CO 430/630 LECTURE 9 SUMMARY

WINTER 2026

Summary

We continued our second pass of labelled counting. We noticed that disjoint union of labelled combinatorial classes gives sum of exponential generating series for essentially the same reason this happened for unlabelled classes and ordinary generating series. Product though is a bit more subtle. We don't want to take the cartesian product of labelled classes because the labels will repeat between the two entries of your ordered pair (unless one of the sides is empty). What we need to do for a pair (a, b) is distribute the labels $\{1, 2, \ldots, w(a) + w(b)\}$ between a and b in a way that preserves the relative ordering of the labellings that a and b already have. We need a few definitions to make that rigorous.

We say a *weak labelling* of an object a which is built of atoms is an injection from the atoms of a to $\mathbb{Z}_{\geq 1}$. We say an *order preserving relabelling* of a labelled object a is a weak labelling of a such that for any two atoms z_1 and z_2 of a, $z_1 \leq z_2$ in the original labelling iff $z_1 \leq z_2$ in the new weak labelling. Given $S \subseteq \mathbb{Z}_{\geq 1}$ with $|S| = w(a)$ there is a unique order preserving relabelling of a with the elements of S and we write this as a_S.

Now we can define the *labelled product*

Definition 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes. Then*

$$\mathcal{A} * \mathcal{B} = \bigcup_{n \geq 0} \left(\bigcup_{S \subseteq \{1,\ldots,n\}} \{(a_S, b_{\{1,\ldots,n\}\setminus S}) : a \in \mathcal{A}_{|S|}, b \in \mathcal{B}_{n-|S|}\} \right)$$

where the unions are disjoint.

Then the exponential generating series of $\mathcal{A} * \mathcal{B}$ is

$$\begin{aligned}
\sum_{n\geq 0} \sum_{S\subseteq\{1,\ldots,n\}} \sum_{(a_S, b_{\{1,\ldots,n\}\setminus S})} \frac{x^{w(a)+w(b)}}{(w(a)+w(b))!} &= \sum_{n\geq 0} \sum_{k=0}^{n} \sum_{\substack{S\subseteq\{1,\ldots,n\} \\ |S|=k}} \sum_{(a_S, b_{\{1,\ldots,n\}\setminus S})} \frac{x^n}{n!} \\
&= \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k b_{n-k}}{n!} x^n \\
&= \sum_{n\geq 0} \sum_{k=0}^{n} \frac{a_k x^k}{k!} \frac{b_{n-k} x^{n-k}}{(n-k)!} \\
&= A(x)B(x)
\end{aligned}$$

We did labelled sequence/Kleene star and got exponential generating series $\frac{1}{1-A(x)}$ by the same argument as in the unlabelled case but now using labelled product.

Set is much easier than in the unlabelled case because the labels avoid any possibility of repetitions so if you want labelled k-element sets of objects from $\mathcal{A}$ you just need to take the k-fold labelled product of $\mathcal{A}$ and then divide out by the permutation action on those k elements, that is the exponential generating series of k-element subsets is $A(x)^k/k!$ and summing over all k we get that the exponential generating series for sets of objects from $\mathcal{A}$ is $\exp(A(x))$.

We saw an example of this last time, permutations were sets of cycles. This is a good illustration of how the definition of labelled product maybe seems a little weird at first but actually often ends up being just what you want. Permutations really are sets of their cycles via their cycle decompositions; the labels just do exactly what you want them to.

That's the end of the second pass for labelled counting. The third pass is combinatorial species. Some motivation is the following: does "built of atoms" feel like a kluge? are you bothered by the arbitrariness of labelling with $\{1, \ldots, n\}$ and do those order preserving relabellings seem messy? In a different but related direction does it bother you that there's no set of all graphs (it's a proper class because the vertex set can be any set and that's too big)? Note that we fix the problem with graphs by thinking about graphs on a fixed vertex set; that's no problem.

Now it is finally time to define combinatorial species.

Definition 2. *A* species*, $\mathcal{A}$, is a rule that assigns*

- *to every finite set X a finite set $\mathcal{A}_X$ called the set of $\mathcal{A}$-*structures *on X and*
- *to every bijection $f : X \to Y$ of finite sets a bijection $f_* : \mathcal{A}_X \to \mathcal{A}_Y$ called* transportation of $\mathcal{A}$-structures along f*,*

subject to

- *if $X \neq Y$ then $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$,*
- *$(Id_X)_* : \mathcal{A}_X \to \mathcal{A}_X$ is the identity, and*
- *if $f : X \to Y$ and $g : Y \to Z$ are bijections then $(g \circ f)_* = g_* \circ f_*$.*

Now this is really a category theory definition. Let SET be the category of finite sets with bijections as morphisms. Then a species is a functor $\mathcal{A} : \text{SET} \to \text{SET}$. For each finite set $X \in \text{SET}$ then $\mathcal{A}_X = \mathcal{A}[X]$ is the set of $\mathcal{A}$-structures on XX and if $f : X \to Y$ is a morphism of SET then $f_* = \mathcal{A}[f] : \mathcal{A}[X] \to \mathcal{A}[Y]$ is the transportation of $\mathcal{A}$-structures along f, so this is just a slightly different notation for the things a functor always has. Furthermore, of the properties the second two are also part of the definition of a functor. On the first one is extra.

Given a species $\mathcal{A}$ we say that $a \in \mathcal{A}_X$ and $b \in \mathcal{B}_Y$ are isomorphic iff there exists a bijection $f : X \to Y$ such that $f_*(a) = b$. Note that if $|X| = |Y| = n$ then $|\mathcal{A}_X| = |\mathcal{A}_Y|$ so we can define $a_n = |\mathcal{A}_X|$ and then define the exponential generating series as before.

We made a table, one of the rows was graphs as a species $\mathcal{G}$ where $\mathcal{G}_X$ is the set of simple graphs on the vertex set X and transportation along $f : X \to Y$ takes $G \in \mathcal{G}_X$ to the graph on Y with edge between y_1 and y_2 iff there is an edge between $f^{-1}(y_1)$ and $f^{-1}(y_2)$. I'm a little lazy and didn't retype the table; here's a picture

Name (Notation)	Structures on X	Transport along $f : X \to Y$	EGF
Sets ($\mathcal{E}$)	X, i.e. $\mathcal{E}_X = \{X\}$	$f_*(X) = Y$	$E(x) = \sum_{n\geq 0} 1\frac{x^n}{n!} = e^x$
Linear Orders ($\mathcal{L}$)	$\mathcal{L}_X = \{(x_1, ..., x_n) : \|X\| = n, X = \{x_1, ..., x_n\}\}$	$f_*(x_1, ..., x_n) = (f(x_1), ..., f(x_n))$	$L(x) = \sum_{n\geq 0} n!\frac{x^n}{n!} = \frac{1}{1-x}$
Endofunctions ($\mathcal{N}$)	$\mathcal{N}_X = \{\alpha : X \to X\}$	$f_*(\alpha) = f \circ \alpha \circ f^{-1}$	$N(x) = \sum_{n\geq 0} n^n \frac{x^n}{n!}$
Permutations ($\mathcal{S}$)	$\mathcal{S}_X = \{\alpha : X \to X \text{ is bijection}\}$	$f_*(\alpha) = f \circ \alpha \circ f^{-1}$	$S(x) = \sum_{n\geq 0} n!\frac{x^n}{n!} = \frac{1}{1-x}$
Cyclic Permutations ($\mathcal{C}$)	$\mathcal{C}_X = \{\alpha : X \to X \text{ is a cycle}\}$	$f_*(\alpha) = f \circ \alpha \circ f^{-1}$	$C(x) = \sum_{n\geq 1}(n-1)!\frac{x^n}{n!} = \log(\frac{1}{1-x})$
Trees ($\mathcal{T}$)	$\mathcal{T}_X = \{\tau \in \mathcal{G}_X : \tau \text{ is a tree}\}$	same as graph	$T(x) = \sum_{n\geq 1} n^{n-2}\frac{x^n}{n!}$
Set Partitions (Π)	Π_X is set partitions of X	$f_*(\{X_1, .., X_k\}) = \{f(X_1), ..., f(X_k)\}$	$\Pi(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!}$

Finally we defined natural transformation. Given $\mathcal{A}$ and $\mathcal{B}$ species, a *natural transformation* $\Phi : \mathcal{A} \to \mathcal{B}$ is a rule which assigns to each finite set X a function $\Phi_X : \mathcal{A}_X \to \mathcal{B}_X$ such that if $f : X \to Y$ is a bijection of sets then the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}_X & \xrightarrow{\Phi_X} & \mathcal{B}_X \\ \downarrow f_* & & \downarrow f_* \\ \mathcal{A}_Y & \xrightarrow{\Phi_Y} & \mathcal{B}_Y \end{array}$$

This is exactly the usual category theoretic notion of natural transformation in this context.

Examples of natural transformations include

- any subspecies inside a species gives a natural transformation
- there's a forgetful natural transformation from any species to sets by just forgetting all the structure other than the underlying set
- for a slightly more specific example, there's a natural transformation from endofunctions to directed graphs by taking $\alpha : X \to X$ to the graph with arcs $(x, \alpha(x))$ for all $x \in X$.

Next time

Next time we'll continue discussing species with a focus on species operations

References

Chapter II of Flajolet and Sedgewick's book *Analytic Combinatorics* is a good place to look for a conventional (not species-theoretic) description of labelled counting.

The standard book on species is *Combinatorial species and tree-like structures* by Bergeron, Labelle, and Leroux.

Another source that is quite close to what we're doing right now is the first three videos from the exponential generating series section of one of Kevin's past offerings of this course `https://www.math.uwaterloo.ca/~kpurbhoo/winter2021-co630/co630-videos.html`.