# Assignment 6

## CO 330 (Fall 2026)

**Due Friday, November 27, 2026 at 3pm Eastern time.**

### Instructions

To get full marks on this assignment you must do questions worth 12 marks total. Any additional questions you do beyond 12 marks-worth will not be counted. Each question has two possible point values, one if you would like it marked for correctness and one ("engagement") if you would like it marked for feedback after you have done some work on it. You will indicate which you want when you submit.

All answers should be justified unless specifically indicated otherwise.

This assignment should be done and submitted individually. You are welcome to discuss with your classmates, but you should not use generative AI for your ideas and you must write up your solutions independently (neither writing it with other people nor with tools such as generative AI). In your write up explicitly acknowledge any discussions or non-class resources and tools that you may have used.

This assignment should be submitted in crowdmark.

### Questions

(1) **{4 marks graded for content, 2 marks graded for engagement}** A **derangement** is a permutation $\sigma$ where there is no $i$ such that $\sigma(i) = i$. Let $\mathcal{S}$ be the combinatorial class of permutations, let $\mathcal{I}$ be the combinatorial class of identity permutations, and let $\mathcal{D}$ be the combinatorial class of derangements, all considered as labelled classes.

(a) Give a combinatorial decomposition involving $\mathcal{S}$, $\mathcal{I}$ and $\mathcal{D}$.

(b) Prove that

$$D(x) = \frac{e^{-x}}{1-x}$$

(here $D(x)$ is the exponential generating function)

(c) Use the previous part to prove that the number of derangements of $n$ is

$$n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

and hence that the probability that a permutation of $n$ is a derangement approaches

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e}$$

as $n \to \infty$.

(2) **{4 marks graded for content, 2 marks graded for engagement}** Find the exponential generating functions for the following classes of set partitions:

(a) Set partitions with all parts of size at most 2.

(b) Set partitions with all parts of size at least 2.

(c) Set partitions where every part has an odd number of elements.
(d) Set partitions where there are an odd number of parts.

You may use that

$$\sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)!} = \sinh(x)$$

the hyperbolic sine function.

(3) **{4 marks graded for content, 2 marks graded for engagement}** A *triangle tree* is a labelled connected graph in which every edge is in exactly one cycle, and this cycle has length 3. Let $\mathcal{T}$ be the labelled combinatorial class of triangle trees.
(a) Give a specification for $\mathcal{T}^\bullet$.
(b) Prove that the number of triangle trees on $n$ vertices is 0 if $n$ is even and is

$$\frac{(2k)!(2k+1)^{k-1}}{k!2^k}$$

if $n = 2k+1$ is odd.

(4) **{4 marks graded for content, 2 marks graded for engagement}** Question 5 from `https://enumeration.ca/extensions/labelled-constructions/`

(5) **{4 marks graded for content, 2 marks graded for engagement}** Given a permutation $\sigma$ of $\{1, \ldots, n\}$ let $w(\sigma) = n$ and let $f(\sigma)$ be the number of fixed points of $\sigma$, that is the number of indices $i$ so that $\sigma(i) = i$. Consider the bivariate generating series

$$S(x, y) = \sum_{\sigma \in \mathcal{S}} \frac{x^{w(\sigma)}}{w(\sigma)!} y^{f(\sigma)}$$

Note that this generating series is exponential in $x$ but ordinary in $y$.
(a) Show that $S(x, y) = \frac{1}{1-x} e^{xy-x}$.
(b) Find the average number of fixed points in a permutation of $\{1, \ldots, n\}$.