CO 330 LECTURE 15 SUMMARY PART 1

FALL 2026

Summary

You spent the first part of the class continuing to work in groups on the binomial identities and their q-analogues. The original three identities were

$$\binom{a+1+b}{b} = \sum_{j=0}^{b} \binom{a+j}{j}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

The group working on the first identitiy used lattice paths, given a lattice path in $\mathcal{L}(a+1,b)$ there is a unique level j where it takes a step from $x=a$ to $x=a+1$. Before that point it is an arbitrary lattice path in $\mathcal{L}(a,j)$ and after that it is and then after that it just goes up. The area then is the area under the smaller path plus the area under that last horizontal step, which is j. So you got

$$\begin{bmatrix} a+1+b \\ b \end{bmatrix}_q = \sum_{j=0}^{b} q^j \begin{bmatrix} a+j \\ h \end{bmatrix}_q$$

Then I asked you to modify the argument to get

$$\begin{bmatrix} a+1+b \\ b \end{bmatrix}_q = \sum_{j=0}^{b} q^{(a+1)(b-j)} \begin{bmatrix} a+j \\ h \end{bmatrix}_q$$

which you did by flipping the picture so now we're considering lattice paths in $\mathcal{L}(b,a+1)$ which you broke at their last vertical step into a lattice path in $\mathcal{L}(j,a)$ and a rectangle of area $(a+1)(b-j)$.

The group doing the second one gave two solutions, one using subsets and one using lattice paths. In both cases they got

$$\sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 = \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

The lattice path proof is in the `enumeration.ca` notes. The subset proof involved breaking a subset of size n of $\{1,\ldots,2n\}$ into two pieces, those elements of the subset which are $\leq n$ giving a subset of size k of $\{1,\ldots,n\}$ for some $0 \leq k \leq n$ and then the rest of the elements, which if you shift them down by n give a subset of size $n-k$ of $\{1,\ldots,n\}$.

The group doing the third one was asked to find two different q-analogues. Thinking of this in terms of subsets we can think of it as taking subsets of size k of $\{1, \ldots, n\}$ and then fixing one of the elements of $\{1, \ldots, n\}$ and splitting the subsets into those which contain the fixed element and those which don't. Taking either 1 or n as the fixed element gives

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_1$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_1$$

Next time

The class continued with integer partitions.

References

- `https://enumeration.ca/extensions/q-analogues/`
- Chapter 5 of `https://melczer.ca/330/WagnerNotes.pdf`