CO 330 LECTURE 19 SUMMARY

FALL 2026

Summary

You worked on two final partition identities together in groups.

The first one takes an arbitary partition λ and decomposes it into its largest square (called the Durfee square, which has size $d(\lambda) \times d(\lambda)$) along with a partition with at most d parts (sitting to the right of the square) and a partition with largest part at most d (sitting below the square). The second of these we can count by the template theorem and then the first of these we can count by taking the conjugate. So you got

$$\prod_{j\geq 1} \frac{1}{1-x^j} - \sum_{d\geq 0} \frac{x^{d^2}}{\left(\prod_{j=1}^{d}(1-x^j)\right)^2}$$

which you upgraded to be bivariate by noticing that we only need to count parts in the square and in the partition that comes below the square, giving

$$\prod_{j\geq 1} \frac{1}{1-yx^j} - \sum_{d\geq 0} \frac{y^d x^{d^2}}{\left(\prod_{j=1}^{d}(1-yx^j)\right)\left(\prod_{j=1}^{d}(1-x^j)\right)}.$$

The second example takes a partition with distinct parts and decomposes it into its largest staircase and then the rest. If you left-justify the rest you get an arbitrary partition with at most as many parts of the staircase. So you got

$$\prod_{j\geq 1}(1+x^j) = \sum_{k\geq 0} \frac{x^{k(k+1)/2}}{\prod_{j=1}^{k}(1-x^j)}$$

which you upgraded to be bivariate by noticing that we only need to count parts in the staircase, giving

$$\prod_{j\geq 1}(1+yx^j) = \sum_{k\geq 0} \frac{y^k x^{k(k+1)/2}}{\prod_{j=1}^{k}(1-x^j)}$$

Next time

We'll start labelled counting and exponential generating series.

References

- `https://enumeration.ca/extensions/partitions/`
- Chapter 9 of `https://melczer.ca/330/WagnerNotes.pdf`