CO 430/630 W26 ASSIGNMENT 1

DUE WEDNESDAY JANUARY 28 AT 10AM IN CROWDMARK

Do any 4 of the 6 problems!

The point of the assignment is for you to learn by doing the problems. I'm sure you can look up answers to many of these in various ways, but you largely defeat the purpose and undercut your own learning if you look things up instead of thinking. Having said that, you may use any resources, animate or inanimate, **provided you credit them**. In particular credit any use of generative AI, even mundane uses like in generating tikz for figures.

(1) Let F be a field with $\mathrm{char}(F) \neq 2$ and let $A(x) \in F[[x]]$. Prove that $A(x)$ has a square root in $F[[x]]$ if and only if $\mathrm{val}_x(A(x)) = 2m$ for some $m \in \mathbb{Z}_{\geq 0}$ and $[x^{2m}]A(x)$ is square in F.

(2) (a) Prove the chain rule for formal derivatives of formal power series. *You may find you want to use other properties you know from calculus; this is a good idea, but you have to prove those too.*

(b) Recall $L(x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n$. Prove $L(\exp(x) - 1) = x$ as formal power series.

(3) (a) Give an example of a sequence $A_0(x, y), A_1(x, y), \ldots \in \mathbb{Q}[[x, y]]$ for which $\sum_{n\geq 0} A_n(x, y)$ converges with respect to val_x but does not converge with respect to val_y.

(b) Define $\mathrm{val}_{x,y} A(x, y) = \min\{i + j : [x^i y^j]A(x, y) \neq 0\}$. Give an example of a sequence $A_0(x, y), A_1(x, y), \ldots \in \mathbb{Q}[[x, y]]$ for which $\sum_{n\geq 0} A_n(x, y)$ converges with respect to $\mathrm{val}_{x,y}$ but does not converge with respect to either of val_x or val_y.

(4) Consider the class of plane rooted trees where the edges can be either red or blue, but at each vertex v, among the edges from v to v's children, all the red edges come to the left of all the blue edges.

(a) Find an equation for the ordinary generating series of this class of trees.

(b) Use LIFT to obtain an expression for the number of trees of this class with n vertices.

(5) (a) Let $\mathcal{A}$ be a combinatorial class. Show that the number of objects in $\mathcal{A}$ of size at most n is $[x^n]\frac{A(x)}{1-x}$.

(b) Give at least three different ways (using the symbolic method, but decomposing differently) to show that the ordinary generating series of weak compositions with two parts (that is, pairs of nonnegative integers) is $\frac{1}{(1-x)^2}$.

(6) Consider the class of plane rooted trees where each vertex has 0, 1, or 2 children. Call this the class of unary-binary trees.

(a) Give an equation for the ordinary generating series of unary-binary trees and solve it to obtain an algebraic expression for this generating series.

(b) Give a second argument to obtain the same answer involving composition with the generating series for binary trees (plane rooted trees where each vertex has 0 or 2 children). If your first argument was of this form, go back and give a different argument for the first part.