CO 330 LECTURE 12 SUMMARY

FALL 2026

Summary

Wednesday before reading week was the midterm, then Friday before reading week we discussed `http://www.ams.org/publications/journals/notices/201610/rnoti-p1190.pdf` and generally lecturing as a pedagogical technique. Today we got back to, well, actually lecturing.

Today we started our next big topic, q-counting. The idea is we're going to upgrade our combinatorial proofs of binomial identities to include parameters. We started with a few classical examples and how we see them combinatorially: We can see

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

as counting subsets of size k of $\{1, 2, \dots, \}$ directly on the left or on the right, either n is not in the subset, leaving a subset of size k of $\{1, \dots, n-1\}$ or n is in the subset and the rest is a subset of size $k-1$ of $\{1, \dots, n-1\}$.

Another way to do this one is to notice that $\binom{n}{k}$ is the number of lattice paths from $(0,0)$ to $(k, n-k)$ using the steps $\uparrow$ and $\rightarrow$ and then for the right hand side, either the last step is $\uparrow$ or the last step is $\rightarrow$ giving the two terms respectively.

Even the binomial coefficients themselves can be thought of bijectively. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, or rewriting $n! = \binom{n}{k}k!(n-k)!$, which we can understand combinatorially by saying that a permutation of n things is a choice of k of the things which will be our first k things, then a permutation of those first k things and a permutation of the remaining $n-k$ things.

The sum in the binomial theorem is itself the generating series for subsets of $\{1, 2, \dots, n\}$ counted by size. The binomial theorem itself is then a product formula for this generating series: $\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^k$.

Now lets start upgrading these. The first step is just to upgrade $n!$. We need some definitions first

Definition 1. *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of $\{1, 2, \dots, n\}$ an* inversion *in σ is a pair (i, j) with $i < j$ and $\sigma_i > \sigma_j$. The number of inversions in σ is denoted inv(σ).*

For example 2431 has 4 inversions.

Definition 2.
- $[k]_q = 1 + q + \cdots + q^{k-1}$. $[0]_q = 0$. *These are the q-natural numbers.*
- $[n]!_q = [n]_q \cdot [n-1]_q \cdot \cdots \cdot [1]_q$. $[0]!_q = 1$. *This is the q-factorial.*

Note that setting $q = 1$ recovers usual natural numbers and usual factorial. It is also sometimes useful to note that $[k]_q = (1-q^k)/(1-q)$.

Now we can upgrade the fact that there are $n!$ permutations of $\{1, 2, \ldots, n\}$ to the fact that the generating series of permutations of $\{1, 2, \ldots, n\}$ counted by inversions is the q-factorial, as the proposition states:

Proposition 3. *Let $\mathcal{S}_n$ be the set of permutations of $\{1, 2, \ldots, n\}$ then*

$$\sum_{\sigma \in \mathcal{S}_n} q^{inv(\sigma)} = [n]!_q$$

The idea of the proof is to upgrade the proof we always knew, but to do so it gets a bit hidden in formality. What we need is a bijection between $\mathcal{S}_n$ and $[n] \times [n-1] \times \cdots \times [1]$ where $[k] = \{1, 2, \ldots, k\}$. The bijection we will use is given $\sigma = \sigma_1 \cdots \sigma_n$ let $f(\sigma) = (s_1, s_2, \ldots, s_n)$ where s_i is the number of indices j with $i \leq j \leq n$ and $\sigma_i \geq \sigma_j$. The inverse map is given $(s_1, s_2, \ldots, s_n)$ recursively define σ_k to be the kth smallest element of $[n] \setminus \{\sigma_1, \ldots, \sigma_{k-1}\}$. We checked that these are well defined and inverses of each other and did an example.

Now the point is that the number of inversions of σ starting at index i where $f(\sigma) = (s_1, s_2, \ldots, s_n)$ is $s_i - 1$ and so $\text{inv}(\sigma) = (s_1 - 1) + (s_2 - 1) + \cdots + (s_n - 1)$. Now we just put this into a generating series.

$$\begin{aligned}
\sum_{\sigma \in \mathcal{S}_n} q^{\text{inv}(\sigma)} &= \sum_{(s_1, \ldots, s_n) \in [n] \times \cdots \times [1]} q^{(s_1-1)+(s_2-1)+\cdots+(s_n-1)} \\
&= \left(\sum_{s_1 \in [n]} q^{s_1 - 1} \right) \left(\sum_{s_2 \in [n-1]} q^{s_2 - 1} \right) \cdots \left(\sum_{s_n \in [1]} q^{s_n - 1} \right) \\
&= [n]_q [n-1]_q \cdots [1]_q \\
&= [n]!_q
\end{aligned}$$

as desired.

Next time

Next time we will do more q-generalizations and hopefully by repeated examples the beauty of this will become clear.

References

- `https://enumeration.ca/extensions/q-analogues/`
- Chapter 5 of `https://melczer.ca/330/WagnerNotes.pdf`, and you'll also want to look back at chapter 2 for the bijective proof that we add in the qs to.