Vanishing and Non-Vanishing Criteria for Branching Schubert Calculus

by

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Abstract

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We investigate several related vanishing problems in Schubert calculus. First we consider the multiplication problem. For any complex reductive Lie group G, many of the structure constants of the ordinary cohomology ring $H^*(G/B;\mathbb{Z})$ vanish in the Schubert basis, and the rest are strictly positive. More generally, one can look at vanishing of Schubert intersection numbers, which generalise the multiplication problem to looking at products of more than two classes. We present a combinatorial game, the "root game", which provides some criteria for determining which of the Schubert intersection numbers vanish. The definition of the root game is manifestly invariant under automorphisms of G, and under permutations of the classes intersected. Although the criteria given by the root game are not proven to cover all cases, in practice they work very well, giving a complete answer to the question for $G = GL(n,\mathbb{C})$, $n \leq 7$.

The root game can be used to study the vanishing problem for multiplication on $H^*(G/P)$ (where $P \subset G$ is a parabolic subgroup) by pulling back the (G/P)-Schubert classes to $H^*(G/B)$. In the case where G/P is a Grassmannian, the Schubert structure constants are Littlewood-Richardson numbers. We show that the root game gives a *necessary and sufficient* rule for non-vanishing of Schubert calculus on Grassmannians. A Littlewood-Richardson number is non-zero if and only if it is possible to win the corresponding root game. More generally, the rule

can be used to determine whether or not a product of several Schubert classes on $Gr_l(n, \mathbb{C})$ is non-zero in a manifestly symmetric way. We give a geometric interpretation of root games for Grassmannian Schubert problems.

Finally, and most generally we look at the vanishing problem for branching Schubert calculus. If $K' \hookrightarrow K$ is an inclusion of compact connected Lie groups, there is an induced map $H^*(K/T) \to H^*(K'/T')$ on the cohomology of the homogeneous spaces. The image of a Schubert class under this map is a positive sum of Schubert classes on K'/T'. We investigate the problem of determining which Schubert classes appear with non-zero coefficient. This problem plays an important role in representation theory and symplectic geometry, as shown in [Berenstein-Sjamaar 2000]. The vanishing problems for multiplication of Schubert calculus can be seen as special cases of the branching problem. We develop root games for branching Schubert calculus, which give a vanishing criterion, and a non-vanishing criterion, for this problem. Again, these two criteria are not enough to give a complete answer to the problem; however they are applicable to a large number of cases. We include a number of examples of root games to illustrate both their simplicity and applicability.

Professor Allen Knutson
Dissertation Committee Chair

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Chapter 1

Introduction

1.1 Schubert calculus

The subject of Schubert calculus primarily deals with the cohomology of complex projective homogeneous spaces. In the most general setting we begin with a complex reductive Lie group G, and a parabolic subgroup $P \subset G$. The space G/P has a transitive G-action; moreover it is a projective variety. Such a space is sometimes called a *generalised flag manifold.*

The main examples motivating this study are the full and partial flag manifolds. The **(full) flag manifold** is the space of all full flags on $\mathbb{C}^n$

$$F = \big(\{0\} = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = \mathbb{C}^n\big),$$

where F_i is a vector subspace of dimension i. The group $GL(n)$ acts transitively on the flag manifold, and the stabiliser of the standard flag

$$\{0\} \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \cdots \subsetneq \langle x_1, \ldots, x_{n-1} \rangle \subsetneq \mathbb{C}^n$$

is the *standard Borel subgroup* $B \subset GL(n)$ (the invertible upper triangular matrices). Thus we identify the flag manifold with the quotient space G/B.

Let $0 < d_1 < \cdots < d_r < n$ be non-negative integers. The **partial flag manifold** of type $(d_1, \ldots, d_r)$ is the space of flags

$$F = \big(\{0\} = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = \mathbb{C}^n\big),$$

where $\dim F_i = d_i$. $GL(n)$ acts transitively on each partial flag manifold, with parabolic stabiliser, so each partial flag manifold is also of the form G/P. In the special case where $r = 1$, the partial flag manifold is just the Grassmannian $Gr_{d_1}(\mathbb{C}^n)$.

There is a natural map from the full flag manifold to each partial flag manifold, obtained by forgetting the subspaces of the wrong dimensions, i.e. the map takes a full flag F to the subflag

$$\{0\} \subsetneq F_{d_1} \subsetneq F_{d_2} \subsetneq \cdots \subsetneq F_{d_r} \subsetneq \mathbb{C}^n.$$

It is well known that the induced map on cohomology is an inclusion, whose image is well understood. Thus to understand the cohomology of any partial flag manifold, it is sufficient to understand the cohomology of the full flag manifold. More generally, it is possible to understand the ring structure of $H^*(G/P)$ for a parabolic subgroup P, in terms of the ring structure of $H^*(G/B)$, where B is a Borel subgroup. For this reason, our attention will be primarily devoted to the study of the spaces G/B.

The most basic problem in Schubert calculus is to effectively compute the structure constants of the cohomology ring. There is a natural basis for $H^*(G/B)$ given by the cohomology classes of the finitely many B-orbits on G/B. Each B-orbit always contains a unique fixed point of the maximal torus $T \subset B$. On the full flag manifold, these torus fixed points are represented by permutation matrices, and so the B-orbits are indexed by the B-orbits of $\{1, \ldots, n\}$. These orbits give an purely even dimensional cellular decomposition of G/B; thus each orbit represents a cohomology class, and set of these classes

$$\{\omega_\pi \mid \pi \in S_n\}$$

is $\mathbb{Z}$-basis for $H^*(G/B)$. Thus we can write a product of two classes as a $\mathbb{Z}$-linear combination of the others,

$$\omega_\pi \cdot \omega_\rho = \sum_{\sigma \in S_n} c_{\pi\rho}^{\sigma} \omega_\sigma.$$

It is known, in principle, how to compute these integers $c_{\pi\rho}^{\sigma}$. The algebra is relatively straightforward and can be easily carried out by a computer, or a highly disciplined undergraduate. There is a well known presentation for the cohomology ring of $GL(n)/B$ in terms of generators and relations:

$$H^*(GL(n)/B) = \mathbb{Z}[x_1, \dots, x_n]/\langle e_1, \dots, e_n \rangle$$

where e_i is the i^{th} elementary symmetric polynomial in $x_1, \dots, x_n$. This is known as the *Borel presentation*, and it generalises naturally to other groups G (at least over $\mathbb{Q}$).

Bernstein-Gel′fand-Gel′fand [BGG73] and Demazure [Dem73] identified the natural Schubert basis in the context of this presentation. Perhaps the simplest description is via divided difference operators. The **divided difference operator** $d_i := \mathbb{Z}[x_1, \dots, x_n] \to \mathbb{Z}[x_1, \dots, x_n]$ is defined by

$$d_i(f) = \frac{f(x_1, \dots x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots x_{i+1}, x_i, \dots, x_n)}{x_i - x_{i+1}}.$$

Note that if f is a polynomial, then so is $d_i(f)$. Now write a permutation $\pi \in S_n$ as

$$\pi = r_{i_1} \cdots r_{i_k} w_0$$

where r_i is the transposition $(i \leftrightarrow i+1)$, and

$$w_0 = \begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ n & n-1 & n-2 & \dots & 3 & 2 & 1 \end{pmatrix}.$$

Divided difference operators give an inductive way to compute representatives of the Schubert classes in the Borel presentation, starting from the class of a point. Lascoux and Schützenberger [LS82] found a "best" representative for the class of a point: $\omega_{w_0} = \prod_{j=1}^{n} x_j^{n-j}$. A representative of the Schubert basis element ω_π, is then simply computed by

$$s_\pi(x_1, \dots, x_n) = d_{i_1} \circ \cdots \circ d_{i_k} \left(\prod_{j=1}^{n} x_j^{n-j} \right).$$

These polynomials s_π are called the **Schubert polynomials**. (We refer to [BH95] for a discussion of Schubert polynomials for the other classical Lie groups.)

It is therefore a simple matter of polynomial and linear algebra to compute the structure constants of the cohomology ring. However, there are two very serious reasons why this is not a satisfactory answer by modern standards. Both problems are concerned with positivity properties of Schubert polynomials which are not reflected in this picture.

The first such positivity fact is that the coefficients of the Schubert polynomials are all positive. From both the Bernstein-Gel′fand-Gel′fand and Demazure constructions it is difficult to see this. Thankfully, this problem has been solved: there are purely combinatorial descriptions of the Schubert polynomials [BJS93, FS94] from which one can easily see the positivity of the coefficients, and a connection with the geometry of Schubert varieties [KM01].

The second is the well known fact that all the structure constants $c_{\pi\rho}^{\sigma}$ are non-negative. This is a consequence of the fact that G/B is a complex variety with a transitive action of a reductive Lie group. By the Kleiman-Bertini theorem [Kle74], general translates of Schubert varieties intersect transversely; these are complex varieties, so the points of intersection are oriented positively. However, the fact that $c_{\pi\rho}^{\sigma} \geq 0$ is not apparent from the algebra, and it remains an unsolved problem to see this simple geometric fact reflected in the algebra. A formula which involves only a sum of positive terms is said to be *manifestly positive*. None of the known formulae for the integers $c_{\pi\rho}^{\sigma}$ are manifestly positive: they all involve alternating sums. Although this may seem inconsequential at first glance, the implication is that although $c_{\pi\rho}^{\sigma}$ are non-negative integers associated to some combinatorial objects (permutations π, ρ, σ), we do not have a combinatorial interpretation for these integers.

In order to fully appreciate this, we should contrast with what is known about the cohomology of Grassmannians. In this case the state of affairs is significantly better. The Schubert cells on a Grassmannian are indexed by partitions (λ, μ, ν), rather than by permutations (π, ρ, σ). There are a number of different descriptions of combinatorial sets $C^\nu_{\lambda\mu}$, such that $c^\nu_{\lambda\mu} = \#C^\nu_{\lambda\mu}$. These include the set of Littlewood-Richardson tableaux (a thorough treatment of these can be found in [Ful97, FG93]), Littelmann paths [Lit95], and Knutson-Tao puzzles [KT99]. Moreover, there is an explicit description of how puzzles are related to the geometry of the Grassmannian [Vak03]. Thus one can see a pleasant connection between the algebra, combinatorics, and geometry of Grassmannians. This is the state of affairs one would like to see for the full flag manifold, and this is generally considered to be a difficult problem.

1.2 Outline of results

One consequence of the lack of manifestly positive formula for $c^\sigma_{\pi\rho}$ is that it is non-trivial to determine whether $c^\sigma_{\pi\rho} = 0$ for any given π, ρ, and σ. A simpler problem therefore is to find a combinatorial rule for answering this question. Even this problem is open, and there are no reasonable conjectures as to what the correct answer might be. However, we will investigate some approaches in this thesis, in order to give some partial answers.

Our principal results are a vanishing criterion, and a non-vanishing criterion. A vanishing criterion is a rule of the form "if *condition X* holds then $c^\sigma_{\pi\rho} = 0$." A non-vanishing criterion is a rule of the form "if *condition Y* holds then $c^\sigma_{\pi\rho} \geq 1$." There are a number of examples of these, scattered throughout the literature. Among the simplest is the Bruhat order condition. The Bruhat order is the partial order of S_n governing which Schubert cells are contained in the closure of other cells. (It can be also described in a purely combinatorial way.) It is an easy fact that $\omega_\pi \cdot \omega_\rho = 0 \in H^*(G/B)$ if and only if $\pi \leq w_0\rho$ in the Bruhat order. In particular, if this relation does not hold, then $c^\sigma_{\pi\rho} = 0$ for all σ. Using Poincaré duality we get two similar inequalities: $\pi \leq \sigma$, and $\rho \leq \sigma$ which are necessary for non-vanishing. Another very simple vanishing condition is Knutson's DC-triviality condition [Knu01]. This states that if that if $\pi(i) < \pi(i+1)$, $\rho(i) < \rho(i+1)$ and $\sigma(i) > \sigma(i+1)$ for some i, then $c^\sigma_{\pi\rho} = 0$. Perhaps the most celebrated vanishing conditions are the Horn inequalities, which are discussed in the next section.

To state our criteria, we will present the *root game*. The rules of the root game, though not entirely unwieldy, are too lengthy to state completely here. In short, a root game begins with a set of *tokens* placed on the positive roots of the Lie algebra of G. The player then tries to move these tokens around according to certain rules, in the hopes that a *winning position* can be reached. Condition Y will then be the condition that it is possible to win the root game corresponding to (π, ρ, σ). Condition X pertains to some, but not all, of the root games which are impossible to win. Unfortunately there will be examples of $(\pi,\ \rho,\ \sigma)$ which will satisfy neither conditions X nor Y, which makes this an incomplete answer. Still, the criteria are applicable in an enormous number of cases, and in a great deal of generality beyond the full flag manifold. Moreover, they are extremely practical for doing computations by hand.

Based on current experimental evidence, one could conjecture that for the full flag manifold, it is possible to win the root game if and only if $c_{\pi\rho}^{\sigma} \geq 1$. There is some weak evidence in support of this conjecture. We checked by computer that it is true for flags in $\mathbb{C}^n$, for $n \leq 7$. This is the largest case which is computationally feasible (we estimate that a complete check of $n = 8$ could take several years of computing time with our current methods). Moreover, we show that it is true for classes pulled back in a certain way from a Grassmannian. However, there is also some evidence against the conjecture; namely it fails for the generalised flag manifold $SO(8, \mathbb{C})/B$. We would prefer to avoid speculating on the reasonableness of such a conjecture, and instead present a number of interesting examples.

In this spirit, Chapter 2 will be devoted to presenting the root game for the full flag manifold, and more generally for G/B, and to proving the vanishing and non-vanishing criteria. We give a number of examples to illustrate the root game in practice, and some of the strengths and weaknesses of the vanishing and non-vanishing criteria. We will see that the vanishing criterion described by the root game generalises DC-triviality, whereas the non-vanishing criterion can be thought of as generalising the Bruhat order criterion. We also compare a few variations on the root game, and briefly discuss some experimental results.

In Chapter 3, we discuss the Grassmannian case. We describe how to obtain a root game from a Grassmannian Schubert problem. We then show that the non-vanishing criterion is actually a necessary and sufficient condition for these Grassmannian root games. The proof makes use of Zelevinsky pictures [Zel81], which are a useful formulation of the Littlewood-Richardson rule. We show that each Zelevinsky picture provides an algorithm for winning the associated root game. Finally we relate the combinatorics to geometry, to show that the non-zero terms appearing in the Schubert expansion of the product $\omega_\lambda \cdot \omega_\mu$ correspond exactly to B-fixed points on a certain variety.

Since root games on Grassmannians only answer the vanishing problem, they are not as powerful as the Littlewood-Richardson rule; nevertheless, they have one superior feature. The *Littlewood-Richardson intersection numbers* $c_{\lambda\mu\nu}$ are equal to certain Littlewood-Richardson coefficients: $c_{\lambda\mu\nu} = c^{\bar{\nu}}_{\lambda\mu}$, where $\bar{\nu}$ is the *complementary partition* to ν (we define this precisely in Section 3.3.1). This quantity represents the number of intersection points of three Schubert varieties (corresponding to λ, μ, and ν) translated into general position. The Littlewood-Richardson intersection numbers are invariant under permutations of the intersected classes (e.g. $c_{\lambda\mu\nu} = c_{\mu\nu\lambda}$, etc.), and also under duality: the operation of taking each partition to its dual. One advantage of root games is that they are also manifestly symmetric under these operations. Other rules, such as the classical Littlewood-Richardson rule and puzzles, do not readily exhibit all of these symmetries.

In Chapter 4, we present the root game in its currently most general context. This is the *branching problem* for Schubert calculus. In short, we consider an inclusion of generalised flag manifolds $i : X \hookrightarrow Y$, and study the induced map on cohomology $i^* : H^*(Y) \to H^*(X)$. Again, there is a positivity theorem for this map: $i^*(\omega_\pi) = \sum_\sigma c^\sigma_\pi \omega_\sigma$ is always a non-negative $\mathbb{Z}$-linear combination of Schubert classes on X, i.e. $c^\sigma_\pi \geq 0$. One can then ask the question: for which π and σ is $c^\sigma_\pi = 0$? In other words, we would like to find a combinatorial rule for determining which classes appear. This is a simpler version of a more general problem, which is to find a combinatorial description of c_π^σ. Again, the algebra is well understood here, but does not reflect the positivity. In fact, the multiplication problem can be seen as special case of the branching problem, so all remarks pertaining to that case apply here as well. We generalise the root game, and hence the vanishing and non-vanishing criteria to give results for the branching problem. We illustrate this with a number of new examples.

1.3 Applications of vanishing criteria

Although the vanishing problem does not yield the actual structure constants, there are nevertheless situations where simply knowing which structure constants vanish is enough information. Perhaps the most famous of these is Horn's problem [Hor62]. For more detailed discussion of the history of this problem, we refer to [Ful00].

Horn's problem concerns the spectra of triples of Hermitian matrices whose sum is zero. Let $\mathcal{H}$ denote the space of $n \times n$ Hermitian matrices. Given $A \in \mathcal{H}$, its eigenvalues (considered with multiplicities)

$$\operatorname{Spec}(A) = \lambda = \{\lambda_1 \geq \cdots \geq \lambda_n\}$$

are always real. Suppose now that $A, B, C \in \mathcal{H}$ satisfy $A + B + C = 0$. Let $\lambda = \operatorname{Spec}(A)$, $\mu = \operatorname{Spec}(B)$, $\mu = \operatorname{Spec}(C)$. Horn's problem is to determine the possible triples (λ, μ, ν) which can arise in this way.

At first glance, it is perhaps somewhat surprising that this question has anything to do with Schubert calculus; nevertheless the vanishing problem for multiplication plays a central role.

One obvious restriction is that the sum of largest eigenvalues of A and B cannot be less than the largest eigenvalue of $-C$. That is $\lambda_1 + \mu_1 \geq -\nu_n$. To see this, if P is any rank one orthogonal projection, then $PAP \leq \lambda_1$ and $PBP \leq \mu_1$. However if P is the projection onto the smallest eigenspace of C, we have $PCP = \nu_n$. Thus $\lambda_1 + \mu_1 \geq PAP + PBP = -PCP = -\nu_n$. Of course, since the problem is symmetrical in A, B, C, we also have $\lambda_1 + \nu_1 \geq -\mu_n$, and $\mu_1 + \nu_1 \geq -\lambda_n$. These inequalities are therefore necessary, but in general not sufficient, to give a solution to the problem.

Nevertheless the solution is given by a system of linear inequalities on (λ, μ, ν). These inequalities naturally arise from the non-vanishing of Schubert calculus. It was shown by Helmke-Rosenthal, Klyachko, and Totaro [HR95, Kly97, Tot94] that there is a necessary inequality corresponding to each non-zero Schubert structure constant for the cohomology rings $H^*(Gr_r(\mathbb{C}^n))$, for $1 \leq r \leq n-1$. Klyachko [Kly97] showed also that these inequalities give a sufficient set. Belkale [Bel01] reduced this to a shorter list of necessary and sufficient inequalities, showing that one needs only those inequalities given by the Schubert structure constants which are equal to 1. Recently, Knutson, Tao, and Woodward, [KTW04] have shown that this is reduced list is in fact a minimal set of inequalities.

Earlier, Horn [Hor62] conjectured a recursive method for computing these inequalities, which was proved by Knutson and Tao [KT99] using the combinatorics of hives and honeycombs. We will not discuss this in any great detail here. However, rephrased in the language of Schubert calculus, Horn's conjecture allows one to determine which Schubert structure constants are non-zero in $H^*(Gr_r(\mathbb{C}^n))$, by knowing non-zero structure constants of $H^*(Gr_d(\mathbb{C}^r))$. Thus the vanishing of Schubert calculus in small Grassmannians is relevant to the vanishing problem on larger Grassmannians. An independent geometric proof of Horn's conjecture has been given by Belkale [Bel02], using techniques which are somewhat similar to those which appear here.

The vanishing problem for branching Schubert calculus has applications to symplectic geometry and representation theory. We let $i : K' \hookrightarrow K$ be an inclusion of compact Lie groups. This induces a surjection $i^* : \mathfrak{k}^* \to (\mathfrak{k}')^*$ on the duals of the Lie algebras. Consider a coadjoint orbit of K, say $K \cdot \lambda$, for λ in the Weyl chamber of K. We consider its image under i^*, in $(\mathfrak{k}')^*$. This is a K'-invariant space, and thus is a union of coadjoint orbits $\bigcup_{\mu \in Z} K' \cdot \mu$, for some subset Z of the K'-Weyl chamber. The problem is to determine Z, i.e. which K'-coadjoint orbits are in the image.

Standard results in symplectic geometry (Kirwan's convexity theorem) tell us that the set Z is a convex polytope inside the Weyl chamber of K'. Berenstein and Sjamaar [BS00] show that there is a necessary and sufficient set of inequalities for this polytope corresponding to the non-vanishing Schubert branching coefficients. Thus the vanishing problem for branching Schubert calculus plays a fundamental role in this problem.

On the representation theory side, the calculation of this polytope can be seen to give an asymptotic solution to the following problem [Hec82, GS82]: given an irreducible K-module V, which irreducible representations of K' appear when V is decomposed as a K'-module? More precisely, let λ and μ denote integral points in the Weyl chambers of K and K' respectively. Then $K' \cdot \mu \subset i^*(K \cdot \lambda)$ if and only there is some integer $N > 0$ such that the irreducible representation $V_{N\lambda}$ has an isotypic component of type $V_{N\mu}$, when decomposed as a K'-module. In the case where $K = U(n) \times U(n)$, and $K' = U(n)$ included diagonally, it turns out one can take $N = 1$. This is the saturation conjecture, which was shown by Fulton [Ful98] and Zelevinsky [Zel99] to imply Horn's conjecture, and proved by Knutson and Tao [KT99]. So, in fact, there is a deep relationship between Klyachko's solution to Horn's problem and this branching picture.

Finally, we note that even vanishing criteria which are not necessary but merely sufficient can be seen to be relevant when computational limitations are taken into consideration. Consider the simplest, most naïve method of determining these inequalities. First, we calculate a complete list of all the Schubert branching coefficients. For each of these, if the coefficient is positive, we include the requisite inequality. Unfortunately, this is somewhat impractical from a computational point of view, as a complete list of all the different Schubert branching problems is quite large, even for relatively low dimensional Lie groups K and K'. It would therefore be useful to know a priori that a large subset of the Schubert branching coefficients are zero. Practically, therefore, vanishing criteria can be used to cut down the space of problems one has to consider in calculating this polytope, or in any other situation where one might want a complete table of Schubert branching coefficients or multiplication structure constants.

Chapter 2

Vanishing of Schubert calculus on G/B

2.1 Preliminaries

2.1.1 General approach

In this chapter, our main objective is to provide some criteria for vanishing of intersection numbers of Schubert varieties on a generalised flag manifold G/B. To understand the cohomology ring $H^*(G/B)$, it is sufficient to count the intersection points of $k \geq 3$ Schubert varieties in general position. Our approach is essentially to fix an intersection point, and determine if the Schubert varieties can be made to intersect transversely. This leads to a linear algebraic criterion (Lemma 2.2.1) which is necessary and sufficient for vanishing of Schubert Calculus.

In Section 2.3, we introduce the *root game* which can sometimes give information about the Schubert intersection number. In some circumstances the root game will tell us that the intersection number is 0 (Theorem 2.1); in other circumstances, the game will tell us that the intersection number is at least 1 (Theorems 2.2 and 2.4). Although these two criteria do not cover all cases, we have confirmed by computer for $G = GL(n)$, $n \leq 7$ that all the remaining cases have intersection number 0. The rules of the game are manifestly symmetric under permutations of the classes intersected, as well as under automorphisms of G. Furthermore, the game is highly amenable to computations by hand and does not involve any Schubert polynomials.

For ease of notation, our presentation will be in terms of intersections of three Schubert varieties; however all theorems and proofs work with any number.

2.1.2 Conventions

Throughout this chapter, let G be a complex connected reductive Lie group. Fix T a maximal torus, B a Borel subgroup, and B_- its opposite, so $T = B \cap B_-$. Let N and N_- denote the corresponding unipotent groups. The Lie algebras of these groups will be denoted $\mathfrak{g}$, $\mathfrak{b}$, etc. Most of our examples use $G = GL(n)$, in which case B and B_- are invertible upper triangular and lower triangular matrices respectively, and T consists of the diagonal matrices in $GL(n)$.

Let Δ denote the root system of G, with Δ_+ and Δ_- the sets of positive and negative roots respectively. For each root $\alpha \in \Delta_+$, we fix a basis vector e_α for the corresponding root space in $\mathfrak{g}$.

Let W denote the Weyl group of G. For $\pi \in W$, let $\tilde{\pi}$ denote some lifting of $\pi \in W = N(T)/T$ to an element of $N(T) \subset G$, and let $[\pi] = \tilde{\pi}B$ denote the corresponding T-fixed point on G/B.

To each $\pi \in W$ we associate the Schubert variety $X_\pi = \overline{B_- \cdot [\pi]}$, the closure of the B_--orbit through $[\pi]$ in G/B. Recall that the length of $\pi \in W$ (as a word in the simple reflections) is the codimension of X_π. We denote the the cohomology class Poincaré dual to the cycle X_π by ω_π.

Let w_0 denote the long element in W. For $x_0, x \in G/B$ we say that x is π-related to x_0 if there is a $g \in G$ such that $gx_0 = w_0$ and $gx \in X_\pi$. Let X_{π,x_0} denote the Schubert variety associated to π based at x_0, that is

$$X_{\pi,x_0} = \{x \in G/B \mid x \text{ is } \pi_i\text{–related to } x_0\}$$

so $X_\pi = X_{\pi,[w_0]}$.

Let $\pi_1, \pi_2, \pi_3 \in W$ whose lengths total $\dim G/B$. Consider the space

$$E = \left\{(x_0, x_1, x_2, x_3) \subset (G/B)^4 \;\middle|\; x_0 \in X_{\pi, x_i},\ i = 1, 2, 3\right\}$$

In other words, x_0 is a point of intersection of the the Schubert varieties X_{π_i, x_i}. If the flags $x_i, i = 1, 2, 3$ are sufficiently generic that the three Schubert varieties intersect in isolated points, then we say that x_0 is a solution to the Schubert problem (π_1, π_2, π_3). Note that the three conditions defining E are transverse, so $\dim E = \dim(G/B)^3$.

There are two forgetful maps from E. We have $p_0 : E \to G/B$ given by $p_0(x_0, x_1, x_2, x_3) = x_0$, and $p_{123} : E \to (G/B)^3$ given by $p_{123}(x_0, x_1, x_2, x_3) = (x_1, x_2, x_3)$. Since E is invariant under the diagonal subgroup $G_\Delta \hookrightarrow G^4$, with $g \cdot p_0^{-1}(x_0) = p_0^{-1}(gx_0)$, the map p_0 is a fibration. Let $U \subset (G/B)^3$ denote the points where p_{123} is finite-to-one, and let $E' = p_{123}^{-1}(U)$.

Let $c_{\pi_1\pi_2\pi_3}$ be the degree of the covering $p_{123}|_{E'}$. Then $c_{\pi_1\pi_2\pi_3}$ is equal to the number of intersection points x_0 of the three Schubert varieties based at a generic triple (x_1, x_2, x_3). (Here we implicitly use the Kleiman-Bertini theorem [Kle74].) Cohomologically,

$$c_{\pi_1\pi_2\pi_3} = \int_{G/B} \omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3}.$$

Note that U (and hence E') may be empty: in fact these sets are empty if and only if the corresponding Schubert problem has no solutions, i.e. if and only if $c_{\pi_1\pi_2\pi_3} = 0$.

Finally, let us recall that the Schubert intersection numbers $c_{\pi_1\pi_2\pi_3}$ determine the Schubert structure constants of the cohomology ring $H^*(G/B)$. Indeed, if we write

$$\omega_{\pi_1} \cdot \omega_{\pi_2} = \sum_{\rho \in W} c^{\rho}_{\pi_1\pi_2} \omega_\rho$$

then

$$\begin{aligned} c_{\pi_1\pi_2\pi_3} &= \int_{G/B} \omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} \\ &= \int_{G/B} \sum_{\rho \in W} c^{\rho}_{\pi_1\pi_2} \omega_\rho \cdot \omega_{\pi_3} \\ &= c^{w_0\pi_3}_{\pi_1\pi_2}. \end{aligned}$$

2.2 A lemma on vanishing

Our first lemma gives a linear algebraic condition for vanishing of Schubert calculus, and is our main technical tool. Although this result is known (Belkale [Bel02] lists this as a standard result, without reference), we are not aware of any proof in the literature.

Let $\mathfrak{n}$ denote the Lie algebra of N. Let P_k be the subspace of $\mathfrak{n}$ generated by the e_α such that $\alpha \in \Delta_+$ and $\pi_k^{-1} \cdot \alpha \in \Delta_-$. Equivalently,

$$P_k = \mathfrak{n} \cap (\pi_k \cdot \mathfrak{b}_-).$$

For $a \in N$, let $a\cdot\ : \mathfrak{n} \to \mathfrak{n}$ denote the adjoint action of N on its Lie algebra.

Lemma 2.2.1. *For $a_1, a_2, a_3 \in N$ generic, $a_1 \cdot P_1 + a_2 \cdot P_2 + a_3 \cdot P_3 = \mathfrak{n}$ if and only if $c_{\pi_1\pi_2\pi_3} = 0$.*

Proof. We'll use the Cartan involution to identify $\mathfrak{n}$ with $\mathfrak{n}_-$, denoted $a \mapsto a^T$, and the Killing form to identify $\mathfrak{n}^*$ with $\mathfrak{g}/\mathfrak{b}$. (For $G = GL(n)$, a^T is the transpose of a, and the identification map $\mathfrak{n} \to \mathfrak{g}/\mathfrak{b}$ is just $a \mapsto a^T/\mathfrak{b}$.) Under these identifications

$$(P_k^\perp)^T = ((\pi_k \cdot \mathfrak{b}_-)/\mathfrak{b})^T = (\pi_k \cdot \mathfrak{b})/\mathfrak{b}_-$$

(here $P^\perp \subset V^*$ is the annihilator of $P \subset V$). So

$$\begin{aligned} a_1 \cdot P_1 + a_2 \cdot P_2 + a_3 \cdot P_3 = \mathfrak{n} &\iff \bigcap_k a_k \cdot P_k^\perp = \{0\} \\ &\iff \bigcap_k a_k^T \cdot (P_k^\perp)^T = \{0\} \\ &\iff \bigcap_k a_k^T \cdot (\pi_k \cdot \mathfrak{b})/\mathfrak{b}_- = \{0\} \end{aligned}$$

Now

$$\begin{aligned} c_{\pi_1\pi_2\pi_3} = 0 &\iff E' = \emptyset \\ &\iff p_0^{-1}(x_0) \cap E' = \emptyset \ \forall x_0 \in G/B \\ &\iff p_{123}(p^{-1}(x_0)) \cap U = \emptyset \ \forall x_0 \in G/B \\ &\iff \bigcup_{x_0 \in G/B} p_{123}(p_0^{-1}(x_0)) \subset (G/B)^3 - U. \end{aligned}$$

But $\bigcup_{x_0 \in G/B} p_{123}(p_0^{-1}(x_0)) = G \cdot p_{123}(p_0^{-1}(x_0))$ for any $x_0 \in G/B$, in particular for $x_0 = [w_0]$. Thus the Schubert problem has intersection number 0 if and only if

$$G \cdot p_{123}(p_0^{-1}([w_0])) \subset (G/B)^3 - U.$$

Since U is a Zariski open dense subset of $(G/B)^3$, this will happen only if $\dim(G \cdot p_{123}(p_0^{-1}([w_0]))) < \dim(G/B)^3$. Conversely if this inequality holds, then p_{123} is not onto, and $c_{\pi_1\pi_2\pi_3} = 0$.

For a point $x = (x_1, x_2, x_3) \in p_{123}(p_0^{-1}([w_0]))$, let

$$S(x) = \left\{ g \in G \;\middle|\; g \cdot x \in p_{123}(p_0^{-1}([w_0])) \right\}.$$

Now $p_{123}(p_0^{-1}([w_0])) = X_{\pi_1} \times X_{\pi_2} \times X_{\pi_3}$, which is B_--invariant and codimension $\dim G - \dim B$ in $(G/B)^3$. Hence $B_- \subset S(x)$, and

$$\operatorname{codim}\left(G \cdot p_{123}(p_0^{-1}(x_0))\right) = \dim S(x) - \dim B_-,$$

for x generic in $p_{123}(p_0^{-1}([w_0]))$.

Let us therefore compute the dimension of $S(x)$ at a generic point

$$x = ([a_1^T\tilde{\pi}_1], [a_2^T\tilde{\pi}_2], [a_3^T\tilde{\pi}_3]) \in p_{123}(p_0^{-1}([w_0])).$$

We have

$$\begin{aligned} g \in S(x) &\iff g \cdot a_k^T\tilde{\pi}_k B \in B_- a_k^T\tilde{\pi}_k B, \ \forall k = 1,2,3 \\ &\iff g \in \bigcap_k B_-\, a_k^T\tilde{\pi}_k\, B\, \tilde{\pi}_k^{-1}(a_k^T)^{-1}. \end{aligned}$$

On the Lie algebra level

$$T_1(S(x)) = \bigcap_k a_k^T\tilde{\pi}_k\, \mathfrak{b}\, \tilde{\pi}_k^{-1}(a_k^T)^{-1} + \mathfrak{b}_-.$$

Thus

$$T_1(S(x))/\mathfrak{b}_- = \bigcap_k a_k^T \cdot (\pi_k \cdot \mathfrak{b})/\mathfrak{b}_-.$$

If this intersection is 0 then $S(x) = B_-$. We now argue that if this last intersection is non-zero then $\dim S(x) > \dim B_-$. Suppose that on the Lie algebra level, this last intersection is non-zero dimensional. Since, the point is $([w_0], x)$ is generic in $p_0^{-1}([w_0])$, we can locally find a smooth, non-vanishing vector field on $p_0^{-1}([w_0])$, generated by some element of $\mathfrak{g} - \mathfrak{b}$ at each point. Flowing along this vector field from $([w_0], x)$ for some time t, lands us at a point $([w_0], x(t)) \in p_0^{-1}([w_0])$, which is also in the G-orbit through $([w_0], x)$. Thus there is some $g(t) \in G$ such that $g(t) \cdot ([w_0], x) = ([w_0], x(t))$. Thus $g(t) \in S(x)$. Moreover by continuity, for t sufficiently small (non-zero), $g(t)$ is not in B_-.

Thus $\dim S(x) = \dim B_-$ if and only if this last intersection is 0, hence if and only if $a_1 \cdot P_1 + a_2 \cdot P_2 + a_3 \cdot P_3 = \mathfrak{n}$. $\square$

Remark 2.2.1. *One undesirable feature of this proof is that it uses the flow along a vector field, which is not an algebraic operation. Thus although the proof is valid over $\mathbb{C}$, it is not valid over an arbitrary algebraically closed field of characteristic zero. In Chapter 4, we give a generalisation of Lemma 2.2.1, and give a proof which is valid for an arbitrary algebraically closed field of characteristic zero.*

2.3 Root games

2.3.1 Weak version of the root game for $G = GL(n)$

Before delving into the root game in its full splendour, we shall first describe a toned down version, and restrict to the case of $G = GL(n)$.

We begin with a set of $\binom{n}{2}$ squares S_{ij}, indexed by $1 \leq i < j \leq n$. We visualise these as the squares above the diagonal inside an $n \times n$ array of squares. In each of the squares we allow tokens to appear. Each token has a label, either 1, 2, or 3, and each kind of token may appear at most once in any particular square. Thus the entries in a square are essentially subsets of $\{1, 2, 3\}$. We'll call a token labeled k a k-token, and write $k \in S_{ij}$ if a k-token appears in square S_{ij}.

Since the Weyl group is S_n we consider the π_k as permutations of the numbers $1, \ldots, n$. The initial configuration of the game is determined by the permutations: for $i < j$ if $\pi_k(i) > \pi_k(j)$ the square S_{ij} includes a k-token in the initial configuration. Otherwise it does not. An example of the initial configuration is shown in Figure 2.1.

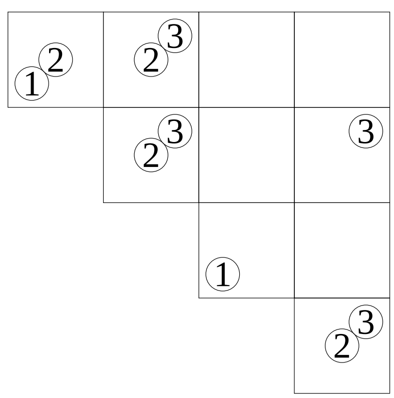


Figure 2.1: Initial position of the game for permutations 21435, 32154, 24153.

From the initial position, the player makes a sequence of moves. A **move** is specified by a pair $[k, S_{ij}]$, where $k \in \{1, 2, 3\}$ is a token label, and S_{ij} is a square in the array. For every l with $j < l \leq n$, if a k-token appears in S_{jl} but not in S_{il}, we move it from S_{jl} to S_{il}. Also for each l with $1 \leq l < i$, if a k-token appears in S_{li} but not in S_{lj}, we move it from S_{li} to S_{lj}. (In Figure 2.2 the dotted lines are drawn so that they intersect in the square S_{ij} and pass through all the tokens and squares involved in the move.)

Definition 2.3.1. *The game is* **won** *if at any point there is exactly one token in each square.*

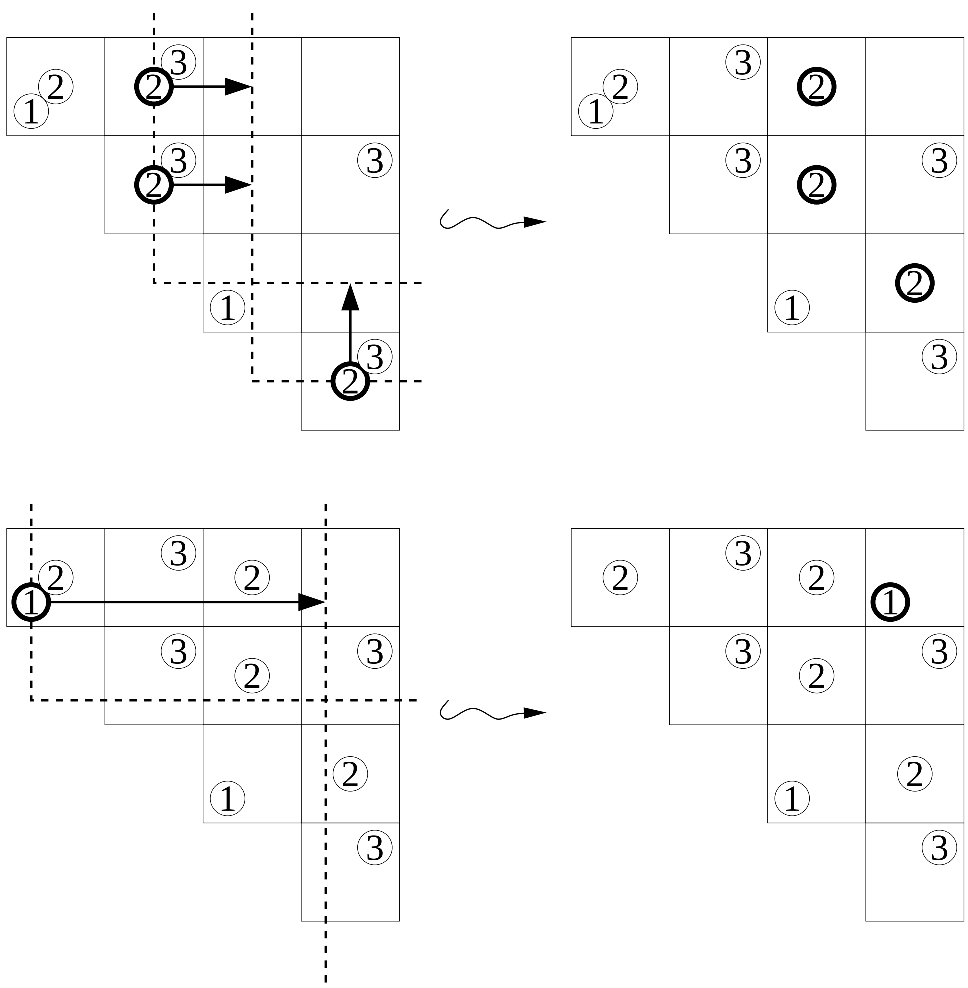


Figure 2.2: Moves $[2, S_{34}]$ and $[1, S_{25}]$ are applied to the initial position in Figure 2.1.

Example 2.3.2. *Figure 2.2 shows a sequence of two moves in the game for the permutations 21435, 32154, 24153, resulting in a win.*

Observe that any token can only ever move upward and to the right. So, for example, if there are two tokens in the upper right corner square, there is no point in proceeding further. More generally, if at some point in the game there is a subset A of the squares, closed under moving upward and to the right (i.e. $(i,j) \in A \implies (i,j') \in A,\ \forall j' > j$ and $(i',j) \in A,\ \forall i' < i$), such that the total number of tokens in all the squares in A is strictly greater than $|A|$, we declare the game to be a loss.

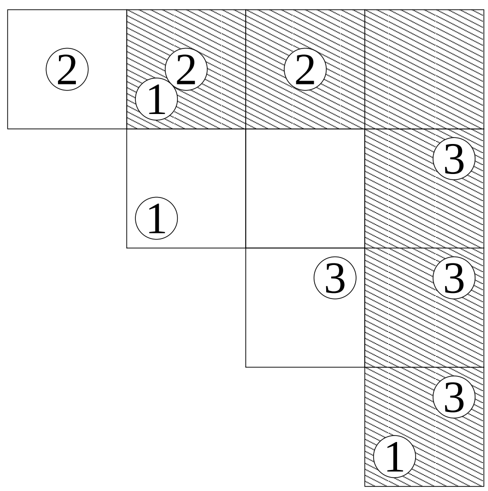


Figure 2.3: The initial position for permutations 23154, 41235, 13542.

Example 2.3.3. *Figure 2.3 shows the initial position of the game for the permutations 23154, 41235, 13542. Since there are 7 tokens in the 6 shaded squares, the game is lost, even before any moves are made.*

In general losing the game does not provide any information—it may simply be the result of bad play. However the case above is exceptional since the game is a loss before any moves are made.

Definition 2.3.4. *If the game is lost before the first move is made, we say the game is* **doomed**.

Theorem 2.1. *If the game is doomed, then* $c_{\pi_1\pi_2\pi_3} = 0$.

But even greater enjoyment can be gained from winning.

Theorem 2.2. *If the game can be won, then* $c_{\pi_1\pi_2\pi_3} \geq 1$.

We defer the proofs until we have presented the root game in its most general context.

2.3.2 Weak version of the root game for general G

Little modification is required for a general group G, although the pictures do sometimes become harder to draw.

The game is played on a set of squares S_α indexed by the positive roots of G. (For $G = GL(n)$, every positive root can be written in the form $\alpha_{ij} = t_i - t_j$ for some $i < j$; the new indexing can be identified with the old via $S_{ij} \leftrightarrow S_{\alpha_{ij}}$.) As before, in each of these squares we allow any combination of the 1,2 or 3-tokens with no label repeated, including the empty combination.

Our initial configuration is such that we have a k-token in square S_α iff $\pi_k^{-1} \cdot \alpha \in \Delta_-$.

The moves are specified by a pair $[k, \beta]$, where $k = 1, 2$ or 3 is a choice of token label, and $\beta \in \Delta_+$. The actual move is made by the following rule. For each pair of positive roots α, α' such that $\alpha' - \alpha = \beta$, if a k-token occurs in the square S_α but not in $S_{\alpha'}$, move it from the first square to the second square.

There is one small caveat: for $G = GL(n)$ it does not make any difference in which order we move these tokens, however for other groups it might. (Figure 2.5 includes an example of this behaviour.) To resolve this ambiguity, we order the relevant α by height and stipulate that we must always move tokens in the highest root squares first.

As before, the game is won if there is exactly one token in each square. For the losing condition, we need the following definition.

Definition 2.3.5. *Let* $A = \{S_\alpha \mid \alpha \in I\}$ *be a subset of the squares. Call* A *an* **ideal subset** *if* I *is closed under raising operations, i.e. If* $\alpha \in I$*, then* $\alpha' \in I$*,* *whenever α', and $\alpha' - \alpha$ are both positive roots. (Equivalently, A is a an ideal subset if and only if $\{e_\alpha | \alpha \in I\}$ span an ideal in the Lie algebra $\mathfrak{n}$.)*

The game is **lost** if there is an ideal subset A such that the the total number of tokens in A is more than $|A|$. Again, the game is **doomed** if this losing condition holds before the first move is made.

Theorems 2.1 and 2.2 hold in this more general setting as well.

2.3.3 An example for $SO(7, \mathbb{C})$

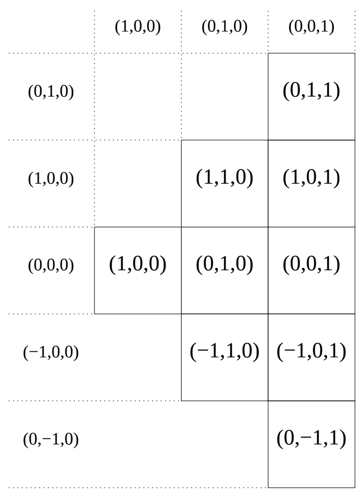


Figure 2.4: A set of squares indexed by the nine positive roots of $SO(7, \mathbb{C})$. Each root is expressed as a sum of a vectors in the left column and a vector top row.

If $G = SO(7, \mathbb{C})$, the root system is $B_3 \subset \mathbb{R}^3$. We'll choose the positive system for which $(0, -1, 1), (-1, 1, 0), (1, 0, 0)$ are the simple roots. To draw pictures of the root games, we must arrange squares in the plane corresponding to the positive roots. There are a number of possible ways to do this, however one which scales easily to the other groups $SO(2n + 1, \mathbb{C})$ is the one shown in Figure 2.4. (This type of arrangement can also be used for $SO(2n, \mathbb{C})$, where the Weyl group is D_n, by deleting the middle row of squares.)

Let $\{x_1, x_2, x_3\}$ denote the standard basis for $\mathbb{R}^3$. An element of Weyl group $W = S_3 \rtimes C_2^3$ can be represented by a permutation $a_1 a_2 a_3$ of 123, where each symbol is either decorated with a bar or not. This permutation acts on $\mathbb{R}^3$ by the matrix whose i^{th} row is x_{a_i} if i is unbarred, and $-x_{a_i}$ if i is barred.

Figure 2.5 gives an example of a game for $SO(7,\mathbb{C})$. Arrows in Figure 2.5 are included not only for all tokens that move, but for all pairs of roots α, α', whose difference is β. Since the game can be won using the moves shown, for $\pi_1 = \bar{1}32$, $\pi_2 = 231$, $\pi_3 = \bar{1}\bar{2}3$ we have $c_{\pi_1\pi_2\pi_3} \geq 1$.

2.3.4 Proof of the vanishing criterion

Proof of Theorem 2.1. At the outset, the set of e_α such that a k-token occurs in square S_α forms a basis for the space P_k. If the the game is doomed because of an ideal subset A, then A's root spaces generate an N-invariant subspace V of $\mathfrak{n}$ such that

$$\sum_{k=1}^{3} \dim(P_k \cap V) > \dim V.$$

Thus for any $a_k \in N$, we have

$$\begin{aligned}
\dim(a_1 \cdot P_1 + a_2 \cdot P_2 + a_3 \cdot P_3)/V &= \dim(a_1 \cdot P_1/V + a_2 \cdot P_2/V + a_3 \cdot P_3/V) \\
&\leq \sum_{k=1}^{3} \dim P_k/V \\
&= \sum_{k=1}^{3} \dim P_k - \dim(P_k \cap V) \\
&< \dim \mathfrak{n} - \dim V \\
&= \dim \mathfrak{n}/V.
\end{aligned}$$

Thus we certainly cannot have $a_1 \cdot P_1 + a_2 \cdot P_2 + a_3 \cdot P_3 = \mathfrak{n}$, hence by Lemma 2.2.1, the intersection number is 0. $\square$

In the case where the game is doomed as a result of an ideal subset A which is maximal (i.e. A consists of all squares except for a single S_α, where α is a simple root), this vanishing condition reduces to the DC-triviality vanishing condition in [Knu01].

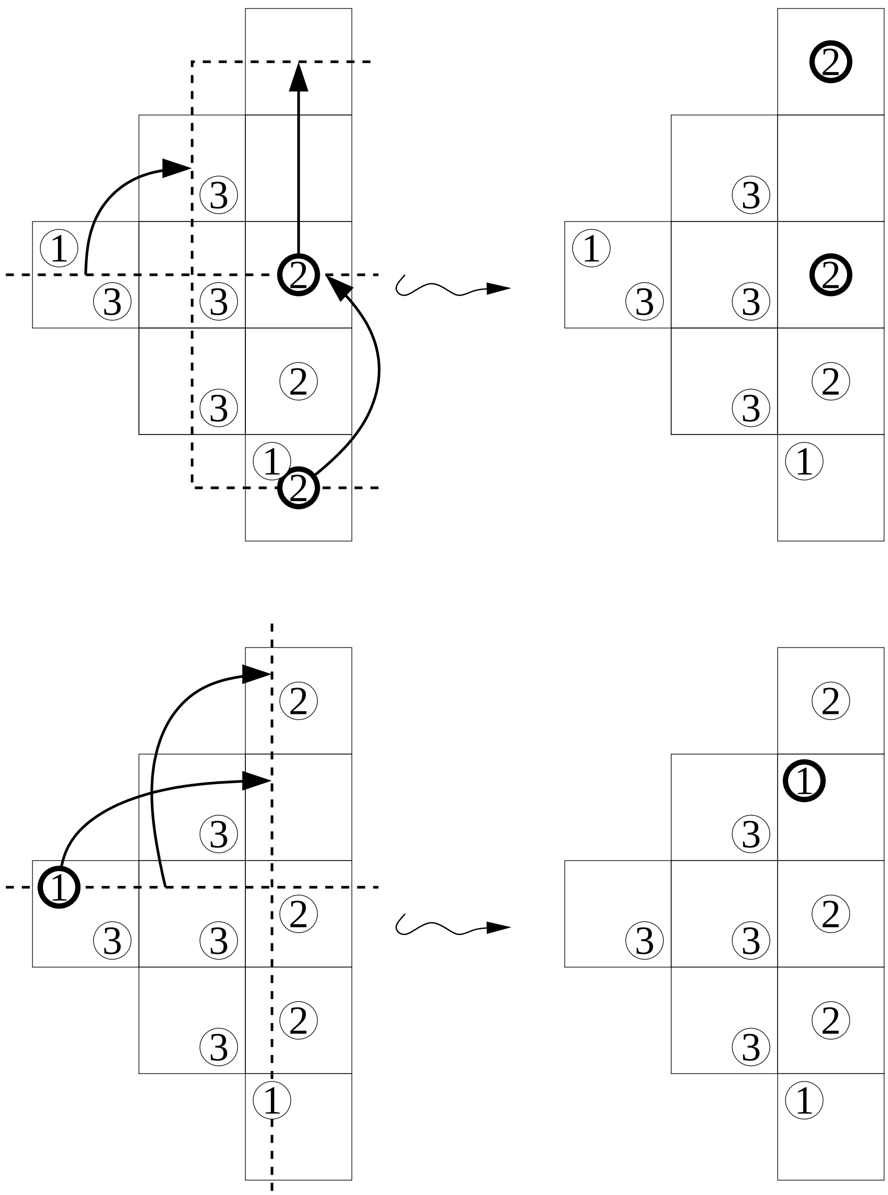


Figure 2.5: A simple game for $SO(7, \mathbb{C})$. In this example, $\pi_1 = \bar{1}32$, $\pi_2 = 231$, $\pi_3 = \bar{1}\bar{2}3$. The root β which is used in each move is the crossing point of the dotted lines.

2.3.5 Relationship with the Bruhat order

We now consider the root game for products of only two Schubert classes. We will show that the root game winning condition is both necessary and sufficient for non-vanishing of products of two Schubert classes. Theorem 2.2 already tells us that winning the root game is sufficient for non-vanishing; thus here we shall only establish necessity. That is, if $\omega_{\pi_1} \cdot \omega_{\pi_2} \neq 0$, then it is possible to win the root game corresponding to (π_1, π_2).

When we are considering the product of only two Schubert classes, the non-vanishing of the product is determined precisely by the Bruhat order. More precisely $\omega_{\pi_1} \cdot \omega_{\pi_2} \neq 0$ if and only if $\pi_1 \leq w_0\pi_2$ in the Bruhat order. In the case where the product is top dimensional, i.e. $l(\pi_1) + l(\pi_2) = \dim G/B$, the fact that we can win the root game is a triviality: we have $\pi_1 \leq w_0\pi_2$ if and only if $\pi_1 = w_0\pi_2$ in which case the set of squares containing a 1-token is the complement of the set of squares containing a 2-token. Thus the initial position of the game is already a winning position.

Less trivial is the case when $\pi_1 < w_0\pi_2$. Notice that here, the total number of tokens will be less than the number of squares. Thus, according to our current definition of winning, it is impossible to win the root game. To accommodate this, we relax the winning condition slightly. We say that the game is won if there is at most one token in each square. It is straightforward to see that a version of Theorem 2.2 still holds with this revised winning condition (see remarks in Section 2.4.2). Moreover, we have the following theorem.

Theorem 2.3. $\omega_{\pi_1} \cdot \omega_{\pi_2} \neq 0$ *if and only if it is possible to win the root game corresponding to* (π_1, π_2).

In fact, we will show that it is possible to do this moving only 1-tokens. We therefore assume in what follows that all moves are moves of 1-tokens; hence we shall specify a move solely by the root β. The following result allows us to argue by induction.

Proposition 2.3.1. *Consider two root games. Let T_1 denote the set of squares containing 1-tokens in the first game, and let T_2 denote set of squares containing 1-tokens in the second game. We perform the same sequence of moves $\beta_1, \ldots, \beta_m$ to each game. If $T_1 \subset T_2$ before the sequence of moves is performed, then $T_1 \subset T_2$ after the moves are performed. Thus if it is possible to win the second game, it is possible to win the first game.*

Proof. This is true for a single move; thus it is true for any sequence. If at the end of this sequence of moves the second game has at at most one token in each square, then the same will be true of the second game. $\square$

If $\pi_1 < w_0\pi_2$, there is a maximal chain in the Bruhat order $\pi_1 = \rho_0 < \rho_1 < \cdots < \rho_m = w_0\pi_2$. Consider the initial position of the game for (ρ_i, π_2). We let T_i denote the set of squares containing 1-tokens in this position. We will show that from the initial position of (ρ_i, π_2), we can perform a sequence of moves so the squares containing 1-tokens are a subset of T_{i+1}.

Let $r_\beta \in W$ denote the reflection in the root β. Since ρ_{i+1} covers ρ_i in the Bruhat order, we can write a reduced expression for $\rho_{i+1} = r_{\alpha_1} \cdots r_{\alpha_l}$, such that

$$\rho_i = r_{\alpha_1} \cdots \widehat{r_{\alpha_s}} \cdots r_{\alpha_l}$$

(here α_j is a simple root). Put $\sigma = r_{\alpha_{s+1}} \cdots r_{\alpha_l}$, $\sigma' = r_{\alpha_1} \cdots r_{\alpha_{s-1}}$, and $\gamma = \alpha_s$. Thus $\rho_i = \sigma'\sigma$, and $\rho_{i+1} = \sigma' r_\gamma \sigma$. Let $\beta = \sigma^{-1}\gamma$ so that $r_\beta = \sigma^{-1} r_\gamma \sigma$. Note that $\rho_{i+1} = \rho_i r_\beta$.

Proposition 2.3.2. *With β as above, we have $\beta \in \Delta_+$, $\rho_i(\beta) \in \Delta_+$, and $\rho_{i+1}(\beta) \in \Delta_-$.*

Proof. Consider the list of roots

$$\beta,\ r_{\alpha_l}(\beta),\ r_{\alpha_{l-1}} r_{\alpha_l}(\beta),\ \ldots\ , r_{\alpha_2} \cdots r_{\alpha_l}(\beta),\ \rho_{i+1}(\beta).$$

Since $r_{\alpha_1} \cdots r_{\alpha_l}$ is a reduced expression for ρ_{i+1}, the roots in this list can switch sign (i.e. from positive to negative, or vice versa) at most once. A subset of this list is

$$\beta,\ \sigma(\beta) = \gamma,\ r_\gamma \sigma(\beta) = -\gamma,\ \rho_{i+1}(\beta).$$

We know that $\gamma \in \Delta_+$, and $-\gamma \in \Delta_-$. So we have located the point where the sign changes. Thus $\beta \in \Delta_+$ and $\rho_{i+1}(\beta) \in \Delta_-$. Furthermore, since $\rho_{i+1}(\beta) = \sigma' r_\gamma \sigma(\beta) = -\sigma'(\gamma)$, and $\rho_i(\beta) = \sigma'\sigma(\beta) = \sigma'(\gamma)$, we have $\rho_i(\beta) \in \Delta_+$. $\square$

Proposition 2.3.3. *Let $\alpha \in \Delta_+$, $\alpha \neq \beta$. If $S_\alpha \in T_i$, then for every root of the form $\alpha - N\beta$, with N a positive integer, we have $\rho_i(\alpha - N\beta) \in \Delta_-$. In particular, if $\alpha - N\beta \in \Delta_+$, then $S_{\alpha - N\beta} \in T_i$.*

Proof. $S_\alpha \in T_i$ if and only if $\rho_i(\alpha) \in \Delta_-$. Thus $\rho_i(\alpha - N\beta) = \rho_i(\alpha) - N\rho_i(\beta) \in \Delta_-$. $\square$

Proposition 2.3.4. *If we repeatedly apply the move β to the initial position of the game for (ρ_i, π_2), we obtain a position such that the set of squares containing 1-tokens is a subset of T_{i+1}.*

Proof. Suppose $\alpha - N\beta < \cdots < \alpha - \beta < \alpha$ are roots, but $\alpha + \beta$ and $\alpha - (N+1)\beta$ are not. We assume $\alpha \neq \pm\beta$. Some of the roots in this sequence are positive, and hence correspond to squares, while others do not. On the other hand some of these roots are inversions for ρ, while others are not. We represent this information in a diagram as follows.

•	•	•	•			
$\alpha-6\beta$	$\alpha-5\beta$	$S_{\alpha-4\beta}$	$S_{\alpha-3\beta}$	$S_{\alpha-2\beta}$	$S_{\alpha-\beta}$	S_α

The dots (•) correspond to the inversions, while the squares correspond to the positive roots. We can think of the dots which are in squares as tokens. The squares are right justified, whereas from Proposition 2.3.3, the inversions are left justified. (Note: this picture is somewhat unrealistic, since we can never actually have $N > 3$. $N = 3$ occurs for G_2, and $N = 2$ occurs in all other non-simply laced types. Otherwise $N = 0$ or 1 are the only possibilities.)

The move β corresponds to shifting each token to the right. If the tokens are already right justified, the move β does nothing. Thus applying β a sufficiently large number of times right justifies all the tokens.

•	•				•	•
$\alpha-6\beta$	$\alpha-5\beta$	$S_{\alpha-4\beta}$	$S_{\alpha-3\beta}$	$S_{\alpha-2\beta}$	$S_{\alpha-\beta}$	S_α

On the other hand, we have

$$\begin{aligned} & S_{\alpha - M\beta} \in T_{i+1} \\ \iff \quad & \rho_{i+1}(\alpha - M\beta) \in \Delta_- \\ \iff \quad & \rho_i r_\beta(\alpha - M\beta) \in \Delta_- \\ \iff \quad & \rho_i(\alpha - (N+1-M)\beta) \in \Delta_-. \end{aligned}$$

Thus the squares of T_{i+1} are found by right justifying all the dots (including those which are not tokens).

			•	•	•	•
$\alpha{-}6\beta$	$\alpha{-}5\beta$	$S_{\alpha-4\beta}$	$S_{\alpha-3\beta}$	$S_{\alpha-2\beta}$	$S_{\alpha-\beta}$	S_α

Thus after the move β is applied several times, the squares containing tokens will be a subset of T_{i+1}. Since every root not equal to $\pm\beta$ is part of such a maximal chain, this completes the proof. (We do not need to concern ourselves with β, since $S_\beta \in T_{i+1}$.) $\square$

Proof of Theorem 2.3. Sufficiency follows from Theorem 2.2. For necessity, we argue by induction. It is certainly possible to win the game for (ρ_m, π_2). Suppose it is possible to win the game for (ρ_{i+1}, π_2), for some $i < m$. By Proposition 2.3.4 we can apply a sequence of moves to the initial position for (ρ_i, π_2) so that the set of squares containing 1-tokens are a subset of T_{i+1}. Therefore by Proposition 2.3.1, it is possible to win the game for (ρ_i, π_2). By induction, it is possible to win the game for (π_1, π_2). $\square$

2.3.6 Converses and counterexamples

The converse of Theorem 2.1 is certainly not true. The first counterexamples in $GL(n)$ occur for $n = 4$. See Figure 2.6.

There are also counterexamples to the converse of Theorem 2.2. The first examples of this in $GL(n)$ occur for $n = 5$.

Example 2.3.6. *The initial position of the game for the permutations 23145, 14253, 41523 shown in Figure 2.7 has only one square with 2 tokens, and one*

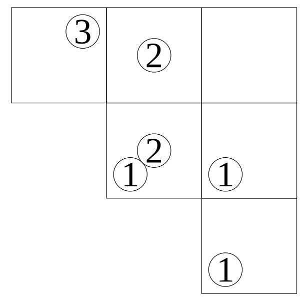


Figure 2.6: The permutations $\pi_1 = 1432$, $\pi_2 = 2314$, $\pi_3 = 2134$ are a counterexample to the converse of Theorem 2.1. The game is not doomed, though $c_{\pi_1\pi_2\pi_3} = 0$. All other $GL(4)$ counterexamples are similar to this one.

empty square. Any effort to rectify this imbalance winds up moving more than just one token, and so the game cannot be won. However the Schubert intersection number for this triple of permutations is 1.

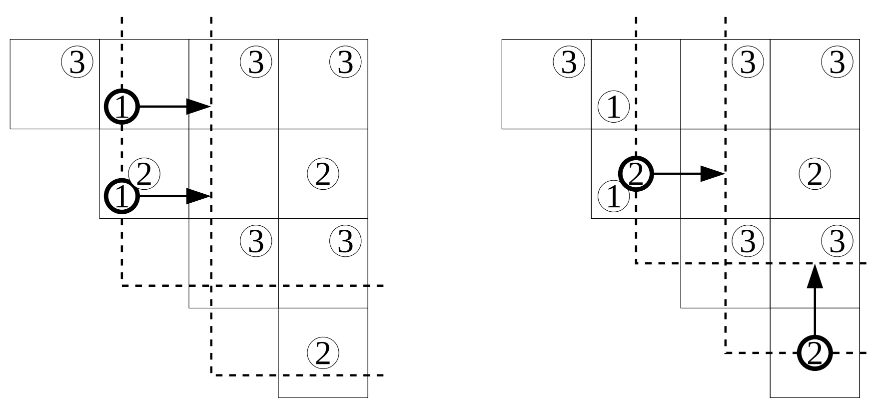


Figure 2.7: The permutations 23145, 14253, 41523 give a counterexample to the converse of Theorem 2.2.

We shall therefore make some refinements to the game which eliminate this last counterexample, as well as many others.

2.3.7 The general game

The general root game is set up identically: we have squares S_α indexed by the positive roots, each can contain any combination of the 1,2, or 3-tokens (with no label repeated), and the initial configuration is the same. Again, we note that the limit to 3 classes is arbitrary: all results naturally generalise to products of arbitrarily many classes.

The difference is that before each and every move, the set of squares is partitioned into "regions". Initially the squares are all in one region. Suppose A is an ideal subset of the squares, with the property that the total number of tokens in the squares of A is exactly equal to $|A|$. For every A with these properties, we subdivide each region R into two regions $R \cap A$ and $R \cap A^c$. (Empty regions produced in this way can be ignored.) Each region will always have the property that the number of tokens in the region is equal to the number of squares in the region.

The moves are more or less as before, except that any move only involves a single region, and no token may cross from one region to another. A move is specified by a triple $[k, \beta, R]$, where $k = 1, 2$ or 3 is a choice of token label, $\beta \in \Delta_+$, and R is a choice of region. Find all pairs of squares $S_\alpha, S'_\alpha \in R$ such that $\alpha' - \alpha = \beta$, and proceeding in order of decreasing height of α, if a k-token occurs in the square S_α but not in $S_{\alpha'}$, move it from the first square to the second square.

As before, to win the game we want exactly one token in each square. An example appears in Figure 2.8. We in invite the reader to replay and win the example from Figure 2.7 under these modified rules.

2.3.8 Proof of the non-vanishing criterion

Theorem 2.4. *If the revised game can be won, then $c_{\pi_1\pi_2\pi_3} \geq 1$.*

The following proves both Theorem 2.4 (directly), and also Theorem 2.2 by ignoring part 3 of the claim within.

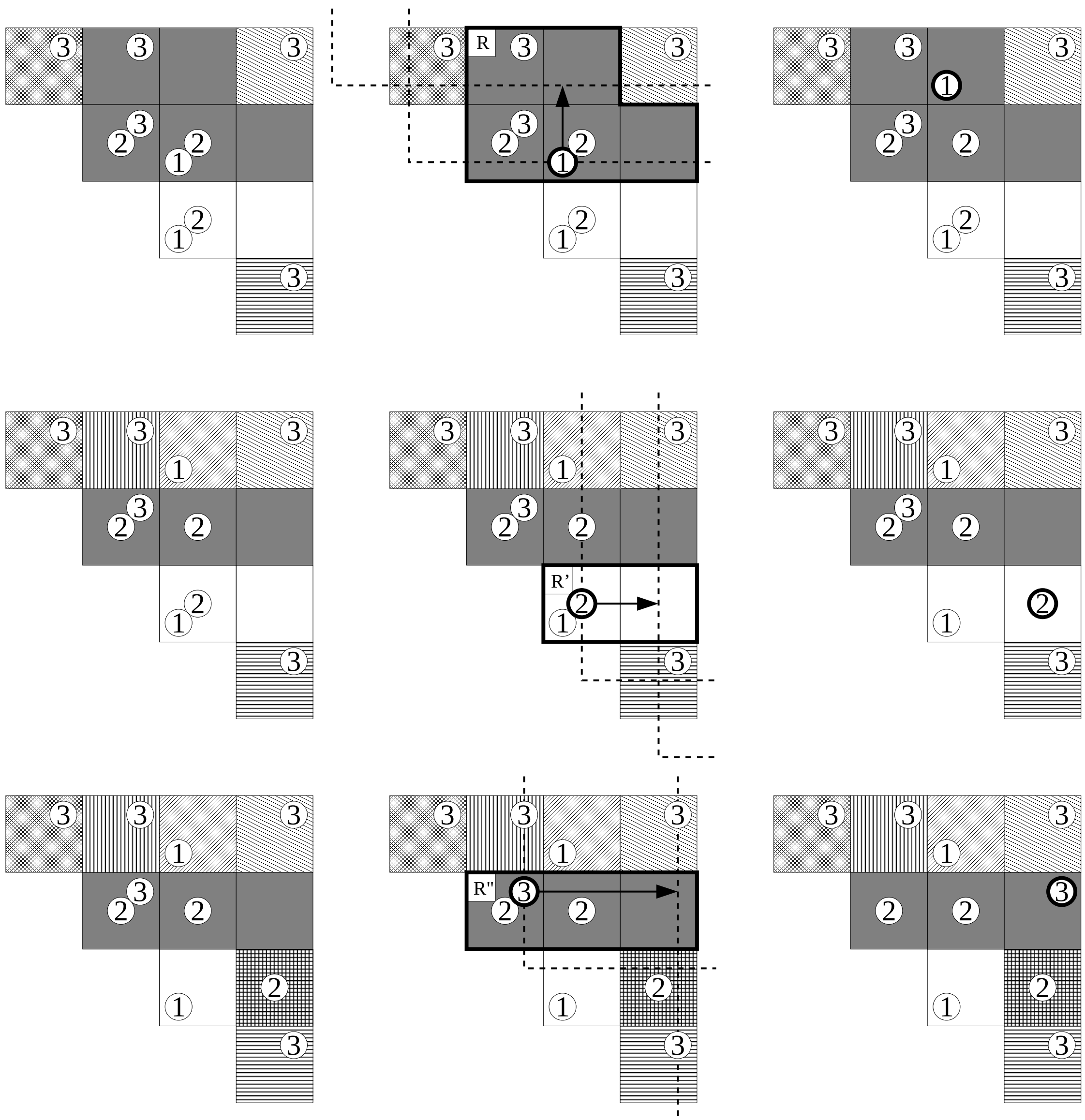


Figure 2.8: The general game, played out for permutations 13425, 41325, 14352. The moves, shown in the centre column, are: $[1, \alpha_{12}, R]$, $[2, \alpha_{45}, R']$, and finally $[3, \alpha_{35}, R'']$. The left column shows the state before the move, in which the set of squares is maximally divided into regions. The right column shows the state immediately after the move, before further subdividing.

Proof. For V a finite dimensional representation of B, let $Gr(V)$ denote the disjoint union of all Grassmannians $Gr_l(V)$. Since V has a B-action, so does $Gr(V)$.

Let $U = (U_1, U_2, U_3) \in Gr(V)^3$. We will call the pair (V, U) "good" if $\dim(U_1) + \dim(U_2) + \dim(U_3) = \dim V$, and there is a point (U_1', U_2', U_3') in the B^3-orbit closure through U such that $U_1' + U_2' + U_3' = V$.

For $U \in Gr(V)^3$, define

$$g(U) = \{(U_1, U_2, U_3) \in B^3 \cdot U \mid U_1 + U_2 + U_3 = V\} \subset Gr(V)^3.$$

Note that the set of $(U_1, U_2, U_3) \in Gr(V)^3$ with $U_1 + U_2 + U_3 = V$ is Zariski open in $Gr(V)^3$. Thus (V, U) is good $\iff$ $g(U)$ is an open dense subset of $B^3 \cdot U$ $\iff$ $g(U) \neq \emptyset$.

At any point in the game, we have the following data:

1. A set of regions, each of which is the difference of two ideal subsets. Hence the corresponding root spaces can be thought of as spanning a subquotient representation V of the B-representation $\mathfrak{n}$.

2. Any subset of $\{1, 2, 3\}$ on each root. For any region corresponding to the subquotient representation $V = V_1/V_2$, these give rise to subspaces $U_k = \mathrm{span}\{\bar{e}_\alpha \mid k \in S_\alpha\}$ (where $\bar{e}_\alpha := e_\alpha + V_2$ is the image of e_α under quotienting by V_2). Thus the arrangement of tokens in squares in the region corresponding to V, describes a T-fixed point $U \in Gr(V)$.

Thus a state of the game can be represented by a set of pairs $\{(V_m, U_m)\}$ where V_m is a B-representation, and $U_m \in Gr(V_m)^3$ is a T-fixed point.

We claim the following:

1. The initial state of the game is given by $(\mathfrak{n}, P)$, where $P = (P_1, P_2, P_3)$.

2. The Schubert intersection number $c_{\pi_1\pi_2\pi_3}$ is non-zero if and only if $(\mathfrak{n}, P)$ is good.

3. Suppose that $\{(V_m, U_m)\}$ is the state of the game before subdividing into regions, and $\{(V_n', U_n')\}$ is the state after. If (V_n', U_n') is good for all n, then each (V_m, U_m) was good for all m.

4. Suppose that $\{(V_m, U_m)\}$ is the state of the game before a move is made, and $\{(V_m, U'_m)\}$ is the state after. If (V_m, U'_m) is good then (V_m, U_m) was good.

5. If $\{(V_m, U_m)\}$ is the state of the game when the game is won, then each (V_m, U_m) is good.

Proof of claims.

1. This is clear from this definition of P_k.

2. Lemma 2.2.1 then says that $c_{\pi_1\pi_2\pi_3}$ is non-zero if and only if for a generic point (P'_1, P'_2, P'_3) in the N^3-orbit ($=B^3$-orbit) through (P_1, P_2, P_3), $P'_1+P'_2+P'_3 = \mathfrak{n}$, in other words if and only if $g((P_1, P_2, P_3))$ is non-empty, or equivalently $\iff$ $(\mathfrak{n}, P)$ is good.

3. Suppose V is a B-invariant subspace of some B-representation V'. Define maps $\sigma_V : Gr(V') \to Gr(V'/V)$, given by $\sigma_V(U) = U/V$, and $\tau_V : Gr(V') \to Gr(V)$ given by $\tau(U) = U \cap V$. Also let $\Sigma_V = \sigma \times \sigma \times \sigma : Gr(V')^3 \to Gr(V'/V)^3$, and $T_V = \tau \times \tau \times \tau : Gr(V') \to Gr(V)^3$. Note σ_V and τ_V are not everywhere continuous, but they are B-equivariant, and continuous on B-orbits.

   Suppose U_k are subspaces of V' with $\dim(U_1)+\dim(U_2)+\dim(U_3) = \dim V'$. By elementary linear algebra, if $\sigma_V(U_1) + \sigma_V(U_2) + \sigma_V(U_3) = V'/V$, and $\tau_V(U_1) + \tau_V(U_2) + \tau_V(U_3) = V$, then $U_1 + U_2 + U_3 = V'$.

   Suppose $(V'/V, \Sigma_V(U))$ and $(V, T_V(U))$ are both good. Then $T_V^{-1}(g(T_V(U))$ and $\Sigma_V^{-1}(g(\Sigma_U))$ are both open dense subsets of $B^3 \cdot U$. Since $g(U)$ contains the intersection of these, (V', U) must be good.

   Let V be the B-invariant subspace of $\mathfrak{n}$ corresponding to a ideal subset. Then the new position of the game after splitting along V is just $\{(V_m \cap \bar{V}, T_{\bar{V}}(U_m))\} \cup \{(V_m/\bar{V}, \Sigma_{\bar{V}}(U_m))\}$ (where $\bar{V}$ is the image under the appropriate quotient map). But if $(V_m \cap \bar{V}, T_{\bar{V}}(U_m))$ and $(V_m/\bar{V}, \Sigma_{\bar{V}}(U_m))$ are both good, then (V_m, U_m) is good.

4. Suppose the move is given by the root α, token label k, and the region corresponding to V_m. We consider the 1-dimensional subgroup of B^3 given by $N_\alpha \hookrightarrow B \hookrightarrow B^3$, where $N_\alpha \cong (\mathbb{C}, +)$ is the exponential of the α root space, and the last inclusion is $b \to (b,1,1), (1,b,1)$ or $(1,1,b)$ depending on k.

   Let $\theta_{\alpha,k} : N_\alpha \to B^3$ denote this composition. We now calculate

   $$\lim_{t\to\infty} \theta_{\alpha,k}(t) \cdot U_m.$$

   Without loss of generality suppose that $k = 1$. Let $U_m = (U_1, U_2, U_3)$ and $U'_m = (U'_1, U_2, U_3)$. We can represent U_1 as $[\bar{e}_{\alpha_1} \wedge \ldots \wedge \bar{e}_{\alpha_l}]$, and U'_1 as $[\bar{e}_{\alpha'_1} \wedge \ldots \wedge \bar{e}_{\alpha'_l}]$, via the Plücker embedding $Gr(V_m) \hookrightarrow P(\bigwedge^* V_m)$. Now

   $$\begin{aligned}\theta_{\alpha,1}(t) \cdot U_m &= \theta_{\alpha,1}(t) \cdot \big([\,\bar{e}_{\alpha_1} \wedge \ldots \wedge \bar{e}_{\alpha_l}], U_2, U_3\big)\\ &= \Big(\big[(\bar{e}_{\alpha_1} + t(\alpha \cdot \bar{e}_{\alpha_1})) \wedge \ldots \wedge (\bar{e}_{\alpha_l} + t(\alpha \cdot \bar{e}_{\alpha_l}))\,\big], U_2, U_3\Big),\end{aligned}$$

   where $\alpha \cdot \bar{e}_{\alpha_i} = \bar{e}_{\alpha_i+\alpha}$, if $\alpha_i + \alpha$ is a root belonging the region corresponding to V_m, and 0 otherwise. In the limit as $t \to \infty$, the only term which survives is the one with the highest power of t, which is precisely

   $$\big([\pm t^{\#\text{tokens that move}} \bar{e}_{\alpha'_1} \wedge \ldots \wedge \bar{e}_{\alpha'_l}], U_2, U_3\big).$$

   Thus

   $$U'_m = \lim_{t\to\infty} \theta_{\alpha,k}(t) \cdot U_m,$$

   is another point in $\overline{B^3 \cdot U_m}$. Thus if (V_m, U'_m) is good, so is (V_m, U_m).

5. In the winning position the point $(V_m, (U_{m1}, U_{m2}, U_{m3}))$ has $\sum_{k=1}^{3} \dim U_{mk} = \dim V_m$ and $U_{m1} + U_{m2} + U_{m3} = V_m$ and thus is good.

$\square$

Thus if the game can be won, all states of the game en route to a winning configuration must be good. This includes the initial state, and hence the Schubert intersection number $c_{\pi_1\pi_2\pi_3}$ is non-zero.

$\square$

2.4 Remarks and variations

2.4.1 Splitting

Let A be an ideal subset. In the rules of the root game, there is a condition for splitting regions along A, namely, that we split along A if and only if the the number of tokens in A equals $|A|$. The astute observer will notice that this condition is never used in the proof. Essentially this means the proof is valid for a variation of the game in which the player has the option to split regions along *any* ideal subset A between moves. That said, we will show now that it is never advantageous to the player to exercise such an option, unless $|A|$ equals the number of tokens in A.

Suppose the rules say not to split along A. If we do split along A there will be too many tokens in one region. Since regions can never be rejoined once they are split, the game cannot be won.

Moreover, suppose the rules say to split along A, and the player chooses not to. Of any move that is made subsequently, one of the following two things must be true: either (1) the same arrangement of tokens could have been reached (possibly using multiple moves) if we had split along A, or (2) the move causes the game to be lost.

In particular, the revised game always performs at least as well as, and sometimes better than the weak version of the game.

2.4.2 Products which are not top degree

It is worth noting that the root game can be used to (sometimes) determine whether three (or more) Schubert varieties in general position intersect non-trivially, even when the sum of their codimensions does not equal $\dim G/B$. On the cohomology level, this means we can use the game to study whether a product of Schubert classes is non-zero, whether or not the product is of top degree. There are two minor modifications to the rules required. One is to change the winning condition to read “at most one token in each square”, rather than “exactly one token in each square”. (The losing and doomed conditions remain as stated previously.) The other is that it is no longer a priori clear exactly when splitting will be strictly advantageous. Therefore, we must remove the rule telling when to split, and instead, allow the option of splitting along any ideal subsets between moves. Under these modifications, if the revised game can be won, the Schubert varieties in question have at least one point of intersection. If the game is doomed, they do not intersect.

The simplest way to see that this is true is to note that $\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} \neq 0$ if and only if we can multiply this product by a product of divisor classes (classes ω_{r_α}, where r_α is a reflection in a simple root α) to obtain

$$\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} \cdot \omega_{r_{\alpha_1}} \cdots \omega_{r_{\alpha_l}} \geq 1 \in H^{\mathrm{top}}(G/B).$$

On the root game level, each of these divisor classes produces a single token with its own label sitting on a simple root. If we consider all possible choices for $\alpha_1, \ldots, \alpha_l$, then after some moves, these new ‘divisor tokens’ can be made to lie in any squares.

Assume that it is possible to win the root game for (π_1, π_2, π_3) according to the more general definition. In the winning position, there will be some empty squares. In the root game corresponding to $(\pi_1, \pi_2, \pi_3, r_{\alpha_1}, \ldots, r_{\alpha_l})$, we first move the divisor tokens to the empty squares, then move the remaining tokens to the winning configuration. This will yield exactly one token in each square. Thus $\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} \cdot \omega_{r_{\alpha_1}} \cdots\ \omega_{r_{\alpha_l}} \geq 1$, and hence $\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} \neq 0$. A similar type of argument shows that a doomed game implies $\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} = 0$.

2.4.3 Converses and computations

It would be quite surprising and remarkable if the converse of Theorem 2.4 were true in any generality. So far, for $GL(n)$, the converse has deftly eluded any counterexamples. In fact the converse of Theorem 2.4 has been affirmed by an exhaustive computer search for $GL(n)$ for $n \leq 7$.

To test the converse, we calculated Schubert intersection numbers using the following recurrence [Knu03]. Let $r = r_\alpha \in W$ be a simple reflection (in the simple root α) such that $\pi_1 r > \pi_1$. Such an r always exists if $\pi_1 \neq w_0$. Then,

$$c_{\pi_1\pi_2\pi_3} = \begin{cases} 0, & \text{if } \pi_2 r > \pi_2 \text{ and } \pi_2 r > \pi_3 \\ c_{\pi_1 r, \pi_2 r, \pi_3}, & \text{if } \pi_2 r < \pi_2 \text{ and } \pi_3 r > \pi_3 \\ c_{\pi_1 r, \pi_2, \pi_3 r}, & \text{if } \pi_2 r > \pi_2 \text{ and } \pi_3 r < \pi_3 \end{cases}$$

or if $\pi_2 r < \pi_2$ and $\pi_3 r < \pi_3$, we use

$$c_{\pi_1\pi_2\pi_3} = c_{\pi_1 r, \pi_2 r, \pi_3} + c_{\pi_1 r, \pi_2, \pi_3 r} + \sum_{\substack{\rho \succ \pi_1,\ \rho \neq \pi_1 r, \\ \rho = \pi_1 r_\beta}} \left(\frac{\alpha - r_\beta \alpha}{\beta} \right) c_{\rho, \pi_2 r, \pi_3}.$$

(Here $\succ$ denotes the Bruhat covering relation, i.e. $\rho \succ \pi \iff \rho > \pi$ and $l(\rho) = l(\pi) + 1$.) Each application of this recurrence increases the length of π_1, so eventually it reduces the calculation of any Schubert intersection number to $c_{w_0,1,1} = 1$.

If the Bruhat covering relation is calculated in advance, this is a remarkably quick way of calculating Schubert intersection numbers. It is significantly faster than doing the corresponding calculations with Schubert polynomials. The speed here is important, because the number of Schubert problems to check gets rapidly large with increasing n. To cut this quantity down from the obvious $|W|^3$-many problems, we first sort the Weyl group elements by their length so that we never need to consider a problem for which $l(\pi_1)+l(\pi_2)+l(\pi_3) \neq \dim(G/B)$. Furthermore, the S_3 symmetry of the Schubert intersection numbers allows us to cut this down further by a factor of 3!.

For each positive Schubert intersection number, we performed a exhaustive search through all possible root games. In each case, the search revealed that it was possible to win the corresponding root game. For $GL(4)$, this calculation takes only seconds. $GL(5)$, runs in about one minute, and $GL(6)$ takes an hour. The same calculation for $GL(7)$ took a total of 76 days computing time on a network of Sun Sparc Ultra 5 machines.

The converse of Theorem 2.4 has also been verified for the exceptional group G_2, as well as for $SO(5)$ and $SO(7)$. (The next smallest exceptional group, F_4, is unfortunately beyond our computational abilities at the moment.)

For the groups $SO(n)$, $n \geq 8$, the converse of Theorem 2.4 is in fact false. For $SO(8)$, we represent an element of the Weyl group $W = S_4 \rtimes C_2^3$ by a permutation $a_1a_2a_3a_4$ of 0123, where each symbol, except 0, is either decorated with a bar or not (0 is always unbarred). This permutation acts on $\mathbb{R}^4$ by the matrix whose i^{th} row is x_{a_i} if i is unbarred, and $-x_{a_i}$ if i is barred. The two counterexamples to the converse of Theorem 2.4 for $SO(8)$ are listed below.

π_1	π_2	π_3
$0\bar{1}32$	$0\bar{2}31$	$03\bar{2}1$
$03\bar{1}2$	$0\bar{2}31$	$0\bar{2}31$

The problem that arises in these examples is that although there exists a T-fixed point (U_1, U_2, U_3) on the appropriate B^3-orbit closure inside $Gr(V)$ with $U_1 + U_2 + U_3 = V$, the moves of the game fail to find it. We are not aware of any examples in which $c_{\pi_1\pi_2\pi_3} \geq 1$ but where there are no suitable T-fixed points on any of the relevant varieties. It therefore seems it would be desirable to be able to describe a larger set of moves—moves which, starting from a T-fixed point on $Gr(V)$ can reach all the other T-fixed points in its B^3-orbit closure.

One special case where the converse of Theorem 2.4 is true is when the classes ω_{π_i} are pulled back from a Grassmannian in an appropriate way. We prove this result in a Chapter 3. Another special case is Theorem 2.3, which tells us that the converse is true for products of only two Schubert classes.

Another modification one might wish to make to the root game is to allow only moves involving tokens labeled 1 and 2. This version looks nicer when viewing Schubert calculus as taking products in cohomology (rather than intersection numbers). Under this weakening, Theorem 2.4 remains true (obviously), but the converse is already false for $GL(n)$. There are no examples of this for $n \leq 5$; however, for $n = 6$ there are a total of four such examples:

π_1	π_2	π_3
145326	321564	315264
154326	312564	315264
514326	152364	135264
154236	312654	315264

It is possible, with still further modifications to the game to dispose of these $GL(6)$ counterexamples. One such modification is to introduce a new kind of move. For this new move, one selects a region R, and a pair of token labels: $k_1 \neq k_2$. The region R must have the property that there is no square in R which contains both a k_1-token and a k_2-token. In this case we may replace every k_2-token with a k_1-token in the same square. In the notation of the proof of Theorem 2.4, if $k_1 = 1, k_2 = 2$, this has the effect of replacing $((U_1, U_2, U_3), V)$ with $((U_1 + U_2, 0, U_3), V)$, provided $U_1 \cap U_2 = 0$; if the latter is good, so is the former, and the proof of Theorem 2.4 will still hold.

2.4.4 Other cohomology theories

The study of Schubert calculus is not limited to the ordinary cohomology of spaces G/B or G/P. One could also apply other cohomology theories to these spaces. Some well known examples include T-equivariant cohomology, K-theory, and quantum cohomology. It is unlikely that the techniques here could apply to either T-equivariant cohomology or K-theory. However, the vanishing of quantum cohomology of homogeneous spaces can be studied using tangent space methods (see [Bel03]), and certainly merits further investigation. We discuss this further in Chapter 5.

Chapter 3

Root games on Grassmannians

3.1 Review of the root game for $GL(n)$

In Chapter 2 we introduced the *root game*, a combinatorial game which often identifies vanishing and non-vanishing structure constants in Schubert calculus on a generalized flag manifold G/B. In this chapter, our goal is to strengthen those results by giving a partial converse to our non-vanishing criterion. We show that a converse of Theorem 2.2 holds when the group G is $GL(n, \mathbb{C})$, and the Schubert classes are pulled back from a Grassmannian. We begin by recalling the relevant material from Chapter 2.

Let $G = GL(n)$. Let B and B_- denote the subgroups of upper and lower triangular matrices respectively, and $T = B \cap B_-$ the standard maximal torus (diagonal matrices). Recall that to each element of the symmetric group $\pi \in S_n$, there is a corresponding T-fixed point π on the flag manifold $Fl(n) = G/B$, and an associated Schubert variety $X_\pi = \overline{B_- \cdot \pi}$. We denote its cohomology class by ω_π.

Given $\pi_1, \ldots, \pi_s \in S_n$, the root game attempts to give some information about the Schubert intersection number $\int_{Fl(n)} \omega_{\pi_1} \cdots \omega_{\pi_s}$, as follows.

We have a set of squares S_{ij}, indexed by $1 \leq i < j \leq n$. In each square we allow tokens to appear. Each token has a label $k \in 1, \ldots, s$, and no square may ever contain two tokens with the same label. We'll call a token labeled k a k-token, and write $k \in S_{ij}$ if a k-token appears in square S_{ij}.

The initial configuration of tokens is as follows: for $i < j$ square S_{ij} includes a k-token in the initial configuration if and only if $\pi_k(i) > \pi_k(j)$.

From the initial position we move the tokens around according to certain rules with the objective of getting exactly one token in each square. However, before each move we have the option of splitting the game into multiple regions. A **region** is just a set of squares. Initially there is a single region which contains all the squares. We define an **ideal subset** of the squares to be a set of squares A such that if $S_{ij} \in A$ then so are all $S_{i'j'}$ with $i \leq i'$ and $j \leq j'$. Given an ideal subset of the squares we can break up a region R into $R \cap A$ and $R \setminus A$. We may repeat this splitting process as many times as desired. It turns out to be advantageous to split if and only if the total number of tokens in all squares in A equals $|A|$. We'll call the process of finding all such A and splitting along them **splitting maximally**.

A **move** is specified by a region R, a token label $k \in \{1, \ldots, s\}$ and a square S_{ij} (which is not necessarily in R). For every l with $j < l \leq n$, if S_{jl} and S_{il} are both in R and a k-token appears in S_{jl} but not in S_{il}, we move it from S_{jl} to S_{il}. Also for each l with $1 \leq l < i$, if S_{li} and S_{lj} are both in R and a k-token appears in S_{li} but not in S_{lj}, we move it from S_{li} to S_{lj}.

The game is **won** when there is exactly one token in each square. The main result from Chapter 2 that we shall need is Theorem 2.4.

Theorem 2.4 (for $GL(n)$). *If the game can be won, then*

$$\int_{Fl(n)} \omega_{\pi_1} \cdots \omega_{\pi_s} \geq 1.$$

In general, we can also use the game to study the cohomology rings of partial flag manifolds by pulling back cohomology classes to the full flag manifold. We shall now investigate this in some detail in the case of the Grassmannian. In particular we prove a version of the converse of Theorem 2.4 for Grassmannian Schubert calculus.

3.2 Associating a game to a Grassmannian Schubert Calculus problem

A **Schubert cell** relative to some base flag $x_0 = V_0 \subset V_1 \subset \cdots \subset V_n$ in the Grassmannian $Gr_l(n)$ is specified by a string $\sigma = \sigma^1 \dots \sigma^n$ of l '1's and $n-l$ '0's. The Schubert cell Y_{σ,x_0} can be described as the subspaces y of $\mathbb{C}^n$ such that $\dim y \cap V_r = \sigma^1 + \cdots + \sigma^r$. We denote the cohomology class of its closure by Ω_σ.

Given a list of $s+2$ 01-strings $\sigma_1, \dots, \sigma_s, \mu, \nu$, we will wish to study the integrals

$$\int \Omega_{\sigma_1} \cdots \Omega_{\sigma_s} \cdot \Omega_\mu \cdot \Omega_\nu$$

We do so by investigating an equivalent problem on a full flag manifold.

Let $N \geq 0$ be an integer. Let $i_1 < \cdots < i_{n-l}$ denote the positions of the '0's, and $j_1 < \cdots < j_l$ denote the positions of the '1's. We define three ways to associate a permutation to a 01-string σ:

$$\begin{aligned}
\pi(\sigma, N) &= i_1 \dots i_{n-l} j_1 \dots j_l (n+1)(n+2) \dots (n+N) \\
\pi'(\sigma, N) &= (i_1 + N) \dots (i_{n-l} + N) 12 \dots N (j_1 + N) \dots (j_l + N) \\
\pi''(\sigma, N) &= i_{n-l} \dots i_1 (N+n) \dots (n+1) j_l \dots j_1
\end{aligned}$$

We produce a list of permutations, $\pi_1, \dots, \pi_{s+2} \in S_{n+N}$:

$$\begin{aligned}
\pi_1 &= \pi(\sigma_1, N) \\
&\vdots \\
\pi_s &= \pi(\sigma_s, N) \\
\pi_{s+1} &= \pi'(\mu, N) \\
\pi_{s+2} &= \pi''(\nu, N)
\end{aligned}$$

Proposition 3.2.1.

$$\int_{Gr_l(n)} \Omega_{\sigma_1} \cdots \Omega_{\sigma_s} \cdot \Omega_\mu \cdot \Omega_\nu = \int_{Fl(n+N)} \omega_{\pi_1} \cdots \omega_{\pi_{s+2}}$$

Proof. If $h \in H^*(X)$ is a Schubert class on X (some partial flag variety), let $h^\vee$ denote the opposite Schubert class. If $h_1, \dots h_r$ are Schubert classes then to say that

$$\int h_1 \cdots h_r = c \tag{3.1}$$

is equivalent to saying that

$$h_1 \cdots \widehat{h_i} \cdots h_r = c\, h_i^\vee + \cdots \tag{3.2}$$

in the Schubert basis. We'll call Equation (3.2) the h_i-special version of Equation (3.1).

For a 01-string σ, let σ_+ denote the string σ followed by N ones, and let $_+\sigma$ denote the string σ preceded by N ones.

Consider the equation in $H^*(Gr_{N+l}(n+N))$:

$$\int_{Gr_{N+l}(n+N)} [Y_{\sigma_{i+}}] \cdots [Y_{\sigma_{i+}}][Y_{+\mu}][Y_{\nu_+}] = c. \tag{3.3}$$

If we take the $[Y_{+\mu}]$-special version of Equation (3.3) and pull it back to $Gr_l(n)$, we get the $[Y_\mu]$-special version of

$$\int_{Gr_l(n)} [Y_{\sigma_1}] \cdots [Y_{\sigma_s}][Y_\mu][Y_\nu] = c$$

On the other hand, if we take the $[Y_{\nu+}]$-special version of Equation (3.3) and pull it back to $Fl(n+N)$ we get the $[X_{\pi_{s+2}}]$-special version of

$$\int_{Fl(n+n)} [X_{\pi_1}] \cdots [X_{\pi_{s+2}}] = c.$$

$\square$

3.3 Converses for Grassmannians

3.3.1 The Grassmannian root game algorithm

Recall the correspondence between 01-strings and Young diagrams. Our Young diagrams will be in the French convention (the rows are left justified and increase in length as we move down). If σ is a 01-string σ, let $r_i(\sigma)$ denote the number of 1s before the i^{th} 0. We associate to σ the Young diagram $\lambda(\sigma)$ whose i^{th} row is $r_i(\sigma)$. We are allowing the possibility that some rows may have length 0.

If λ is a Young diagram, let $N + \lambda$ denote the Young diagram obtained by adding N squares to each row of λ *including all rows which contain* 0 *squares.*

Take $\sigma_1, \ldots, \sigma_s, \mu, \nu$ to be 01-strings representing Schubert classes in $Gr_l(n)$ (if $s = 1$, we'll write σ instead of σ_1), and associate permutations $\pi_1, \ldots, \pi_{s+2}$ as before.

Theorem 3.1. *Take N suitably large ($N \geq n - l$ will always suffice). The root game corresponding to $(\pi_1, \ldots, \pi_{s+2})$ can be won if and only if*

$$\int_{Gr_l(n)} \Omega_{\sigma_1} \cdots \Omega_{\sigma_s} \cdot \Omega_\mu \cdot \Omega_\nu \geq 1$$

Moreover, only moves involving tokens labelled $1, \ldots, s$ are required.

Proof. $\Rightarrow$ This follows from Proposition 3.2.1 and Theorem 2.4.

$\Leftarrow$ We shall first address the case where $s = 1$.

The initial positions of the 1-tokens are in the shape of the Young diagram $\lambda_1 = \lambda(\sigma)$. The initial positions of the 2-tokens are in the shape of a Young diagram $\lambda_2 = N + \lambda(\mu)$. The squares that do *not* contain a 3-token are also in the shape of a Young diagram, which we'll denote $\bar{\lambda}_3$. See Figure 3.1.

If $\lambda_2 \not\subseteq \bar{\lambda}_3$, then it is a basic fact that $\Omega_\mu \cdot \Omega_\nu = 0 \in H^*(Gr_l(n))$. So we may assume that no square contains both a 2-token and a 3-token. The squares which contain neither a 2-token nor a 3-token are empty squares (since N is suitably large), and are in the shape of a skew-diagram $\bar{\lambda}_3/\lambda_2$.

At the outset of the game, some immediate splitting occurs. Each square containing a 3-token becomes a one-square region of its own. The remaining squares are those of $\bar{\lambda}_3$, which also form an unsplittable region. In what follows there will always be at most one unsolved region, which we shall refer to as the "big region".

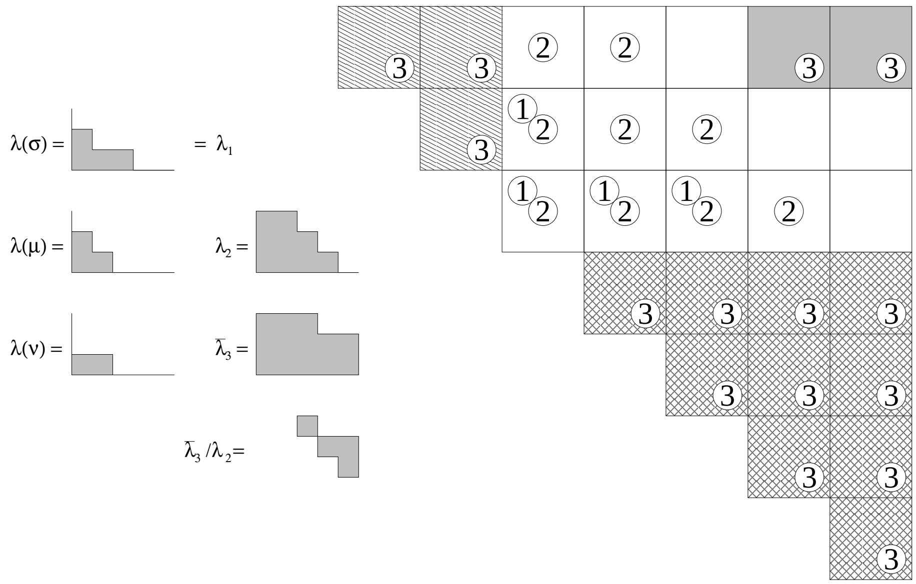


Figure 3.1: Initial position of the game for $\sigma = 010110$, $\mu = 010101$, $\nu = 001101$, with $N = 2$

Definition 3.3.1 (Zelevinsky [Zel81]). *A* **picture** *between two (French) skew diagrams is a bijection between their boxes with the property that if box A is weakly above and weakly right of box B in one diagram, then the corresponding boxes A' and B' are in lexicographic order (S_{ij} precedes $S_{i'j'}$ if $i < i'$ or $i = i'$ and $j < j'$) in the other diagram.*

Now $\int_{Gr_l(n)} \Omega_\sigma \cdot \Omega_\mu \cdot \Omega_\nu$ is given by the Littlewood-Richardson coefficient $c^{\bar{\lambda}_3}_{\lambda_1 \lambda_2}$, which can be described [RW84, Zel81] as the number of pictures between λ_1 and $\bar{\lambda}_3/\lambda_2$. In particular, if this number is non-zero, there exists such a picture. We pick one, and denote by f the map it defines from the squares of σ to the squares of $\bar{\lambda}_3/\lambda_2$. Each of these squares in question is in fact also some square S_{ij} in the big region of the game.

We now use f to construct a root game which transports each 1-token to a square of $\bar{\lambda}_3/\lambda_2$.

At each point in the game there will be a number associated to each unplaced 1-token (i.e. one which has not already reached its final destination) and each square of $\bar{\lambda}_3/\lambda_2$. Let t be a 1-token, whose initial square is S in σ, and whose current square is S_{ij}. Let $S_{i'j'} = f(S)$. We associate to both t and to the square $S_{i'j'}$ the number $i - i'$.

The essential properties of this numbering scheme are the following:

Claim Initially, the numbers on the unplaced 1-tokens are are

a) weakly increasing in each row, and

b) weakly decreasing in each column

In $\bar{\lambda}_3/\lambda_2$ the numbers are

c) weakly decreasing in each row, and

d) weakly decreasing in each column.

The proof is by a straightforward induction, proceeding right to left along each row of $\bar{\lambda}_3/\lambda_2$, beginning with the bottom row, and proceeding upward.

Note that since the number associated to the lower-leftmost token is at least 0 the number associated to each token/square is non-negative. Call an empty square of $\bar{\lambda}_3/\lambda_2$, or 1-token in a square of λ_2, **ready** if it has the number 0 associated to it.

It will be evident that the algorithm below will preserve the properties in the claim.

The Grassmannian root game algorithm (GRGA)

1. If any of the 1-tokens are ready, go to step 2. Otherwise, perform a sequence of moves to shift all unplaced 1-tokens up one square. (Choosing N sufficiently large ensures it is possible to do this.) This will decrease the number associated to each 1-token by 1. Repeat this step until some 1-token is ready.

2. Scan through the columns of $\bar{\lambda}_3/\lambda_2$, beginning with the rightmost column and proceeding to the left. Within each column locate the topmost square which does not already contain a 1-token. Let S be the *first* ready square which we encounter in this way.

3. Find a ready token t in the same row as the square S. Make the unique move which causes t to move into S. This may cause other tokens to move as well.

4. Repeat steps 1 through 3 until every square of $\bar{\lambda}_3/\lambda_2$ contains a 1-token.

For $s = 1$ the game will be won at this point. However in order to make use of this algorithm effectively in the case where $s > 1$, we will require the following additional step.

5. Split in a minimal way so that every 1-token is in a one-square region of its own.

A small example of the GRGA and all of its the moves from beginning to end is illustrated in Example 3.3.2, at the end of this proof.

We now show that the algorithm will win the game.

- *A move from step 3 causes only ready tokens to move.* Because S is the top square in its column, no tokens above t move. Because of part (b) in the claim, all tokens below t are ready.

- *Only ready squares are filled.* The algorithm attempts to fill the rightmost squares first. If there is a ready square in the some column, the topmost empty square without a token in that column will also be ready (by part (d) of the claim), and thus the algorithm will never fill a square left of it first. However, by part (c) of the claim the ready squares are rightmost in their row. Thus the only way a non-ready square could be filled is if there were more ready tokens than ready squares in some row. But this is impossible.

- *The move from step 3 is always possible.* An empty square S is ready if and only if the 1-token which began in the square $f^{-1}(S)$ is in the same row as

S. Since only ready squares are filled by ready tokens, the number of ready squares and ready tokens in any given row is always equal. So if there is a ready square S in some row, there is also a ready token t in that row, and t is left of S.

- *Every ready square eventually gets filled (by a ready token).* First every ready square in the rightmost column is filled; then the next rightmost column, and so on.

Now because step 1 decreases the number associated to each square by 1, every square of $\bar{\lambda}_3/\lambda_2$ is ready at some point; thus the algorithm puts a 1-token in each square of $\bar{\lambda}_3/\lambda_2$, at which point the game is won.

For $s > 1$, we proceed by induction. Suppose $\int \omega_{\pi_1} \cdots \omega_{\pi_{s+2}} \neq 0$. Then we can write

$$\omega_{\pi_2} \cdots \omega_{\pi_{s+1}} = c\omega_\rho + \cdots \tag{3.4}$$

in the Schubert basis such that $c > 0$ and

$$\int \omega_{\pi_1} \cdot \omega_\rho \cdot \omega_{\pi_{s+2}} \neq 0.$$

By the $s = 1$ argument we can win the game corresponding to (π_1, ρ, π_{s+2}), only moving 1-tokens. It is easy to see that exactly the same sequence of moves and splittings (in step 5), can be made in the game for $(\pi_1, \pi_2, \ldots, \pi_{s+1}, \pi_{s+2})$, and that it causes the 1-tokens to end up in exactly the same final destinations. This sequence of moves no longer wins the game; however, suppose we now replace each 1-token by an $(s+2)$-token. This has no effect whatsoever on the game, because after step 5 every 1-token is in its own one-square region. But now we have precisely reached the initial position of the game corresponding to $\pi_2, \ldots, \pi_{s+1}, w_0\rho$. This is again a game associated to a Grassmannian problem, and by Equation 3.4 the Schubert intersection number is non-zero. By induction, there is a sequence of moves to win this new game. Thus by concatenating the two sequences of moves, one can win the original game. $\square$

3.3.2 An example

Example 3.3.2. *Figures 3.2 and 3.3 illustrate the GRGA in action. This example is on $Gr_4(7)$, with $\sigma = 1010101$, $\mu = 1001011$, $\nu = 0100111$, and $N = 3$. Only the 1-tokens are shown in this diagram (the number is the number associated to that token, not the "token label"). Each 1-token is given a shading and the corresponding square under f in $\bar{\lambda}_3/\lambda_2$ is shaded similarly. Only the squares in the upper right 3×7 rectangle are shown here, as these are the only ones relevant to the movement of the 1-tokens. Each unshaded square actually contains a 2-token. The two darkly shaded squares in the upper right corner contain 3-tokens, as do each of the squares not shown in this diagram, but these squares are not part of the big region.*

3.3.3 Remarks

In step 3 of the GRGA, there is a somewhat canonical choice for the token t, namely the leftmost ready token in its row. If we use this choice of t, one can verify that the game actually transports the 1-token which is initially in square S to the square $f(S)$.

In the GRGA, we do not split maximally before every move. It can never be harmful to split maximally between moves, however in this case, very little changes if one does. In particular, note that in the $s = 1$ case it is possible to win the game without any further splitting beyond that which occurs at the outset. In Section 3.4 we will make use of this fact.

The root game can be used to determine whether or not $\Omega_{\sigma_1} \cdots \Omega_{\sigma_s} \cdot \Omega_\mu \cdot \Omega_\nu = 0$, even if the cohomological degree of the product is not $\dim_{\mathbb{R}} Gr_l(n)$. To do this, we modify the game by changing the winning condition to read "at most one token in each square", rather than "exactly one token in each square". Once we do this, the above product will be non-zero if and only if the modified game can be won (see Section 2.4.2). However, under these more general circumstances, there is no longer an easy necessary and sufficient condition indicating when splitting is

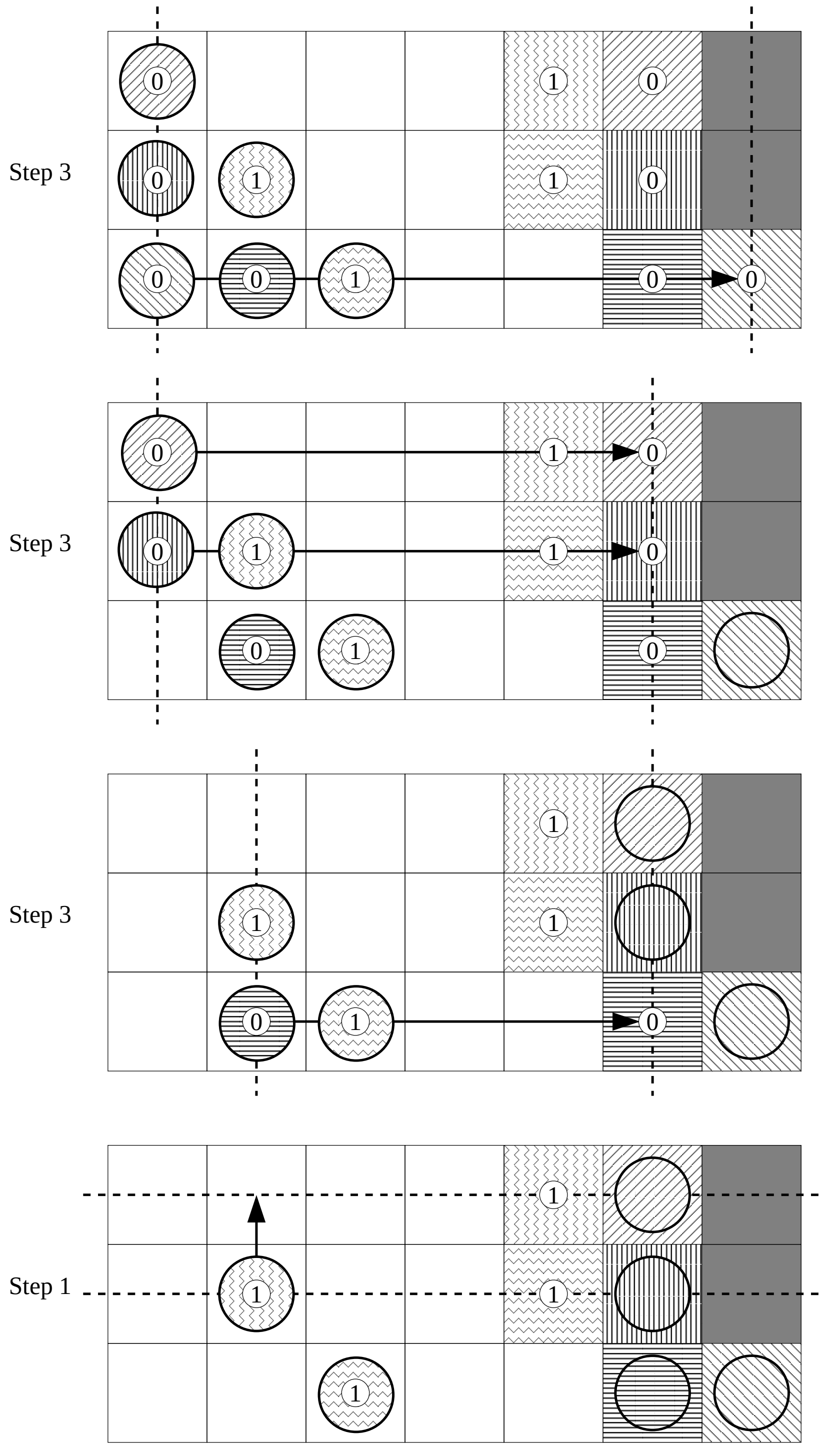


Figure 3.2: The Grassmannian root game algorithm. Here $\sigma = 1010101$, $\mu = 1001011$, $\nu = 0100111$, and $N = 3$.

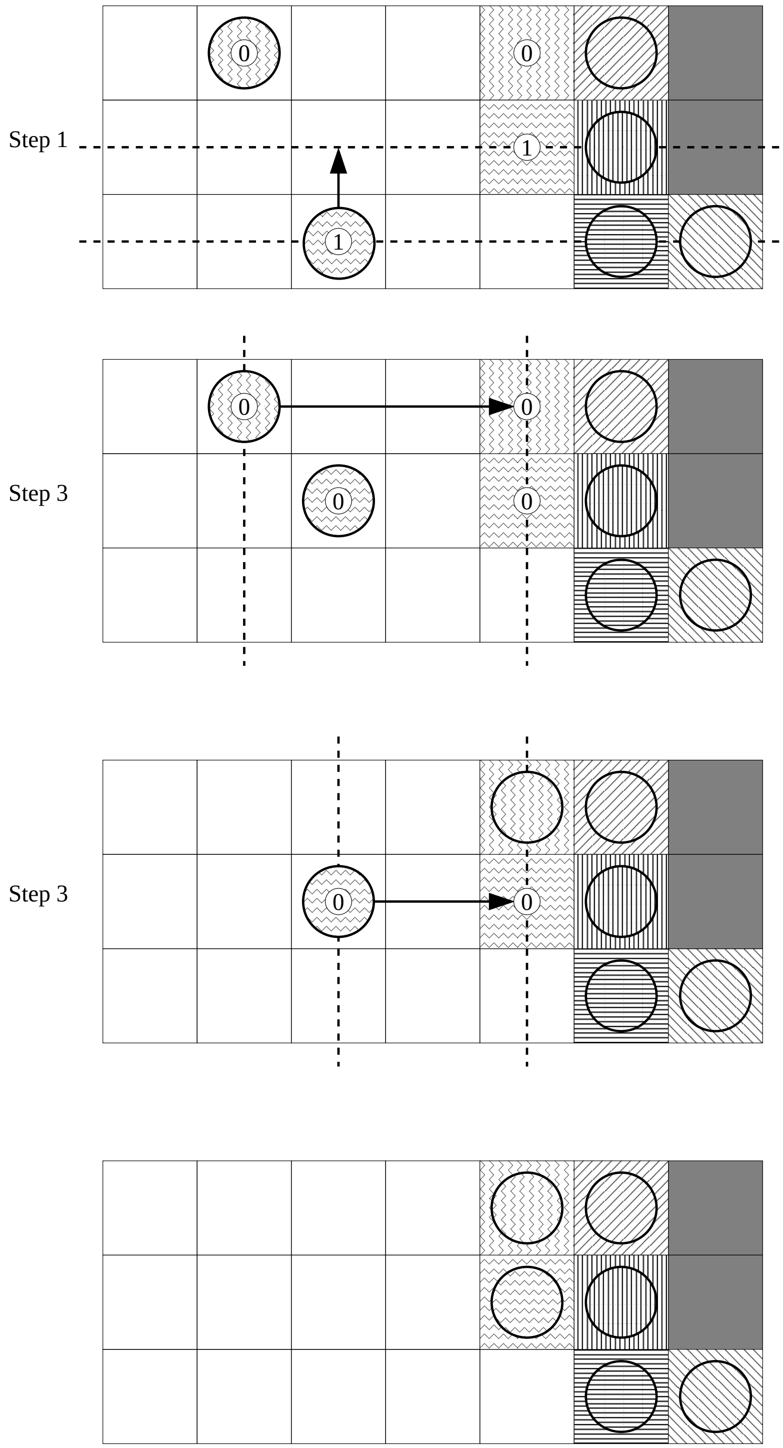


Figure 3.3: Continuation of Figure 3.2

advantageous.

One of the unfortunate features of this presentation is the asymmetry in the way the permutations $\pi_1, \ldots, \pi_{s+2}$ are defined. However, as the proof is valid for arbitrary s, we can produce a symmetrical game by taking $\sigma_1, \ldots, \sigma_s$ to be arbitrary, and $\nu = \mu = 0 \ldots 01 \ldots 1$, so that $\Omega_\nu = \Omega_\mu = 1 \in H^*(Gr_l(n))$. Compare Figures 3.1 and 3.4.

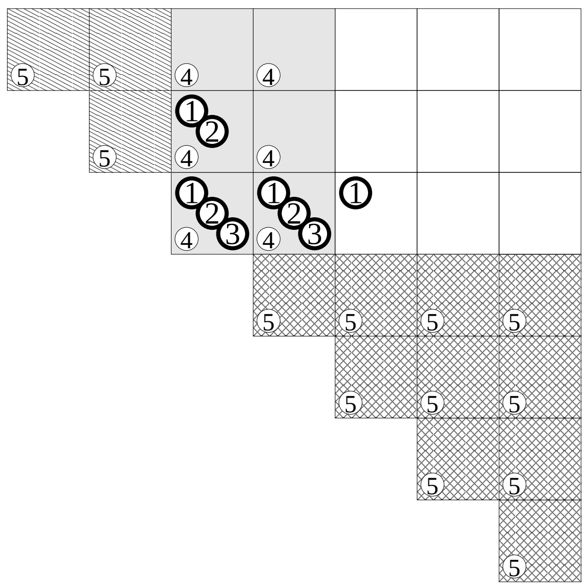


Figure 3.4: Initial position of the game for $\sigma_1 = 010110$, $\sigma_2 = 010101$, $\sigma_3 = 001101$, $\mu = \nu = 000111$, with $N = 2$. Squares are shaded if they contain a 4-token or a 5-token. Contrast with Figure 3.1.

The result would be nicer if we could take $N = 0$ in the theorem. Although we are not aware of any example which prove that this is not the case, the GRGA simply falls apart if N is too small. There are several problems which occur with trying to follow a similar approach. The most serious of these is that a token may be to the right of the square for which it is destined (according to the chosen Zelevinsky picture). We instead prove a geometrical analogue of Theorem 3.1 for all N, which we describe in the next section.

3.4 The geometry of root games on Grassmannians

3.4.1 T-fixed points on B-orbit closures

For simplicity of argument we shall once again assume $s = 1$. Let B denote the standard Borel subgroup of $GL(n+N)$ (upper triangular matrices), and let B_- denote its opposite (lower triangular matrices). For any B-module V let $Gr(V)$ be the disjoint union of all Grassmannians $Gr_k(V)$, $0 \leq k \leq \dim V$.

In Chapter 2, Section 2.3.8, we show that the position of tokens in a single region of the game corresponds to a pair (U, V) where V is a B-module, and $U = (U_1, U_2, U_3)$ is a T-fixed point on $Gr(V)^3$ (or equivalently the U_k are T-invariant subspaces of V).

A move in the game (or a sequence of moves without splitting regions) takes the pair (U, V) to the pair (U', V) where U' is in the B^3-orbit closure through U. If it happens that $\dim U_1' + \dim U_2' + \dim U_3' = \dim V$ and $U_1' + U_2' + U_3' = V$ then that region is won. If U' satisfies these properties we'll call it **transverse**. Thus solving a region provides a road map to locating a transverse point $U' \in \overline{B^3 \cdot U}$. Most importantly,

Fact 3.4.1. *For each region of the game consider the associated pair (U, V). If for each region in the game there exists T-fixed point $U' \in \overline{B^3 \cdot U}$ which is transverse, then*

$$\int_{Fl(n)} \omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} \geq 1$$

This is true at any stage in the game. However from here on in we shall restrict our attention to the state of the game before the first move is made, but after splitting. In which case, Fact 3.4.1 is just one direction of the following result.

Fact 3.4.2. *Consider the regions which occur at the outset of the game, and the associated pairs (U, V). Then*

$$\int_{Fl(n)} \omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} \geq 1$$

if and only if for each region of the game there exists a transverse point $U' \in \overline{B^3 \cdot U}$.

The details of both of these facts are spelled out in the proof of Theorem 2.4 including how the pair (U, V) is associated to an arrangement of tokens in a region.

The primary question concerning the game is therefore how specialised can we make the point U', and still have Fact 3.4.2 be true. There are three levels of specialisation that we could request of this transverse point U'.

1. U' is any transverse point in $\overline{B^3 \cdot U}$.
2. U' is a T-fixed transverse point in $\overline{B^3 \cdot U}$.
3. U' is a (T-fixed) transverse point in $\overline{B^3 \cdot U}$, where (U', V) comes from a position in the game for (π_1, π_2, π_3).

Level 1 is always possible, if $\int_{Gr_l(n)} \Omega_\sigma \cdot \Omega_\mu \cdot \Omega_\nu \geq 1$. This is the content of Fact 3.4.2.

For Grassmannian Schubert calculus, and N sufficiently large, we can actually demand any of these levels of specialisation. Level 3, which is the most specialised, is equivalent to asking for the game to be won without splitting any regions (after the initial splitting). By Theorem 3.1, we know this is possible if $\int_{Gr_l(n)} \Omega_\sigma \cdot \Omega_\mu \cdot \Omega_\nu \geq 1$.

Level 2 is equivalent to asking for a converse to Fact 3.4.1. Since Level 2 is less specialised than Level 3, this converse is also true for Grassmannian Schubert problems if N is sufficiently large. Our goal in this section is to show the that the converse of Fact 3.4.1 is true in the case of Grassmannian Schubert problems even for $N = 0$. This is the content of Theorem 3.2.

To make this more concrete, we now explicitly describe the initial pair (U, V), for a game associated to σ, μ, ν. (Since there is only one unsolved region R in the initial configuration of tokens, we only need one pair to describe the state.) Let M_{n+N} be the space of $(n+N) \times (n+N)$ matrices, having basis e_{ij}, on which B acts by conjugation. Let $R_{i,j}$ denote the B-submodule of M_{n+N} generated by the entries in the upper right $i \times j$ rectangle. Let Q be the B-submodule of $R_{l,n+N-l}$ generated by e_{ij} such that S_{ij} contains a 3-token. Then V is the quotient $R_{l,n+N-l}/Q$. Note V has a basis $\{\tilde{e}_{ij} := e_{ij} + Q | S_{ij} \in R\}$. The point $U \in Gr(V)^3$ is described as follows: U_k is the span of those $\tilde{e}_{ij}$ such that S_{ij} initially contains a k-token. Note that $U_3 = \{0\}$, so we may ignore U_3 altogether.

Theorem 3.2. *For every $N \geq 0$,*

$$\int_{Gr_l(n)} \Omega_\sigma \cdot \Omega_\mu \cdot \Omega_\nu \geq 1$$

if and only if with (U, V) as above, there is a transverse T-fixed point U' in $\overline{B^3 \cdot U}$. Moreover U' exists such that $U'_2 = U_2$.

Proof. $\Leftarrow$ This follows from Proposition 3.2.1 and Fact 3.4.1.

$\Rightarrow$ We know the result is true for N sufficiently large, as this is the geometrical analogue of Theorem 3.1. We use this fact deduce the result for other values of N.

Assume

$$\int_{Gr_l(n)} \Omega_\sigma \cdot \Omega_\mu \cdot \Omega_\nu \geq 1.$$

For any two choices of N, say N_1 and N_2, there is a linear map ϕ_{N_1,N_2} between $V^{(N=N_1)}$ and $V^{(N=N_2)}$, given by $\phi_{N_1,N_2} : \tilde{e}_{ij} \mapsto \tilde{e}_{ij'}$ where $j' = j + N_2 - N_1$ (or 0 if $S_{ij'} \notin R$). Note that $d = \dim U_1$ is independent of N. Thus ϕ_{N_1,N_2} induces a map ϕ_* from $\left\{X \in Gr_d(V^{(N=N_1)}) | X \cap \ker \phi_{N_1,N_2} = \{0\}\right\}$ (a dense open subset of $Gr_d(V^{(N=N_1)})$) to $Gr_d(V^{(N=N_2)})$, given by

$$\phi_*(X) = \text{Image } \phi_{N_1,N_2}|_X.$$

Moreover if $U'^{(N=N_1)} = \left(U_1'^{(N=N_1)}, U_2^{(N=N_1)}, \{0\}\right)$ is transverse then so is the point

$$\left(\phi_*(U_1'^{(N=N_1)}), U_2^{(N=N_2)}, \{0\}\right) \in Gr(V^{(N=N_2)})^3.$$

If the result holds for $N = N_1$ *and* this latter point lies in the B^3-orbit closure through $U^{(N=N_2)}$, then the result will be true for $N = N_2$ as well.

It suffices to show the result for $N = 0$. This is because for ϕ_{0,N_2} the map ϕ_* is an inclusion of the B-orbit closure through $U_1^{(N=0)}$ into the B-orbit closure though $U_1^{(N=N_2)}$.

The result holds for N sufficiently large. Thus there exists a transverse $U'^{(N=N_1)}$ for some N_1. We now consider the map $\phi_{N_1,0}$. Note that $U_1'^{(N=N_1)}$ is in the domain of the induced map ϕ_*. We show that ϕ_* takes a dense subset of $B \cdot U_1^{(N=N_1)}$ to a dense subset of $B \cdot U_1^{(N=0)}$.

To see this we consider the B-orbit not on $U_1 \in Gr(V)$, but lifted to a point $\tilde{U}_1 \in Gr(R_{l,n+N-l})$. ($\tilde{U}_1$ is defined in the same way as U_1: as the span of e_{ij} such that S_{ij} initially contains a 1-token.) It suffices to show that ϕ_* takes a dense subset of $B \cdot \tilde{U}_1^{(N=N_1)}$ to a dense subset of $B \cdot \tilde{U}_1^{(N=0)}$.

Let $L \cong GL(l) \times GL(N+n-l)$ be the subgroup of $GL(n+N)$ of block diagonal matrices of type $(l, n+N-l)$. Now L also acts on $R_{l,n+N-l}$, and $\tilde{U}_1$ is fixed by $B_- \cap L$. Since $(B \cap L) \cdot (B_- \cap L)$ is dense in L, it follows that the orbit $B \cdot \tilde{U}_1$ is dense in $L \cdot \tilde{U}_1$.

Thus in fact it suffices to show that ϕ_* takes a dense subset of $L^{(N=N_1)} \cdot \tilde{U}_1^{(N=N_1)}$ to a dense subset of $L^{(N=0)} \cdot \tilde{U}_1^{(N=0)}$. But this is true, as

$$\phi_*\left(\begin{bmatrix} A^{l\times l} & 0 & 0 \\ 0 & B^{n\times N_1} & C^{n\times n} \\ 0 & D^{N_1\times N_1} & E^{N_1\times n} \end{bmatrix} \tilde{U}_1^{(N=N_1)} \right) = \begin{bmatrix} A^{l\times l} & 0 \\ 0 & C^{n\times n} \end{bmatrix} \tilde{U}_1^{(N=0)}.$$

Thus $U_1'^{(N=0)} = \phi_*(U_1'^{(N=N_1)})$ lies in $\overline{B \cdot U_1^{(N=0)}}$, as required. □

An immediate consequence of this theorem is the following.

Corollary 3.4.3. *Fix σ and ν, and write*

$$\Omega_\sigma \cdot \Omega_\nu = \sum_\mu c_{\sigma\nu\mu} \Omega_\mu^\vee.$$

The non-zero structure constants $c_{\sigma\nu\mu}$ correspond precisely to the B-fixed points on $\overline{B \cdot U_1} \subset Gr(V)$.

Proof. The B-fixed points on $Gr(V)$ are the T fixed points which are complementary to U_2 for some choice of μ. By Theorem 3.2 such a point exists for a given μ if and only if

$$c_{\sigma\mu\nu} = \int_{Gr_l(n)} \Omega_\sigma \cdot \Omega_\mu \cdot \Omega_\nu \neq 0.$$

□

3.4.2 The moment polytope

Recall that the action of a Lie group K on a symplectic manifold (M, ω) is **Hamiltonian** if there is there is a map $w : M \to \mathfrak{k}^*$ satisfying

$$\omega^{-1}(\xi^{\#}) = d\langle w(m), \xi \rangle$$

(here $\xi^{\#}$ represents the vector field on M generated by $\xi \in \mathfrak{k}$). The map w is called the **moment map** for the K-action. If M is compact, and K is a real torus, the image of the moment map $w(M)$ is a polytope called the **moment polytope** of M. It is well known that the moment polytope is equal to the convex hull of the w-images of K-fixed points on M (Atiyah/Guillemin-Sternberg [Ati82, GS82]).

Put $u = \dim U_1$. Let Z^ν_σ denote the B-orbit closure $\overline{B \cdot U_1} \subset Gr_u(V)$. This depends on both σ and ν (as V is defined in terms of ν and U_1 is defined in terms of σ), but not on μ.

Let now $T_{\mathbb{R}}$ denote the real maximal torus inside $B \subset GL(n)$ which acts on V. Recall that the Plücker embedding of

$$Gr_u(V) \hookrightarrow \mathbb{P}^{\binom{\dim V}{u}-1}$$

induces a canonical symplectic form on $Gr_u(V)$, by pulling back the Fubini-Study form. The action of $T_{\mathbb{R}}$ on $Gr_u(V)$ is Hamiltonian, and preserves Z^ν_σ; let w denote the moment map for this action. Thus we may consider the restriction of w to Z^ν_σ, and so Z^ν_σ has a moment polytope, which we denote Δ^ν_σ.

The B-fixed points on Z^ν_σ correspond to non-vanishing Littlewood-Richardson numbers. However, they are special for another reason as well. We will show that they are all extremal points (vertices) of Δ^ν_σ. Moreover, it is easy to identify which vertices of the moment polytope are the images of B-fixed points. Thus, we do not need a complete description of Z^ν_σ to determine non-vanishing Littlewood-Richardson numbers in this way: we only need the moment polytope Δ^ν_σ.

Let us begin by identifying the image of a $T_{\mathbb{R}}$-fixed point (equivalently a T-fixed point) under w. The Fubini-Study symplectic form has the property that the image of a T-fixed point on $l \in \mathbb{P}^N$ is the T-weight of the line l. It follows from basic properties of moment maps and of the Plücker embedding that the image of a T-fixed point $P \in Gr_u(V)$, is the sum of the T-weights of P. That is if we decompose

$$P \cong \bigoplus_{\alpha \in T^*} (\mathbb{C}_\alpha)^{n_\alpha}$$

as a T-representation (where $\mathbb{C}_\alpha$ denotes the T-irrep of weight α), then

$$w(P) = \sum_{\alpha \in T^*} n_\alpha \alpha.$$

Let $x_1, \ldots, x_n$ denote the standard basis for $\mathbb{R}^n$. We identify $\mathfrak{t}^*_{\mathbb{R}}$ with the subspace of $\mathbb{R}^n$, $\{(a_1, \ldots, a_r) \mid \sum_{i=1}^n a_i = 0\}$. For a T-fixed point P, let $\hat{P}$ denote the set of (i, j) such that $\tilde{e}_{ij} \in P$. Thus

$$P = \operatorname{span}\{\tilde{e}_{ij} \mid (i, j) \in \hat{P}\}.$$

We therefore have

$$w(P) = \sum_{(i,j) \in \hat{P}} x_i - x_j.$$

Whenever P is a T-fixed point, we will also define a complementary function

$$\bar{w}(P) = \sum_{(i,j) \in \hat{V} \setminus \hat{P}} x_i - x_j.$$

In terms of the familiar root game diagrams, we can view P as an arrangement of tokens inside the region R. We place a token in the square S_{ij} iff $(i, j) \in \hat{P}$. Then

$$w(P)_k = \begin{cases} \#\text{ of tokens in row } k, & \text{if } k \leq n - l \\ -(\#\text{ of tokens in column } k), & \text{if } k > n - l \end{cases}$$

whereas $\bar{w}(P)$ counts the number of empty squares in the rows and columns respectively. The B-fixed points are those T-fixed points for which all the tokens are justified upward and to the right.

Theorem 3.3. *Let P_0 be a T-fixed point of Z^ν_σ, and put*

$$\bar{w}(P_0) = (p_1, \ldots, p_{n-l}, -q_{n-l+1}, \ldots, -q_n).$$

1. *If P_0 is a B-fixed point then $w(P_0)$ is a vertex of Δ^ν_σ.*

2. *If $w(P_0)$ is a vertex of Δ^ν_σ, the P_0 is a B-fixed point if and only if $p_1 \leq \cdots \leq p_{n-l}$, $q_{n-l+1} \geq \cdots \geq q_n$, and these are dual partitions.*

Proof. We will show that these statements are true for the moment polytope of $Gr_u(V)$, and therefore also for Δ^ν_σ.

First suppose P_0 is a B-fixed point. We would like to show that $w(P_0)$ is a vertex. To this end, we must find a linear functional $\xi : \mathfrak{t}^*_{\mathbb{R}} \to \mathbb{R}$, such that $\xi(w(\cdot))$ attains its maximum only at P. Given a candidate ξ, we know a priori that ξ attains its maximum on a $T_{\mathbb{R}}$-invariant subset of $Gr_u(V)$; thus it is sufficient to show that $\xi(w(P_0)) > \xi(w(P))$ for all other $T_{\mathbb{R}}$-fixed points P.

Let $\rho = (\frac{n}{2}, \frac{n}{2} - 1, \ldots, 1 - \frac{n}{2}, -\frac{n}{2})$ (this is the *Weyl vector* for $GL(n)$). We claim that the linear functional $\xi(a) = a \cdot (\rho - \bar{w}(P_0))$ is maximised only at P_0. We have

$$\begin{aligned}\xi(w(P_0)) - \xi(w(P)) &= \sum_{(i,j)\in \hat{P}_0} \xi(x_i - x_j) - \sum_{(i,j)\in \hat{P}} \xi(x_i - x_j) \\ &= \sum_{(i,j)\in \hat{P}_0 \setminus \hat{P}} \xi(x_i - x_j) + \sum_{(i,j)\in \hat{P}\setminus \hat{P}_0} -\xi(x_i - x_j).\end{aligned}$$

Now,

$$\begin{aligned}\xi(x_i - x_j) &= (x_i - x_j) \cdot (\rho - \bar{w}(P_0)) \\ &= \rho_i - \rho_j - \bar{w}(P_0)_i + \bar{w}(P_0)_j \\ &= (j - i) - (p_i + q_j).\end{aligned}$$

It is easy to interpret this quantity combinatorially in terms of the tokens for P_0. Consider a walk from the lower left square of R to the square S_{ij}, always moving upward or to the right. Such a walk passes through $j - i$ squares. On the other hand since P_0 is a B-fixed point, the tokens are justified upward and to the right, so consider the empty squares encountered on such a walk. If S_{ij} contains a token, there are at least $p_i + q_j$ empty squares encountered, and at most $j - i - 1$, so $p_i + q_j \leq j - i - 1$. On the other hand if S_{ij} does not contain a token, then all squares are empty on this walk, and their number is at least $p_i + q_j - 1$, so

$p_i + q_j - 1 \geq j - i$. Thus we have $\xi(x_i - x_j) = (j-i) - (p_i + q_j) > 0$ if $(i,j) \in \hat{P}_0$, and $\xi(x_i - x_j) < 0$ otherwise.

From here we easily see that $\xi(w(P_0)) - \xi(w(P)) > 0$ if $P \neq P_0$, thus proving that $w(P_0)$ is a vertex.

For the second statement, the $\Rightarrow$ direction is clear. On the other hand, for every dual pair of partitions, $p_1 \leq \cdots \leq p_{n-l}$, $q_{n-l+1} \geq \cdots \geq q_n$ with $\sum p_i = \dim V - u$, there is a B-fixed point $P \in Gr_u(V)$ with $\bar{w}(P) = (p_1, \ldots, p_{n-l}, -q_{n-l+1}, \ldots, -q_n)$. But we have just shown that there is a linear functional ξ such that $\xi(w(\cdot))$ is maximised uniquely at P. So if $\bar{w}(P_0) = (p_1, \ldots, p_{n-l}, -q_{n-l+1}, \ldots, -q_n)$, then $P = P_0$, so P_0 is a B-fixed point. □

Corollary 3.4.4. *Let $(t_1, \ldots, t_n) = \sum_{(i,j)\in\hat{V}} x_i - x_j$. For fixed σ and ν, the set of μ such that $c_{\sigma\nu\mu} \geq 1$ is in bijective correspondence with the set of vertices $(x_1, \ldots, x_n) \in \Delta_\sigma^\nu$ which satisfy the following conditions:*

1. *$t_1 - x_1 \leq \cdots \leq t_{n-l} - x_{n-l}$.*
2. *$x_{n-l+1} - t_{n-l+1} \geq \cdots \geq x_n - t_n$.*
3. *The partitions $\lambda(x) = (t_1 - x_1, \ldots, t_{n-l} - x_{n-l})$ and $\lambda^\vee(x) = (x_{n-l+1} - t_{n-l+1}, \ldots, x_n - t_n)$ are dual.*

Moreover, $\lambda(x)$ is the partition associated to μ under the bijective correspondence.

Proof. If P is a B-fixed point in Z_σ^ν, then $\bar{w}(P) = (t_1, \ldots, t_n) - w(P)$. Thus if x is a vertex of Δ_σ^ν, then $w^{-1}(x)$ is B-fixed point if and only if conditions 1–3 hold. If μ is such that $c_{\sigma\nu\mu} \geq 1$, then $\mu \leftrightarrow x$ if and only if we have $w^{-1}(x) + U_2 = V$. From this, it is straightforward to check that $\lambda(x)$ is indeed the partition associated to μ. □

Chapter 4

The vanishing problem for branching Schubert calculus

4.1 Preliminaries

4.1.1 Branching Schubert calculus

Our objective in this chapter is to show that all of the results of Chapter 2 hold in an even more general setting, which we call *branching Schubert calculus*. Let $i : K' \hookrightarrow K$ be an inclusion of compact connected Lie groups. Let T' be a maximal torus of K', and let T extend the image $i(T')$ to a maximal torus of K. Then we obtain an inclusion $i : K'/T' \hookrightarrow K/T$ (which we also denote by i, in a mild abuse of notation). Hence there is a map on cohomology $i^* : H^*(K/T) \to H^*(K'/T')$. The problem of branching Schubert calculus is to determine the map i^* in the Schubert basis, i.e. given a Schubert class $\omega \in H^*(K/T)$ we would like to express $i^*(\omega) \in H^*(K'/T')$ in the Schubert basis of the later.

The coefficients which appear in such an expression are always non-negative integers. Although there are formulae for these integers, it is not known how to determine them combinatorially, or even how to determine which terms appear. Our goal in this paper is simply to investigate the latter problem, and to obtain some widely applicable criteria for determining which terms appear.

Our motivation for this work comes from a paper of Berenstein and Sjamaar [BS00], in which they use the vanishing problem for branching Schubert calculus to calculate the K' moment polytope of a K coadjoint orbit. They show that each non-vanishing branching coefficient gives rise to an inequality satisfied by the moment polytope. Moreover, all together, the complete list of non-vanishing branching coefficients gives a sufficient set of inequalities to determine this polytope.

This turns out to be equivalent to an asymptotic representation theory question [Hec82, GS82] (for more of this picture see also [GLS96]). Let λ and μ be dominant weights for K and K' respectively. Let V_λ denote the irreducible K-representation with highest weight λ; similarly let V'_μ denote the irreducible K'-representation with highest weight μ. When V_λ is decomposed as a K'-module, we would like to know whether a component of type V'_μ appears. The asymptotic version of this problem is the following: does there exists a positive integer N, such that the K-module $V_{N\lambda}$ has a component of type $V'_{N\mu}$, when decomposed as a K'-module? This latter problem is equivalent to determining whether the point μ lies in the K'-moment polytope for the K-coadjoint orbit through λ. Thus the non-vanishing branching coefficients give an answer to this asymptotic representation theory question as well.

There are a few important things to note about the vanishing problem for branching Schubert calculus. The first is that this situation actually generalises the vanishing problem for multiplication in Schubert calculus. Indeed if $i : K' \hookrightarrow K = K' \times K'$ is the diagonal inclusion, then the map i^* is just the cup product in cohomology. Moreover, this can be used to give multiplication of more than two terms by considering $K' \hookrightarrow K' \times \cdots \times K'$.

The second is that it suffices to solve the following apparently simpler problem: determine which Schubert classes are in the kernel of i^*. In the case of vanishing for multiplication of Schubert classes, this is a familiar fact: we can determine which structure constants of the cohomology ring are zero, based on the which triple products vanish. More generally, suppose we wish to know whether the Schubert class $\sigma \in H^*(K'/T')$ appears in the expansion of $i^*(\omega)$, for some Schubert class $\omega \in H^*(K/T)$. To do this we find the dual Schubert class $\sigma^\vee$ under the Poincaré pairing and consider the integral

$$\int_{K'/T'} \sigma^\vee \cdot i^*(\omega).$$

If this integral is non-zero (and therefore positive), then σ appears in the expansion of $i^*(\omega)$ (with coefficient equal to $\int_{K'/T'} \sigma^\vee \cdot i^*(\omega)$) otherwise it does not. Now consider the inclusion $j = id \times i : K' \hookrightarrow K' \times K$. We have the Schubert class

$$(\sigma^\vee, \omega) \in H^*(K'/T') \times H^*(K/T) = H^*(K' \times K/T_{K\times K'}).$$

But now $\sigma^\vee \cdot i^*(\omega) = j^*(\sigma^\vee, \omega)$ so

$$\int_{K'/T'} \sigma^\vee \cdot i^*(\omega) = \int_{K'/T'} j^*(\sigma^\vee, \omega).$$

Thus it is sufficient to know whether $j^*(\sigma^\vee, \omega) = 0$, for any given pair $(\sigma^\vee, \omega)$ (together with $\deg \sigma^\vee + \deg \omega = \dim K/T + \dim K'/T'$, this is a necessary and sufficient condition).

We will find it more convenient to formulate our problem in this latter way: given $i : K' \hookrightarrow K$ as before, determine which Schubert classes on K/T are in the kernel of i^*.

4.1.2 Conventions

In order to study Schubert classes on K/T and K'/T', it is convenient to pass to a complex picture.

Let G and H be complex connected reductive Lie groups which are the complexifications of K and K' respectively. The inclusion $K' \hookrightarrow K$ induces a natural inclusion $H \hookrightarrow G$.

Choose a Borel subgroup $B_0 \subset G$, and consider the H-orbits on G/B_0 of minimal dimension. Each such orbit is closed, therefore, compact, and so is H/P for some parabolic subgroup $P \subset H$. Choose a point x_0 on such an orbit. The stabiliser of x_0 inside H, H_{x_0}, is conjugate to P, whereas the stabiliser of x_0 inside G, G_{x_0}, is conjugate to B_0. Thus $H_{x_0} \subset G_{x_0}$ is solvable, but H/H_{x_0} is compact, hence H_{x_0} is a Borel subgroup of H.

Let $B = G_{x_0}$ and $B' = H_{x_0}$. We can now identify T with K_{x_0} (the stabiliser of x_0 inside K), and T' with K'_{x_0}. Moreover, we have G/B naturally isomorphic to K/T, and H/B' naturally isomorphic to K'/T'. Thus we have naturally extended $T' \hookrightarrow T$ to an inclusion of Borel subgroups $B' \hookrightarrow B$.

To complete this picture, we will also wish to consider the complex tori $T'_{\mathbb{C}}$ and $T_{\mathbb{C}}$, which will denote the complexifications of T' and T respectively. Also let N' and N denote the corresponding unipotent subgroups of B' and B. Of course, $T' \hookrightarrow T$, and $N' \hookrightarrow N$.

Let Δ denote the root system of G, and Δ' the root system of H. The positive and negative roots of Δ (with respect to the choice of B) are denoted Δ_+ and Δ_- respectively. For each root $\alpha \in \Delta$, we fix a basis vector e_α for the corresponding root space in $\mathfrak{g}$. Likewise, for each root $\beta \in \Delta'$, we fix a basis vector e'_β for the corresponding root space in $\mathfrak{h}$.

We will denote the Lie algebras of these groups by the corresponding fraktur letters, e.g. $\mathrm{Lie}(B) = \mathfrak{b}$, $\mathrm{Lie}(N') = \mathfrak{n}'$, etc.

Consider the tangent spaces to x_0 in G/B and H/B'. These are $\mathfrak{g}/\mathfrak{b}$ and $\mathfrak{h}/\mathfrak{b}'$ respectively. Thus we have a natural inclusion $\mathfrak{h}/\mathfrak{b}' \hookrightarrow \mathfrak{g}/\mathfrak{b}$. We use the Cartan involution to identify $\mathfrak{n}$ with $\mathfrak{n}_-$, denoted $a \mapsto a^T$, and the Killing form to identify $\mathfrak{n}$ with $(\mathfrak{g}/\mathfrak{b})^*$. Similarly, we identify the dual of $\mathfrak{h}/\mathfrak{b}'$ with $\mathfrak{n}'$. Thus we obtain a linear map

$$\phi : \mathfrak{n} \to \mathfrak{n}'$$

which is adjoint to the inclusion of tangent spaces. Essentially ϕ encodes all the information about the inclusion $K' \hookrightarrow K$.

Note that since x_0 is a T'-fixed point, the map ϕ is T'-equivariant. Thus, it takes the T-weight spaces to T'-weight spaces, and induces a map

$$\hat{\phi} : \Delta \to \Delta' \cup \{0\}$$

defined by the condition that

$$\hat{\phi}(\alpha) = \begin{cases} 0, & \text{if } \phi(e_\alpha) = 0 \\ \beta, & \text{where } 0 \neq \phi(e_\alpha) \text{ is in the } \beta\text{-weight space} \end{cases}$$

We will be interested in subsets of $\mathcal{T} \subset \Delta$ with the following properties.

Definition 4.1.1. *Suppose $\mathcal{T} \subset \Delta$ satisfies*

1. $0 \notin \phi(\mathcal{T})$*, and*
2. $\hat{\phi}|_{\mathcal{T}}$ *is injective.*

We call such a subset $\mathcal{T}$ **injective***. Equivalently $\mathcal{T} \subset \Delta$ is injective if $\phi|_{\langle e_\alpha \,|\, \alpha \in \mathcal{T}\rangle}$ is an injective linear map.*

4.1.3 Schubert varieties

Let $W = N(T)/T$ denote the Weyl group of G. For $\pi \in W$, let $[\pi]$ denote the corresponding T-fixed point on G/B, and let $\tilde{\pi}$ denote some lifting of $\pi \in W$ to an element of $N(T) \subset G$.

Let w_0 denote the long element in W. For $\pi \in W$, let $\pi' = w_0\pi$. To each $\pi \in W$ we associate the Schubert variety $X_\pi = \overline{B \cdot [\pi']}$, the closure of the B-orbit through $[\pi']$ in G/B. This is the Schubert variety based at the point $x_0 = [1]$. (We are defining our Schubert varieties to be B-orbits rather than B_--orbits. This is why our orbit is through $[\pi']$ rather than $[\pi]$. Since $X_\pi = \tilde{w}_0 \cdot \overline{B_-[\pi]}$, the cohomology class of the B-orbit through $[\pi']$ is the same as the class of the B_--orbit through $[\pi]$. This change of notation from Chapter 2 makes our lives easier, since Schubert varieties of G/B and H/B' now have the same base point.) The length of $\pi \in W$ is the complex codimension of X_π.

More generally, for $y, y_0 \in G/B$ we say that y is π-related to z if there is a $g \in G$ such that $g \cdot y_0 = x_0$ and $g \cdot y \in X_\pi$. Let X_{π,y_0} denote the Schubert variety associated to π based at y_0, that is

$$X_{\pi,y_0} = \{y \in G/B \mid y \text{ is } \pi\text{-related to } y_0.\}$$

so $X_\pi = X_{\pi,x_0}$.

Let ω_π denote the cohomology class of the Schubert cell X_π. In this paper, we shall be investigating the question of whether $i^*(\omega_\pi) = 0$, for $\pi \in W$. We will assume that $\pi \in W$ is an element whose length $l(\pi) \leq \dim H/B'$ (all dimensions are over $\mathbb{C}$ unless otherwise specified). If $l(\pi) > \dim H/B'$ then $i^*(\omega_\pi) = 0$ for dimensional reasons. We are primarily interested in the case where $l(\pi) = \dim H/B'$, however almost everything in this paper holds for all $\pi \in W$.

4.1.4 An example of the branching root game

In Section 4.2 we will give the complete rules of the root game, and a number of examples. Here, we will simply describe a few of the salient features, and give a simple example. As before, the root game is played on a set of *squares* corresponding to the positive roots of G. Each square can be empty or contain a *token* (of which there is now only one type), and there can never be more than one token in any square; thus the arrangement of the tokens $\mathcal{T}$ can be viewed as a subset of Δ_+. We then perform a sequence of *moves*. Possible moves are specified by an element $\beta \in \Delta_+$. A move causes some subset of the tokens to be relocated to different squares.

The object of the game is to perform a sequence of moves such that the eventual arrangement of tokens $\mathcal{T}$ is injective (Definition 4.1.1). Thus in order to play the root game in practice, we shall need to compute the map $\hat{\phi}$ in a number of examples. If we can reach such a position we say that the game can be won. Our non-vanishing criterion (Theorem 4.2) states that if the game can be won, then $i^*(\omega_\pi) \neq 0$.

Example 4.1.2. *We consider the example of* $i : SO(5,\mathbb{C}) \hookrightarrow SL(5)$*. In this case, the squares correspond to* $(A_4)_+ = \{x_j - x_i \mid 1 \leq i < j \leq 5\}$*, which we arrange in a staircase shape (see Example 4.2.1). The map* $\hat{\phi}$ *corresponds to folding along the antidiagonal (see Example 4.2.7), as shown below.*

$$\begin{array}{cccc} & SL(5) & & \\ \hline A & B & C & 0 \\ & D & 0 & C \\ & & D & B \\ & & & A \end{array} \longrightarrow \begin{array}{cc} \multicolumn{2}{c}{SO(5)} \\ & C \\ D & B \\ & A \end{array}$$

The squares on the antidiagonal map to 0 *under* $\hat{\phi}$.

If $\pi = 31254$*, then the initial arrangement of tokens is*

$$\begin{array}{|c|c|c|c|} \hline \spadesuit & \clubsuit & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \bullet \\ \hline \end{array}$$

This is not a winning position, since the upper left corner square and lower right corner square map both contain tokens, and to the same element under $\hat{\phi}$*. However, from here we can move to a winning position. The move corresponding to the root* $x_4 - x_3$ *will cause two of the tokens to move (as we shall see in Section 4.2). The* $\bullet$*-token will move up one square, and* $\clubsuit$*-token will move right one square.*

$$\begin{array}{|c|c|c|c|} \hline \spadesuit & & \clubsuit & \\ \hline & & & \\ \hline & & & \bullet \\ \hline & & & \\ \hline \end{array}$$

We see now that this arrangement of tokens is injective, since there are no tokens on the antidiagonal, and after folding along the antidiagonal, there is at most one token in each square.

$$\begin{array}{cc} & \times \\ \times & \clubsuit \\ & \bullet \\ & \spadesuit \end{array}$$

(The '×'s denote the antidiagonal whose image is $\{0\}$ *under* $\hat{\phi}$*.)*

We conclude that $i^*(\omega_{31254}) \neq 0$.

4.2 Root games for branching Schubert calculus

4.2.1 The roots of G and H

The **position** in a root game consists of the following data:

- A partition of the set of positive roots, i.e. $\mathcal{R} = \{R_1, \ldots, R_s\}$, such that $\Delta_+ = \coprod_{i=1}^{s} R_i$. Each R_i is called a **region**.
- A subset $\mathcal{T}$ of the positive roots, which we call the **arrangement of tokens**.

We visualise this information as follows. To each positive root $\alpha \in \Delta_+$, we assign a square S_α. If $\alpha \in \mathcal{T}$, we place a token in that square, otherwise the square is empty. We view the tokens as physical objects which can be picked up from one square, and placed in another.

The regions are just sets of the squares. As such, if R is a region, we will sometimes write $S_\alpha \in R$ rather than $\alpha \in R$.

In the abstract, these squares serve no real purpose. However, in practice they provide a concrete way of visualising the positive roots of G. We now give a few examples, illustrating how this can be accomplished in types A, B, and D. The rationale for the arrangements in Examples 4.2.2 and 4.2.3 comes from the calculations in Examples 4.2.6 and 4.2.7.

In the following examples $x_1, \ldots, x_n$ is an orthonormal basis for $\mathbb{R}^n$.

Example 4.2.1. *If $G = SL(n)$, The root system $\Delta = A_{n-1}$ is $\{\alpha_{ij} = x_j - x_i \mid i \neq j\}$. The positive roots are those for which $i < j$. We can view our squares corresponding to the positive roots as being arranged inside an $n \times n$ array of squares. Let AS_{ij} denote the square in position $(i.j)$. The relevant squares are squares AS_{ij} (the square in position (i,j)), where $1 \leq i < j \leq n$. Thus the positive root α_{ij}, with $i < j$ is assigned to the square AS_{ij}.*

$$\Delta(SL(6))_+ = (A_5)_+ = \begin{array}{|c|c|c|c|c|} \hline \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ \hline \multicolumn{1}{c|}{} & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ \cline{2-5} \multicolumn{2}{c|}{} & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ \cline{3-5} \multicolumn{3}{c|}{} & \alpha_{45} & \alpha_{46} \\ \cline{4-5} \multicolumn{4}{c|}{} & \alpha_{56} \\ \cline{5-5} \end{array}$$

Example 4.2.2. *If $G = SO(2n)$, the root system $\Delta = D_n$ is*

$$\{(-1)^\varepsilon x_i + (-1)^\delta x_j \mid i \neq j\}.$$

The positive roots are of two types:

$$\{\beta ij = x_j - x_i \mid i < j\} \cup \{\beta'_{ij} = x_j + x_i \mid i < j\}$$

We view the squares as two copies of the $SL(n)$ picture: one copy for the β_{ij}, and the second copy for the β'_{ij}. We arrange these two copies inside a $2n \times n$ array of squares (denoted DS_{ij}) as follows: the root β_{ij} corresponds to the square $DS_{n+i,j}$. The root β'_{ij} corresponds to the square $DS_{n+1-i,j}$.

$$\Delta(SO(10))_+ = (D_5)_+ = \begin{array}{cccc} & & & \beta'_{45} \\ & & \beta'_{34} & \beta'_{35} \\ & \beta'_{23} & \beta'_{24} & \beta'_{25} \\ \beta'_{12} & \beta'_{13} & \beta'_{14} & \beta'_{15} \\ \beta_{12} & \beta_{13} & \beta_{14} & \beta_{15} \\ & \beta_{23} & \beta_{24} & \beta_{25} \\ & & \beta_{34} & \beta_{35} \\ & & & \beta_{45} \end{array}$$

Example 4.2.3. *If $G = SO(2n+1)$, the root system $\Delta = B_n$ is*

$$\{(-1)^{\varepsilon} x_i + (-1)^{\delta} x_j \mid i \neq j\} \cup \{\pm x_i\}.$$

The positive roots are of three types:

$$\Delta_+ = \{\gamma_{ij} = x_j - x_i \mid i < j\} \cup \{\gamma'_{ij} = x_j + x_i \mid i < j\} \cup \{\gamma^\circ_j = x_j\}.$$

We arrange the squares inside a $(2n+1) \times n$ array of squares (denoted BS_{ij}) as follows: the root γ_{ij} corresponds to the square $BS_{n+1+i,j}$. The root γ'_{ij} corresponds to the square $BS_{n+1-i,j}$. The root γ°_j corresponds to the square $BS_{n+1,j}$.

$$\Delta(SO(9))_+ = (B_4)_+ = \begin{array}{cccc} & & & \gamma'_{34} \\ & & \gamma'_{23} & \gamma'_{24} \\ & \gamma'_{12} & \gamma'_{13} & \gamma'_{14} \\ \gamma^\circ_1 & \gamma^\circ_2 & \gamma^\circ_3 & \gamma^\circ_4 \\ & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ & & \gamma_{23} & \gamma_{24} \\ & & & \gamma_{34} \end{array}$$

The rules of the game heavily involve the map $\hat{\phi}$. Thus before proceeding further, we compute this map in a number of important examples.

Example 4.2.4. *If $H \hookrightarrow G_1, \ldots, H \hookrightarrow G_s$, then $H \hookrightarrow G = G_1 \times \cdots \times G_s$ via the diagonal map. Let $\hat{\phi}_i : \Delta(G_i)_+ \to \Delta' \cup \{0\}$ denote the map on root systems for $H \hookrightarrow G_i$. The positive roots of G are $\Delta_+ = \Delta(G_1)_+ \sqcup \cdots \sqcup \Delta(G_s)_+$, and*

$$\hat{\phi} : \Delta_+ \to \Delta'_+ \cup \{0\}$$

is simply given by $\hat{\phi}(\alpha) = \phi_i(\alpha)$ if $\alpha \in \Delta(G_i)_+$.

In particular, if $H = G_1 = \cdots G_s$, then each $\hat{\phi}_i$ is just the identity map. Thus this example allows us to deal with the vanishing problem for multiplication of Schubert calculus.

Example 4.2.5. *If $H = SL(k) \hookrightarrow G = SL(n)$ is the inclusion*

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix}$$

then

$$\hat{\phi}(\alpha_{ij}) = \begin{cases} \alpha_{ij} \in \Delta', & \textit{if } j \leq k \\ 0, & \textit{otherwise} \end{cases}$$

We now consider the inclusion of $SO(n, \mathbb{C}) \hookrightarrow G = SL(n)$. We begin with the case where n is even. Let R denote the $n/2 \times n/2$ matrix with 1 on the antidiagonal, and 0 everywhere else. We take as our maximal torus of $SO(n, \mathbb{R})$ the subgroup

$$T' = \{ \begin{pmatrix} A & BR \\ -RB & RAR \end{pmatrix} \in GL(n, \mathbb{R}) \mid A,\, B \text{ are diagonal}\}.$$

The complexification of T' is a complex maximal torus in $SO(n, \mathbb{C})$.

The difficulty here is that the standard maximal torus of $SO(n)$ is not a subgroup of the standard torus of $SL(n)$. To handle this, we instead use a conjugate subgroup of $SO(n)$. Let

$$U = \begin{pmatrix} I & iR \\ iR & I \end{pmatrix}.$$

Example 4.2.6. *Let $H = U\,SO(n)\,U^{-1} \hookrightarrow G = SL(n)$, where $n = 2m$. One can easily verify that a maximal torus of H, is the set of invertible diagonal matrices*

$$U\,T'_{\mathbb{C}}\,U^{-1} = \Big\{\lambda = \begin{pmatrix} \lambda_m & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \lambda_m^{-1} \end{pmatrix} \in SL(n)\Big\}.$$

The Lie algebra $\mathfrak{h} = \{(a_{ij}) \in \mathfrak{sl}(n) \mid a_{ij} = -a_{n+1-i\ j}\}$, is the set of $n \times n$ matrices which are skew symmetric about the antidiagonal. And $\mathfrak{n}'$ is simply the set of upper triangular matrices in $\mathfrak{h}$.

Let E_{ij} denote the matrix with a 1 in the i, j position, and 0 everywhere else. We see that for $i < j$,

$$\lambda E_{ij}\lambda^{-1} = \begin{cases} \lambda_{m+1-i}\lambda_{j-m}E_{ij}, & \text{if } i+j > n+1,\ i \le m \\ \lambda_{m+1-i}\lambda_{j-m}E_{ij}, & \text{if } i+j \le n,\ j > m \\ \lambda_{m+1-i}\lambda_{m+1-j}^{-1}E_{ij}, & \text{if } j \le m \\ \lambda_{i-m}^{-1}\lambda_{j-m}E_{ij}, & \text{if } i > m \\ 0, & \text{if } i+j = n+1 \end{cases}$$

Thus $\hat{\phi}$ is given by

$$\hat{\phi}(\alpha_{ij}) = \begin{cases} \beta'_{m+1-i,j-m} & \text{if } i+j > n+1,\ i \le m \\ \beta'_{j-m,m+1-i} & \text{if } i+j \le n,\ j > m \\ \beta_{m+1-j,m+1-i}, & \text{if } j \le m \\ \beta_{i-m,j-m}, & \text{if } i > m \\ 0, & \text{if } i+j = n+1 \end{cases}$$

In terms of the arrangement of squares (described in Examples 4.2.1 and 4.2.2), the map $\hat{\phi}$ is symmetrical about the antidiagonal, with the antidiagonal itself map-

β_{34}	β_{24}	β_{14}	β'_{14}	β'_{24}	β'_{34}	
	β_{23}	β_{13}	β'_{13}	β'_{23}		β'_{34}
		β_{12}	β'_{12}		β'_{23}	β'_{24}
				β'_{12}	β'_{13}	β'_{14}
				β_{12}	β_{13}	β_{14}
					β_{23}	β_{24}
						β_{34}

Figure 4.1: The map $\hat{\phi} : \Delta(SL(8))_+ \to \Delta(SO(8))_+ \cup \{0\}$. The root $\hat{\phi}(\alpha)$ is written in the square corresponding to α. Empty squares are mapped to 0.

ping to 0*. Moreover, below the antidiagonal (i.e. for $i + j > n + 1$), we simply have $\hat{\phi}(AS_{ij}) = DS_{i-m,j}$. See Figure 4.1.*

The analysis for n odd is very similar, changing T' and U to

$$T' = \left\{ \left(\begin{array}{ccccccc} & & & 0 & & & \\ & A & & \vdots & & BR & \\ & & & 0 & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & 0 & & & \\ & -RB & & \vdots & & RAR & \\ & & & 0 & & & \end{array} \right) \in GL(n, \mathbb{R}) \;\middle|\; A,\, B \text{ are diagonal} \right\},$$

$$U = \left(\begin{array}{ccccccc} & & & 0 & & & \\ & I & & \vdots & & iR & \\ & & & 0 & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & 0 & & & \\ & iR & & \vdots & & I & \\ & & & 0 & & & \end{array} \right).$$

Example 4.2.7. *Let $H = U\,SO(n)\,U^{-1} \hookrightarrow G = SL(n)$, where $n = 2m-1$ is odd.*

$$U\,T'_{\mathbb{C}}\,U^{-1} = \left\{\lambda = \begin{pmatrix} \lambda_{m-1} & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & 1 & & & \\ & & & & \lambda_1^{-1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda_{m-1}^{-1} \end{pmatrix} \in SL(n)\right\}.$$

As in the case where n is even, $\mathfrak{h} = \{(a_{ij}) \in \mathfrak{sl}(n) \mid a_{ij} = -a_{n+1-i,j}$, is the set of $n \times n$ matrices which are skew symmetric about the antidiagonal, and $\mathfrak{n}' = \mathfrak{b} \cap \mathfrak{h}$. The map $\hat{\phi}$ is given by

$$\hat{\phi}(\alpha_{ij}) = \begin{cases} \gamma_i^\circ & \text{if } j = m \\ \gamma_j^\circ & \text{if } i = m \\ \gamma'_{m-i,j-m} & \text{if } i+j > n+1,\ i < m \\ \gamma'_{j-m,m-i} & \text{if } i+j \leq n,\ j > m \\ \gamma_{m-j,m-i}, & \text{if } j < m \\ \gamma_{i-m,j-m}, & \text{if } i > m \\ 0, & \text{if } i+j = n+1 \end{cases}$$

More simply, in terms of the arrangement of squares (see Examples 4.2.1 and 4.2.3), we have that $\hat{\phi}$ is symmetrical about the antidiagonal, and identically zero on the antidiagonal. Below the antidiagonal $\hat{\phi}(AS_{ij}) = BS_{i,j-m}$. See Figure 4.2.

Example 4.2.8. *Let G be the complex form of G_2, and $H = SL(3)$. The map $i : H \hookrightarrow G$ is defined on the level of roots: A_2 includes into G_2 as the long roots. Since $SL(3)$ is simply connected, this defines a homomorphism on the Lie groups (and this map is an inclusion). The map $\hat{\phi} : (G_2)_+ \to (A_2)_+ \cup \{0\}$ is therefore*

$$\hat{\phi}(\alpha) = \begin{cases} 0, & \text{if } \alpha \text{ is a short root of } G_2 \\ \alpha, & \text{if } \alpha \text{ is a long root of } G_2. \end{cases}$$

γ_{23}	γ_{13}	γ_3°	γ_{13}'	γ_{23}'	
	γ_{12}	γ_2°	γ_{12}'		γ_{23}'
		γ_2°		γ_{12}'	γ_{13}'
			γ_1°	γ_2°	γ_3°
				γ_{12}	γ_{13}
					γ_{23}

Figure 4.2: The map $\hat{\phi} : \Delta(SL(7))_+ \to \Delta(SO(7))_+ \cup \{0\}$. The root $\hat{\phi}(\alpha)$ is written in the square corresponding to α. Empty squares are mapped to 0.

We arrange the squares of G in a linear fashion, with the short simple root at the bottom, and the long simple root on the left. The map $\hat{\phi}$ and the arrangement of squares for G_2 are both illustrated in Figure 4.3.

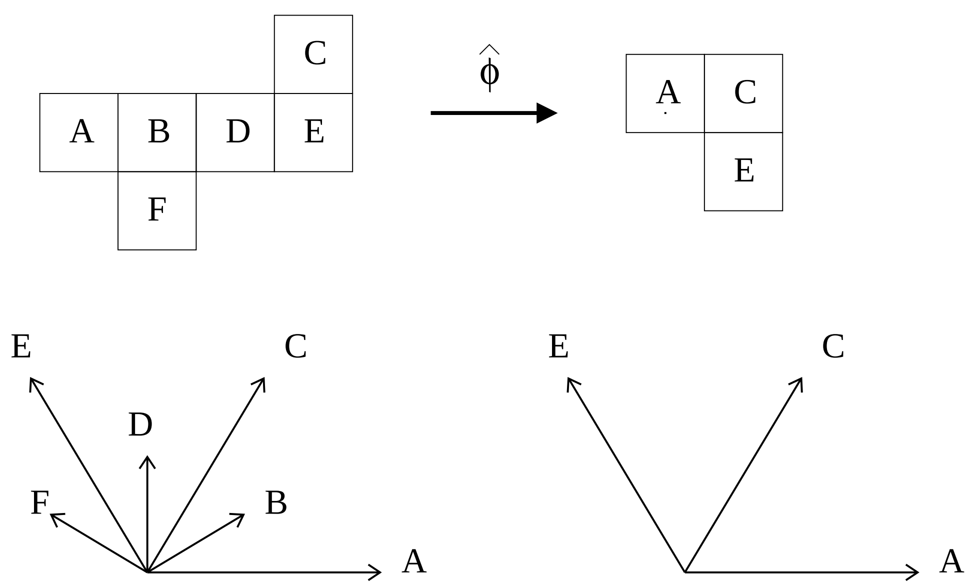


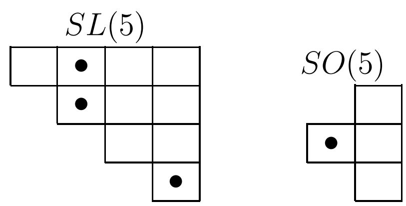
Figure 4.3: The map $\hat{\phi} : (G_2)_+ \to (A_2)_+ \cup \{0\}$, and the corresponding arrangement of squares.

4.2.2 Initial configuration, moving and splitting

The game always begins with a single region $R_1 = \Delta_+$, which contains all the squares. The initial arrangement of tokens is the inversion set of π, i.e.

$$\mathcal{T} = \{\alpha \in \Delta_+ \mid \pi \cdot \alpha \in \Delta_-\}.$$

Example 4.2.9. *If $G = SL(5) \times SO(5)$, $\pi = (23154, r_{\gamma_1^\circ})$ where $r_{\gamma_1^\circ})$ is the reflection in the simple root γ_1°. Then the initial position is as shown below:*



From here we perform a sequence of *splittings* and *moves*. A splitting is an operation which refines the partition $\mathcal{R} = \{R_1, \dots, R_s\}$ as follows: given a set of squares $A \subset \Delta_+$ (with certain additional properties), we subdivide each region according to its intersection with A. More precisely, we produce

$$\mathcal{R}' = \{R_1 \cap A, R_1 \cap A^c,\ R_2 \cap A, R_2 \cap A^c,\ \dots,\ R_s \cap A, R_s \cap A^c\}.$$

(Empty regions have no effect on the game, thus we may discard any copies of the empty set produced in this way.) This operation is called **splitting along** A. The subsets $A \subset \Delta$ which can be legally used for splitting are called *splitting subsets*. A move is an operation which changes the arrangement of tokens. Each move applies only to tokens in one region R_i. The move causes some of the tokens in the region R_i to relocate to new squares in within the region R_i. Moves and splittings may be performed in any order.

To specify these more precisely, we need the following definitions:

Definition 4.2.10. *Let $I \subset \Delta_+$ be a subset of the positive roots of G. Call I an* **ideal subset** *if it is closed under raising operations, i.e. if $\alpha \in I$, then $\alpha' \in I$, whenever α', and $\alpha' - \alpha$ are both positive roots. (Equivalently, I is a an ideal subset if and only if $\{e_\alpha | \alpha \in I\}$ span an ideal in the Lie algebra $\mathfrak{n}$.)*

Definition 4.2.11. *Let $A \subset \Delta_+$ be a subset of the positive roots of G. We call A a* **splitting subset** *if A is an ideal subset, and $\hat{\phi}(A^c) \cap \hat{\phi}(A) \subset \{0\}$.*

Example 4.2.12. *For $SO(n) \hookrightarrow SL(n)$, a set $A \subset \Delta_+$ is an ideal subset if whenever it contains a square S, it contains all squares above and to the right of S. A is a splitting subset if it is an ideal subset which is symmetrical about the antidiagonal.*

Splitting along A (as described above) is permissible whenever A is a splitting set. However, when $l(\pi) = \dim H/B'$, one can determine a priori whether splitting will help us win the root game. It turns out that if $l(\pi) = \dim H/B'$, the splitting is advantageous if and only if $|\mathcal{T} \cap A| = |\hat{\phi}(A) \setminus \{0\}|$.

A **move** is specified by a pair $[\beta, R]$, where $\beta \in \Delta_+$, and R is a choice of region. To execute the move, we find all pairs of squares $S_\alpha, S'_\alpha \in R$ such that $\alpha' - \alpha = \beta$. We then order the relevant S_α according to the height of the root α. Proceeding in order of decreasing height of α, we move the tokens as follows: if a token appears in the square S_α but not in $S_{\alpha'}$, move the token up from the first square to the second square.

4.2.3 Vanishing and non-vanishing criteria

To make this game worthwhile, it must be possible to win. Recall the definition of an injective subset of Δ (Definition 4.1.1).

Definition 4.2.13. *The game is* **won** *if the arrangement of tokens $\mathcal{T}$ is injective.*

Though there is no shame in losing, it is worth noting that from certain positions it may be impossible for victory to be attained. In particular, we observe that if A is an ideal subset, then any token which begins its move in A must remain in A. Thus $|\mathcal{T} \cap A|$ can increase, but never decrease from a sequence of moves. Suppose then, at some position in the game, there is an ideal subset A such that $|\mathcal{T} \cap A| > |\hat{\phi}(A) \setminus \{0\}|$. Then $\mathcal{T}$ is not injective, and will never be injective, i.e. the game cannot be won. In such a position, we declare the game to be **lost**.

There is one special case of this which is particularly important: that is the situation when the game is lost before any moves are made.

Definition 4.2.14. *The game is* **doomed** *if it is lost in the initial token arrangement.*

The losing condition in general does not provide any information. However, for those special times when the game is doomed, we have the following result.

Theorem 4.1. *If the game is doomed, then* $i^*(\omega_\pi) = 0$.

Example 4.2.15. *Let* $G = SL(n)$ *and* $H = SO(n)$. *Let* $\pi : \{1, \dots, n\} \to \{1, \dots, n\} \in S_n$. *If* $\pi(n) < \pi(1)$ *then* $i^*(\omega_\pi) = 0$.

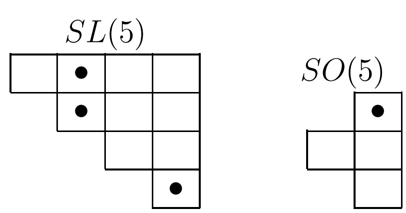
To see this, observe that $A = \{\alpha_{1n}\}$ *is a splitting subset, whose image under* $\hat{\phi}$ *is* $\{0\}$. *Thus* $|\hat{\phi}(A) \setminus \{0\}| = 0$. *If* $\pi_n < \pi_1$, *then* $\alpha_{1n} \in \mathcal{T}$, *so* $|\mathcal{T} \cap A| = 1$ *and the game is doomed.*

Likewise, we also gain information if it is possible to win the game.

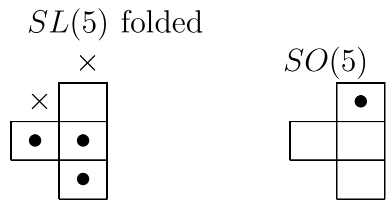
Theorem 4.2. *If the game can be won, then* $i^*(\omega_\pi) \neq 0$.

Example 4.2.16. *If* $\mathfrak{h}$ *a is* T*-invariant subspace of* $\mathfrak{g}$, *and* $\hat{\phi}^{-1}(\{0\})$ *is an ideal subset, then the initial position is a winning position if and only if the game is not doomed, giving a simple necessary and sufficient condition for* $i^*(\omega_\pi) = 0$. *Unfortunately this only occurs when the Dynkin diagram of* H *is obtained by deleting some of the vertices of* G*'s Dynkin diagram. Some common examples include* $SL(k) \hookrightarrow SL(n)$, $SO(2k+1) \hookrightarrow SO(2n+1)$ *and* $SO(2k) \hookrightarrow SO(2n)$, *for* $k < n$.

Example 4.2.17. *Let* $G = SL(5) \times SO(5)$, $\pi = (23154, r_{\gamma_1^\circ})$. *The initial position is shown in Example 4.2.9. We can win the game with one move, and no splittings. The move corresponds to the root* $\gamma_2^\circ \in (B_2)_+$. *This causes the token on the* $SO(5)$ *part to move from* γ_1° *to* γ_{12}'.



To see that this is a winning position, we fold the $SL(5)$ picture along the antidiagonal (this is $\hat{\phi} : (A_4)_+ \to (B_2)_+$).



(The '$\times$'s denote the diagonal of the folding map.) We then superimpose the two $(B_2)_+$ pictures which this folding produces (this is $\hat{\phi} : (B_2)_+ \times (B_2)_+ \to (B_2)_+$). Since no tokens overlap in this process, or appear on the diagonal of the folding map ($= \hat{\phi}^{-1}(\{0\})$), this is a winning position.

Example 4.2.18. *Let $G = SL(7) \times SO(7)$, $H = SO(7)$, $\pi = (1425736, \bar{2}3\bar{1})$, where $\bar{2}3\bar{1}$ is the $SO(7)$ Weyl group element represented by the matrix*

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Figure 4.4 shows a sequence of splittings and moves lead to a winning position. Squares belonging to the same region are similarly shaded. For each move, the relevant region is outlined, and the relevant root is indicated by an asterisk in the corresponding square.

Remark 4.2.19. *Although Theorems 4.1 and 4.2 are applicable for a large number of $\pi \in W$, they do not cover all cases. In particular the converses are not true in general. For examples of the failure of the converses for the multiplicative problem, see Chapter 2.*

Example 4.2.20. *Let $G = G_2 \times SL(3)$, and $H = SL(3)$ including diagonally, where $SL(3) \hookrightarrow G_2$ is as described in Example 4.2.8. We consider all possible $\pi \in W$, with $l(\pi) = 3 = \dim H/B'$. There are such 11 such π in total. Of these, 3 associated games are doomed. These are $\pi = (r_1, 231)$, $\pi = (r_2r_1, 132)$,*

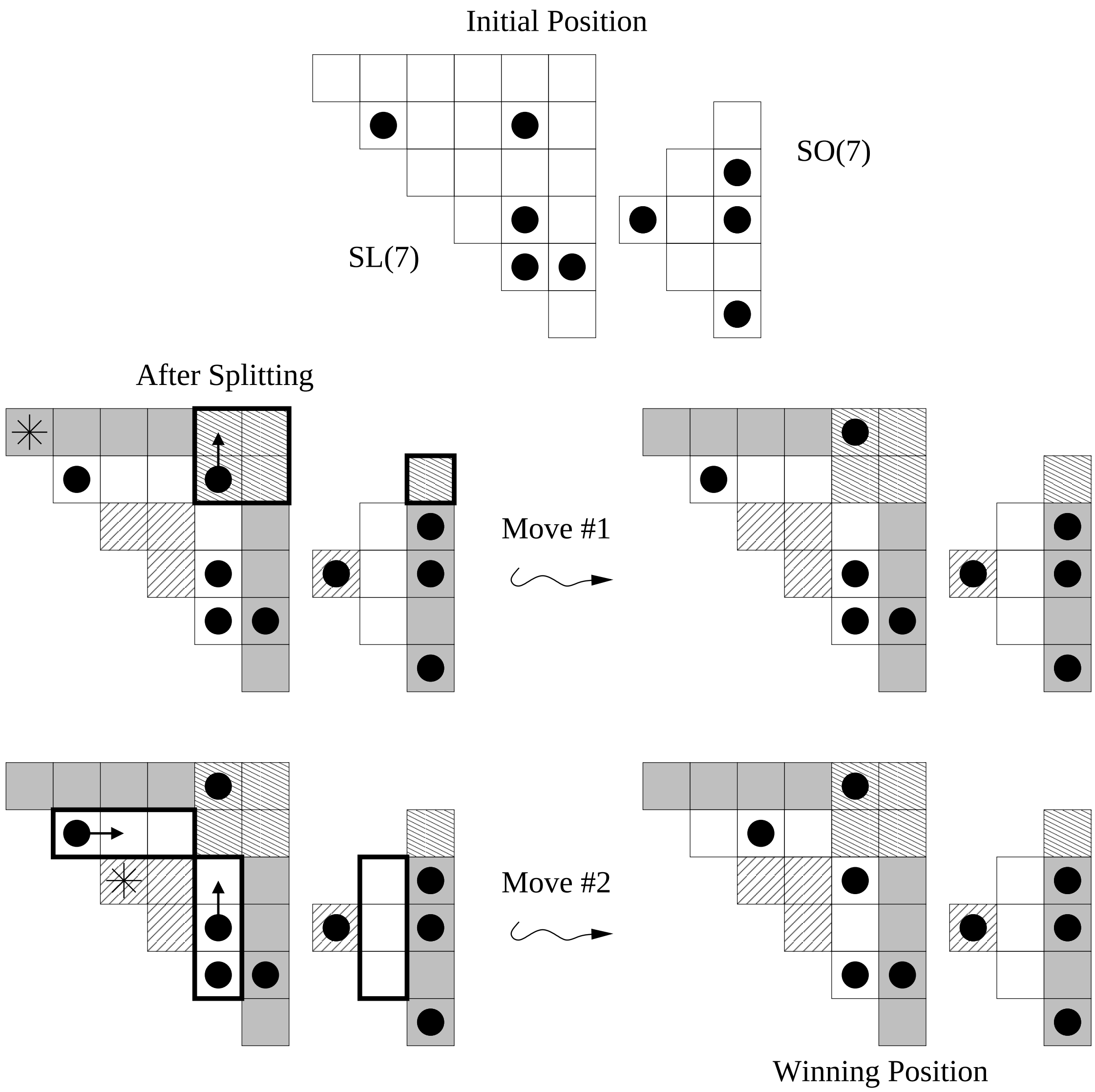


Figure 4.4: A sequence of moves in the root game for $SO(7) \hookrightarrow SL(7) \times SO(7)$, $\pi = (1425736, \bar{2}3\bar{1})$. The bold outline indicates which region is being used in each move, and the $*$ indicates which root is being used.

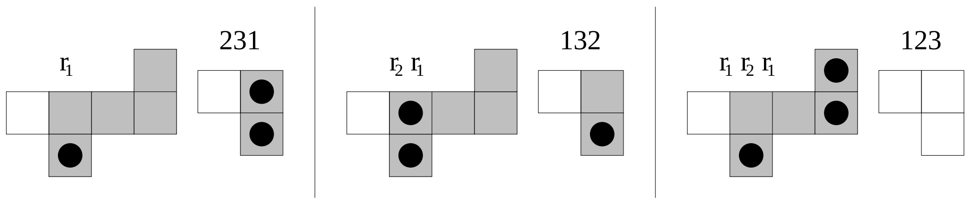


Figure 4.5: The 3 games which are doomed for $SL(3) \hookrightarrow G_2 \times SL(3)$. The shaded squares indicate a minimal ideal subset A for which $|\mathcal{T} \cap A| > |\hat{\phi}(A) \setminus \{0\}|$.

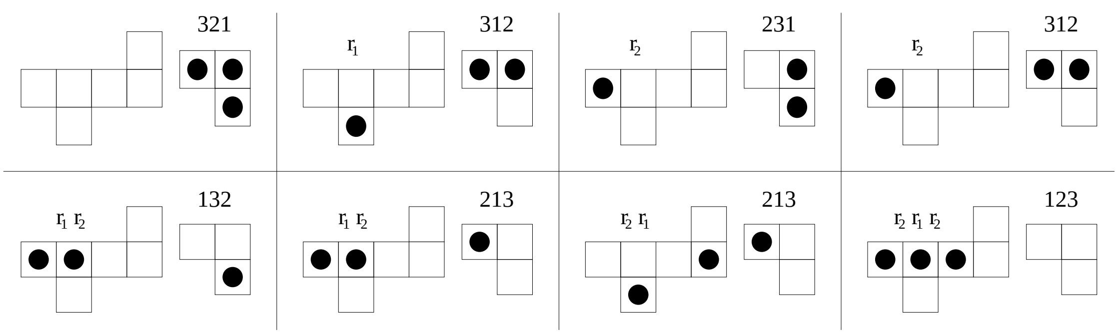


Figure 4.6: The 8 games which are not doomed for $SL(3) \hookrightarrow G_2 \times SL(3)$. Each of these games can be won.

and $\pi = (r_1 r_2 r_1, 123)$, where r_1 and r_2 represent reflections in the short and long simple roots respectively. These are shown in Figure 4.5. The remaining 8 games are shown in Figure 4.6. One can check that each of these can be won. Figure 4.7 shows a sequence of moves from the initial position of these games, $\pi = (r_1 r_2, 213)$, to a winning position. Thus the root game gives a complete answer to the vanishing problem for branching $SL(3) \hookrightarrow G_2$.

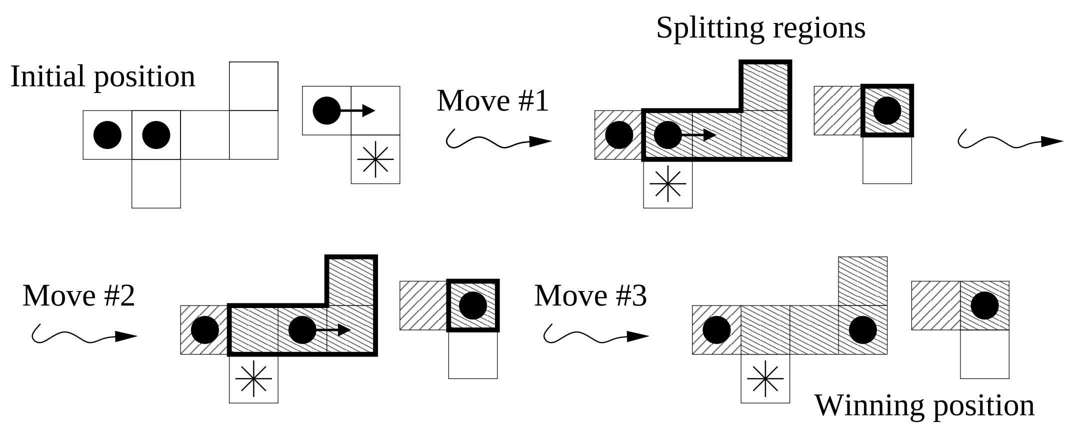


Figure 4.7: A sequence of moves in the root game for $SL(3) \hookrightarrow G_2 \times SL(3)$, $\pi = (r_1 r_2, 213)$. After the first move, we split into three regions, indicated by the different shading of squares. The bold outline indicates which region is being used in each move, and the $*$ indicates which root is being used.

4.3 A vanishing lemma for branching

Consider the space

$$E = \{(y,z) \in G/B \times H/B' \mid y \in X_{\pi,z}\} \subset G/B \times H/B'$$

Note that because of the length condition on π, $\dim E \geq \dim G/B$.

There are two projection maps from E. We have $p_H : E \to H/B'$ given by $p_0(y,z) = z$, and $p_G : E \to G/B$ given by $p_G(y,z) = y$. The map p_H is a fibration, and moreover is H-equivariant.

We claim that $i^*(\omega_\pi)$ is given by the class of the generic fibre of p_G. (When $l(\pi) = \dim H/B'$, $\int_{(H/B')} i^*(\omega_\pi)$ is the number of points of the generic fibre of p_G.) This is because of the Kleiman-Bertini theorem [Kle74], which says that (in characteristic zero) a Schubert variety X_{π,y_0} for a generic y_0 intersects H/B transversely. The class of the intersection is therefore equal to $i^*(\omega_\pi)$. But this intersection is just $p_H(p_G^{-1}(y_0))$.

Thus we see that $i^*(\omega_\pi) = 0$ if and only if the generic fibre of p_G is empty. Equivalently, let $U \subset G/B$ denote the points over which p_G is finite-to-one, and let $E' = p_G^{-1}(U)$. Then $i^*(\omega_\pi) = 0$ if and only if U, and hence E', are empty.

In studying the vanishing problem for multiplication in Schubert calculus, our approach was to fix a point of intersection of three (or more) Schubert varieties, and determine whether the tangent spaces can be made to intersect transversely (Lemma 2.2.1). Our next lemma generalises this idea to the branching problem. It is the main technical tool for developing vanishing and non-vanishing conditions.

Let $Q \subset \mathfrak{n}$ be the subspace generated by the e_α such that $\alpha \in \Delta_+$ and $\pi^{-1} \cdot \alpha \in \Delta_-$. Equivalently,

$$Q = \mathfrak{n} \cap (\pi \cdot \mathfrak{b}_-).$$

For $a \in N$, let $a\cdot\ : \mathfrak{n} \to \mathfrak{n}$ denote the adjoint action of N on its Lie algebra.

Lemma 4.3.1. *The following are equivalent:*

1. $i^*(\omega_\pi) = 0$.

2. $\phi|_{a \cdot Q}$ *is injective for some* $a \in N$.

3. $\phi|_{a \cdot Q}$ *is injective for generic* $a \in N$.

Proof. The equivalence of conditions 2 and 3 is clear, as the maps $\phi|_A$ are injective for a Zariski open set of subspaces A. We now show the equivalence of 1 and 3.

$$\begin{aligned} i^*(\omega_\pi) = 0 &\iff E' = \emptyset \\ &\iff p_H^{-1}(z) \cap E' = \emptyset \ \forall z \in H/B' \\ &\iff p_G(p_H^{-1}(z)) \cap U = \emptyset \ \forall z \in H/B' \\ &\iff \bigcup_{z \in H/B'} p_H(p_G^{-1}(z)) \subset (G/B) \setminus U. \end{aligned}$$

But $\bigcup_{z\in H/B'} p_G(p_H^{-1}(z)) = H \cdot p_G(p_H^{-1}(z))$ for any $z \in H/B'$, in particular for $z = x_0$, in which case we have $p_G(p_H^{-1}(x_0)) = X_\pi$. Thus $i^*(\omega_\pi) = 0$ if and only if

$$H \cdot X_\pi \subset (G/B) \setminus U.$$

Since U is a Zariski open dense subset of G/B, this will happen only if $\dim(H \cdot X_\pi) < \dim G/B$. Conversely, if this inequality holds, then p_G is not onto, and $i^*(\omega_\pi) = 0$.

For a point $y \in X_\pi$, let

$$S(y) = \left\{ h \in H \;\middle|\; h \cdot y \in p_G(p_H^{-1}(x_0)) \right\}.$$

Now X_π is B-invariant, hence B'-invariant; thus $B' \subset S(y)$ for all y. Given this, the expected dimension of $S(y)$ is $\dim H/B' - l(\pi) - \dim B'$. This is the dimension $S(y)$ would be if $H \cdot X_\pi$ were dense. In general, we have

$$\operatorname{codim}\big(H \cdot X_p i\big) = \dim S(y) - \text{expected dim } S(y),$$

for y generic in X_π.

Let us therefore compute the actual dimension of $S(y)$ at a generic point $y = a[\pi'] \in X_\pi$. We have

$$\begin{aligned} h \in S(y) &\iff h \cdot a\tilde{\pi}' B \in B a \tilde{\pi}' B \\ &\iff h \in B\, a\tilde{\pi}' B\, (a\tilde{\pi}')^{-1} \cap H. \end{aligned}$$

We now pass to the Lie algebra level, by considering the tangent space to $S(y)$ at the identity element of H.

$$\begin{aligned} T_1(S(y)) &= \big(a\tilde{\pi}'\,\mathfrak{b}\,(\tilde{\pi}')^{-1}a^{-1} + \mathfrak{b}\big) \cap \mathfrak{h} \\ &= \big(a \cdot (\pi' \cdot \mathfrak{b}) + \mathfrak{b}\big) \cap \mathfrak{h} \\ &= \big(a \cdot (\pi' \cdot \mathfrak{b}) + \mathfrak{b}\big) \cap \mathfrak{h} \end{aligned}$$

Thus

$$\begin{aligned} T_1(S(x))/\mathfrak{b}' &= \big(a \cdot (\pi' \cdot \mathfrak{b})\big)/\mathfrak{b} \cap \mathfrak{h}/\mathfrak{b}' \\ &= (a \cdot Q)^{\perp} \cap \mathfrak{h}/\mathfrak{b}' \end{aligned}$$

This intersection is of the expected dimension (i.e. transverse) if and only if $\phi|_{a\cdot Q}$ is injective.

To complete the proof, we must argue that $S(y)$ is of the expected dimension if and only if this is true on the lie algebra level of the expected dimension. One direction is clear: if $T_1(S(y))/\mathfrak{b}'$ is of the expected dimension, then so is $S(y)$. The reverse direction is true by Kleiman-Bertini, which tells that if $g \cdot H$ and X_π have a point of intersection for generic $g \in G$ they intersect transversely at that point. If y is such a point of intersection then $S(y)$ will be of the expected dimension, as will $T_1(S(y))/\mathfrak{b}'$.

Thus we have established the equivalence of 1 and 3. □

4.4 Proofs

4.4.1 Proof of the vanishing criterion

The vanishing criterion (Theorem 4.1) is an easy consequence of Lemma 4.3.1.

Proof. (Theorem 4.1) At the outset, $\{e_\alpha \mid \alpha \in \mathcal{T}\}$ is a basis for the space Q. If the game is doomed then there is an ideal subset A such that

$$|\mathcal{T} \cap A| > |\hat{\phi}(A) \setminus \{0\}|.$$

Let $S \subset \mathfrak{n}$ be the ideal generated by $\{e_\alpha \mid \alpha \in A\}$. Then $\dim(Q \cap S) = |\mathcal{T} \cap A|$, and $\dim \phi(S) = |\hat{\phi}(A) \setminus \{0\}|$. Moreover, as S is N-invariant, we have that

$$\dim(a \cdot Q \cap S) = \dim(Q \cap S) > \dim \phi(S)$$

for all $a \in N$. It follows that $\phi|_{(a \cdot Q \cap S)}$ is not injective, and thus $\phi|_{a \cdot Q}$ is not injective. Therefore, by Lemma 4.3.1, $i^*(\omega_\pi) = 0$. $\square$

4.4.2 Proof of the non-vanishing criterion

The proof of the non-vanishing criterion (Theorem 4.2) is more involved. It requires us to translate the combinatorics of the root game into a geometrical framework.

Proof. (Theorem 4.2) We are interested in the following general setup. Let V be a finite dimensional representation of B, and V' be a finite dimensional representation of B'. Suppose we have a B'-equivariant map $\psi : V \to V'$.

Let $Gr(V)$ denote the disjoint union of all Grassmannians

$$Gr(V) = \coprod_{l=0}^{\dim V} Gr_l(V).$$

Since V has a B-action, so does $Gr(V)$.

Let $U \in Gr(V)$ be a subspace of V. We call the quadruple (U, V, V', ψ) **good** if there is a point $\tilde{U}$ in the B-orbit closure through U such that $\psi|_U : U \to V'$ is an injective linear map. We define

$$g(U) = \{\tilde{U} \in B \cdot U \mid \psi|_{\tilde{U}} \text{ is injective}\} \subset Gr(V).$$

Note that the set of $U \in Gr(V)$ with $\phi|_U$ injective is Zariski open in $Gr(V)^3$. Thus (U, V, V', ψ) is good $\iff$ $g(U)$ is an open dense subset of $B \cdot U$, i.e. $\iff$ $g(U) \neq \emptyset$.

Each position in the game consists of a set of regions $\mathcal{R}$, and an arrangement of tokens $\mathcal{T}$. The data of a single region, together with the set of tokens contained in that regions give will give rise to a quadruple (U, V, V', ψ). The region itself will determine V, V' and ψ. The subspace U will be determined by the arrangement of tokens.

We note that each region R is in fact a difference of two ideal subsets $R = A_1 \setminus A_2$. Now the root spaces of A_1 and A_2 span ideals S_1 and S_2 of $\mathfrak{n}$. Thus the corresponding root spaces of R can be thought of as spanning the subquotient representation S_1/S_2 of $\mathfrak{n}$. Thus we take $V = S_1/S_2$, and note that it has a basis $\bar{e}_\alpha$ corresponding to $\alpha \in R$.

We let $V' = \phi(S_1)/\phi(S_2)$. Note that since S_1 and S_2 are B-modules, they are also also B'-modules, and since ϕ is B'-equivariant, V' is a B'-module. We note that V' has a basis corresponding to $\hat{\phi}(R)$. Moreover, A_1 and A_2 are more than ideal subsets: they are splitting subsets, and thus two distinct regions R_1 and R_2 actually give rise to disjoint V'.

The map $\psi : V \to V'$ descends from $\phi : \mathfrak{n} \to \mathfrak{n}'$. Finally we take

$$U = \operatorname{span}\{\bar{e}_\alpha \mid \alpha \in \mathcal{T} \cap R\} \subset V.$$

Since each we get one such quadruple for each region, the entire state of the game is in fact given by a list of quadruples, one for each region in $\mathcal{R}$:

$$\{(U_1, V_1, V_1', \psi), \ldots, (U_s, V_s, V_s', \psi_s)\}.$$

We abbreviate this as $\{(U_m, V_m, V_m', \psi_m)\}$, where the subscript (in this case m) is understood to index the regions $R_m \in \mathcal{R}$.

We claim the following:

1. $i^*(\omega_\pi)$ is non-zero if and only if the initial state of the game is good.

2. Suppose that $\{(U_m, V_m, V_m', \psi_m)\}$ is the state of the game before a splitting, and and $\{(\tilde{U}_n, \tilde{V}_n, \tilde{V}_n', \tilde{\psi}_n)\}$ is the state after. If $(\tilde{U}_n, \tilde{V}_n, \tilde{V}_n', \tilde{\psi}_n)$ is good for all n, then (U_m, V_m, V_m', ψ_m) is good for all m.

3. Suppose that $\{(U_m, V_m, V_m', \psi_m)\}$ is the state of the game before a move is made, and $\{(\tilde{U}_m, V_m, V_m', \psi_m)\}$ is the state after. If $(\tilde{U}_m, V_m, V_m', \psi_m)$ is good for all m, then (U_m, V_m, V_m', ψ_m) is good for all m.

4. If $\{(U_m, V_m, V'_m, \psi_m)\}$ is the state of the game when the game is won, then each (U_m, V_m, V'_m, ψ_m) is good.

Proof. (of claims)

1. From the definition of the initial position, we see that this the initial state of the game is given by $(Q, \mathfrak{n}, \mathfrak{n}', \phi)$. The claim is therefore simply a restatement of Lemma 4.3.1.

2. Let $V = S_1/S_2$ correspond to the region $R \in \mathcal{R}$, where $S_1 \supset S_2$ are ideals of the B-module $\mathfrak{n}$ generated by splitting subsets A_1 and A_2. Let A be an third splitting subset which generates an ideal S. We assume that $A_1 \supset A \supset A_2$, and thus $S_1 \supset S \supset S_2$. (If this is not the case, we should replace A with $A_1 = (A \cap S_1) \cup S_2$. A_1 is also a splitting set, and $R \cap A_1 = R \cap A$, so splitting R along A is the same as splitting R along A_1.) Let $\tilde{S} = S/S_2$. Thus $V/\tilde{S} = S_1/S$ is quotient of V (as a B-module), and $\tilde{S}$ is a submodule.

   We also obtain sub- and quotient representations of $V' = \psi(V)$; namely, $\psi(\tilde{S})$ is a B'-submodule of V' and $V'/\psi(\tilde{S})$ is a quotient module. Moreover, there is are natural induced maps from ψ:

   $$\psi_\sigma : V/\tilde{S} \to V'/\psi(\tilde{S})$$

   and

   $$\psi_\tau : \tilde{S} \to \psi(\tilde{S}).$$

   If the triple (V, V', ψ) represents the region R, then the two regions obtained by splitting R along A are represented by the two triples $(V/\tilde{S}, V'/\psi(\tilde{S}), \psi_\sigma)$ and $(\tilde{S}, \psi(\tilde{S}), \psi_\tau)$.

   Define functions $\sigma : Gr(V) \to Gr(V/\tilde{S})$, given by $\sigma(U) = U/\tilde{S}$, and $\tau : Gr(V) \to Gr(\tilde{S})$ given by $\tau(U) = U \cap \tilde{S}$. Note σ_V and τ_V are not everywhere continuous, but they are B-equivariant, and hence continuous on B-orbits.

   Suppose $U \in Gr(V)$. By elementary linear algebra, if

   $$\psi_\sigma|_{\sigma(U)} : \sigma(U) \to V'/\psi(\tilde{S})$$

is injective, and

$$\psi_\tau|_{\tau(U)} : \tau(U) \to \psi(\tilde{S})$$

is injective, then $\psi|_U : U \to V'$ is injective.

Suppose $(\sigma(U), V/\tilde{S}, V'/\psi(\tilde{S}), \psi_\sigma)$ and $(\tau(U), \tilde{S}, \psi(\tilde{S}), \psi_\tau)$ are both good. Then $\tau^{-1}(g(\tau(U))$ and $\sigma^{-1}(g(\sigma(U)))$ are both open dense subsets of $B \cdot U$. Since $g(U)$ contains the intersection of these, (U_m, V_m, V'_m, ψ_m) must be good.

Thus we have shown that if the position after splitting is good, then so was the position before splitting, as required.

3. Suppose the move is given by the root α, and the region R corresponding to (V, V', ψ). Let $U \subset V$ correspond to the arrangement of tokens before the move, and $\tilde{U} \subset V$ correspond to the arrangement of tokens after.

   We consider the 1-dimensional subgroup of B given by $\theta_\alpha : N_\alpha \hookrightarrow B$, where $N_\alpha \cong (\mathbb{C}, +)$ is the exponential of the α root space.

   We now calculate

$$\lim_{t\to\infty} \theta_\alpha(t) \cdot U.$$

   We can represent U as $[\bar{e}_{\alpha_1} \wedge \ldots \wedge \bar{e}_{\alpha_l}]$, and $\tilde{U}$ as $[\bar{e}_{\alpha'_1} \wedge \ldots \wedge \bar{e}_{\alpha'_l}]$, via the Plücker embedding $Gr(V) \hookrightarrow P(\bigwedge^* V)$. Now

$$\begin{aligned}\theta_\alpha(t) \cdot U &= \theta_\alpha(t) \cdot [\bar{e}_{\alpha_1} \wedge \ldots \wedge \bar{e}_{\alpha_l}],\\ &= [\bar{e}_{\alpha_1} + t(\alpha \cdot \bar{e}_{\alpha_1}) \wedge \ldots \wedge \bar{e}_{\alpha_l} + t(\alpha \cdot \bar{e}_{\alpha_l})]\end{aligned}$$

   where $\alpha \cdot \bar{e}_{\alpha_i} = \bar{e}_{\alpha_i+\alpha}$, if $\alpha_i + \alpha$ is a root belonging the region corresponding to V_m, and 0 otherwise. In the limit as $t \to \infty$, the only term which survives is the one with the highest power of t, which is precisely

$$[\pm t^{\#\text{tokens that move}} \bar{e}_{\alpha'_1} \wedge \ldots \wedge \bar{e}_{\alpha'_l}].$$

   Thus

$$\tilde{U} = \lim_{t\to\infty} \theta_\alpha(t) \cdot U.$$

But this is just another point in $\overline{B \cdot U}$. Thus if $(\tilde{U}, V, V', \psi)$ is good, so is (U, V, V', ψ).

4. In a winning position the set $\mathcal{T}$ is injective. It follows that within each region R_m, $\psi|_U : U \to V'$ is injective, where U is the span of the root spaces of $\mathcal{T} \cap R$. In particular each (U_m, V_m, V'_m, ψ_m) is good.

$\square$

The result now follows easily from the four claims. If a game can be won, the winning position is good (claim part 4). Thus all positions (whether attained by moves or splittings) leading up to the winning position are good (claim parts 2 and 3). In particular, the initial position is good, which implies that $i^*(\omega_\pi) = 0$ (claim part 1). $\square$

Chapter 5

Future directions

Obviously the big outstanding problem in this work is to determine in which cases the non-vanishing criterion is necessary as well as sufficient. We have already shown that it is necessary and sufficient in two special cases. The first case is for products of two Schubert classes, in which the root game can be seen to be equivalent to the Bruhat order. The second case, discussed in Chapter 3, is when the classes are pulled back from a Grassmannian. Here we will briefly discuss a few special cases where we might hope to be able to obtain further results of this sort.

5.1 Quantum Schubert calculus

One natural next step in continuing this work would be to study vanishing criteria for *quantum* Schubert calculus of the Grassmannian. This is determined by the cohomology of two-step flag manifolds [BKT03], and the root game provides a conjectural rule for determining the answer. Root games pulled back from two-step flag manifolds bear a strong resemblance to those pulled back from a Grassmannian: the root games can be thought of as three Grassmannian problems which might interact in interesting ways. It may be possible to gain a complete understanding of these, particularly when the dimension of one of the subspaces of the two step flag manifold is small.

The quantum cohomology of Grassmannians is of particular interest, since there is already a conjectural combinatorial rule for the structure constants of the cohomology of two-step flag manifolds. The rule is known as Knutson's 012-puzzle conjecture, and it has been verified by Anders Buch for two-step flag manifolds in $\mathbb{C}^n$ for all $n \leq 16$. As well, Belkale has proved the quantum analogue of the Horn conjecture [Bel03], which would provide a natural place with which to compare the result.

5.2 Minuscule flag manifolds

Minuscule flag manifolds are those generalised flag varieties G/P for which the representation of the Levi subgroup $L \subset P$ on the exterior algebra of the tangent space $\bigwedge^*(\mathfrak{g}/\mathfrak{p})$ is multiplicity free. (There are many other equivalent definitions.)

The most familiar example is the Grassmannian $Gr_l(n)$, in which case the Levi subgroup is $GL(l) \times GL(n-l)$ acting on $\mathfrak{g}/\mathfrak{p} \cong \mathrm{Hom}(\mathbb{C}^l, \mathbb{C}^{n-l})$ in the usual way. However, minuscule flag manifolds exist for all the classical Lie groups, as well as the exceptional groups E_6 and E_7. For example, the Lagrangian Grassmannian is a minuscule flag manifold for $G = Sp(2n)$. These should be the simplest cases in which to study types B, C and D-Schubert calculus. A positive rule for the cohomology of Lagrangian Grassmannian is known [Ste89].

Kostant's work on Lie algebra cohomology [Kos63], and Stembridge's results on minuscule representations [Ste03] suggest that it should be possible to gain a complete understanding of the cohomology of minuscule flag manifolds. Root games appear to fit very naturally into this context. This would therefore be a natural place to try to gain a fuller understanding of the non-vanishing condition given by root games.

5.3 Horn inequalities and B-varieties

Using techniques in the tangent space, some of which are similar to those in this thesis, Belkale has given a geometric proof Horn's conjecture [Bel02]. Horn's conjecture, as we recall from Chapter 1 gives a recursive method for finding the set of inequalities which determine the non-vanishing Schubert structure constants in the cohomology of a Grassmannian.

In Chapter 3, we showed that the non-vanishing Schubert structure constants in the product $\Omega_\pi \cdot \Omega_\rho = \sum_\sigma c^\sigma_{\pi\rho}\Omega_\sigma$ correspond to B-fixed points on a certain B-variety $Z_\pi^{\rho^\vee}$, depending on π and ρ. In fact, we showed that it is possible to identify the non-vanishing coefficients simply from the T-moment polytope of this variety.

Thus we have two quite different classes of polytopes from which one can determine non-vanishing of Schubert calculus, both arising from the geometry of the tangent spaces of Schubert varieties. It is therefore natural to wonder whether the inequalities defining the moment polytope are connected to Horn's inequalities. At the moment, it is unclear what the precise connection could be. It appears to be more natural to view Horn's inequalities in terms of the vanishing criterion in the root game, rather than the non-vanishing criterion. A connection between these two should therefore go a long way towards a deeper understanding of the vanishing of Schubert calculus.

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