Solutions to Assignment 5
CO 330 (Fall 2026)

Question 1. Let $\mathcal{A}$ be the set of partitions where each part occurs at most 3 times and let $\mathcal{B}$ be the set of partitions where each even part occurs at most once (and no constraint on odd parts). Showing equinumerosity of $\mathcal{A}$ and $\mathcal{B}$ via generating series is one of the problems of the form of assignment 4 question 4. For this question, give a weight-preserving bijection between these two sets.

Solutions. In one way or another, all of the solutions that students wrote rely on the bijection from Lecture 18 between partitions into distinct parts and partitions into odd parts: repeatedly halve even parts $2k \mapsto (k, k)$ until all parts are odd, and in reverse, repeatedly merge equal pairs $(k, k) \mapsto 2k$ until all parts are distinct.

For the three solutions, we let $\mathcal{P}$, $\mathcal{O}$, and $\mathcal{E}$ denote the sets of all partitions, partitions into odd parts, and partitions into even parts, respectively, and we let $\mathcal{M}_S$ denote the set of partitions in $\mathcal{M}$ where every part has multiplicity in S. For example, $\mathcal{A} = \mathcal{P}_{\{0,1,2,3\}}$ and $\mathcal{B} \cong \mathcal{O} \times \mathcal{E}_{\{0,1\}}$.

First solution. We construct a weight-preserving bijection from $\mathcal{A}$ to $\mathcal{B}$ via the chain of bijections

$$\mathcal{A} = \mathcal{P}_{\{0,1,2,3\}} \longrightarrow \mathcal{O}_{\{0,1,2,3\}} \times \mathcal{E}_{\{0,1\}} \times \mathcal{E}_{\{0,2\}} \longrightarrow \mathcal{O}_{\{0,1,2,3\}} \times \mathcal{E}_{\{0,1\}} \times \mathcal{O}_{4\mathbb{N}} \longrightarrow \mathcal{B}.$$

First, we map $\lambda \in \mathcal{P}_{\{0,1,2,3\}}$ to $(\alpha, \beta, \gamma) \in \mathcal{O}_{\{0,1,2,3\}} \times \mathcal{E}_{\{0,1\}} \times \mathcal{E}_{\{0,2\}}$ by placing all odd parts of λ into α and decomposing the multiplicity in $\{0, 1, 2, 3\}$ of each even part of λ uniquely as a number in $\{0, 1\}$ (its multiplicity in β) plus a number in $\{0, 2\}$ (its multiplicity in γ); in reverse, we recover λ by combining the parts of α, β, and γ.

Next, we map $(\alpha, \beta, \gamma) \in \mathcal{O}_{\{0,1,2,3\}} \times \mathcal{E}_{\{0,1\}} \times \mathcal{E}_{\{0,2\}}$ to $(\alpha, \beta, \omega) \in \mathcal{O}_{\{0,1,2,3\}} \times \mathcal{E}_{\{0,1\}} \times \mathcal{O}_{4\mathbb{N}}$ by leaving α and β unchanged and mapping γ to ω as follows. Every even positive integer can be written uniquely as $2^s r$ with $s \geq 1$ and r odd; we obtain ω from γ by replacing each pair $(2^s r, 2^s r)$ with 2^{s+1} copies of r (since $s \geq 1$, the count 2^{s+1} is divisible by 4, so $\omega \in \mathcal{O}_{4\mathbb{N}}$). Conversely, in $\omega \in \mathcal{O}_{4\mathbb{N}}$, each odd part r has multiplicity $4c_r$, which expands uniquely in binary as $4c_r = \sum_{s \geq 1} b_s 2^{s+1}$ with $b_s \in \{0, 1\}$, and we recover $\gamma \in \mathcal{E}_{\{0,2\}}$ by including the pair $(2^s r, 2^s r)$ whenever $b_s = 1$. Weight is preserved because $2 \cdot 2^s r = 2^{s+1} r$, and uniqueness of both representations guarantees that these maps are well-defined and mutual inverses.

(An equivalent way to define this bijection is to repeatedly halve pairs of equal even parts $(2k, 2k) \mapsto (k, k, k, k)$ until all parts are odd, and in reverse to repeatedly merge four equal parts $(k, k, k, k) \mapsto (2k, 2k)$ until every part occurs fewer than four times.)

Finally, we map $(\alpha, \beta, \omega) \in \mathcal{O}_{\{0,1,2,3\}} \times \mathcal{E}_{\{0,1\}} \times \mathcal{O}_{4\mathbb{N}}$ to $(\delta, \beta) \in \mathcal{O} \times \mathcal{E}_{\{0,1\}}$ by combining α and ω into δ, and then combine δ and β into a single partition $\mu \in \mathcal{B}$; in reverse, given $\mu \in \mathcal{B}$, we separate its even parts into β and its odd parts into δ, and decompose the multiplicity of each odd part in δ uniquely by the division algorithm as a remainder in $\{0, 1, 2, 3\}$ (its multiplicity in α) plus a multiple of 4 (its multiplicity in ω).

Each of these bijections preserves the weight of the partition, and composing them yields a weight-preserving bijection from $\mathcal{A}$ to $\mathcal{B}$.

For example, tracing $\lambda = (8, 6, 4^2, 2^3, 1) \in \mathcal{A}$ (of weight $n = 29$) through the chain gives

$$\underbrace{(8, 6, 4^2, 2^3, 1)}_{\mathcal{A}} \mapsto \underbrace{\big((1),\, (8, 6, 2),\, (4^2, 2^2)\big)}_{\mathcal{O}_{\{0,1,2,3\}} \times \mathcal{E}_{\{0,1\}} \times \mathcal{E}_{\{0,2\}}} \mapsto \underbrace{\big((1),\, (8, 6, 2),\, (1^{12})\big)}_{\mathcal{O}_{\{0,1,2,3\}} \times \mathcal{E}_{\{0,1\}} \times \mathcal{O}_{4\mathbb{N}}} \mapsto \underbrace{(8, 6, 2, 1^{13})}_{\mathcal{B}}.$$

Sketch of second solution. We obtain a second weight-preserving bijection from $\mathcal{A}$ to $\mathcal{B}$ via the chain

$$\mathcal{A} = \mathcal{P}_{\{0,1,2,3\}} \;\longrightarrow\; \mathcal{P}_{\{0,1\}} \times \mathcal{P}_{\{0,2\}} \;\longrightarrow\; \mathcal{O} \times \mathcal{E}_{\{0,1\}} \;\cong\; \mathcal{B}.$$

For the first map, split every multiplicity in $\{0, 1, 2, 3\}$ uniquely into $\{0, 1\} + \{0, 2\}$ to separate $\mathcal{A}$ into $\mathcal{P}_{\{0,1\}} \times \mathcal{P}_{\{0,2\}}$ (in reverse, combine the parts). Next, map $\mathcal{P}_{\{0,1\}} \to \mathcal{O}$ directly by repeatedly halving even parts until all parts are odd (in reverse, merge equal pairs), and map $\mathcal{P}_{\{0,2\}} \to \mathcal{E}_{\{0,1\}}$ by merging each pair $(j, j) \mapsto 2j$ (in reverse, halve each even part $2j \mapsto (j, j)$). Finally, combine the odd and even parts of $\mathcal{O} \times \mathcal{E}_{\{0,1\}}$ into a single partition in $\mathcal{B}$ (in reverse, separate into odd and even parts).

For example, tracing the same partition $\lambda = (8, 6, 4^2, 2^3, 1) \in \mathcal{A}$ (of weight $n = 29$) through this chain gives

$$\underbrace{(8, 6, 4^2, 2^3, 1)}_{\mathcal{A}} \mapsto \underbrace{\big((8, 6, 2, 1),\, (4^2, 2^2)\big)}_{\mathcal{P}_{\{0,1\}} \times \mathcal{P}_{\{0,2\}}} \mapsto \underbrace{\big((3^2, 1^{11}),\, (8, 4)\big)}_{\mathcal{O} \times \mathcal{E}_{\{0,1\}}} \mapsto \underbrace{(8, 4, 3^2, 1^{11})}_{\mathcal{B}}.$$

Sketch of third solution. Let $\mathcal{C}$ be the set of partitions with no parts divisible by 4, and let $2\mathcal{O}$ be the set of partitions into parts of the form $2r$ with r odd. We obtain a third weight-preserving bijection from $\mathcal{A}$ to $\mathcal{B}$ via the chain

$$\mathcal{A} = \mathcal{P}_{\{0,1,2,3\}} \;\longrightarrow\; \mathcal{C} \;\cong\; \mathcal{O} \times 2\mathcal{O} \;\longrightarrow\; \mathcal{O} \times \mathcal{E}_{\{0,1\}} \;\cong\; \mathcal{B}.$$

For the first map $\mathcal{A} \to \mathcal{C}$, repeatedly split each part divisible by 4 via $4k \mapsto (k, k, k, k)$ until no part is divisible by 4 (in reverse, repeatedly merge four equal parts $(k, k, k, k) \mapsto 4k$ until every part occurs at most three times, corresponding to the base-4 expansion of each multiplicity). Since every part in $\mathcal{C}$ is either odd or twice an odd integer, separating these two types identifies $\mathcal{C}$ with $\mathcal{O} \times 2\mathcal{O}$. Next, leave the component in $\mathcal{O}$ unchanged and map $2\mathcal{O}$ to $\mathcal{E}_{\{0,1\}}$ using the inverse bijection $\mathcal{O} \to \mathcal{P}_{\{0,1\}}$ scaled by 2: repeatedly merge equal pairs $(2k, 2k) \mapsto 4k$ until all even parts are distinct (in reverse, repeatedly halve even parts divisible

by 4 until every even part is twice an odd integer). Finally, combine the parts of the pair in $\mathcal{O} \times \mathcal{E}_{\{0,1\}}$ into a single partition in $\mathcal{B}$.

For example, tracing the same partition $\lambda = (8, 6, 4^2, 2^3, 1) \in \mathcal{A}$ (of weight $n = 29$) through this chain gives

$$\underbrace{(8, 6, 4^2, 2^3, 1)}_{\mathcal{A}} \mapsto \underbrace{(6, 2^7, 1^9)}_{\mathcal{C}} \mapsto \underbrace{\big((1^9),\, (6, 2^7)\big)}_{\mathcal{O} \times 2\mathcal{O}} \mapsto \underbrace{\big((1^9),\, (8, 6, 4, 2)\big)}_{\mathcal{O} \times \mathcal{E}_{\{0,1\}}} \mapsto \underbrace{(8, 6, 4, 2, 1^9)}_{\mathcal{B}}.$$

Comments.

• It can be helpful to define "middle steps" in the bijections, to make each individual map easier to describe and to verify that it is well-defined and invertible. You also gain the advantage of being able to precisely describe the intermediate sets (which are also equinumerous with $\mathcal{A}$ and $\mathcal{B}$).

Question 2. Question 7 from `https://enumeration.ca/extensions/partitions/`:

Fix positive integers m and n. A *monotonically increasing function* from $[n]$ to $[m]$ is any function $\phi\colon \{1,\dots,n\} \to \{1,\dots,m\}$ such that $\phi(k) \le \phi(k+1)$ for all $k = 1, 2, \dots, n-1$. Give a bijection between the set of monotonic increasing functions from $[n]$ to $[m]$ and partitions with at most n parts where each part is at most $m-1$ (including the empty partition).

Solution. Let $\mathcal{F}_{n,m}$ be the set of monotonically increasing functions $\phi\colon [n] \to [m]$, and let $\mathcal{Y}_{m-1,n}$ be the set of partitions with at most n parts, each at most $m-1$. As in Assignment 4 Question 2, we pad each $\lambda \in \mathcal{Y}_{m-1,n}$ with trailing zeros to write $\lambda = (\lambda_1, \dots, \lambda_n)$ with $m-1 \ge \lambda_1 \ge \cdots \ge \lambda_n \ge 0$.

Define $\Phi\colon \mathcal{F}_{n,m} \to \mathcal{Y}_{m-1,n}$ and $\Psi\colon \mathcal{Y}_{m-1,n} \to \mathcal{F}_{n,m}$ by

$$\Phi(\phi) = \big(m-\phi(1),\ m-\phi(2),\ \dots,\ m-\phi(n)\big) \qquad \text{and} \qquad \Psi(\lambda)(k) = m - \lambda_k \quad (1 \le k \le n),$$

dropping trailing zeros from $\Phi(\phi)$ to form a standard partition. Because $1 \le \phi(1) \le \cdots \le \phi(n) \le m$ if and only if $m-1 \ge m-\phi(1) \ge \cdots \ge m-\phi(n) \ge 0$, both maps are well-defined, and $\Psi(\Phi(\phi))(k) = m - (m - \phi(k)) = \phi(k)$ and $\Phi(\Psi(\lambda))_k = m - (m - \lambda_k) = \lambda_k$, so Φ and Ψ are mutual inverses.

Comments.

• Another equally valid bijection is $\widetilde{\Phi}(\phi) = (\phi(n) - 1, \phi(n-1) - 1, \dots, \phi(1) - 1)$, dropping trailing zeros to form a partition.

Question 3. *Note that this question cannot be submitted for engagement!* The two sets of notes we are using (`https://enumeration.ca/extensions/labelled-constructions/` and `https://melczer.ca/330/WagnerNotes.pdf` chapter 11), use different notation for operators/constructions on labelled combinatorial classes. Make a dictionary relating the notation between the two sources, that is, for each construction give a translation for what it is in the notation of the one source and in the notation of the other source. If there are slight discrepancies, give the translation that works the bulk of the time but also clarify how the edge cases differ.

Solution. In addition to comparing enumeration.ca and Wagner's notes, the table below also compares them to the lecture summaries on the course website (`https://www.math.uwaterloo.ca/~kayeats/teaching/co330_f26.html`).

Wagner	**enumeration.ca**	**Lectures**	**Comments**
$\mathcal{A} \oplus \mathcal{B}$	$\mathcal{A} + \mathcal{B}$	$\mathcal{A} \cup \mathcal{B}$	In lectures, we require disjointness.
$\mathcal{A} * \mathcal{B}$	$\mathcal{A} \times \mathcal{B}$	$\mathcal{A} * \mathcal{B}$	
$\mathcal{A}^k$	$\mathcal{A}^k$	$\mathcal{A}^k$	
$\mathcal{A}^*$	$\mathrm{SEQ}(\mathcal{A})$	$\mathsf{Seq}(\mathcal{A})$	
$\mathcal{A}^\bullet$	not mentioned	$\mathcal{A}^\bullet$	Called "rooting" in Wagner, "pointing" in lectures.
$\mathcal{E}_0$	ϵ	$\mathcal{E}$	
$\mathcal{E}_1 = \mathcal{X}$	$\mathcal{Z}$	$\mathcal{Z}$	
$\mathcal{E}_k[\mathcal{A}]$	$\mathrm{SET}_k(\mathcal{A})$	$\mathsf{Set}_k(\mathcal{A})$	
$\mathcal{E}[\mathcal{A}]$	$\mathrm{SET}(\mathcal{A})$	$\mathsf{Set}(\mathcal{A})$	
$\mathcal{C}[\mathcal{A}]$	$\mathrm{CYC}(\mathcal{A})$	$\mathsf{Cyc}(\mathcal{A})$	
$\mathcal{B}[\mathcal{A}]$	not mentioned	not mentioned	

Comments.

• The notation $\mathcal{A}^*$ is a bit hard to find in the Wagner's textbook (I think it initially shows up in Definition 4.11) but it is there and is also common notation.

Question 4. Question 1 from `https://enumeration.ca/extensions/labelled-constructions/`:

Find the EGFs of the following labelled classes. Express your answer in closed form.

(a) Permutations that are involutions (applying the permutation twice gives the identity).

(b) Derangements with an even number of cycles (i.e., permutations with no fixed points that contain an even number of cycles).

(c) Permutations where every cycle has even length and there are no cycles of size less than 4.

(d) The class of permutations where one element in each cycle is coloured red (when thinking of a permutation as a set of disjoint cycles).

Solution.

(a) A permutation is an involution if and only if each of its cycles has length 1 or 2. The class of involutions is

$$\mathcal{I} = \mathsf{Set}\big(\mathsf{Cyc}_1(\mathcal{Z}) \cup \mathsf{Cyc}_2(\mathcal{Z})\big).$$

Since the EGF of $\mathsf{Cyc}_1(\mathcal{Z}) \cup \mathsf{Cyc}_2(\mathcal{Z})$ is $x + \frac{x^2}{2}$, the EGF of $\mathcal{I}$ is

$$I(x) = \exp\left(x + \frac{x^2}{2}\right).$$

(b) A derangement has no cycles of length 1, so every cycle has length at least 2. The class of derangements with an even number of cycles is

$$\mathcal{D}_{\text{even}} = \mathsf{Set}_{\text{even}}\big(\mathsf{Cyc}_{\geq 2}(\mathcal{Z})\big), \qquad \text{where} \qquad \mathsf{Cyc}_{\geq 2}(\mathcal{Z}) = \bigcup_{k=2}^{\infty} \mathsf{Cyc}_k(\mathcal{Z}).$$

The class of allowed cycles has EGF

$$C_{\geq 2}(x) = C(x) - C_1(x) = \ln \frac{1}{1-x} - x.$$

To find the generating function for $\mathsf{Set}_{\text{even}}(\mathcal{A})$, recall that the unrestricted set operator $\mathsf{Set}(\mathcal{A}) = \bigcup_{k=0}^{\infty} \mathsf{Set}_k(\mathcal{A})$ has EGF $\exp(A(x)) = \sum_{k=0}^{\infty} \frac{A(x)^k}{k!}$. Replacing $A(x)$ with $-A(x)$ gives $\exp(-A(x)) = \sum_{k=0}^{\infty} (-1)^k \frac{A(x)^k}{k!}$. Adding these two series cancels out all odd terms, thus

$$\frac{\exp\big(A(x)\big) + \exp\big(-A(x)\big)}{2} = \sum_{m=0}^{\infty} \frac{A(x)^{2m}}{(2m)!},$$

which isolates the even terms and yields the EGF of $\mathsf{Set}_{\text{even}}(\mathcal{A})$. Substituting $\mathcal{A} = \mathsf{Cyc}_{\geq 2}(\mathcal{Z})$, and thus substituting $A(x) = C_{\geq 2}(x) = \ln \frac{1}{1-x} - x$, the EGF of $\mathcal{D}_{\text{even}}$ is

$$D_{\text{even}}(x) = \frac{\exp\big(C_{\geq 2}(x)\big) + \exp\big(-C_{\geq 2}(x)\big)}{2} = \frac{1}{2}\left(\frac{e^{-x}}{1-x} + (1-x)e^x\right).$$

(c) The cycles must have even lengths of at least 4. The class of such permutations is

$$\mathcal{P} = \mathsf{Set}\big(\mathsf{Cyc}_{\text{even},\geq 4}(\mathcal{Z})\big), \qquad \text{where} \qquad \mathsf{Cyc}_{\text{even},\geq 4}(\mathcal{Z}) = \bigcup_{k=2}^{\infty} \mathsf{Cyc}_{2k}(\mathcal{Z}).$$

With the same cancelling out trick and removing the contribution of Cyc_2, the EGF of the allowed cycles is

$$C_{\text{even},\geq 4}(x) = \frac{C(x)+C(-x)}{2} - C_2(x) = \frac{1}{2}\left(\ln\frac{1}{1-x} + \ln\frac{1}{1+x}\right) - \frac{x^2}{2} = \frac{1}{2}\ln\frac{1}{1-x^2} - \frac{x^2}{2}.$$

Applying the Set operator, the EGF of $\mathcal{P}$ is

$$P(x) = \exp\big(C_{\text{even},\geq 4}(x)\big) = \exp\left(\frac{1}{2}\ln\frac{1}{1-x^2} - \frac{x^2}{2}\right) = \frac{e^{-x^2/2}}{\sqrt{1-x^2}}.$$

(d) Distinguishing one element in each cycle corresponds to pointing each cycle, $\mathsf{Cyc}(\mathcal{Z})^\bullet$. The class of such permutations is

$$\mathcal{P}_{\text{red}} = \mathsf{Set}\big(\mathsf{Cyc}(\mathcal{Z})^\bullet\big).$$

By the pointing rule, $\mathsf{Cyc}(\mathcal{Z})^\bullet$ has EGF

$$C_{\text{red}}(x) = x\left(\ln\frac{1}{1-x}\right)' = \frac{x}{1-x}.$$

Applying the Set operator gives

$$P_{\text{red}}(x) = \exp\big(C_{\text{red}}(x)\big) = \exp\left(\frac{x}{1-x}\right).$$

Comments.

• (a) is pretty standard, the only tricky thing is noticing that involutions are permutations with only cycles of length 1 or 2.
• (b) is a bit tricky. The trick is to notice that $\frac{\exp(A(x))+\exp(-A(x))}{2}$ cancels out the odd terms, and thus cancels out when you get an odd number of cycles.
• (c) is also somewhat standard. Here the trick is to note that $C_{\text{even}}(x) = \frac{C(x)+C(-x)}{2}$. Also note the difference between this question and the last. Before we wanted to remove all of the odd terms **after** applying exp, whereas here we want to do it before.
• (d) This is again pretty standard, the difficulty mostly being in understanding how to have distinguished elements.

Question 5.

(a) Give an example of a combinatorial class built of atoms with the property that for each size there are the same number of labelled objects as unlabelled objects of that size.

(b) Give an example of a combinatorial class built of atoms with the property that for each size n, there are $n!$ times as many labelled objects as unlabelled objects of size n.

(c) Give an example of a combinatorial class where the number of labelled objects compared to unlabelled objects is intermediate between the extremes of the previous parts. Plot the first 100 terms of the counting sequence for the labelled and unlabelled version of this class.

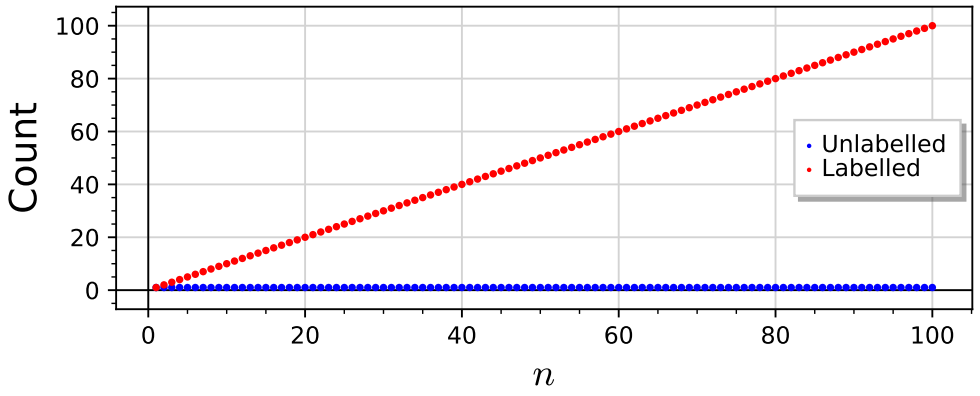
Solution.

(a) Consider sets of atoms, $\mathsf{Set}(\mathcal{Z})$. For each size $n \geq 0$, there is only one unlabelled set of n indistinguishable atoms, and because the order of elements in a set does not matter, there is also only one labelled set of size n (namely $\{1, 2, \ldots, n\}$). Thus, there are the same number of labelled and unlabelled objects of every size.

(b) Consider sequences of atoms, $\mathsf{Seq}(\mathcal{Z})$. For each size $n \geq 0$, there is only one unlabelled sequence of n indistinguishable atoms, whereas there are $n!$ labelled sequences of size n (the linear orderings of $\{1, 2, \ldots, n\}$). Hence, for every size $n \geq 0$, there are $n!$ times as many labelled objects as unlabelled objects of size n.

(c) There are many examples of classes with intermediate behaviour. For a simple example, consider pointed sets of atoms, $\mathsf{Set}(\mathcal{Z})^\bullet$. For $n = 0$, there are zero pointed sets in both settings. For each size $n \geq 1$, there is only one unlabelled pointed set of n indistinguishable atoms (one distinguished atom and $n-1$ other atoms), whereas there are n labelled pointed sets of size n (the set $\{1, 2, \ldots, n\}$ together with a choice of one of its n elements to distinguish). Hence, for each size $n \geq 1$, the ratio of labelled to unlabelled objects is n, which satisfies $1 < n < n!$ for all $n \geq 3$ and is thus strictly intermediate between the two extremes in parts (a) and (b).

Below is a plot of the first 100 nonzero terms ($1 \leq n \leq 100$) for $\mathsf{Set}(\mathcal{Z})^\bullet$.



This plot can be produced in SageMath using the following code.

```
p = list_plot([(n, 1) for n in [1..100]], legend_label="Unlabelled", zorder=3)
p += list_plot([(n, n) for n in [1..100]],
              color="red", legend_label="Labelled", zorder=3)
p.save("q5_pointed_plot.pdf", frame=True, gridlines=True,
      gridlinesstyle=dict(color="lightgray", linestyle="-"),
      axes_labels=["$n$", "Count"], xmin=-2, xmax=103, ymin=-5, ymax=105,
      figsize=[6, 2.6])
```