CO 330 LECTURE 6 SUMMARY

FALL 2026

Summary

You did some examples in groups about validity of formal power series operations. The key thing is composition. Given $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$ define $A(B(x)) = \sum_{n\geq 0} a_n B(x)^n$ but this is only valid when either $b_0 = 0$ or $A(x)$ is a polynomial.

The more conventional way to capture these kinds of ideas is to define a notion of convergence in the ring of formal power series. Specifically, we have the following definition

Definition 1. *Say a sequence $A_0(x), A_1(x), \ldots$ of formal power series in $R[[x]]$* converges *if for all $n \in \mathbb{Z}_{\geq 0}$ there exists an $L \in \mathbb{Z}_{\geq 0}$ and $c_n \in R$ such that $[x^n]A_k(x) = c_n$ for $k \geq L$ in which case we write* $\lim_{n\to\infty} A_n(x) = \sum_{i\geq 0} c_i x^i$.

The idea here is that a sequence converges if for each n the coefficient of x^n eventually stabilizes. Don't confuse this notion of convergence in the ring of formal power series with convergence of a series. There are lots of other topology words we could say here, which you can look up if they interest you. This is a particularly nice framework for making sense of infinite sums and products, just define them to be the limit of the finite partial sums or partial products when those converge and not defined otherwise. This does also tie to the notion of valid because if we truncate the input series then we can apply the operator to those truncated series to get a sequence of output series and those will converge precisely when the operation was valid.

We ended the class with you doing some key examples to illustrate the notion of convergence in the ring of formal power series. Some of them can be tricky!

Next time

Next time we'll be back to combinatorics with combinatorial constructions.

References

Valid is a quirky thing I like to do as a lead in to the notion of convergence, so you won't really find that anywhere, but you can read about convergence in the following places (including examples and counterexamples if you missed the ones at the end of class).

- "Formal power series as a metric space" in `https://enumeration.ca/toolbox/generating-functions/`
- Definition 7.10 and surrounding of `https://melczer.ca/330/WagnerNotes.pdf`.