CO 430/630 W26 ASSIGNMENT 1

SOLUTIONS

(1) The interesting direction is the $\Leftarrow$ direction so let's start with that. It feels like this is a Hensel's lemma question, which it is, and you might first be tempted to set it up as $F(t,x) = x^2 - A(t)$, but this doesn't satisfy the hypotheses for Hensel's lemma and the problem comes down to the fact that we need to normalize $A(x)$ by its lowest order term. So, let $[x^m]A(x) = a^2$ and note that $A(x)/(a^2x^{2m}) - 1 \in F[[x]]_+$. Then set

$$F(t,x) = (1+x)^2 - \frac{A(t)}{a^2t^{2m}}$$

So $F(0,0) = 1 - 1 = 0$ and $F'(t,x) = \frac{d}{dx}F(t,x) = 2(1+x)$ so $F'(0,0) = 2$ which is invertible since $\mathrm{char}(F) \neq 2$. So by Hensel's lemma we get $f(t) \in F[[x]]_+$ such that $0 = F(t, f(t)) = (1+f(t))^2 - A(t)/(a^2t^{2m})$. Therefore $ax^m(1+f(x))$ is a square root of $A(x)$ in $F[[x]]$.

Now for the $\Rightarrow$ direction, suppose $g(x) \in F[[x]]$ is a square root of $A(x)$. Then $2\mathrm{val}_x(g(x)) = \mathrm{val}_x(A(x))$ so letting $m = \mathrm{val}_x(g(x)) \in \mathbb{Z}_{\geq 0}$ we have $\mathrm{val}_x(A(x)) = 2m$ as desired. Further $[x^{2m}](g(x)^2) = ([x^m]g(x))^2$ giving $[x^{2m}]A(x) = ([x^m]g(x))^2$ which is a square in F.

(2) (a) Let $A(x) = \sum_{n\geq 0} a_nx^n$ and $B(x) = \sum_{n\geq 0} b_nx^n$.
Let's prove the sum and product rule first as they are each pretty straightforward but then we'll have them at hand.

$$\frac{d}{dx}(A(x)+B(x)) = \sum_{n\geq 1} n(a_n+b_n)x^{n-1} = \sum_{n\geq 0}(na_n + nb_n)x^{n-1} = A'(x) + B'(x)$$

Additionally, we get an infinite sum rule because by definition an infinite sum is defined as the limit of the partial sums and this limit converges if the coefficients stabilize. So to calculate any particular coefficient in the derivative of the infinite sum, it suffices to take a sufficiently large partial sum wherein that coefficient stabilizes, which is just saying that the derivative of the limit of the partial sums is the limit of the derivative of the partial sums. Now use the finite sum rule in each partial sum and take the limit to obtain the expected infinite sum.

$$\begin{aligned}\frac{d}{dx}(A(x)B(x)) &= \sum_{n\geq 1} n\left(\sum_{k=0}^{n} a_kb_{n-k}\right)x^{n-1}\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} ka_kb_{n-k}\right)x^{n-1} + \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k(n-k)b_{n-k}\right)x^{n-1}\\ &= A'(x)B(x) + A(x)B'(x)\end{aligned}$$

Now the chain rule won't be too messy to deal with.

$$\begin{aligned}\frac{d}{dx}(A(B(x))) &= \frac{d}{dx}\sum_{n\geq 0} a_n B(x)^n \\ &= \sum_{n\geq 0} a_n \frac{d}{dx} B(x)^n \text{ as noted above} \\ &= \sum_{n\geq 1} a_n n B'(x) B(x)^{n-1} \text{ by repeated application of the product rule} \\ &= B'(x)\sum_{n\geq 1} nB(x)^{n-1} \\ &= B'(x)A'(B(x))\end{aligned}$$

(b) First note that $L(x)$ and $\exp(x)-1$ both have zero constant term so $L(\exp(x)-1)$ also has zero constant term. It suffices then to check that $\frac{d}{dx}(L(\exp(x)-1) = 1$. This will be tidier than trying to do something more direct.
First calculate

$$\frac{d}{dx}L(x) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}nx^{n-1} = \sum_{n\geq 1}(-1)^{n-1}x^{n-1} = \frac{1}{1+x}$$

and

$$\frac{d}{dx}\exp(x) = \sum_{n\geq 1}\frac{1}{n!}nx^{n-1} = \sum_{n\geq 1}\frac{1}{(n-1)!}x^{n-1} = \exp(x)$$

so

$$\frac{d}{dx}L(\exp(x)-1) = \frac{1}{1+(\exp(x)-1)}\exp(x) = 1$$

by the chain rule.

(3) (a) Let $A_n(x,y) = x^n y$. Then $\sum_{n\geq 0} A_n(x,y)$ converges to $\frac{y}{1-x}$ in val_x because

$$\mathrm{val}_x\left(\sum_{n=0}^{N} A_n(x,y) - \frac{y}{1-x}\right) = \mathrm{val}_x\left(\sum_{n=0}^{N} x^n y - \frac{y}{1-x}\right) = N+1 \to_{N\to\infty} \infty.$$

However, $\sum_{n\geq 0} A_n(x,y)$ does not converge in val_y as the coefficient of $[y]$ in the partial sum $\sum_{n=0}^{N} A_n(x,y)$ is $\frac{1-x^{N+1}}{1-x}$ which does not stabilize.

(b) We can do something similar here. Let $A_0(x,y) = x+y$, $A_1(x,y) = xy$ and $A_n(x,y) = x^n y + y^n x$. Now $\sum_{n\geq 0} A_n(x,y)$ does not converge in val_y for the same reason as in the previous part as the coefficient of $[y]$ in $\sum_{n\geq 0} A_n(x,y)$ is identical in this part and in the previous part. Similarly $\sum_{n\geq 0} A_n(x,y)$ does not converge in val_x because the $A_n(x,y)$ are all symmetric in x and y so the analogous argument holds. The sum does however, converge in $\mathrm{val}_{x,y}$ to $\frac{x}{1-y} + \frac{y}{1-x} - xy$ because

$$\mathrm{val}_{x,y}\left(\sum_{n=0}^{N} A_n(x,y) - \frac{x}{1-y} - \frac{y}{1-x} + xy\right) = N+2 \to_{N\to\infty} \infty$$

(4) (a) Let $\mathcal{A}$ be the class of these trees with w the weight function which gives the number of vertices. We see that each element of $\mathcal{A}$ is a root along with a sequence of some number of subtrees connected by red edges followed by a sequence of some number of subtrees connected by blue edges, where all the subtrees are also elements of $\mathcal{A}$. In reverse, given a root and two sequences of elements of $\mathcal{A}$, build a new element of $\mathcal{A}$ by connecting those from the first sequence (in order) to the root with red edges and then, moving leftward, connecting all those from the second sequence (in order) to the root with blue edges. The maps thus described are inverses and so we have

$$\mathcal{A} \leftrightharpoons \{\bullet\} \times (\mathcal{A}^*)^2$$

Then by the product and sequence rules we have

$$A(x) = \frac{x}{(1 - A(x))^2}$$

(b) By LIFT for $n \geq 1$ we have

$$[x^n]A(x) = \frac{1}{n}[\lambda^{n-1}](1-\lambda)^{-2n} = \frac{1}{n}\binom{3n-2}{n-1}$$

(5) (a)

$$[x^n]\frac{A(x)}{1-x} = [x^n]A(x)(1+x+x^2+\cdots) = \sum_{i=0}^{n}[x^n]A(x)x^i = \sum_{i=0}^{n}[x^{n-i}]A(x) = a_0+a_1+a_2+\cdots+a_n$$

as desired.

(b) Let $\mathcal{P}$ be the set of weak compositions with two parts.
The first decomposition that comes to my mind is that such a composition is an ordered pair of its two parts so $\mathcal{P} \leftrightharpoons \mathbb{Z}^2_{\geq 0}$, and hence $P(x) = \frac{1}{(1-x)^2}$.
We could also use the previous part of the question (which is why it was there) and say we can make a composition with two parts by first just taking weak compositions with one part, and now if we consider all weak compositions of one part with size up to n, then we can uniquely complete each of those with a second part so they add to n, and hence by the previous part of the question $P(x) = \frac{\frac{1}{1-x}}{1-x}$.
We could also see an element of $\mathcal{P}$ as follows. Take a nonnegative integer m, viewed as a sequence of m dots, then apply the pointing operator to split the sequence of dots into three parts, the part before the dot we're pointing at, which is a nonnegative integer, the pointed dot itself, and the part after which is another nonnegative integer. The parts other than the pointed dot itself give an element of $\mathcal{P}$ so we have $\mathcal{P} \times \{\bullet\} \leftrightharpoons \mathcal{Z}^\bullet_{\geq 0}$, which gives $xP(x) = x\frac{d}{dx}\left(\frac{1}{1-x}\right)$ giving one more time the expression we know for $P(x)$.
For a fourth, you could apply the result from the first part of the question twice starting with $\bullet$.
Did anyone find something other than those four?

(6) Let $\mathcal{U}$ be the class of unary-binary trees with the weight function which counts the number of vertices.

(a) With the usual decomposition of a class of trees in terms of the root and the subtrees at the children of the root we have

$$\mathcal{U} \leftrightharpoons \{\bullet\} \cup \{\bullet\} \times \mathcal{U} \cup \{\bullet\} \times \mathcal{U}^2.$$

So $U(x) = x + xU(x) + xU(x)^2$. We can solve this using the quadratic formula to get

$$U(x) = \frac{-(x-1) \pm \sqrt{(x-1)^2 - 4x^2}}{2x}.$$

Now expanding out the first terms we see that the square root begins with 1 and so the + root has first term $2/2x$ so is not a formal power series. Thus the solution we want is the − root, that is,

$$U(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}.$$

As it turns out this is the generating series for Motzkin numbers, which are a nice sequence that's not quite as ubiquitous as the Catalan numbers, but still shows up a lot.

(b) Now let's think about it another way. Let $\mathcal{T}$ be the combinatorial class of binary trees counted by number of vertices. Consider a unary-binary tree t, we can contract up any paths of vertices with a single child to obtain a binary tree t'. To reconstruct t from t' we need the information of how many single-child vertices were contracted into each vertex – there must be at least one for each vertex so this is the information of an element of $\mathbb{Z}_{>0}$ for each vertex of t'. This we can represent as a composition of $\mathcal{T}$ with $\mathbb{Z}_{>0}$. In particular $\mathcal{U} \leftrightharpoons \mathcal{T} \circ \mathbb{Z}_{>0}$ and so $U(x) = T(x/(1-x))$.
To finish things off, we need an expression for $T(x)$. You might know this in which case you can just use it, but let's derive it as it is similar but slightly simpler than the previous part: $\mathcal{T} \leftrightharpoons \{\bullet\} \cup \{\bullet\} \times \mathcal{T}^2$ as we saw in class so $T(x) = x + xT(x)^2$, so by the quadratic formula

$$T(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x}$$

whereas in the previous part we must take the − root as the other one is not a formal power series. Therefore

$$U(x) = \frac{1 - \sqrt{1 - 4\left(\frac{x}{1-x}\right)^2}}{2\frac{x}{1-x}}$$

$$= \frac{1 - x - \sqrt{(1-x)^2 - 4x^2}}{2x} = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}$$

as we had before.