CO 330 LECTURE 21 SUMMARY

FALL 2026

Summary

In order to get to the labelled product we gave a slightly more formal definition of labelled counting (if you still find this unsatisfying, you'll probably like combinatorial species, but that's not in this course).

Let $\mathcal{A}$ be an unlabelled combinatorial class built by a specification using $\mathcal{E}$, $\mathcal{Z}$, and admissible operations. Then for $a \in \mathcal{A}$ $w(a)$ is the number of copies of $\mathcal{Z}$ in the construction of a. Call these the *atoms* of a.

- A *labelling* of $a \in \mathcal{A}$ is a bijection from the atoms of a to $\{1, \ldots, w(a)\}$.
- A *weak labelling* of $a \in \mathcal{A}$ is an injection from the atoms of a to $\mathcal{Z}_{\geq 1}$.

An *order preserving relabelling* of a weakly labelled object a is a wek labelling of a such that for any two atoms z_1 and z_2 of a, $z_1 < z_2$ in the original weak labelling if and only if $z_1 < z_2$ in the new weak labelling.

Given $S \subseteq \mathcal{Z}_{\geq 1}$ with $|S| = w(a)$, a a labelled object, there is a unique order preserving relabelling of a with elements of S. Write a_S for this relabelled a.

Now we can define the labelled product.

Definition 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be labelled combinatorial classes. Then*

$$\mathcal{A} * \mathcal{B} = \bigcup_{n \geq 0} \left(\bigcup S \subseteq \{1, \ldots, n\} \left\{ (a_S, b_{\{1,\ldots,n\} \setminus S}) : a \in A_{|S|}, b \in B_{n-|S|} \right\} \right)$$

We did an example because this is something looks worse written out than it really is. The point is we have to split the labels $\{1, \ldots, n\}$ among the a part and the b part compatibly with the original labellings of a and b. We did a calculation to check that the exponential generating series

$$(A * B)(x) = A(x)B(x)$$

which worked out nicely because the choice of subset gave us a binomial coefficient that made the factorials in the exponential generating series do the right thing.

From this, when $\mathcal{C}_0 = \emptyset$, we can define

$$\mathsf{Seq}(\mathcal{C} = \mathcal{E} \cup \mathcal{C} \cup \mathcal{C} * \mathcal{C} \cup \mathcal{C} \cup \mathcal{C} \cup \mathcal{C} \cup \cdots$$

which has exponential generating series $1/(1 - C(x))$. We'll write restrictions as subscripts as before, eg $\mathsf{Seq}_k(\mathcal{C}) = \mathcal{C} * \cdots * \mathcal{C}$ (k times).

We also define

$$\mathsf{Set}_k(\mathcal{C})$$

to be the combiantorial class of k-element sets of obejcts from $\mathcal{C}$ and when $\mathcal{C}_0 = \emptyset$

$$\mathsf{Set}(\mathcal{C}) = \bigcup_{k \geq 0} \mathsf{Set}_k(\mathcal{C}).$$

We checked that if $\mathcal{A} = \mathsf{Set}_k(\mathcal{C})$ and $\mathcal{B} = \mathsf{Set}(\mathcal{C})$ then the exponential generating series are

$$A(x) = \frac{C(x)^k}{k!} \quad \text{and} \quad B(x) = \exp(C(x))$$

The key to this proof is that similarly to counting subsets there is a bijection $\mathcal{A} \times \mathcal{S}_k \leftrightharpoons \mathsf{Seq}_k(\mathcal{C})$. This is much simpler than in the unlabelled case because the labels mean that there are no identical copies in the sequence.

Next time

More constructions and then examples

References

- `https://enumeration.ca/extensions/labelled-constructions/`
- Chapter 11 of `https://melczer.ca/330/WagnerNotes.pdf`