CO 330 LECTURE 2 SUMMARY

FALL 2026

Summary

We talked about bijections today. First we noticed we've already seen bijections used for counting, for example

$$\mathcal{B} \fallingdotseq \{\bullet\} \cup \{\bullet\} \times \mathcal{B} \times \mathcal{B}$$

from last class. We reminded ourselves of the definition of bijection and inverse

Definition 1. *Let A and B be sets. Then $f : A \to B$ is a* bijection *if it is* injective *(for all $a, a' \in A$, $a \neq a' \Rightarrow f(a) \neq f(a')$) and* surjective *(for all $b \in B$ there exists an $a \in A$ such that $f(a) = b$).*

Definition 2. *Suppose $f : A \to B$ then $g : B \to A$ is the inverse of f if $g(f(a)) = a$ for all $a \in A$ and $f(g(b)) = b$ for all $b \in B$.*

The point for counting is that when there's a bijection between two sets then they have the same size. Furthermore, in combinatorics it is often nicest to prove bijections by describing the map, describing its purported inverse, then checking that it really is the inverse. When you do that don't forget to check well-definedness, particularly that the maps really do map where you claim they do. The fact that this works relies on the following lemma.

Lemma 3. *$f : A \to B$ has an inverse if and only if f is a bijection.*

I think most instructors do cover that in 239, but see the references if you haven't seen it or if you want to see it again.

Then we talked about bijections for combinatorial proofs, that is when you want to prove that LHS = RHS for some likely algebraic formula, and you do it by finding a set A for which $|A| =$ LHS and a set B for which $|B| =$ RHS and then show A and B are in bijection. As a variant maybe you find one set A so that one way of counting it gives $|A| =$ LHS and another way of counting it gives $|A| =$ RHS. In either case you've proved LHS = RHS in a way that is both often fun and can provide really different insight than an algebraic proof. We did one example with a binomial identitity. In a month or so we'll upgrade these kinds of proofs with parameters when we do q-counting.

It's useful for combinatorial proofs to have a good vocabulary of things counted by common numbers and sequences, so you can think of those pieces when you are trying to build your proof. There's one such table in the references which aligns closely with what we put on the board, but there are many more.

Finally, we noted that often what we want is not just a bijection but a *weight preserving bijection.*

Definition 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes. $f : \mathcal{A} \to \mathcal{B}$ is a* weight preserving bijection *if it is a bijection between the sets $\mathcal{A}$ and $\mathcal{B}$ and $f(a) = b \Rightarrow w(a) = w(b)$.*

A weight preserving bijection restricts to a bijection from $\mathcal{A}_n$ to $\mathcal{B}_n$ for all n and so $a_n = b_n$ for all n and so $A(x) = B(x)$. In fact the converse holds, but not necessarily in a satisfying way. Suppose $\mathcal{A}$ and $\mathcal{B}$ are combinatorial classes with $A(x) = B(x)$. Then the coefficients of $A(x)$ and $B(x)$ agree and hence $|\mathcal{A}_n| = |\mathcal{B}_n|$ for all n. So these are finite set of the same size and so there is a bijection between them. Gluing such bijections for each n we get a weight preserving bijection between $\mathcal{A}$ and $\mathcal{B}$. However, if you just take any old bijection on each size then putting them together isn't necessarily a nice or insightful bijection. Finding a nice one takes thought (and is usually one of the fun parts).

As an example I left you with the puzzle of finding a nice bijection between binary rooted trees and the well formed parenthesizations that were mentioned in the review session. Note that with the most obvious weight functions, you'll actually get an offset in weight. You can either just make a bijection that's not weight preserving but transforms the weight in a nice controlled way (someone asked a question about this and we talked about it a little) or you can adjust the weight functions so that it lines up exactly.

Next time

Next time we'll get into the formalism of formal power series.

References

- This most closely followed `https://enumeration.ca/toolbox/bijections/`
- Another source is chapters 1-4 of `https://melczer.ca/330/WagnerNotes.pdf`.