CO 330 LECTURE 4 SUMMARY

FALL 2026

Summary

First we showed that formal Laurent series are exactly what we need to be able to invert all nonzero formal power series. We needed a few things in order to say this algebraically.

Lemma 1. *Let R be an integral domain. Then $A(x) = \sum_{n\geq 0} a_n x^n \in R[[x]]$ has a multiplicative inverse in $R[[x]]$ if and onyl if a_0 is invertible in R.*

You might have essentially proved this in 239/249 (though without saying integral domain!) what you do is suppose you have $B(x) = \sum_{n\geq 0} b_n x^n$, multiply out $A(x)B(x) = 1$ and then equate coefficients.

This next is really a definition/theorem the proof of which we left to your future algebra classes if you're interested

Definition 2. *Let R be an integral domain. The field of fractions of R, $\mathsf{Frac}(R)$ is the smallest field containing R and can be conceretely constructed by $\mathsf{Frac}(R) = R \times (R \setminus \{0\})/\sim$ where $(a,b) \sim (c,d)$ if and only if $ad = bc$ and where we define the operations $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and $\frac{a}{b} \cdot \frac{c}{d} = \frac{a\cdot c}{b\cdot d}$, using the notation $\frac{a}{b}$ for the ordered pair (a,b).*

The point here is that you can use exactly the elementary school notion of fractions at this level of generality. It is analogous to how you build $\mathbb{Q}$ from $\mathbb{Z}$.

Proposition 3. *Let K be a field. The field of fractions of $K[[x]]$ is $K((x))$.*

Proof. The first step of the proof is to show that $K[[x]]$ is an integral domain so the notion of field of fractions makes sense. This is on your assignment.

Next take $(f,g) \in K[[x]] \times (K[[x]] \setminus \{0\})$ Since g is not zero it has at least one nonzero coefficient and so we can let n be the smallest power of x with a nonzero coefficient in g. Then write $g = x^n \tilde{g}$ so $\tilde{g}$ is invertible in $K[[x]]$ by the lemma above. Then $(f,g) = (f, x^n\tilde{g}) \sim (f\tilde{g}^{-1}, x^n)$. We can view this as an element of $K((x))$ by taking the series expansion of $f\tilde{g}^{-1}$ and shifting down all the exponents by n.

Furthermore, if $f = 0$ then $(f,g) \sim (0,1)$, while if $f \neq 0$ let m be the smallest power of x with a nonzero coefficient in g and write $f = x^m\tilde{f}$ Then if $m \geq n$ we have $(f,g) \sim (x^{m-n}\tilde{f}\tilde{g}^{-1}, 1)$ and if $m < n$ then we have $(f,g) \sim (\tilde{f}\tilde{g}, x^{n-m})$. In this way every (f,g) has exactly one element in its $\sim$ equivalence class of the form $(h,1)$ with $h \in K[[x]]$ or (h, x^d) with $h \in K[[x]]$ having nonzero constant coefficient and $d \geq 1$, and these elements are in bijection with the elements of $K((x))$. Furthermore the operations agree, which I left to you as an exercise. $\square$

Next we looked at the generalized binomial theorem. First we upgraded our definition of binomial coefficients

Definition 4. *In* $\mathbb{Q}[y]$ *define*

$$\binom{y}{n} = \frac{y(y-1)\cdots(y-(n-2))(y-(n-1))}{n!}$$

for $n \in \mathbb{Z}_{\geq 0}$.

We noted that if you plug in a positive integer for y this agrees with your first definition of binomial coefficients. Next we defined taking a yth power of $(1+x)$.

Definition 5. $(1+x)^y = \exp(y\log(1+x)) \in (\mathbb{Q}[y])[[x]]$

However this definition depends on known that exp and log mean as formal power series, knowing that they satisfy the expected operations, and knowing when we can take compositions of formal power series. We'll talk about those things next time.

Theorem 6 (Generalized binomial theorem)**.**

$$(1+x)^y = \sum_{n=0}^{\infty} \binom{y}{n} x^n$$

in $(\mathbb{Q}[y])[[x]]$.

Next time

Next time we'll talk about exp and log and when formal power series operations are valid.

References

- `https://enumeration.ca/toolbox/generating-functions/`
- The beginning of chapter 7 of `https://melczer.ca/330/WagnerNotes.pdf`.