CO 330 LECTURE 1 SUMMARY PART 2

FALL 2026

Summary

We began the content with two motivating examples, binary rooted trees and ternary rooted trees. For the purposes of this lecture our rooted trees had order information of the children (call this a plane tree or an ordered tree) and the binary trees are the kind with 0 or 2 children at each vertex while the ternary ones are the kind with 0 or 3 children at each vertex.

We wrote out the first few examples by hand, and you did some of it yourselves. From this we saw that the counting sequences were $0, 1, 0, 1, 0, 2, 0, 5, \cdots$ and $0, 1, 0, 0, 1, 0, 0, 3, 0, 0, 12, \cdots$ for the binary and ternary cases respectively where these are counted by number of vertices.

Letting $\mathcal{B}$ be the set of binary rooted trees and $\mathcal{T}$ be the set of ternary rooted trees, we saw the decompositions

$$\mathcal{B} \leftrightharpoons \{\bullet\} \cup \{\bullet\} \times \mathcal{B} \times \mathcal{B}$$

and

$$\mathcal{T} \leftrightharpoons \{\bullet\} \cup \{\bullet\} \times \mathcal{T} \times \mathcal{T} \times \mathcal{T}$$

where $\leftrightharpoons$ indicates a bijection (in fact as someone asked afterwards, it should be a *weight preserving* bijection, which we'll get back to on Monday) and the unions are disjoint. Then from the sum and product lemmas of math 239 we have

$$B(x) = x + xB(x)^2 \qquad \text{and} \qquad T(x) = x + xT(x)^3$$

where $B(x)$ and $T(x)$ are the generating series for $\mathcal{B}$ and $\mathcal{T}$ respectively. We will go from decompositions to equations for generating series a lot in this course. We only used 239 things to do this, but if it isn't feeling comfortable for you, take the time to do some review.

Now the equation for $B(x)$ is quadratic so we can solve with the quadratic formula to obtain

$$B(x) = \frac{1 \pm \sqrt{1 - 4x^2}}{2x}$$

but which root do we want? Now $\sqrt{1 - 4x^2} = (1 - 4x^2)^{1/2}$ is a formal power series which starts with constant term 1, which we can see by the binomial theorem. If we take the plus root we get that $B(x)$ starts with $2/2x$ which is not a formal power series, so we must have

$$B(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x}$$

We left this example here for the moment, but you can extract coefficients from this (using the binomial theorem) and you will get, as you might have guessed from the initial terms, the Catalan numbers (interspersed with 0s).

Then we set up the definitions and notation we will use for this

Definition 1. *A* combinatorial class *is a set* $\mathcal{C}$ *with a* weight *function* $\omega : \mathcal{C} \to \mathbb{Z}_{\geq 0}$ *such that for all* $n \in \mathbb{Z}_{\geq 0}$ *the set* $\mathcal{C}_n = \{c \in \mathcal{C} | \omega(c) = n\}$ *is finite.*

Definition 2. *Given a combinatorial class* $\mathcal{C}$ *the* ordinary generating series *of* $\mathcal{C}$ *is*

$$C(x) = \sum_{c \in \mathcal{C}} x^{\omega(c)}$$

You also know from math 239 that $C(x)$ can be written $C(x) = \sum_{n \geq 0} c_n x^n$ where $c_n = |\mathcal{C}_n|$. This notation where the curly capital letter is the class and the corresponding capital italic letter is the generating series while the lowercase italic letter (subscripted) gives the counting sequence will be used without comment in the course.

Next time

Next time we'll talk about three ways bijections come up in enumeration with some examples.

References

- This most closely followed the first part ("Introduction" and the beginning of "Combinatorial classes") of `https://enumeration.ca/`
- Another source is chapters 1 and 4 of the old CO 330 course notes `https://melczer.ca/330/WagnerNotes.pdf`. A different kind of binary rooted tree can be found at the beginning of chapter 6 in those notes – one ends up with the same counts despite the difference in definition, can you see why?