CO 330 LECTURE 18 SUMMARY

FALL 2026

Summary

We gave a weight-preserving bijection between partitions with odd parts and partitions with distinct parts.

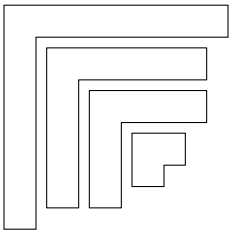
The bijection is, given a partition with distinct parts, take the largest even part, replace it by two parts of half the size and repeat until no even parts remain. In the other direction, take the smallest pair of repeated parts and replace them by a single part of the same size, and repeat until no repeated parts remain. You don't have to do the splitting or combining in this order, but describing it this way makes it clearer that they are mutual inverses. Another way you might see this bijection written is using the binary expansion of the even parts or of the number of copies of the repeated parts. It amounts to the same thing.

Next we looked at another bijection that's even more visual. First we need a definition.

Definition 1. *Given a partition λ, the* conjugate *partition is the partition associated to the transpose of the Ferrer's diagram of λ. We denote the conjugate partition of λ by $\tilde{\lambda}$.*

We did an example and noted that $n(\lambda) = n(\tilde{\lambda})$, $k(\lambda) = \tilde{\lambda}_1$, $\lambda_1 = k(\tilde{\lambda})$, $(\tilde{\tilde{\lambda}}) = \lambda$ and $d(\lambda) = d(\tilde{\lambda})$ where $d(\lambda)$ is the number of boxes on the main diagonal.

Let $\mathcal{SC}$ be the set of self-conjugate partitions (i.e. $\lambda = \tilde{\lambda}$). Let $\mathcal{OD}$ be the set of partitions with odd and distinct parts. There's a bijection between these which we can summarize with this schematic



You can check out either set of notes for the details. We can find the generating series for $\mathcal{OD}$ using the template theorem.

$$OD(x) = \prod_{i \geq 1} (1 + x^{2i-1}).$$

For the self-conjugate partitions, it is not easy to see how to find the generating series directly, but via the bijection with $\mathcal{OD}$ we now know that $SC(x) = OD(x)$. We also thought about how the parameters carry through.

Next time

You'll do two more examples where the picture tells the story to obtain partition identities.

References

- https://enumeration.ca/extensions/partitions/
- Chapter 9 of https://melczer.ca/330/WagnerNotes.pdf