CO 430/630 LECTURE 4 SUMMARY

WINTER 2026

Summary

Today we talked about operations on combinatorial classes and how they correspond to operations on the generating series. The first three were more of a review; if you don't already know them check out your favorite book on generating functions (I like the first chapter of Flajolet and Sedgewick for this material, see the references below).

Let $\mathcal{A}$ and $\mathcal{B}$ be two combinatorial classes with weight functions w_A and w_B respectively. Then

- the disjoint union $\mathcal{A} \cup \mathcal{B}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset$ is a combinatorial class with weight function $w : \mathcal{A} \cup \mathcal{B} \to \mathcal{Z}_{\geq 0}$ given by $w|_{\mathcal{A}} = w_A$ and $w|_{\mathcal{B}} = w_B$ and with ordinary generating series $A(x) + B(x)$,
- the cartesian product $\mathcal{A} \times \mathcal{B}$ is a combinatorial class with weight function $w : \mathcal{A} \times \mathcal{B} \to \mathcal{Z}_{\geq 0}$ given by $w((a, b)) = w_A(a) + w_B(b)$ and with ordinary generating series $A(x)B(x)$,
- provided $\mathcal{A}_0 = \emptyset$ the sequence operator $\mathcal{A}^* = \bigcup_{k \geq 0} A^k$ gives a combinatorial class with weight function $w((a_1, a_2, \ldots, a_k)) = w_A(a_1) + \cdots + w_A(a_k)$ and with ordinary generating series $\frac{1}{1-A(x)}$.

This is already enough to do some examples and we did binary strings, binary trees counted by vertices and by leaves, plane trees counted by vertices, and finally with a bivariate generating series plane trees counted by both vertices and leaves. All those tree examples are well suited to using LIFT to compute the coefficients explicitly and we did so. If you missed the examples check Flajolet and Sedgewick or your favorite source, or check with a friend. The specific examples aren't so important as having experience with the approach in a variety of examples.

Next we revisited the Dirichlet set up. What if we were in the cartesian product set up as above but the natural weight function for our situation was multiplicative rather than additive, so $w((a, b)) = w_A(a) \cdot w_B(b)$. Then the ordinary generating series will not be well behaved but the Dirichlet generating series will be.

Definition 1. *Given $\mathcal{A}$ a combinatorial class with weight function $w : \mathcal{A} \to \mathbb{Z}_{>0}$ the Dirichlet generating series of $\mathcal{A}$ is*

$$\sum_{a \in \mathcal{A}} w(a)^{-s} = \sum_{n \geq 1} a_n n^{-s}$$

Note that it is important that the weight function is going to *positive* integers.

Now suppose $\mathcal{A}$ has a multiplicative weight function over products and further has unique factorization into some indecomposable elements. Let $\mathcal{P}$ be the set of indecomposable elements. Then we have the following proposition.

Proposition 2. *With setup as above*

$$\sum_{n\geq 1} a_n n^{-s} = \prod_{n\geq 2}(1-n^{-s})^{-p_n}$$

We sketched the proof. First define val as before (it really only cared about coefficients not the kind of series) $\text{val}_s(\sum_{n\geq 1} a_n n^{-s}) = \min\{m : a_m \neq 0\}$ and then can define convergence as before and again the intuition is that convergence is when the coefficients stabilize. Then we see the product above is convergent. Now we can just multiply out and interpret

$$\begin{aligned}\prod_{n\geq 2}(1-n^{-s})^{-p_n} &= \prod_{p\in P}(1-w(p)^{-s})^{-1} \\ &= \prod_{p_1^{k_1}p_2^{k_2}\cdots} (w(p_1)^{k_1}w(p_2)^{k_2}\cdots)^{-s} \\ &= \prod_{p_1^{k_1}p_2^{k_2}\cdots} (w(p_1^{k_1}p_2^{k_2}\cdots))^{-s} \\ &= \sum_{a\in\mathcal{A}} w(a)^{-s}\end{aligned}$$

where we used that $(1-w(p)^{-s})^{-1}$ is the Dirichlet generating series for $\{p\}^*$, and the last line is by unique factorization.

The first example is when $\mathcal{A} = \mathcal{Z}_{\geq 1}$ and $\mathcal{P}$ is the set of primes and then the Dirichlet generating series is the Riemann zeta function and the proposition gives its Euler product form. Another example is finite abelian groups which decompose into finite cyclic groups of size p^n. Then if a_n is the number of finite abelian groups of size n the proposition tells us

$$\sum_{n\geq 1} a_n n^{-s} = \prod_{\substack{p \text{ prime} \\ n\geq 1}} (1-(p^n)^{-s})^{-1}$$

The last thing we did this lecture was the multiset operator. As a motivating example, suppose you don't want your rooted trees to have any order structure on the children at each vertex. In this context, instead of a sequence of subtrees at the root you have a multiset of subtrees at the root. How do we deal with this? The Euler product construction above gives the idea but now we're in the formal power series context instead of the Dirichlet context. Specifically let $\mathcal{A}$ be a combinatorial class and say we're interested in $\mathsf{MSet}(\mathcal{A})$, the combinatorial class of multisets of elements of $\mathcal{A}$, where we need to assume $\mathcal{A}_0 = \emptyset$. Without loss of generality we can choose a convenience order, first order all the elements of $\mathcal{A}_1$, then all the elements of $\mathcal{A}_2$ etc, and put the elements of our multiset by first all the copies of the first element, then all the copies of the second element and so on. Then we have

$$\mathsf{MSet}(\mathcal{A}) \leftrightharpoons \prod_{a\in\mathcal{A}}\{a\}^*$$

(with the order implicit in the product). Writing $\mathcal{B} = \mathsf{MSet}(\mathcal{A})$ we have

$$B(x) = \prod_{a\in\mathcal{A}} \frac{1}{1-x^{w(a)}} = \prod_{n\geq 1}\frac{1}{(1-x^n)^{a_n}}$$

giving the product form of the multiset formula. We can rewrite this a bit more

$$\begin{aligned}
B(x) &= \prod_{n\geq 1} \frac{1}{(1-x^n)^{a_n}} \\
&= \exp\left(\log\left(\prod_{n\geq 1} \frac{1}{(1-x^n)^{a_n}}\right)\right) \\
&= \exp\left(\sum_{n\geq 1} a_n \log((1-x^n)^{-1})\right) \\
&= \exp\left(\sum_{n\geq 1} a_n \sum_{i\geq 1} \frac{x^{ni}}{i}\right) \\
&= \exp\left(\sum_{i\geq 1} \frac{1}{i}\left(\sum_{n\geq 1} a_n x^{ni}\right)\right) \\
&= \exp\left(\sum_{i\geq 1} \frac{A(x^i)}{i}\right)
\end{aligned}$$

where we used some properties of the formal power series of exp and log which we haven't proved, but which you can prove and should as an exercise (one bit is one assignment 1). This is the multiset formula. Notice how it has a different flavour than the formulas we've had so far since it isn't a composition where you plug $A(x)$ into another series on account of the x^is showing up.

Then for rooted trees specifically with no plane structure, letting that class be $\mathcal{T}$ we'd have $\mathcal{T} \leftleftharpoons \{\bullet\} \times \mathsf{MSet}(\mathcal{T})$ and so $T(x) = x\exp(\sum_{i\geq 1} T(x^i)/i)$.

Next time

Next time you'll work on a combinatorial proof of LIFT.

References

Chapter I of Flajolet and Sedgewick's book *Analytic Combinatorics* gives a good presentation of this material in much this language, (but not the Dirichlet part).