CO 330 LECTURE 16 SUMMARY

FALL 2026

Summary

We continued talking about integer partitions.

Given an integer partition λ we draw its *Ferrers diagram* or *Young diagram* or just *shape* as a diagram of $\bullet$ (in the Ferrer's case) or $\square$ (in the Young case) of left justified rows with λ_1 boxes in the first row, λ_2 boxes in the second row, etc.

We drew an example, then made a few remarks. This is so-called English notation and other orientations of the diagram are also used. We'll use matrix indexing, so the top left box is $(1,1)$, the next one to its right is $(1,2)$ and so on. We can also see the shape as the boundary of a lattice path.

Together we counted the number of partitions of n up to $n = 7$. The resulting sequence is A000041 in the OEIS. Note that unlike compositions there is no clear pattern in these numbers. However, finding the generating series is not bad.

Let $\mathcal{Y}$ be the combinatorial class of partitions. Flipping the order of the partition we can think of it as a sequence of 1s followed by a sequence of 2s followed by a sequence of 3s etc. So

$$Y(x) = \prod_{i \geq 1} \frac{1}{1 - x^i}.$$

However, that argument wasn't entirely rigorous because a partition needs to have a finite number of parts so we can't actually write $\mathcal{Y} = \mathsf{Seq}(1) \times \mathsf{Seq}(2) \times \cdots$ because the infinite cartesian product would allow infinite sequences. We can fix it two ways. If we consider partitions with all parts $\leq N$ then it really is the case that

$$\mathcal{Y}_{\text{parts } \leq N} = \mathsf{Seq}(1) \times \mathsf{Seq}(2) \times \cdots \times \mathsf{Seq}(N)$$

and hence

$$Y_{\text{parts } \leq N}(x) = \prod_{i=1}^{N} \frac{1}{1 - x^i}$$

and now take the limit. We also see that the infinite product converges this way since parts of size $> N$ can only contribute to coefficients for powers of x greater than N.

Another approach that can be useful is to work with multiplicity vectors. We used $\mathcal{M}$ for the set of infinite sequences $m = (m_1, m_2, \ldots)$ with the m_i nonnegative integers and where only finitely many of the m_i are nonzero. We build the multiplicity vector of a partition λ by letting m_i be the number of is in λ. Then $k(\lambda) = \sum_{i \geq 1} m_i$ and $n(\lambda) = \sum_{i \geq 1} i m_i$. You can check this is a bijection between $\mathcal{Y}$ and $\mathcal{M}$ and from here we can write down the generating function and obtain the product formula. This is the approach both the sets of notes we're following use, so you can check them out for more details.

Our final observation today is that by the same arguments (whichever one you like best!) if we let $Y(x,y) = \sum_{\lambda \in \mathcal{Y}} x^{n(\lambda)} y^{k(\lambda)}$ then

$$Y(x,y) = \prod_{i \geq 1} \frac{1}{1 - yx^i}$$

Next time

We will continue discussing integer partitions.

References

- `https://oeis.org/A000041`
- `https://enumeration.ca/extensions/partitions/`
- Chapter 9 of `https://melczer.ca/330/WagnerNotes.pdf`