CO 330 LECTURE 13 SUMMARY

FALL 2026

Summary

We continued q-counting today.

The usual binomial theorem is a product formula for the generating series of subsets. We can get such a product formula when we count our subsets by both size and sum, namely $\mathrm{sum}(S) = \sum_{s\in S} s$. Then calculate using indicator vectors

$$\begin{aligned}\sum_{S\subseteq\{1,\dots,n\}} x^{|S|}q^{\mathrm{sum}(S)} &= \sum_{w_1\cdots w_n\in\{0,1\}^n} x^{w_1+w_2+\cdots+w_n}q^{1w_1+2w_2+\cdots+nw_n}\\ &= \left(\sum_{w_1\in\{0,1\}} x^{w_1}q^{w_1}\right)\left(\sum_{w_2\in\{0,1\}} x^{w_2}q^{2w_2}\right)\cdots\left(\sum_{w_n\in\{0,1\}} x^{w_n}q^{nw_n}\right)\\ &= (1+xq)(1+xq^2)\cdots(1+xq^n).\end{aligned}$$

The last thing we need for a q-binomial theorem is the q-analogue of $\binom{n}{k}$. I'll spoil it by giving the definition:

Definition 1.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}$$

the q-binomial coefficients or Gaussian binomial coefficients.

The next step it to revisit the combinatorial proof of $n! = \binom{n}{k}k!(n-k)!$. We again need to formalize a bit. We want a bijection $\mathcal{S}_n \leftrightharpoons B(n,k)\times\mathcal{S}_k\times\mathcal{S}_{n-k}$ where $B(n,k)$ is the set of size k subsets of $\{1,\dots,n\}$. The map is $\sigma = \sigma_1\cdots\sigma_n$ maps to (S,τ,η) where $S = \{\sigma_1,\dots,\sigma_k\}$, τ is the normalization of $\sigma_1\cdots\sigma_k$ (that is keep the same relative order but shift the numbers down to be $1,\dots,k$) and η is the normalization of $\sigma_{k+1}\cdots\sigma_n$. We checked this and did an example.

What happens with inversions under this map.

- If $i<j\le k$ is an inversion of σ then it is also an inversion of τ and vice versa.
- If $k<i<j$ is an inversion of σ then it is also an inversion of η (shifting indices down by k) and vice versa.
- if $i\le k<j$ is an inversion of σ then there is a pair (a,b) with $a\in S$, $b\in\{1,\dots,n\}\setminus S$ and $a>b$, and every such pair gives such an inversion.

Then count the number of such pairs. Writing $S=\{s_1,\dots,s_k\}$ with $s_1<s_2<\cdots<s_k$ we see that there are s_1-1 elements in $[n]\setminus S$ which are less than s_1, s_2-2 elements in $[n]\setminus S$ which are less than s_2 and so on. So the total number of such pairs is

$$(s_1-1)+(s_2-1)+\cdots+(s_k-k) = \mathrm{sum}(S) - \frac{k(k+1)}{2}.$$

Putting this all together

$$[n]!_q = \sum_{\sigma\in\mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{(S,\tau,\eta)} q^{\mathrm{sum}(S)-\frac{k(k+1)}{2}+\mathrm{inv}(\tau)+\mathrm{inv}(\eta)} = \left(\sum_{S\in B(n,k)} q^{\mathrm{sum}(S)}\right) q^{-\frac{k(k+1)}{2}}[k]!_q[n-k]!_q$$

so

$$\left(\sum_{S\in B(n,k)} q^{\mathrm{sum}(S)}\right) = q^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Those extra powers of q sometimes show up in q-generalizations. You can't see them when $q = 1$ so you just have to be careful to keep your eyes out for where they need to be added. Finally, then,

$$(1+xq)(1+xq^2)\cdots(1+xq^n) = \sum_{S\subseteq[n]} x^{|S|}q^{\mathrm{sum}(S)} = \sum_{k=0}^{n}\left(\sum_{S\in B(n,k)} q^{\mathrm{sum}(S)}\right)x^k = \sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

which is the q-binomial theorem.

Next time

Next time we'll talk about q-counting of lattice paths.

References

- `https://enumeration.ca/extensions/q-analogues/`
- Chapter 5 of `https://melczer.ca/330/WagnerNotes.pdf`