

CO 330 LECTURE 10 SUMMARY

FALL 2026

SUMMARY

First we set up for our multivariate LIFT example. There's an important general principle here. Suppose we're counting our objects from a class $\mathcal{A}$ by some primary weight $w(a)$ and some secondary parameter $\ell(a)$ and we want to find the average value of the secondary parameter among objects of weight n . We can do this using the bivariate generating series

$$A(x, y) = \sum_{a \in \mathcal{A}} x^{w(a)} y^{\ell(a)}$$

Then

$$\frac{\partial}{\partial y} A(x, y)|_{y=1} = \sum_{a \in \mathcal{A}} \ell(a) x^{w(a)} = \sum_{n \geq 0} \left(\sum_{a \in \mathcal{A}_n} \ell(a) \right) x^n$$

so the average value of the secondary parameter among all objects of weight n is

$$\frac{[x^n] \frac{\partial}{\partial y} A(x, y)|_{y=1}}{[x^n] A(x, 1)}$$

There are lots of examples where both of these coefficient extractions can be nicely done with LIFT, one of which we did at the start of class. Specifically starting from $C(x, y) = xy + 2xC(x, y) + xC(x, y)^2$ we obtained $[x^n]C(x, 1) = \frac{1}{n} \binom{2n}{n-1}$. I'd been worried about a typo here, but it is correct. You can rewrite $\frac{1}{n} \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}$ which is Catalan. Then we also obtained $[x^n] \frac{\partial}{\partial y} C(x, y)|_{y=1} = \binom{2(n-1)}{n-1}$. The trick here is that you can flip the order of the coefficient extraction and the derivative (twice! make sure you see why), so that the LIFT is easy and the extraction afterwards is also easy.

Note that

$$\lim_{n \rightarrow \infty} \frac{\binom{2(n-1)}{n-1}}{\frac{1}{n+1} \binom{2n}{n}} = \lim_{n \rightarrow \infty} \frac{n^2 + n}{4n - 2} = \frac{n}{4}$$

in our particular case that meant on average a quarter of the internal vertices of our rooted binary trees are cherries.

Next we proved LIFT. The proof we followed is in both sets of notes we're primarily referencing. See below for the exact lemmas.

The idea of this proof is to work with a formal residue operator, that is $[x^{-1}]L(x)$ where $L(x)$ is a formal Laurent series. This particular coefficient is especially well-behaved for calculations involving derivatives and compositions and things like that. That's just what we need. If you've done some analysis and know how nice and how useful residues are, it shouldn't be too surprising that their formal version is also useful.

Having said that the proof isn't very combinatorial in the sense that there isn't a nice construction or interpretation. If you would like to see a more combinatorial proof see

chapter 13 of the older notes <https://melczer.ca/330/WagnerNotes.pdf> (but we won't cover it in this class, so you aren't responsible for it.)

NEXT TIME

We'll discuss using computer algebra systems to do this kind of combinatorics.

REFERENCES

- <https://enumeration.ca/toolbox/lift/> the three lemmas are Lemma 1, Lemma 2, and the Change of Variables Lemma.
- <https://melczer.ca/330/WagnerNotes.pdf> here the three lemmas are Lemmas 8.3, 8.4, and 8.5 and the proof itself is on pages 78 and 79.