CO 330 LECTURE 22 SUMMARY

FALL 2026

Summary

We finished our labelled constructions.

Very similiarly to Set, given a labelled combinatorial class $\mathcal{C}$ we define

$$\mathsf{Cyc}_k(\mathcal{C})$$

to be the combiantorial class of k-element (directed) cycles of obejcts from $\mathcal{C}$ and when $\mathcal{C}_0 = \emptyset$

$$\mathsf{Cyc}(\mathcal{C}) = \bigcup_{k\geq 0} \mathsf{Cyc}_k(\mathcal{C}).$$

We checked that if $\mathcal{A} = \mathsf{Cyc}_k(\mathcal{C})$ and $\mathcal{B} = \mathsf{Cyc}(\mathcal{C})$ then the exponential generating series are

$$A(x) = \frac{C(x)^k}{k} \quad \text{and} \quad B(x) = \log\left(\frac{1}{1-C(x)}\right).$$

The proof is also similar to the Set case but with a cyclic action isntead of a permutation action.

We've alredy seen this in the example of permutations, which we can now rewrite with this notation as the set of permutations: $\mathcal{S} = \mathsf{Seq}(\mathcal{Z})$ so $S(x) = 1/(1-x)$ and the set of permutations that are a single cycle: $\mathcal{C} = \mathsf{Cyc}(\mathcal{Z})$ so $C(x) = \log(1/(1-x))$ but also $\mathcal{S} = \mathsf{Set}(\mathcal{C})$ (the cycle decomposition of permutations), which is consistent as $S(x) = \exp(\log(1/(1-x))) = 1/(1-x)$.

Whenever we have connected somethings (eg connected graphs), then Set brings us from the connected ones to all of them (eg all graphs), and this is exp at the level of the exponential generating series.

We have one more construction. Again let $\mathcal{C}$ be a labelled combinatorial class. Then the pointed version of $\mathcal{C}$, denoted $\mathcal{C}^\bullet$ is the set of pairs of an element $c \in \mathcal{C}$ and an atom of c. (Usually we draw this by drawing c and marking the special atom with an arrow or a circle or similar.) Then we saw that $C^\bullet(x) = x\frac{d}{dx}(C(x))$.

A nice example of all this is set partitions. Given a set S, a set partition of S is a decomposition of S into disjoint nonempty subsets (called parts) whose union is S where the subsets have no additional structure on them (just a set of subsets).

We wrote down the set partitions of $\{1,2,3\}$ and you worked out the set partitions of $\{1,2,3,4\}$. The number of set partitions of $\{1,2,\ldots,n\}$ into k parts is denoted $\left\{ {n \atop k} \right\}$ and these numbers are called the Sirling numbers of the second kind.

We can specify set partitions as labelled objects. I didn't emphasize in class, but someone asked at the end, but the magic here is that the labelling will just be the elements of

$\{1, 2, \ldots, n\}$ so we don't need to be clever to keep track of it. The parts are simply nonempty sets of atoms $\mathsf{Set}_{\geq 1}(\mathcal{Z}$ and then set partitions with k parts are sets of k parts, namely

$$\mathsf{Set}_k(\mathsf{Set}_{\geq 1}(\mathcal{Z}))$$

while all set partitions are

$$\mathsf{Set}(\mathsf{Set}_{\geq 1}(\mathcal{Z}))$$

so the exponential generating series are

$$\Pi_k(x) = \frac{(e^x - 1)^k}{k!}$$

and

$$\Pi(x) = e^{e^x - 1}$$

respectively. The coefficients of the latter are called the Bell numbers. From the former we can compute (note I made a mistake and didn't take out the $n!$ at the end)

$$\frac{1}{n!}\left\{ {n \atop k} \right\} = [x^n]\Pi_k(x) = \frac{1}{k!}[x^n] \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{(k-j)x} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(k-j)}{n!}$$

and so the correct final formula is

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)$$

Next time

Trees as one final example that I will do and then you will do some examples.

References

- `https://enumeration.ca/extensions/labelled-constructions/`
- Chapter 11 of `https://melczer.ca/330/WagnerNotes.pdf`