CO 330 LECTURE 3 SUMMARY

FALL 2026

Summary

First we talked about bijections between binary rooted trees and well formed parenthesizations, which was the question we ended with last time. The first thing you could do is do it recursively. There's no choice in the base case (one object in each case), but then both of them have a decomposition where outside the base case you make the new object out of two smaller objects: $\mathcal{B} \leftrightharpoons \{\bullet\} \cup \{\bullet\} \times \mathcal{B} \times \mathcal{B}$ and $\mathcal{W} \leftrightharpoons \epsilon \cup (\mathcal{W})\mathcal{W}$, so we can recursively define a bijection by taking a binary rooted tree, inductively mapping the left and right children to well formed parenthesizations and then gluing those two parenthesizations together via $(\mathcal{W})\mathcal{W}$. That is correct but not entirely satisfing so it is nice to find a direct bijection. In this case the way to do it is by tree traversals. There are a few possibilities. One way is to take a counterclockwise path around the outside of the tree (implicitly an infix traversal) and every time you meet an internal vertex from the left side put a (while every time you meet an internal vertex from the bottom put a).

Next we started the formalism of formal power series. We defined ring, ring of polynomials, ring of formal power series, and ring of formal Laurent series. The latter three definitions are almost the same, for the first the expressions start at 0 and are of finite length, for the second they start at 0 and are infinite, for the last they can start at any integer.

I'll let you look up ring in the references (you know many examples from math 135/145 even though one doesn't say the word ring in that course), but we do need a bit of ring vocabulary. A ring R is *commutative* if $a \cdot b = b \cdot a$ for all $a, b \in R$. A ring has *no zero divisors* if $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$. A commutative ring with no zero divisors is called an *integral domain*. A element a in a commutative ring R is *invertible* if there exists a $b \in R$ such that $a \cdot b = 1$. An integral domain with all nonzero elements invertible is called a *field*.

Here are the definitions of polynomials, formal power series, and formal Laurent series:

Definition 1. *Let R be an integral domain. The ring of* polynomials *over R, denoted $R[x]$ is the set of expressions of the form*

$$a_0 + a_1x + \cdots a_nx^n$$

with $a_0, a_1, \ldots, a_n \in R$ and $n \in \mathbb{Z}_{\geq 0}$ with operations

$$(a_0+a_1x+\cdots a_nx^n)+(b_0+b_1x+\cdots b_mx^m) = (a_0+b_0)+(a_1+b_1)x+\cdots+(a_{\max(m,n)}+b_{\max(m,n)})x^{\max(m,n)}$$

where $a_\ell = 0$ for $\ell > n$ and $b_\ell = 0$ for $\ell > m$, and

$$(a_0 + a_1x + \cdots a_nx^n)(b_0 + b_1x + \cdots b_mx^m) = \sum_{k=0}^{m+n} \left(\sum_{\ell=0}^{k} a_\ell b_{k-\ell} \right) x^k$$

Definition 2. *Let R be an integral domain. The ring of* formal power series *over R, denoted $R[[x]]$ is the set of expressions of the form*

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

with $a_0, a_1, a_2, \ldots \in R$ and with operations

$$(\sum_{n=0}^{\infty} a_n x^n) + (\sum_{m=0}^{\infty} b_m x^m) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

and

$$(\sum_{n=0}^{\infty} a_n x^n)(\sum_{m=0}^{\infty} b_m x^m) = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} a_\ell b_{k-\ell} \right) x^k$$

Definition 3. *Let R be an integral domain. The ring of* formal Laurent series *over R, denoted $R((x))$ is the set of expressions of the form*

$$\sum_{n=I}^{\infty} a_n x^n$$

with $a_0, a_1, a_2, \ldots \in R$ and $I \in \mathbb{Z}$, with operations

$$(\sum_{n=I}^{\infty} a_n x^n) + (\sum_{m=J}^{\infty} b_m x^m) = \sum_{k=\min(I,J)}^{\infty} (a_k + b_k) x^k$$

where $a_\ell = 0$ for $\ell < I$ and $b_\ell = 0$ for $\ell < J$, and

$$(\sum_{n=I}^{\infty} a_n x^n)(\sum_{m=J}^{\infty} b_m x^m) = \sum_{k=I+J}^{\infty} \left(\sum_{\ell=I}^{k-J} a_\ell b_{k-\ell} \right) x^k$$

Throughout, these definitions make sense in and of themselves at a purely formal level, but they are also good definitions in a more human sense because they line up with the calculus functions that are written using the same or similar notation when those calculus functions are defined. That is why we use this kind of notation for formal power series instead of just writing them as sequences of coefficients and defining the ring operations directly on these sequences.

Next time

Next time we'll prove that formal Laurent series are the field of fractions of formal power series and proceed to the generalized binomial theorem.

References

- `https://enumeration.ca/toolbox/generating-functions/`
- The beginning of chapter 7 of `https://melczer.ca/330/WagnerNotes.pdf`.