CO 330 LECTURE 17 SUMMARY

FALL 2026

Summary

We gave the template theorem for generating series of integer partitions.

Theorem 1. *For each $j \in \mathbb{Z}_{\geq 1}$ let $M_j \subseteq \mathbb{Z}_{\geq 0}$ be nonempty and suppose that for all but finitely many j, $0 \in M_j$. Let $\mathcal{A}$ be the class of integer partitions where the number of parts equal to j is in M_j for each $j \geq 1$. Then*

$$A(x,y) = \sum_{\lambda \in \mathcal{A}} x^{n(\lambda)} y^{k(\lambda)} = \prod_{j \geq 1} \sum_{m \in M_j} x^{jm} y^m$$

This generalizes where we ended last time and essentially the same proof works. Note that what parts are allowed restricts which j occur in the product (because if $M_j = \{0\}$ we might as well not even write that factor) while how may of a part can occur restricts what the factor looks like.

As two examples of using the template theorem we have that the generating series for partitions with odd parts is

$$\prod_{j \text{ odd}} \frac{1}{1 - yx^j} = \prod_{i \geq 1} \frac{1}{1 - yx^{2i-1}}$$

and the generating series for partitions with distinct parts is

$$\prod_{j \geq 1} (1 + yx^j).$$

We also did a more ridiculous example and then you exchanged some further examples with each other.

Let's look a little more at those two examples. Let $\mathcal{O}$ be the combinatorial class of partitions with odd parts and let $\mathcal{D}$ be the combinatorial clas sof partitions with distinct parts. Considering the univariate generating series $O(x)$ and $D(x)$, by some algebraic manipulations we saw $O(x) = D(x)$ (you can find the details on p88 in the notes by David Wagner). This tells us that the number of partitions of n with odd parts is the same as the number of partitions of n with distinct parts. Can we also prove this bijectively? We put some small examples on the board so as to start building an intuition.

Next time

We will give the bijection between partitions with odd and distinct parts and then talk about more identities and bijections for partitions.

References

- https://enumeration.ca/extensions/partitions/
- Chapter 9 of https://melczer.ca/330/WagnerNotes.pdf