Midterm

Solutions

Instructions:

- All answers must be justified, unless otherwise stated.
- Unless otherwise stated, you may use any result proved or stated in class but you should be explicit about which result you are using.
- For full marks you should answer **any three of the questions**. If you answer more than three only the first three will be graded.
- Please put your solutions in the space provided. If you need more space use the last page and clearly indicate the problem number your solution corresponds to.

1. (5 points) *(Answer any three questions.)*

   Let R be an integral domain in which the positive integers have multiplicative inverses. Prove that for $a, b \in R$

   $$\exp((a+b)x) = \exp(ax)\exp(bx)$$

   as formal power series.

$$\exp(ax)\exp(bx) = \left(\sum_{n\geq 0} \frac{a^n}{n!}x^n\right)\left(\sum_{n\geq 0}\frac{b^n}{n!}x^n\right) \quad \text{(def of exp for formal pow series)}$$

$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{a^k}{k!}\frac{b^{n-k}}{(n-k)!}\right)x^n \quad \text{(def of product of formal pow series)}$$

$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\right)\frac{x^n}{n!}$$

$$= \sum_{n\geq 0}(a+b)^n\frac{x^n}{n!} \quad \text{(binomial theorem)}$$

$$= \exp((a+b)x) \quad \text{(def of exp)}$$

2. *(Answer any three questions.)*

(a) (3 points) Suppose $\mathcal{A}$ and $\mathcal{B}$ are combinatorial classes and $\mathcal{B}$ has no elements of weight 0. Show that $\mathcal{A} \leftrightharpoons \mathsf{Seq}(\mathcal{B})$ if and only if $\mathcal{A} \leftrightharpoons \mathcal{E} \cup \mathcal{A} \times \mathcal{B}$.

Observe $\mathrm{Seq}(\mathcal{B}) = \mathcal{E} \cup \mathrm{Seq}_{\geq 1}(\mathcal{B})$ as every sequence is either empty or has length ≥ 1

$= \mathcal{E} \cup (\mathrm{Seq}(\mathcal{B})) \times \mathcal{B}$ as every sequence of length ≥ 1 is a sequence of length one less followed by a last element

so $\mathcal{A} \rightleftharpoons \mathrm{Seq}(\mathcal{B})$ implies $\mathcal{A} \rightleftharpoons \mathcal{E} \cup (\mathrm{Seq}(\mathcal{B})) \times \mathcal{B}$ implies $\mathcal{A} \rightleftharpoons \mathcal{E} \cup \mathcal{A} \times \mathcal{B}$

furthermore $\mathcal{A} \rightleftharpoons \mathcal{E} \cup \mathcal{A} \times \mathcal{B}$ implies $\mathcal{A} \rightleftharpoons \mathcal{E} \cup (\mathcal{E} \cup \mathcal{A} \times \mathcal{B}) \times \mathcal{B} = \mathcal{E} \cup \mathcal{B} \cup \mathcal{A} \times \mathcal{B}^2$

which implies $\mathcal{A} \rightleftharpoons \mathcal{E} \cup \mathcal{B} \cup (\mathcal{E} \cup \mathcal{A} \times \mathcal{B}) \times \mathcal{B}^2 = \mathcal{E} \cup \mathcal{B} \cup \mathcal{B}^2 \cup \mathcal{A} \times \mathcal{B}^3$

continuing likewise (formally you can do a quick induction but not necessary for an exam)

$\mathcal{A} \rightleftharpoons \mathrm{Seq}(\mathcal{B})$

(b) (2 points) Give an example (from class or describe it) of a combinatorial class that can be decomposed as in part a. Be explicit about what both $\mathcal{A}$ and $\mathcal{B}$ are.

Any of our Catalan or Motzkin examples give something that will work well here.

For instance $\mathcal{B}$ could be plane rooted trees and then $\mathcal{A}$ is ordered forests (aka lists) of plane rooted trees.

3. *(Answer any three questions.)*

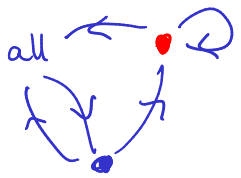
Let $\mathcal{T}$ be the class of plane rooted trees with red and blue vertices where each red vertex has an even number of blue children and an odd number of red children with all blue children to the left of all red children, and each blue vertex has at most 2 children of any colour.

(a) (3 points) Give a specification for $\mathcal{T}$.

$$\mathcal{T} = \mathcal{T}_\bullet \cup \mathcal{T}_\circ$$
$$\mathcal{T}_\bullet \rightleftharpoons \mathcal{Z}_\bullet \times \mathrm{Seq}(\mathcal{T}_\bullet \times \mathcal{T}_\bullet) \times \mathcal{T}_\bullet \times \mathrm{Seq}(\mathcal{T}_\circ \times \mathcal{T}_\circ)$$
$$\mathcal{T}_\circ \rightleftharpoons \mathcal{Z}_\circ \cup \mathcal{Z}_\circ \times \mathcal{T} \cup \mathcal{Z}_\circ \times \mathcal{T}^2$$

did you notice the question really should have swapped these (or other possible fixes) as otherwise no finite tree can have any red vertices. Fortunately you can still answer it as is

(b) (1 point) What is the dependency digraph of your specification?



(c) (1 point) What can you say about the colours of the leaves of a tree in $\mathcal{T}$?

All leaves are blue because every red vertex has an odd number of red children, hence has at least one red child, hence is not a leaf.

but note the remark above — oops.

4. (5 points) *(Answer any three questions.)*

Fix $c \in \mathbb{Z}_{>0}$. For each $n \in \mathbb{Z}_{\geq 0}$ find the number of plane rooted trees with n vertices where each vertex has a number of children divisible by c. Leave your binomial coefficients with nonnegative arguments.

Let $\mathcal{T}$ be the class of these trees, then

$$\mathcal{T} \rightleftharpoons \mathcal{Z} \times \mathrm{Seq}(\mathcal{T}^c)$$

so $T(x) = \dfrac{x}{1 - T(x)^c}$

using LIFT we get that the number we want is

$$[x^n]\, T(x) = \frac{1}{n}[u^{n-1}](1-u^c)^{-n}$$

$$= \begin{cases} \frac{1}{n}\binom{-n}{\frac{n-1}{c}}(-1)^{\frac{n-1}{c}} & \text{if } c \mid n-1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{n}\binom{n+\frac{n-1}{c}-1}{\frac{n-1}{c}} & \text{if } c \mid n-1 \\ 0 & \text{otherwise} \end{cases}$$

by the negative binomial theorem.

Extra space for solutions.