CO 430/630 LECTURE 2 SUMMARY

WINTER 2026

Summary

We continued by using the valuation we'd defined last time to define a topology on $R[[x]]$.

Definition 1. *Take* $0 < \epsilon < 1$ *and define* $\|A(x)\|_\epsilon = \epsilon^{val_x(A(x))}$ *for* $A(x) \neq 0$ *and* $\|0\|_\epsilon = 0$ *as well as the metric* $d_\epsilon(A(x), B(x)) = \|A(x) - B(x)\|_\epsilon$.

These have the expected properties except that the norm is not a vector space norm because scalars don't pull out. Additionally we get a stronger property than the triangle inequality:

$$\|A(x) + B(x)\|_\epsilon \leq \max\{\|A(x)\|_\epsilon, \|B(x)\|_\epsilon\}$$

and analogously for d_ϵ, so we are in a non-archimedean setting here.

The point of this is that it gives us a nice way to formalize a notion of convergence. Informally this is convergence when coefficients eventually stabilize. Formally:

Definition 2. *For* $A_i \in R[[x]]$ *define* $\lim_{n\to\infty} A_n(x) = A(x)$ *iff*

$$\lim_{n\to\infty} val_x(A_n(x) - A(x)) = \infty$$

in $\mathbb{R}$.

We did some examples illustrating the difference between this and the calculus notion of convergence.

We can use this to tell when formal power series operations are valid. For example we defined infinite sums and infinite products of formal power series as the limits of the finite sums/products.

Composition of formal power series ends up being a nice special case of infinite sum. Let $A(x) = \sum_{n\geq 0} a_n x^n$. Then $A(B(x)) = \sum_{n\geq 0} a_n B(x)^n$ converges iff $\mathrm{val}_x(a_n B(x)^n) \to_{n\to\infty} \infty$ which happens iff a_n is eventually 0 (so $A(x)$ is a polynomial) or $B(x)$ has zero constant term. We write $R[[x]]_+$ for the set of elements of $R[[x]]$ with zero constant term.

Be careful in the multivariate case that you know what valuation you are working with.

There are many formal power series and formal power series operations and properties thereof that you should know, which we did not go into detail about. See Kevin's videos "More FPS operations" and "Special FPS" (see reference below for link) for some more details. Here is the basic checklist: for $A(x), B(x) \in R[[x]]$ with $A(x) = \sum_{n\geq 0} a_n x^n$

- $A(x), B(x)$ are compositional inverses if $A(B(x)) = B(A(x)) = x$.
- The formal derivative is $\frac{d}{dx}A(x) = A'(x) = \sum_{n\geq 1} n a_n x^{n-1}$.
- If $\mathbb{Q} \subseteq R$ then the formal integral is $\int_x A(x) = \sum_{n\geq 0} \frac{a_n}{n+1} x^{n+1}$.
- $\exp(x) = \sum_{n\geq 0} \frac{1}{n!} x^n \in \mathbb{Q}[[x]]$.
- $L(x) = \log(1+x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n \in \mathbb{Q}[[x]]$.

- $B(x,y) = (1+x)^y = \sum_{n\geq 0} \binom{y}{n} x^n \in \mathbb{Q}[y][[x]]$.

Then we finished the class with Hensel's lemma. The statement is

Proposition 3. *Let $F(t,x) \in R[[t,x]]$ and $F'(t,x) = \frac{d}{dx}F(t,x)$. Suppose $F(0,0) = 0$ and $F'(0,0)$ is invertible. Then there exists a unique $f(t) \in R[[t]]_+$ such that $F(t, f(t)) = 0$.*

The idea of the proof is to use linear approximation and Newton's method in this context. Define $f_0(t) = 0$ and

$$f_{n+1}(t) = f_n(t) - \frac{F(t, f_n(t))}{F'(t, f_n(t))}$$

and then check (with linear approximation and val it isn't too bad, but easy to make a small mistake as we did once and corrected) that $\lim_{n\to\infty} f_n(t)$ exists, will serve as our $f(t)$, and no other such is possible. If you missed it you can find it in Kevin's videos or ask a friend.

Next time

Next time we will use Hensel's lemma to show when a compositional inverse exists and then move on other kinds of formal series (formal Laurent series, formal Dirichlet series) before finally starting to use these to count things.

References

Already mentioned are the videos from one of Kevin's offerings `https://www.math.uwaterloo.ca/~kpurbhoo/winter2021-co630/co630-videos.html`.

Another reference is the book Combinatorial Theory by Martin Aigner.