CO 330 LECTURE 20 SUMMARY

FALL 2026

Summary

We started labelled counting. We started by drawing all the rooted trees up to three vertices on the board (there's one with one vertex, two with two vertices, and 9 with three vertices). To label a rooted tree on n vertices we give a bijection between the vertices of the tree and $\{1, \ldots, n\}$. You noticed that there are at most $n!$ labellings of a rooted tree on n vertices and at least n labellings.

We noticed we can also label graphs, and we can think of permutations as labellings of a row of dots.

The correct notion of generating series for labelled objects is the exponential generating series. Let $\mathcal{A}$ be a labelled combinatorial class then the *exponential generating series* is

$$A(x) = \sum_{a \in \mathcal{A}} \frac{x^{w(a)}}{w(a)!} = \sum_{n \geq 0} \frac{a_n}{n!} x^n$$

Note we are using the same notation for ordinary generating series as for exponential generating series, so we'll have to write enough words that it is clear in any given context.

The exponential generating series for permutations is $P(x) = \sum_{n \geq 0} \frac{n!}{n!} x^n = \frac{1}{1-x}$. As another example we did cycles (these are non-empty and directed) and by elementary counting we realized there are $(n-1)!$ of them of size n so the exponential generating series is $C(x) = \sum_{n \geq 1} \frac{(n-1)!}{n!} x^n = \sum_{n \geq 1} \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right)$. We observed here that

$$P(x) = \exp(C(x))$$

and this is our first example of a much more general phenomenon. We can see a permutation by its cycle decomposition, so we can think of a permutation as a labelled collection of cycles. In this way the cycles are the "connected" objects here and the permutations are all (not necessarily connected) objects. Exponentiation captures this connected-to-general relationship across labelled objects as we'll prove next week. It is much simpler than MSet which plays the same role for unlabelled objects.

Next we started talking about labelled specifications. We saw that $\mathcal{E}$ and $\mathcal{Z}$ work exactly as before and their exponential generating series are the same as their ordinary generating series. We saw that disjoint union also works the same as before and the exponential generating series of the disjoint union is the sum of the exponential generating series of the two pieces for the same reasons as before.

We ended the class by noticing that cartesian product does not work out of the box. If we take an ordered pair of two labelled objects then the pair contains repeated labels so it is not itself a labelled object. Next time we will fix this problem and give a more rigorous definition of what labelled object even means.

Next time

Labelled product and more labelled operations.

References

- https://enumeration.ca/extensions/labelled-constructions/
- Chapter 11 of https://melczer.ca/330/WagnerNotes.pdf