$, we define

$$\phi(\lambda) = \rightarrow^{a-\lambda_1} \uparrow \rightarrow^{\lambda_1-\lambda_2} \uparrow \cdots \uparrow \rightarrow^{\lambda_{b-1}-\lambda_b} \uparrow \rightarrow^{\lambda_b}.$$

Because $a \geq \lambda_1 \geq \cdots \geq \lambda_b \geq 0$, each exponent is a nonnegative integer, and the total number of horizontal steps is

$$(a-\lambda_1) + (\lambda_1 - \lambda_2) + \cdots + (\lambda_{b-1} - \lambda_b) + \lambda_b = a.$$

Since there are b steps $\uparrow$ and a steps $\rightarrow$, we have $\phi(\lambda) \in L(a,b)$. Notice that when λ has $k < b$ parts, $\lambda_b = 0$ and the path ends with $\rightarrow^0 = \epsilon$ after the final $\uparrow$, so it terminates by stepping vertically along the right boundary $x = a$.

Conversely, given any path $P \in L(a,b)$, we define $\psi(P) = (\mu_1, \ldots, \mu_b)$, where μ_i is the number of $\rightarrow$ steps following the i-th $\uparrow$ step (ordered from bottom to top). Because every $\rightarrow$ step occurring after the $(i+1)$-st $\uparrow$ step also occurs after the i-th $\uparrow$ step, we have $a \geq \mu_1 \geq \cdots \geq \mu_b \geq 0$, so $\psi(P) \in \mathcal{Y}_{a,b}$. It is straightforward to check that ϕ and ψ are inverses of each other. To see that ϕ is weight-preserving, observe that the region beneath P decomposes into b horizontal strips: for each $i \in \{1, \ldots, b\}$, the region to the right of the i-th $\uparrow$ step (between heights $i-1$ and i) extends from the path to the right wall $x = a$, having height 1 and width λ_i, and so has area λ_i. Summing over all b vertical steps yields

$$\operatorname{area}(\phi(\lambda)) = \sum_{i=1}^{b} \lambda_i = n(\lambda).$$

Therefore, ϕ is a weight-preserving bijection, which proves

$$\sum_{\lambda \in \mathcal{Y}_{a,b}} q^{n(\lambda)} = \sum_{P \in L(a,b)} q^{\operatorname{area}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q.$$

Solution via recursion. Let $F(a,b) = \sum_{\lambda \in \mathcal{Y}_{a,b}} q^{n(\lambda)}$. We prove that $F(a,b) = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$ by induction on $a+b$. For the base cases, if $a = 0$ or $b = 0$, then $\mathcal{Y}_{a,b}$ contains only the empty partition of size 0, so $F(0,b) = 1 = \begin{bmatrix} b \\ 0 \end{bmatrix}_q$ and $F(a,0) = 1 = \begin{bmatrix} a \\ a \end{bmatrix}_q$.

Now suppose $a, b \geq 1$ and the identity holds for all smaller values of $a+b$. We partition $\mathcal{Y}_{a,b}$ into two disjoint subsets according to whether the first part λ_1 is less than a or equal to a:

- If $\lambda_1 < a$ (including $\lambda = \emptyset$), then every part is at most $a-1$, so $\lambda \in \mathcal{Y}_{a-1,b}$. These partitions contribute $F(a-1,b)$.
- If $\lambda_1 = a$, removing the first part leaves a partition in $\mathcal{Y}_{a,b-1}$ of size $n(\lambda) - a$. Conversely, prepending a part of size a to any partition in $\mathcal{Y}_{a,b-1}$ yields a unique partition in $\mathcal{Y}_{a,b}$ with first part a. These partitions contribute $q^a F(a,b-1)$.

Combining both cases and applying the inductive hypothesis gives

$$F(a,b) = F(a-1,b) + q^a F(a,b-1) = \begin{bmatrix} a+b-1 \\ a-1 \end{bmatrix}_q + q^a \begin{bmatrix} a+b-1 \\ a \end{bmatrix}_q = \begin{bmatrix} a+b \\ a \end{bmatrix}_q,$$

where the final equality is the q-Pascal recurrence. By induction, $F(a,b) = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$ for all nonnegative integers a and b.

Question 3. Let $pe(n)$ be the number of partitions of n with an even number of parts, and let $po(n)$ be the number of partitions of n with an odd number of parts. Let $od(n)$ be the number of partitions of n which have odd and distinct parts. Show that for all $n \in \mathbb{Z}_{\geq 0}$,

$$pe(n) - po(n) = (-1)^n od(n).$$

Solution. By the Template Theorem (Lecture 17), the bivariate generating series for the class $\mathcal{P}$ of all integer partitions is

$$P(x,y) = \sum_{\lambda \in \mathcal{P}} x^{n(\lambda)} y^{k(\lambda)} = \prod_{j=1}^{\infty} \frac{1}{1-yx^j}.$$

Setting $y = -1$, each partition λ contributes $(-1)^{k(\lambda)} x^{n(\lambda)}$, which is $x^{n(\lambda)}$ if λ has an even number of parts and $-x^{n(\lambda)}$ if λ has an odd number of parts. Hence,

$$\sum_{n=0}^{\infty} \big(pe(n) - po(n)\big) x^n = P(x,-1) = \prod_{j=1}^{\infty} \frac{1}{1+x^j}.$$

Similarly, by the Template Theorem with $M_{2k-1} = \{0,1\}$ and $M_{2k} = \{0\}$ for all $k \geq 1$, the generating series for partitions with odd and distinct parts is

$$OD(x) = \sum_{n=0}^{\infty} od(n) x^n = \prod_{k=1}^{\infty} (1 + x^{2k-1}).$$

Substituting $-x$ for x, we have

$$\sum_{n=0}^{\infty} (-1)^n od(n) x^n = OD(-x) = \prod_{k=1}^{\infty} \left(1 + (-x)^{2k-1}\right) = \prod_{k=1}^{\infty} (1 - x^{2k-1}).$$

To see that the two infinite products are equal in $\mathbb{Z}[[x]]$, we use $1 + x^j = \frac{1-x^{2j}}{1-x^j}$ to write

$$\prod_{j=1}^{\infty} \frac{1}{1+x^j} = \prod_{j=1}^{\infty} \frac{1-x^j}{1-x^{2j}} = \frac{\prod_{j=1}^{\infty}(1-x^j)}{\prod_{k=1}^{\infty}(1-x^{2k})} = \prod_{k=1}^{\infty} (1 - x^{2k-1}),$$

where the even-indexed factors $(1-x^{2k})$ cancel from the numerator and denominator. Equating the coefficient of x^n on both sides gives $pe(n) - po(n) = (-1)^n od(n)$ for all $n \in \mathbb{Z}_{\geq 0}$.

Comments.

• One student came up with a direct bijective proof via a sign-reversing involution on $\mathcal{P}_n \setminus \mathcal{OD}_n$: let x be the largest even part and k be the largest repeated part (whichever exist); if $2k > x$ (or x does not exist), merge two copies of k into $2k$, and otherwise split x into two copies of $x/2$. This function is an involution and it changes the number of parts by ± 1, giving a bijection between $\mathcal{PE}_n \setminus \mathcal{OD}_n$ and $\mathcal{PO}_n \setminus \mathcal{OD}_n$. The result follows by analyzing the different parities of n.

Question 4. Make another problem like chapter 9 question 4 or question 5 from `https://melczer.ca/330/WagnerNotes.pdf` or equivalently like question 2 or question 3 of `https://enumeration.ca/extensions/partitions/`, and also write a solution to your problem. That is, find two classes of partitions which are equinumerous by the template theorem and some algebraic manipulations and justify that this is true. We can try as a class to find as many different examples as we can.

Solution. Here are several examples of equinumerous partition classes found by the class:

• (Glaisher's theorem) For any fixed integer $r \geq 2$, the class of partitions in which each part occurs at most $r-1$ times and the class of partitions with no part divisible by r both have generating function

$$\prod_{j=1}^{\infty} \left(1 + x^j + \cdots + x^{(r-1)j}\right) = \prod_{r \nmid j} \frac{1}{1-x^j}.$$

• The class of partitions in which even parts occur at most three times and odd parts occur any number of times and the class of partitions with no part divisible by 8 both have generating function

$$\left(\prod_{j \text{ odd}} \frac{1}{1-x^j}\right) \prod_{j \text{ even}} \left(1 + x^j + x^{2j} + x^{3j}\right) = \prod_{8 \nmid j} \frac{1}{1-x^j}.$$

• The class of partitions in which parts congruent to $1 \pmod 4$ occur at most once and the class of partitions with no parts congruent to $2 \pmod 8$ both have generating function

$$\left(\prod_{j \equiv 1 \pmod 4} (1+x^j)\right) \prod_{j \not\equiv 1 \pmod 4} \frac{1}{1-x^j} = \prod_{j \not\equiv 2 \pmod 8} \frac{1}{1-x^j}.$$

• The class of partitions in which parts greater than 6 are distinct and the class of partitions with no even parts greater than or equal to 14 both have generating function

$$\left(\prod_{j=1}^{6} \frac{1}{1-x^j}\right) \prod_{j=7}^{\infty} (1+x^j) = \left(\prod_{j=1}^{13} \frac{1}{1-x^j}\right) \prod_{k=7}^{\infty} \frac{1}{1-x^{2k+1}}.$$

• The class of partitions in which each part occurs 0, 1, 6, or 7 times and the class of partitions with no parts congruent to 2 or $4 \pmod 6$ and where parts divisible by 6 occur at most once both have generating function

$$\prod_{j=1}^{\infty} \left(1 + x^j + x^{6j} + x^{7j}\right) = \left(\prod_{j \not\equiv 0,2,4 \pmod 6} \frac{1}{1-x^j}\right) \prod_{6 \mid j} (1+x^j).$$

• The class of partitions in which odd parts occur an even number of times and even parts occur 0 or $k \equiv 2 \pmod 3$ times, and the class of partitions in which each part has multiplicity 0, 2, $6m+4$, or $6m+6$ ($m \geq 0$), both have generating function

$$\left(\prod_{j \text{ odd}} \frac{1}{1-x^{2j}}\right) \prod_{j \text{ even}} \left(1 + \frac{x^{2j}}{1-x^{3j}}\right) = \prod_{j=1}^{\infty} \left(1 + x^{2j} + \frac{x^{4j}}{1-x^{6j}} + \frac{x^{6j}}{1-x^{6j}}\right).$$

Question 5. The notes by David Wagner, `https://melczer.ca/330/WagnerNotes.pdf`, in chapter 10 contain a proof of the Euler pentagonal number theorem via a sign-reversing involution on certain partitions. The proof is 5.5 pages long (with some details left to exercises, but also with a full page example figure included in the proof). Read the proof and summarize the *key ideas* in about half a page. You are not trying to give a re-proof, or necessarily make fully rigorous statements, but you are tasked with communicating what the key ideas are and how they fit together.

Euler's Pentagonal Number Theorem.

$$\prod_{j=1}^{\infty}(1-x^j) = \sum_{h=-\infty}^{\infty}(-1)^h x^{h(3h-1)/2}.$$

Solution. The idea of this proof is to first simplify the expression until we are just counting some set of objects that we know about, and then use clever combinatorics to give a more explicit formula. In a bit more detail, let $\mathcal{D}$ be the class of partitions with distinct parts. By the Template Theorem, we have

$$\sum_{\lambda\in\mathcal{D}} x^{n(\lambda)}y^{k(\lambda)} = \prod_{j=1}^{\infty}(1+x^j y).$$

Setting $y=-1$ gives us $\prod_{j=1}^{\infty}(1-x^j)$, so we need only show that the coefficients of the LHS are given by the pentagonal numbers.

To do this, notice that in the sum $\sum_{\lambda\in\mathcal{D}} x^{n(\lambda)}(-1)^{k(\lambda)}$, the coefficient of x^n simply counts the difference between the number of partitions of n in $\mathcal{D}$ with an even number of parts and those with an odd number of parts. So we need to find this difference.

How we do this is by defining a function that is "almost a bijection" between the two sets: a function on $\mathcal{D}$ that is its own inverse, and almost always sends a partition with an even number of parts to one with an odd number of parts and vice versa. Such a map is called a sign-reversing involution, and with it, we need only count the fixed points of this map, because all paired partitions cancel.

The actual map is defined by placing X's and O's in the Ferrers diagram of the partition, and depending on how these X's and O's interact, we construct a new partition. It turns out that the fixed points of this map are precisely those whose sizes are the pentagonal numbers, with each pentagonal number appearing once (with sign $(-1)^k$). This completes the proof.

Comments.

- It is very easy to get bogged down in the details here. You should give an overview of what is happening in the proof, rather than focusing on any details. Some examples are given below.
- Try not to give exact details on the definition of what a sign reversing involution is.
- Try not to give exact details for how the X's and O's are placed, and how the map is precisely defined.