

An Axiomatic Duality Framework for the Theta Body and Related Convex Corners

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Abstract Lovász theta function and the related theta body of graphs have been in the center of the intersection of four research areas: combinatorial optimization, graph theory, information theory, and semidefinite optimization. In this paper, utilizing a modern convex optimization viewpoint, we provide a set of minimal conditions (axioms) under which certain key, desired properties are generalized, including the main equivalent characterizations of the theta function, the theta body of graphs, and the corresponding antiblocking duality relations. Our framework describes several semidefinite and polyhedral relaxations of the stable set polytope of a graph as generalized theta bodies. As a by-product of our approach, we introduce the notion of “Schur Lifting” of cones which is dual to PSD Lifting (more commonly used in SDP relaxations of combinatorial optimization problems) in our axiomatic generalization. We also generalize the notion of complements of graphs to diagonally scaling-invariant polyhedral cones. Finally, we provide a weighted generalization of the copositive formulation of the fractional chromatic number by Dukanovic and Rendl from 2010.

Keywords Lovász theta number · Semidefinite programming · Copositive programming · Stability number · (fractional) Chromatic number

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1 Introduction

The Lovász theta function is one of the most elegant highlights in combinatorial and semidefinite optimization. First introduced in the seminal paper by Lovász [32] to solve a problem in information theory, the theta function was further developed in the 1980's along with applications of the ellipsoid method [22, 23], leading to the definition of the object known as the theta body of a graph as a semidefinite relaxation of its stable set polytope. This relaxation is tight for perfect graphs, and it leads to the only known (strongly) polynomial algorithm for finding optimal stable sets and colorings in such graphs. Since then, the theory surrounding the Lovász theta function has been further extended [34, 16, 37, 25], and it has been used in the design of approximation algorithms [28, 29, 2, 11], in complexity theory [47, 15, 5, 4], in information theory [38, 45, 12], and in extremal geometry [3].

For a graph $G = (V, E)$, the *theta body* of G may be defined as the set

$$\text{TH}(G) = \left\{ x \in \mathbb{R}^V : \exists X \in \mathbb{S}^V, X_{ii} = x_i \forall i \in V, X_{ij} = 0 \forall ij \in E, \begin{bmatrix} \mathbb{1} & x \\ x & X \end{bmatrix} \in \mathbb{S}_+^{\{0\} \cup V} \right\}, \quad (1)$$

where \mathbb{S}^V denotes the set of $V \times V$ symmetric matrices and $\mathbb{S}_+^{\{0\} \cup V}$ is the set of symmetric positive semidefinite matrices on the index set $\{0\} \cup V$; we assume that 0 is not an element of V . The *stable set polytope* of G , denoted by $\text{STAB}(G)$, is defined as the convex hull of incidence vectors of stable sets of G ; a set $S \subseteq V$ is *stable* in G if no edge of G joins two elements of S . It is not hard to check that, if x is the incidence vector of a stable set of G , then $X := xx^\top$ satisfies the constraints on the RHS of (1), so $x \in \text{TH}(G)$. Thus, the theta body is a relaxation of $\text{STAB}(G)$, and the *theta function*

$$\vartheta(G) = \max \left\{ \sum_{i \in V} x_i : x \in \text{TH}(G) \right\} \quad (2)$$

is an upper bound for the size of a largest stable set in G . It is also convenient to define a weighted version $\vartheta(G; w)$ of the theta function, by multiplying x_i in the objective function of (2) by some weight $w_i \geq 0$ for each $i \in V$.

Part of the broad applicability of the theta function owes to its multitude of equivalent formulations, which led Goemans to the conclusion that “it seems all paths lead to $\vartheta!$ ” [20]. For instance, the eigenvalue formulation

$$\vartheta(G) = \min \left\{ \lambda_{\max}(A + \mathbb{1}\mathbb{1}^\top) : A \in \mathbb{S}^V, A_{ij} \neq 0 \implies ij \in E \right\}, \quad (3)$$

where λ_{\max} extracts the largest eigenvalue and $\mathbb{1}$ is the vector of all-ones, is central for the approximate (vector) coloring algorithm of Karger, Motwani and Sudan [28]. Similarly, the non-convex formulation

$$\vartheta(G) = \max \left\{ \lambda_{\max}(B) : B \in \mathbb{S}_+^V, B_{ii} = 1 \forall i \in V, B_{ij} = 0 \forall ij \in E \right\}$$

essentially says that $\vartheta(G)$ is the best lower bound for the chromatic number of G from a family of bounds due to Hoffman [26]; the *chromatic number* of G is the minimum size of a partition of V into stable sets of G . The description (1) itself is also especially well suited for proving that all (nontrivial) facets of $\text{TH}(G)$ are determined by *clique inequalities* (see, e.g., [43, Theorem 67.13]), i.e., inequalities of the form $\sum_{i \in K} x_i \leq 1$ for some clique K of G ; a set $K \subseteq V$ is a *clique* of G if every pair of elements of K is an edge of G .

Most of these alternative formulations also apply to some classical variants of ϑ , such as the functions ϑ' and ϑ^+ defined as in (2) over corresponding variants of the theta body, denoted by $\text{TH}'(G)$ and $\text{TH}^+(G)$, respectively (we define them in Section 2); see [39, 42, 46]. Many of such wealth of interesting characterizations arguably come from Semidefinite Programming (SDP) Strong Duality. A particularly illuminating manifestation of this duality is the identity (see, e.g., [43, Theorem 67.12])

$$\text{abl}(\text{TH}(G)) = \text{TH}(\overline{G}), \quad (4a)$$

that is, the *antiblocker* of $\text{TH}(G)$ is the theta body of \overline{G} , the complement of G . Antiblocking duality is the notion of duality most appropriate for a class of convex sets known as convex corners (defined in Section 2), which include all variants of theta bodies, as well as $\text{STAB}(G)$. Similar instances of (4a) include

$$\text{abl}(\text{TH}'(G)) = \text{TH}^+(\overline{G}), \quad (4b)$$

$$\text{abl}(\text{STAB}(G)) = \text{QSTAB}(\overline{G}); \quad (4c)$$

here, $\text{QSTAB}(G) \subseteq [0, 1]^V$ is a classical polyhedral relaxation for $\text{STAB}(G)$ determined by clique inequalities (we define it in Section 7). The relaxations in (4) are related by the following chain:

$$\text{STAB}(G) \subseteq \text{TH}'(G) \subseteq \text{TH}(G) \subseteq \text{TH}^+(G) \subseteq \text{QSTAB}(G) \subseteq \text{FRAC}(G); \quad (5)$$

here, $\text{FRAC}(G) \subseteq [0, 1]^V$ is determined by edge inequalities, i.e., clique inequalities where the clique is a single edge (we define it in Section 4.2). The beautiful, striking relationships in (4) continue to manifest themselves in certain lift-and-project methods [1, 17, 31].

In this paper, we define a notion of generalized theta bodies and develop a duality theory that: (1) describes all sets in (5) as theta bodies, (2) extends many of the equivalent formulations for ϑ to the corresponding generalized theta functions, and (3) extends the antiblocker relations (4) to generalized theta bodies. We shall parameterize a generalized theta body, henceforth called just theta body, by two convex cones. One of them, which we denote by \mathbb{A} , shall encode the adjacency constraints $X_{ij} = 0$ in (1), i.e., those constraints shall be replaced with “ $X \in \mathbb{A}$ ”. The other cone, which we denote by $\widehat{\mathbb{K}}$, will replace $\mathbb{S}_+^{\{0\} \cup V}$ in (1); it essentially constrains how X and xx^\top are related, so we may think of $\widehat{\mathbb{K}}$ as a cone that encodes quadratic relations. By varying these cones over some natural families, we shall obtain a description of all sets in (5) as theta bodies, as well as the corresponding antiblocking duality relations that links them in pairs as in (4). We thus unify the description of all these relaxations and show that the ingenious though *ad hoc* description (1) is in fact quite central, powerful, and natural. The corresponding theory shall also make clearer the key role played by the positive semidefinite cone in this generalized context, leading to the most striking of the duality relations (4), namely, the extremely symmetric relation (4a). We refer the reader to Fig. 2 for a schematic diagram of the duality relations exposed by our framework.

The key axiom we shall need require from our parameter cones \mathbb{A} and $\widehat{\mathbb{K}}$ will be their *diagonally scaling invariance*, i.e., these cones must be closed under simultaneous left- and right-multiplications by any (and the same) nonnegative diagonal matrix. We shall then generalize the proofs of equivalence of several formulations for ϑ to rely (essentially) solely on this invariance property. This axiomatic approach also allows us to gauge the full power of the existing proof methods; for instance, some of the equivalent formulations for ϑ shall only work when the cone $\widehat{\mathbb{K}}$ is the positive semidefinite cone $\mathbb{S}_+^{\{0\} \cup V}$.

Another advantage of unifying the equivalent formulations of ϑ is that it provides less error-prone proofs of equivalent formulations of the variants ϑ' and ϑ^+ . Many such formulations are listed in the literature without proof, with the apparently implicit suggestion that, to prove them, it suffices to repeat and slightly adapt the corresponding proofs for ϑ . While this may be true in most cases, it has already led to some inaccuracies in the literature, as pointed out by the authors in [9, Sec. 4.1].

Finally, we are able to extend the copositive formulation of the fractional chromatic number by Dukanovic and Rendl [14] to the weighted case, and we provide a unified treatment of weighted generalizations of the convex quadratic characterization of ϑ by Luz and Schrijver [37] to all of ϑ , ϑ' , and ϑ^+ .

1.1 Organization of the text

The classical monograph [23], which develops much of the theory surrounding the theta function, defines weighted parameters $\vartheta_i(G; w)$ for each $i \in \{1, \dots, 4\}$ and shows that they are all equal to $\vartheta(G; w)$ by proving the chain of inequalities

$$\vartheta(G; w) \leq \vartheta_1(G; w) \leq \vartheta_2(G; w) \leq \vartheta_3(G; w) \leq \vartheta_4(G; w) \leq \vartheta(G; w) \quad (6)$$

for a nonnegative weight function w on V ; see also [30, Sec. 5]. For instance, ϑ_2 is the (weighted generalization of the) formulation on the RHS of (3). In this paper, we shall generalize these parameters to arbitrary theta bodies and prove that they are all equal under some mild assumptions. As in [23], this shall establish an antiblocking relation like those in (4).

As we briefly hinted just before (5), we shall parameterize our theta bodies using two cones, which we usually denote by \mathbb{A} and $\widehat{\mathbb{K}}$; e.g., the membership constraints $X_{ij} = 0$ for every $ij \in E$ in (1) are replaced with the single membership constraint “ $X \in \mathbb{A}$ ”. One slightly confusing issue is the fact that the cones \mathbb{A} and $\widehat{\mathbb{K}}$ do not live in the same dimension: \mathbb{A} lives in \mathbb{S}^V , whereas $\widehat{\mathbb{K}}$ lies in the higher-dimensional cone $\mathbb{S}^{\{0\} \cup V}$, where 0 is assumed not to be in V . Throughout the paper, we label subsets of $\mathbb{S}^{\{0\} \cup V}$ with a wide hat, as in $\widehat{\mathbb{K}}$, and elements of such sets with a hat, e.g., $\hat{X} \in \widehat{\mathbb{K}}$.

We shall see later that it is rather natural and convenient to define a “lifting” of a cone \mathbb{K} in \mathbb{S}^V to a higher-dimensional cone in $\mathbb{S}^{\{0\} \cup V}$ in a systematic way. In fact, we shall define two such lifting operators for a cone $\mathbb{K} \subseteq \mathbb{S}^V$, denoted by $\text{Psd}(\cdot)$ and $\text{Schur}(\cdot)$, in such a way that the antiblocker of a theta body parameterized by $\text{Psd}(\mathbb{K})$ is parameterized by $\text{Schur}(\cdot)$ applied to the dual cone of \mathbb{K} . Similarly, we will define a notion of duality for the cone \mathbb{A} , that encodes adjacencies, and the “dual” of \mathbb{A} will be denoted by $\bar{\mathbb{A}}$ to match the occurrence of complements of graphs in (4).

With these remarks in mind, we may now describe the organization of the paper. We list basic terminology and notation in Subsection 1.2. We then define arbitrary theta bodies and prove their most basic properties in Section 2. In Section 3, we study the structure of the cones \mathbb{A} that make sense in our theory, namely, the ones that are diagonally scaling-invariant and polyhedral; we also quickly develop their duality theory. Next, in Section 4, we describe the lifting operators for cones mentioned above, and we prove a weak duality theorem. Some basic reformulations of antiblocking duality are recalled and adapted to our context in Section 5. The latter three sections also prove a number of equivalences among the ϑ_i ’s corresponding to (6). These equivalences are put together in Section 6 to prove our generalization of the equivalence (6), the duality relations (4), and further ϑ results. In Sections 7 and 9, we study properties of some theta bodies defined over some specific cones, namely, the copositive and completely positive cones, and the semidefinite cone. In between those sections, we describe in Section 8 a further characterization of ϑ related to Hoffman bounds for the chromatic number of a graph.

1.2 Notation

We set the following notation. Throughout the paper, V shall denote a finite set. We assume throughout that $0 \notin V$, as we will often adjoin 0 to V to form an index set $\{0\} \cup V$. The family of subsets of V of size 2 is denoted by $\binom{V}{2}$. If $E \subseteq \binom{V}{2}$, we set $\bar{E} := \binom{V}{2} \setminus E$. For distinct $i, j \in V$, we denote $ij := \{i, j\}$. The *standard basis vectors* of \mathbb{R}^V are $\{e_i : i \in V\}$. We adopt *Iverson notation*: for a predicate P , we denote

$$[P] := \begin{cases} 1 & \text{if } P \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

If P is false, then we consider $[P]$ to be “strongly zero,” in the sense that we sometimes write expressions of the form $[x \neq 0](1/x)$ that evaluate to 0 if $x = 0$.

Most of the rest of our notation is listed over Tables 1, 2, and 3.

Table 1: Special sets.

\mathbb{R}_+	$:=$	$\{x \in \mathbb{R} : x \geq 0\}$
\mathbb{R}_{++}	$:=$	$\{x \in \mathbb{R} : x > 0\}$
\mathbb{S}^V	$:=$	the space of symmetric $V \times V$ matrices
\mathbb{S}_+^V	$:=$	the cone of positive semidefinite matrices in \mathbb{S}^V
$\mathbb{S}_{\geq 0}^V$	$:=$	$\{X \in \mathbb{S}^V : X \geq 0\}$, the cone of entrywise nonnegative matrices in \mathbb{S}^V
\mathcal{C}_V	$:=$	$\{X \in \mathbb{S}^V : h^\top X h \geq 0 \forall h \in \mathbb{R}_+^V\}$, the cone of <i>copositive matrices</i>
\mathcal{C}_V^*	$:=$	the cone of <i>completely positive matrices</i> , i.e., the dual cone of \mathcal{C}_V

Table 2: Notation for vectors and matrices.

$\langle X, Y \rangle$	$:=$	$\text{Tr}(XY^\top)$, the <i>trace inner-product</i> on \mathbb{S}^V
diag	$:=$	the linear map that extracts the diagonal of a matrix
Diag	$:=$	the adjoint of diag
$X[U]$	$:=$	the principal submatrix of $X \in \mathbb{R}^{V \times V}$ indexed by $U \subseteq V$
$\lambda_{\max}(X)$	$:=$	the largest eigenvalue of $X \in \mathbb{S}^V$
$\lambda_{\min}(X)$	$:=$	the smallest eigenvalue of $X \in \mathbb{S}^V$
I	$:=$	the identity matrix in appropriate dimension
A^\dagger	$:=$	the <i>Moore-Penrose pseudoinverse</i> of $A \in \mathbb{R}^{V \times W}$; see [27]
$\mathbf{1}$	$:=$	the vector of all-ones in the appropriate space
$\mathbf{1}_U$	$:=$	the <i>incidence vector</i> of $U \subseteq V$ in $\{0, 1\}^V$
\sqrt{w}	$:=$	the componentwise square root of $w \in \mathbb{R}_+^V$, i.e., $[\sqrt{w}]_i := \sqrt{w_i}$ for every $i \in V$
$x \oplus y$	$:=$	the direct sum of vectors $x \in \mathbb{R}^V$ and $y \in \mathbb{R}^W$
$\text{supp}(x)$	$:=$	$\{i \in V : x_i \neq 0\}$, the <i>support</i> of $x \in \mathbb{R}^V$
$x \odot y$	$:=$	the <i>Hadamard product</i> of $x, y \in \mathbb{R}^V$, i.e., $[x \odot y]_i := x_i y_i$ for every $i \in V$

Table 3: Notation for Convex Analysis, mostly following [41].

\mathbb{E}	$:=$	an <i>Euclidean space</i> , i.e., a finite-dimensional real vector space equipped with an Euclidean inner-product
\mathbb{E}^*	$:=$	the <i>dual space</i> of \mathbb{E}
\mathbb{K}^*	$:=$	$\{y \in \mathbb{E}^* : \langle x, y \rangle \geq 0 \forall x \in \mathbb{K}\}$, the <i>dual cone</i> of a convex cone $\mathbb{K} \subseteq \mathbb{E}$
$\delta^*(w \mathcal{C})$	$:=$	$\sup\{\langle w, x \rangle : x \in \mathcal{C}\}$, the <i>support function</i> of $\mathcal{C} \subseteq \mathbb{E}$ defined for each $w \in \mathbb{E}^*$
$\text{abl}(\mathcal{C})$	$:=$	$\{y \in \mathbb{R}_+^V : \langle y, x \rangle \leq 1 \forall x \in \mathcal{C}\}$, the <i>antiblocker</i> of $\mathcal{C} \subseteq \mathbb{R}_+^V$
$\text{cl}(\mathcal{C})$	$:=$	the <i>closure</i> of $\mathcal{C} \subseteq \mathbb{E}$
$\text{int}(\mathcal{C})$	$:=$	the <i>interior</i> of $\mathcal{C} \subseteq \mathbb{E}$
$\text{Aut}(\mathcal{C})$	$:=$	$\{\mathcal{T} : \mathbb{E} \rightarrow \mathbb{E} : \mathcal{T} \text{ nonsingular linear map, } \mathcal{T}(\mathcal{C}) = \mathcal{C}\}$, the <i>automorphism group</i> of $\mathcal{C} \subseteq \mathbb{E}$

We will often consider subsets of \mathbb{S}^V and $\mathbb{S}^{\{0\} \cup V}$. Subsets of the latter, as well as their elements, shall be decorated with a hat, e.g., $\hat{X} \in \hat{\mathbb{K}} \subseteq \mathbb{S}^{\{0\} \cup V}$; see Fig. 1 for related notational conventions. If $\mathbb{K} \subseteq \mathbb{E}$ is a pointed closed convex cone with nonempty interior, then \mathbb{K} defines a partial order $\succeq_{\mathbb{K}}$ on \mathbb{E} : we write $x \succeq_{\mathbb{K}} y$ to mean that $x - y \in \mathbb{K}$. We set $\succeq := \succeq_{\mathbb{S}_+^V}$.

Elements	Sets	Spaces
$\hat{X} = \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix}$	$\in \hat{\text{TH}}(\mathbb{A}, \hat{\mathbb{K}}) \subseteq \hat{\mathbb{K}} \subseteq \mathbb{S}^{\{0\} \cup V}$	
X	$\in \mathbb{A}, \mathbb{A}_{E^+, E^-} \subseteq \mathbb{S}^V$	
x	$\in \text{TH}(\mathbb{A}, \hat{\mathbb{K}}) \subseteq \mathbb{R}_+^V$	

Fig. 1: We denote elements of \mathbb{R}^V by lowercase letters and symmetric $V \times V$ matrices by uppercase letters. We denote symmetric matrices with index set $\{0\} \cup V$ by a hatted uppercase letter, with analogous versions of that same letter in the corresponding block decomposition.

$$\begin{array}{c}
\hat{\text{TH}}(G) \subseteq \mathbb{S}_+^{\{0\} \cup V} \longleftrightarrow \hat{\text{TH}}(\bar{G}) \subseteq \mathbb{S}_+^{\{0\} \cup V} \\
\mathbb{S}_+^{\{0\} \cup V} \longleftrightarrow \mathbb{S}_+^{\{0\} \cup V} \\
G \longleftrightarrow \bar{G} \\
\text{TH}(G) \subseteq \mathbb{R}_+^V \longleftrightarrow \underbrace{\text{TH}(\bar{G})}_{\text{abl}(\text{TH}(G))} \subseteq \mathbb{R}_+^V \\
\text{(a)}
\end{array}
\quad \Bigg| \quad
\begin{array}{c}
\hat{\text{TH}}(\mathbb{A}, \text{Psd}(\mathbb{K})) \subseteq \mathbb{S}^{\{0\} \cup V} \longleftrightarrow \hat{\text{TH}}(\bar{\mathbb{A}}, \text{Schur}(\mathbb{K}^*)) \subseteq \mathbb{S}^{\{0\} \cup V} \\
\text{Psd}(\mathbb{K}) \subseteq \mathbb{S}_+^{\{0\} \cup V} \xleftrightarrow[\text{duality}]{\text{cone}} \text{Schur}(\mathbb{K}^*) \supseteq \mathbb{S}_+^{\{0\} \cup V} \\
\mathbb{K} \subseteq \mathbb{S}_+^V \xleftrightarrow[\text{duality}]{\text{cone}} \mathbb{K}^* \supseteq \mathbb{S}_+^V \\
\mathbb{A} = \mathbb{A}_{E^+, E^-} \subseteq \mathbb{S}^V \xleftrightarrow[\text{duality}]{\bar{\cdot}} \bar{\mathbb{A}} = \mathbb{A}_{\bar{E}^+, \bar{E}^-} \subseteq \mathbb{S}^V \\
\text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})) \subseteq \mathbb{R}_+^V \xleftrightarrow[\text{duality}]{\text{abl}(\cdot)} \text{cl}(\text{TH}(\bar{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))) \subseteq \mathbb{R}_+^V \\
\text{(b)}
\end{array}$$

Fig. 2: Part (a) shows the previously known relations involving the theta body of a graph and its complement, as well as their building blocks. The operation in all arrows is graph complementation; the only known corresponding duality operation was antiblocking duality, at the bottom row. Part (b) describes a more complete picture unraveled by our framework, which uncovers the appropriate notions of duality at every level. These duality relations are proved en route to Theorem 14.

2 Theta Bodies

For each $h \in \mathbb{R}^V$, define the *diagonal scaling map* $\mathcal{D}_h: \mathbb{R}^{V \times V} \rightarrow \mathbb{R}^{V \times V}$ as

$$\mathcal{D}_h(X) := \text{Diag}(h)X\text{Diag}(h) \quad \forall X \in \mathbb{R}^{V \times V}.$$

Note that each entry $[\mathcal{D}_h(X)]_{ij}$ is the componentwise product $X_{ij}[hh^\top]_{ij}$ for every $X \in \mathbb{S}^V$ and $h \in \mathbb{R}^V$. A subset \mathbb{K} of \mathbb{S}^V is called *diagonally scaling-invariant* if $\mathcal{D}_h(\mathbb{K}) \subseteq \mathbb{K}$ for every $h \in \mathbb{R}_+^V$. The cones $\mathbb{S}_+^V, \mathbb{S}_{\geq 0}^V$,

$\mathcal{C}_V, \mathcal{C}_V^*$ are all examples of diagonally scaling-invariant subsets of \mathbb{S}^V . Some other important examples are the sets of the form

$$\mathbb{A}_{E^+, E^-} := \{X \in \mathbb{S}^V : X_{ij} \geq 0 \forall ij \in E^+, X_{ij} \leq 0 \forall ij \in E^-\}, \quad (7)$$

where $E^+, E^- \subseteq \binom{V}{2}$. Clearly, every diagonally scaling-invariant set is a cone, and since the map \mathcal{D}_h is self-adjoint, diagonal scaling invariance is preserved under duality. Moreover, if $\mathbb{K} \subseteq \mathbb{S}^V$ is diagonally scaling-invariant, then $\{\mathcal{D}_h : h \in \mathbb{R}_{++}^V\} \subseteq \text{Aut}(\mathbb{K})$.

For sets $\mathbb{A} \subseteq \mathbb{S}^V$ and $\widehat{\mathbb{K}} \subseteq \mathbb{S}^{\{0\} \cup V}$, define

$$\widehat{\text{TH}}(\mathbb{A}, \widehat{\mathbb{K}}) := \left\{ \hat{X} \in \widehat{\mathbb{K}} : \hat{X}_{00} = 1, \hat{X}e_0 = \text{diag}(\hat{X}), \hat{X}[V] \in \mathbb{A} \right\} \quad (8)$$

and

$$\text{TH}(\mathbb{A}, \widehat{\mathbb{K}}) := \left\{ \text{diag}(\hat{X}[V]) : \hat{X} \in \widehat{\text{TH}}(\mathbb{A}, \widehat{\mathbb{K}}) \right\}.$$

We are interested in sets of the form $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$, where \mathbb{A} and $\widehat{\mathbb{K}}$ are diagonally scaling-invariant convex cones with a few extra properties. The most important known examples of sets of this form are the theta body $\text{TH}(G)$ of a graph $G = (V, E)$ and its variants $\text{TH}'(G)$ and $\text{TH}^+(G)$. In fact, we define

$$\begin{aligned} \text{TH}(G) &:= \text{TH}(\mathbb{A}_{E, E}, \mathbb{S}_+^{\{0\} \cup V}), \\ \text{TH}'(G) &:= \text{TH}(\mathbb{A}_{E \cup \bar{E}, E}, \mathbb{S}_+^{\{0\} \cup V}), \\ \text{TH}^+(G) &:= \text{TH}(\mathbb{A}_{\emptyset, E}, \mathbb{S}_+^{\{0\} \cup V}). \end{aligned} \quad (9)$$

It thus makes sense to call sets of the form $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ as *theta bodies* (the terminology “theta bodies” was also used for another generalization of the theta body by [21]; our definition and approach are very different). To avoid confusion, whenever we refer to the specific theta body $\text{TH}(G)$, we shall call it *the* theta body of G . In the remainder of this section we shall prove that, under certain simple hypotheses, every theta body is a *convex corner*, i.e., a compact, lower-comprehensive convex subset of the nonnegative orthant with nonempty interior. Recall that a subset \mathcal{C} of \mathbb{R}_+^V is called *lower-comprehensive* if, for any $x, y \in \mathbb{R}_+^V$, the chain of relations $0 \leq y \leq x \in \mathcal{C}$ implies $y \in \mathcal{C}$. In what follows, the extra hypotheses (10) and (11) on \mathbb{A} and $\widehat{\mathbb{K}}$ may be thought of as requiring that \mathbb{A} is not “too small”, and that $\widehat{\mathbb{K}}$ is neither “too small” nor “too big.”

Proposition 1 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ and $\widehat{\mathbb{K}} \subseteq \mathbb{S}^{\{0\} \cup V}$ be diagonally scaling-invariant closed convex cones. Suppose that \mathbb{A} satisfies*

$$\text{Im}(\text{Diag}) \subseteq \mathbb{A}, \quad (10)$$

and suppose that $\widehat{\mathbb{K}}$ satisfies

$$\widehat{\mathbb{K}} \supseteq \left\{ (e_0 + e_i)(e_0 + e_i)^\top : i \in V \right\} \quad (11a)$$

and

$$\text{diag}(\{ \hat{X} \in \widehat{\mathbb{K}} : \hat{X}_{00} = 1, \hat{X}e_0 = \text{diag}(\hat{X}) \}) \subseteq [0, 1]^{\{0\} \cup V}. \quad (11b)$$

Then $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ is a convex lower-comprehensive subset of $[0, 1]^V$ with nonempty interior. In particular, $\text{cl}(\text{TH}(\mathbb{A}, \widehat{\mathbb{K}}))$ is a convex corner.

Proof Convexity of the projection $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ follows from that of $\widehat{\text{TH}}(\mathbb{A}, \widehat{\mathbb{K}})$. It is clear from (11b) that $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}}) \subseteq [0, 1]^V$. To prove that the convex set $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ is lower-comprehensive, it suffices to show that if $x \in \text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ then $x - x_i e_i \in \text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ for each $i \in V$. Let $\hat{X} \in \widehat{\text{TH}}(\mathbb{A}, \widehat{\mathbb{K}})$ such that $x = \text{diag}(\hat{X}[V])$. Let $i \in V$. Set $\hat{Y} := \mathcal{D}_{1 \oplus h}(\hat{X}) \in \widehat{\mathbb{K}}$ for $h := \mathbb{1} - e_i \in \mathbb{R}^V$. Then diagonal scaling invariance of \mathbb{A} and $\widehat{\mathbb{K}}$ imply that $\hat{Y} \in \widehat{\text{TH}}(\mathbb{A}, \widehat{\mathbb{K}})$. Thus, $x - x_i e_i = \text{diag}(\hat{Y}[V]) \in \text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$. This proves that $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ is lower-comprehensive. It remains to show that $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ has nonempty interior. Let $i \in V$. By (10), we have $e_i e_i^\top \in \mathbb{A}$. Thus, $(e_0 + e_i)(e_0 + e_i)^\top \in \widehat{\text{TH}}(\mathbb{A}, \widehat{\mathbb{K}})$ by (11a) whence $e_i \in \text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$. Now convexity of $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ implies that $\frac{1}{n} \mathbb{1} \in \text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$, where $n := |V|$. Since $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ is lower-comprehensive, we find that $\frac{1}{2n} \mathbb{1} \in \text{int}(\text{TH}(\mathbb{A}, \widehat{\mathbb{K}}))$. \square

The reason for using the hypothesis (10) rather than the slightly weaker condition $\text{Diag}(\mathbb{R}_+^V) \subseteq \mathbb{A}$ shall be made clearer in Section 3, where we develop a notion of duality for such cones. As for the closedness of theta bodies, we are not aware of any example of a theta body that is not closed, so the closure operator in Proposition 1 shall remain as a minor nuisance. We shall now see that, under a mild condition on the cone $\widehat{\mathbb{K}}$, the theta body $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ is actually closed, and hence a convex corner itself.

Corollary 2 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ and $\widehat{\mathbb{K}} \subseteq \mathbb{S}^{\{0\} \cup V}$ be diagonally scaling-invariant closed convex cones such that (10) and (11a) hold. If*

$$\widehat{\mathbb{K}} \subseteq \left\{ \hat{X} \in \mathbb{S}^{\{0\} \cup V} : \hat{X}[S] \succeq 0, \forall S \in \binom{\{0\} \cup V}{2} \right\}, \quad (12)$$

then (11b) holds and $\widehat{\text{TH}}(\mathbb{A}, \widehat{\mathbb{K}})$ is compact. In particular, $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ is closed, and hence a convex corner.

Proof Let $\widehat{\mathbb{M}}$ be the set of all \hat{X} in the RHS of (12) such that $\hat{X}_{00} = 1$ and $\hat{X}e_0 = \text{diag}(\hat{X})$. Then $\widehat{\mathbb{M}}$ is bounded. To see this, first use sets $S \in \binom{\{0\} \cup V}{2}$ containing 0 to show that $\text{diag}(\widehat{\mathbb{M}}) \subseteq [0, 1]^{\{0\} \cup V}$. Thus, (11b) holds. Next, use (12) with sets $S \in \binom{V}{2}$ to show that all off-diagonal entries of $\hat{X} \in \widehat{\mathbb{M}}$ have absolute value bounded above by 1. Since $\widehat{\text{TH}}(\mathbb{A}, \widehat{\mathbb{K}}) \subseteq \widehat{\mathbb{M}}$, it follows that $\widehat{\text{TH}}(\mathbb{A}, \widehat{\mathbb{K}})$ is compact. Now closedness of $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ follows from the fact that $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ is a linear image of the compact set $\widehat{\text{TH}}(\mathbb{A}, \widehat{\mathbb{K}})$. The rest follows from Proposition 1. \square

For most of the theta bodies in this paper, the cone $\widehat{\mathbb{K}}$ shall be a subset of $\mathbb{S}_+^{\{0\} \cup V}$, and hence (12) shall be satisfied; this already includes the theta bodies described in (9). An important diagonally scaling-invariant closed convex cone which does not satisfy (12) is the cone of copositive matrices; indeed, note that $\mathcal{C}_{\{0\} \cup V} \supseteq \mathbb{S}_{\geq 0}^{\{0\} \cup V}$. We shall deal with theta bodies arising from the copositive cone in Section 7, where we shall prove directly that the corresponding theta body is closed.

3 Polyhedral Diagonally Scaling-Invariant Cones

When studying a theta body $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$, we think of \mathbb{A} as an ‘‘elementary’’ cone, while $\widehat{\mathbb{K}}$ is (potentially) a ‘‘sophisticated’’ cone. In the most important instances of theta bodies, namely the ones described in (9), the cone \mathbb{A} is polyhedral, whereas $\widehat{\mathbb{K}}$ is the nonlinear cone $\mathbb{S}_+^{\{0\} \cup V}$. In general, it makes sense to focus on the case where \mathbb{A} is polyhedral. At any rate, when defining a theta body $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$, any trace of ‘‘non-polyhedrality’’ may be (and should be) ‘‘pushed’’ away from \mathbb{A} and into $\widehat{\mathbb{K}}$. We shall show next that requiring a closed convex cone to be both diagonally scaling-invariant and polyhedral severely constrains its structure.

We shall need a family of cones slightly more refined than the cones \mathbb{A}_{E^+, E^-} defined in (7). Let $V^+, V^- \subseteq V$ and $E^+, E^- \subseteq \binom{V}{2}$. Define

$$\mathbb{A}_{V^+, V^-, E^+, E^-} := \{ X \in \mathbb{A}_{E^+, E^-} : \text{diag}(X[V^+]) \geq 0, \text{diag}(X[V^-]) \leq 0 \}.$$

Clearly, every set of this form is diagonally scaling-invariant and polyhedral. In fact, every polyhedral diagonally scaling-invariant cone is of this form:

Proposition 3 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant closed convex cone. If \mathbb{A} is polyhedral, then \mathbb{A} is of the form $\mathbb{A} = \mathbb{A}_{V^+, V^-, E^+, E^-}$ for some subsets $V^+, V^- \subseteq V$ and $E^+, E^- \subseteq \binom{V}{2}$.*

Proof Let $\text{Sym}: \mathbb{R}^{V \times V} \rightarrow \mathbb{S}^V$ denote the orthogonal projection onto \mathbb{S}^V , that is,

$$\text{Sym}(X) := \frac{1}{2}(X + X^T) \quad \forall X \in \mathbb{R}^{V \times V}. \quad (13)$$

It suffices to show that

$$\text{every extreme ray of } \mathbb{A}^* \text{ is of the form } \pm \mathbb{R}_+ \text{Sym}(e_i e_j^T) \text{ for some } i, j \in V. \quad (14)$$

We first show that,

$$\text{if } \mathbb{R}_+ X \text{ is an extreme ray of } \mathbb{A}^*, \text{ then } |\text{supp}(\text{diag}(X))| \leq 1. \quad (15)$$

Suppose that $\mathbb{R}_+ X$ is an extreme ray of \mathbb{A}^* such that $X_{ii} \neq 0 \neq X_{jj}$ for distinct $i, j \in V$. Since \mathbb{A}^* is also diagonally scaling-invariant, we have $\{\mathcal{D}_h : h \in \mathbb{R}_{++}^V\} \subseteq \text{Aut}(\mathbb{A}^*)$. For $t \in \mathbb{R}_{++}$, define $h(t) := t e_i + t^{-1} e_j + \mathbb{1}_{V \setminus \{i, j\}}$. Thus, $\{\mathbb{R}_+ \mathcal{D}_{h(t)}(X) : t \in \mathbb{R}_{++}\}$ is an infinite set of extreme rays of \mathbb{A}^* . This contradicts the fact that \mathbb{A}^* is polyhedral and thus proves (15).

To prove (14), let $\mathbb{R}_+ X$ be an extreme ray of \mathbb{A}^* . Let us show that

$$X_{ij} \text{Sym}(e_i e_j^T) \in \mathbb{A}^* \quad \forall i, j \in V. \quad (16)$$

Let $i, j \in V$. If $i = j$ then (16) holds by diagonal scaling invariance of \mathbb{A} , so assume $i \neq j$. By (15), at most one of X_{ii} and X_{jj} is nonzero. We may assume by symmetry that $X_{ii} = 0$. For $t \in \mathbb{R}_{++}$, define $h(t) := t e_i + t^{-1} e_j$ and note that $\mathcal{D}_{h(t)}(X) \in \mathbb{A}^*$ for every $t \in \mathbb{R}_{++}$. By driving t to ∞ we find that $2X_{ij} \text{Sym}(e_i e_j^T) = \lim_{t \rightarrow \infty} \mathcal{D}_{h(t)}(X)$ lies in the closed set \mathbb{A}^* . This proves (16).

Since

$$X = \sum_{i, j \in V} 2^{[i \neq j]} X_{ij} \text{Sym}(e_i e_j^T) \quad (17)$$

and $\mathbb{R}_+ X$ is an extreme ray of \mathbb{A}^* , it follows from (16) that at most one of the terms in the RHS of (17) is nonzero. This proves (14) and concludes the proof. \square

It follows that a polyhedral diagonally scaling-invariant cone $\mathbb{A} \subseteq \mathbb{S}^V$ such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$ must have the form shown in (7). We shall focus our attention on such cones for the first argument of $\widehat{\text{TH}}$ for the remainder of the paper.

We shall define some alternative notions of “duality” for the cones \mathbb{A} and $\widehat{\mathbb{K}}$ to describe more conveniently the antiblockers of theta bodies. The appropriate duality notion for the cones $\mathbb{A} \subseteq \mathbb{S}^V$ is defined as follows:

$$\overline{\mathbb{A}} := \text{Im}(\text{Diag}) - \mathbb{A}^*.$$

It is easy to check that, for every $E^+, E^- \subseteq \binom{V}{2}$, we have

$$\overline{\mathbb{A}_{E^+, E^-}} = \mathbb{A}_{\overline{E^+}, \overline{E^-}}$$

and, for every polyhedral diagonally scaling-invariant cone $\mathbb{A} \subseteq \mathbb{S}^V$, we have $\overline{\overline{\mathbb{A}}} = \mathbb{A} + \text{Im}(\text{Diag})$, so this operation is an involution when restricted to cones that contain $\text{Im}(\text{Diag})$.

4 Liftings of Cones

In this section, we define two operators that lift a cone in \mathbb{S}^V to a cone in $\mathbb{S}^{\{0\} \cup V}$, as we briefly mentioned in Subsection 1.1. Note that, in the definition (8), whenever we test membership of a matrix \hat{X} in $\widehat{\mathbb{K}}$, the 0th column of \hat{X} is completely determined by $\hat{X}[V]$. Thus, it makes some sense to define a lifting of a cone \mathbb{K} in \mathbb{S}^V as a cone in $\mathbb{S}^{\{0\} \cup V}$ in such a way that the 0th column is strongly related to \mathbb{K} . We shall need two such liftings, which are studied in the next two subsections. We shall also see a glimpse of the duality relation involving these two liftings in a Weak Duality theorem, as well as a natural description of the set $\text{FRAC}(G)$ (that appeared in (5)) as a theta body using one of the liftings.

4.1 PSD Liftings of Cones

In this subsection, we define the PSD lifting of a cone \mathbb{K} in \mathbb{S}^V and we prove that, under mild hypotheses, the support function of a theta body defined over the PSD lifting of \mathbb{K} can be formulated as a simple conic optimization problem over $\widehat{\mathbb{K}}$. This shall correspond to the equivalence $\vartheta_3 = \vartheta_4$ in (6).

Let $\mathbb{K} \subseteq \mathbb{S}^V$. Define the *PSD lifting* of \mathbb{K} as

$$\text{Psd}(\mathbb{K}) := \left\{ \hat{X} \in \mathbb{S}_+^{\{0\} \cup V} : \hat{X}[V] \in \mathbb{K} \right\}. \quad (18)$$

Note that if \mathbb{K} is diagonally scaling-invariant, then so is $\text{Psd}(\mathbb{K})$. Moreover, $\text{Psd}(\mathbb{S}_+^V) = \mathbb{S}_+^{\{0\} \cup V}$. Before using PSD liftings, we shall need the following straightforward weighted generalization of [19, Proposition 9].

Lemma 4 *Let $\mathbb{M} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant closed convex cone. Suppose that*

$$\text{diag}(\mathbb{M}) \subseteq \mathbb{R}_+^V, \quad (19a)$$

$$\text{if } X_{ii} = 0 \text{ for some } X \in \mathbb{M} \text{ and } i \in V, \text{ then } X e_i = 0, \quad (19b)$$

$$\{X \in \mathbb{M} : \text{Tr}(X) = 1\} \text{ is compact.} \quad (19c)$$

Let $w \in \mathbb{R}_+^V$. Let X^* be an optimal solution of

$$\max \left\{ \sqrt{w}^\top X \sqrt{w} : \text{Tr}(X) = 1, X \in \mathbb{M} \right\}, \quad (20)$$

and suppose that $\sqrt{w}^\top X^* \sqrt{w} > 0$. Set

$$d := \text{diag}(X^*),$$

$$\bar{X} := \text{Diag}(\sqrt{d})^\dagger X^* \text{Diag}(\sqrt{d})^\dagger,$$

$$\lambda := \lambda_{\max}(\mathcal{D}_{\sqrt{w}}(\bar{X})).$$

Then

$$\text{supp}(d) \subseteq \text{supp}(w), \quad (21a)$$

$$\mathcal{D}_{\sqrt{w}}(\bar{X}) \sqrt{d} = \lambda \sqrt{d}, \quad (21b)$$

$$\lambda = \sqrt{w}^\top X^* \sqrt{w}, \quad (21c)$$

$$X^* \sqrt{w} = \lambda \text{Diag}(\sqrt{w})^\dagger d. \quad (21d)$$

Proof We first show (21a). Let $i \in \text{supp}(d)$, so that $X_{ii}^* > 0$. Suppose that $w_i = 0$. If $X_{ii}^* = 1$, then $X^* = e_i e_i^\top$ by (19) whence $\sqrt{w}^\top X^* \sqrt{w} = 0$. If $X_{ii}^* < 1$, then $(1 - X_{ii}^*)^{-1} \mathcal{D}_{\mathbb{1} - e_i}(X^*)$ is feasible for (20) with objective value $(1 - X_{ii}^*)^{-1} \sqrt{w}^\top X^* \sqrt{w}$, hence strictly larger than the objective value of X^* . In either case, we get a contradiction. This proves (21a).

If $d_i = 0$ for some $i \in V$, we are done by induction on $|V|$. Thus, from (21a) we may assume that

$$\text{supp}(d) = \text{supp}(w) = V.$$

Define $d^{-1/2} \in \mathbb{R}^V$ such that $d^{-1/2} \odot \sqrt{d} = \mathbb{1}$, so $\bar{X} = \mathcal{D}_{d^{-1/2}}(X^*) \in \mathbb{M}$ and $\text{diag}(\bar{X}) = \mathbb{1}$. For every $h \in \mathbb{R}_+^V$ with $\|h\| = 1$, the point $\mathcal{D}_h(\bar{X})$ is feasible for (20) with objective value $\sqrt{w}^\top \mathcal{D}_h(\bar{X}) \sqrt{w} = h^\top \mathcal{D}_{\sqrt{w}}(\bar{X}) h$. Since $X^* = \mathcal{D}_{\sqrt{d}}(\bar{X})$ is optimal for the optimization problem (20), it follows that \sqrt{d} is an optimal solution for $\max\{h^\top \mathcal{D}_{\sqrt{w}}(\bar{X}) h : h \in \mathbb{R}_+^V, \|h\| = 1\}$. In fact, since $[\sqrt{d}]_i > 0$ for all $i \in V$, we find that \sqrt{d} is a local optimal solution for $\max\{h^\top \mathcal{D}_{\sqrt{w}}(\bar{X}) h : h \in \mathbb{R}^V, \|h\| = 1\}$, hence also a global one (note that the sign of h is unconstrained here). Thus, $\mathcal{D}_{\sqrt{w}}(\bar{X}) \sqrt{d} = \lambda \sqrt{d}$. This proves (21b). Now we unroll:

$$\begin{aligned} \lambda d &= \lambda \text{Diag}(\sqrt{d}) \sqrt{d} = \text{Diag}(\sqrt{d}) \mathcal{D}_{\sqrt{w}}(\mathcal{D}_{d^{-1/2}}(X^*)) \sqrt{d} = \text{Diag}(\sqrt{d}) \mathcal{D}_{d^{-1/2}}(\mathcal{D}_{\sqrt{w}}(X^*)) \sqrt{d} \\ &= \text{Diag}(\sqrt{d}) \text{Diag}(d^{-1/2}) \mathcal{D}_{\sqrt{w}}(X^*) \text{Diag}(d^{-1/2}) \sqrt{d} = \text{Diag}(\sqrt{w}) X^* \text{Diag}(\sqrt{w}) \mathbb{1} = \text{Diag}(\sqrt{w}) X^* \sqrt{w}. \end{aligned}$$

This proves (21d). Finally, $\lambda = \lambda \text{Tr}(X^*) = \lambda \mathbb{1}^\top d = \mathbb{1}^\top \text{Diag}(\sqrt{w}) X^* \sqrt{w} = \sqrt{w}^\top X^* \sqrt{w}$ so (21c) is proved. \square

We can now show that the support function of some theta bodies of the form $\text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K}))$, which shall correspond to ϑ_4 in (6), may be formulated as a conic optimization problem over \mathbb{K} ; the latter shall correspond to ϑ_3 from (6). Note that the next result does not make use of Duality Theory.

Theorem 5 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ and $\mathbb{K} \subseteq \mathbb{S}_+^V$ be diagonally scaling-invariant closed convex cones such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$ and $\text{Diag}(\mathbb{R}_+^V) \subseteq \mathbb{K}$. Let $w \in \mathbb{R}_+^V$. Then*

$$\delta^*(w \mid \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K}))) = \max \left\{ \langle \sqrt{w} \sqrt{w}^\top, X \rangle : \langle I, X \rangle = 1, X \in \mathbb{A}, X \in \mathbb{K} \right\}. \quad (22)$$

Moreover, both optimization problems in (22) have optimal solutions.

Proof We begin by proving ‘ \leq ’. Let $y \in \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K}))$ and let $\hat{Y} \in \widehat{\text{TH}}(\mathbb{A}, \text{Psd}(\mathbb{K}))$ such that $y = \text{diag}(Y)$ for $Y := \hat{Y}[V]$. We will show that there exists a feasible solution X for the RHS of (22) with objective value at least $\langle w, y \rangle$. We may assume that $\langle w, y \rangle > 0$; otherwise, take $X = e_i e_i^\top$ for any $i \in V$. Set $h := \langle w, y \rangle^{-1/2} \sqrt{w} \geq 0$ and $X := \mathcal{D}_h(Y) \in \mathbb{A} \cap \mathbb{K}$. Then $\text{Tr}(X) = \langle w, y \rangle^{-1} \text{Tr}(\mathcal{D}_{\sqrt{w}}(Y)) = \langle w, y \rangle^{-1} \langle \sqrt{w} \odot \sqrt{w}, \text{diag}(Y) \rangle = 1$, whence X is feasible on the RHS of (22). Moreover,

$$\begin{bmatrix} 1 & (h \odot y)^\top \\ h \odot y & X \end{bmatrix} = \mathcal{D}_{\mathbb{1} \oplus h} \left(\begin{bmatrix} 1 & y^\top \\ y & Y \end{bmatrix} \right) \in \mathcal{D}_{\mathbb{1} \oplus h}(\text{Psd}(\mathbb{K})) \subseteq \text{Psd}(\mathbb{K}) \subseteq \mathbb{S}_+^{\{0\} \cup V}.$$

Thus, by Schur complement, we get $X \succeq \langle w, y \rangle^{-1} (\sqrt{w} \odot y) (\sqrt{w} \odot y)^\top$ and so

$$\sqrt{w}^\top X \sqrt{w} \geq \langle w, y \rangle^{-1} \sqrt{w}^\top (\text{Diag}(\sqrt{w}) y) (\text{Diag}(\sqrt{w}) y)^\top \sqrt{w} = \frac{1}{\langle w, y \rangle} \langle w, y \rangle^2.$$

This completes the proof of ‘ \leq ’.

Now we prove ‘ \geq ’. For that, we will show that,

$$\text{if } X \in \mathbb{A} \cap \mathbb{K} \text{ and } X\sqrt{w} \geq 0, \text{ then } \sqrt{w}^\top X\sqrt{w} \leq [\text{Tr}(X)]\langle w, y \rangle \text{ for some } y \in \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})). \quad (23)$$

So, let $X \in \mathbb{A} \cap \mathbb{K}$ such that $X\sqrt{w} \geq 0$. We may assume that $\sqrt{w}^\top X\sqrt{w} > 0$; otherwise take $y = 0$. Since $X \in \mathbb{K} \subseteq \mathbb{S}_+^V$, there exists $B \in \mathbb{R}^{V \times V}$ such that $X = B^\top B$. Define

$$\begin{aligned} c &:= (\sqrt{w}^\top X\sqrt{w})^{-1/2} B\sqrt{w}, \\ d &:= \text{diag}(X), \\ \tilde{B} &:= B[\text{Diag}(\sqrt{d})]^\dagger, \\ \bar{B} &:= \tilde{B} \text{Diag}(\tilde{B}^\top c), \\ y &:= \bar{B}^\top c = \text{Diag}(\tilde{B}^\top c)\tilde{B}^\top c = (\tilde{B}^\top c) \odot (\tilde{B}^\top c). \end{aligned}$$

We will show that

$$y \in \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})). \quad (24)$$

Set $Y := \bar{B}^\top \bar{B}$ and note that

$$\hat{Y} := \begin{bmatrix} 1 & y^\top \\ y & Y \end{bmatrix} = \begin{bmatrix} 1 & c^\top \bar{B} \\ \bar{B}^\top c & \bar{B}^\top \bar{B} \end{bmatrix} = \begin{bmatrix} c^\top \\ \bar{B}^\top \end{bmatrix} \begin{bmatrix} c & \bar{B} \end{bmatrix} \in \text{Psd}(\mathbb{K});$$

to see that Y lies in $\mathbb{K} \cap \mathbb{A}$, note that $Y = \mathcal{D}_h(X)$ for some $h \geq 0$ since

$$\tilde{B}^\top c \geq 0,$$

which follows from $(\sqrt{w}^\top X\sqrt{w})^{1/2} \tilde{B}^\top c = [\text{Diag}(\sqrt{d})]^\dagger B^\top B\sqrt{w} = [\text{Diag}(\sqrt{d})]^\dagger X\sqrt{w} \geq 0$. Finally,

$$\begin{aligned} \text{diag}(Y) &= \text{diag}(\bar{B}^\top \bar{B}) = \text{diag}\left(\text{Diag}(\tilde{B}^\top c)\tilde{B}^\top \tilde{B} \text{Diag}(\tilde{B}^\top c)\right) \\ &= (\tilde{B}^\top c) \odot \text{diag}(\tilde{B}^\top \tilde{B}) \odot (\tilde{B}^\top c) \\ &= (\tilde{B}^\top c) \odot \text{diag}\left([\text{Diag}(\sqrt{d})]^\dagger B^\top B [\text{Diag}(\sqrt{d})]^\dagger\right) \odot (\tilde{B}^\top c) \\ &= (\tilde{B}^\top c) \odot \mathbb{1}_{\text{supp}(d)} \odot (\tilde{B}^\top c) = (\tilde{B}^\top c) \odot (\tilde{B}^\top c) = y, \end{aligned}$$

where we used for the second-to-last equation the fact $(\tilde{B}^\top c)_j = e_j^\top \tilde{B}^\top c = e_j^\top [\text{Diag}(\sqrt{d})]^\dagger B^\top c = 0^\top B^\top c = 0$ which is implied by $d_j = 0$. Thus, $\hat{Y} \in \widehat{\text{TH}}(\mathbb{A}, \text{Psd}(\mathbb{K}))$, and (24) is proved.

We also have $\tilde{B} \text{Diag}(\sqrt{d}) = B[\text{Diag}(\sqrt{d})]^\dagger \text{Diag}(\sqrt{d}) = B \text{Diag}(\mathbb{1}_{\text{supp}(d)}) = B$ since $d_i = 0$ implies $Be_i = 0$. Thus,

$$\begin{aligned} \sqrt{w}^\top X\sqrt{w} &= \left(\frac{\sqrt{w}^\top B^\top B\sqrt{w}}{(\sqrt{w}^\top X\sqrt{w})^{1/2}} \right)^2 = \left(\sqrt{w}^\top B^\top c \right)^2 \\ &= \left(\sqrt{w}^\top \text{Diag}(\sqrt{d})\tilde{B}^\top c \right)^2 = \left(\sqrt{d}^\top \text{Diag}(\sqrt{w})\sqrt{y} \right)^2 \\ &\leq \|\sqrt{d}\|^2 \|\text{Diag}(\sqrt{w})\sqrt{y}\|^2 = [\text{Tr}(X)]\langle w, y \rangle. \end{aligned}$$

This completes the proof of (23).

Let X be an optimal solution for the RHS of (22); the latter set is compact since it is a closed subset of $\{X \in \mathbb{S}_+^V : \text{Tr}(X) = 1\}$. By Lemma 4, we have $X\sqrt{w} \geq 0$. Thus, $\delta^*(w | \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K}))) \geq \sqrt{w}^T X \sqrt{w}$ by (23) and the proof of ‘ \geq ’ is complete.

The latter paragraph showed that the RHS is attained. Attainment of the LHS also follows by compactness by Corollary 2. \square

4.2 Schur Liftings of Cones

In this subsection, we define a lifting operator that is, in a sense, dual to the PSD lifting introduced in the previous subsection. After proving that the Schur lifting of certain convex cones are also convex, we shall prove a Weak Duality result relating both liftings, and then we shall describe $\text{FRAC}(G)$ as a theta body over the Schur lifting of a simple cone.

Let $\mathbb{K} \subseteq \mathbb{S}^V$. Define the *Schur lifting* of \mathbb{K} as

$$\text{Schur}(\mathbb{K}) := \left\{ \begin{bmatrix} x_0 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{\{0\} \cup V} : X \in \mathbb{K}, x_0 \in \mathbb{R}_+, x_0 X \succeq_{\mathbb{K}} x x^T \right\}.$$

Note that $\text{Schur}(\mathbb{S}_+^V) = \mathbb{S}_+^{\{0\} \cup V}$. It is instructive to rewrite the PSD lifting $\text{Psd}(\mathbb{K})$ in the following format similar to $\text{Schur}(\mathbb{K})$:

$$\text{if } \mathbb{K} \subseteq \mathbb{S}_+^V, \text{ then } \text{Psd}(\mathbb{K}) = \left\{ \begin{bmatrix} x_0 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{\{0\} \cup V} : X \in \mathbb{K}, x_0 \in \mathbb{R}_+, x_0 X \succeq x x^T \right\};$$

note the difference in the last (conic) inequality.

Whereas the expression (18) makes it clear that the PSD lifting of a closed convex cone is convex, the same can not be said about the Schur lifting. We shall now show that, under certain simple conditions, the Schur lifting of a convex cone is also convex, and in fact it satisfies the properties (11) of the cone $\widehat{\mathbb{K}}$ in Proposition 1:

Theorem 6 *Let $\mathbb{K} \subseteq \mathbb{S}^V$ be a closed convex cone such that $\mathbb{K} \supseteq \mathbb{S}_+^V$ and $\text{diag}(\mathbb{K}) \subseteq \mathbb{R}_+^V$. Then $\text{Schur}(\mathbb{K})$ is a closed convex cone that satisfies (11); if moreover \mathbb{K} is diagonally scaling-invariant, then so is $\text{Schur}(\mathbb{K})$. In particular, if $\mathbb{A} \subseteq \mathbb{S}^V$ is a diagonally scaling-invariant closed convex cone such that (10) holds, then $\text{cl}(\text{TH}(\mathbb{A}, \text{Schur}(\mathbb{K})))$ is a convex corner contained in $[0, 1]^V$.*

Proof Closedness of $\text{Schur}(\mathbb{K})$ follows from that of \mathbb{K} . Using $\text{diag}(\mathbb{K}) \subseteq \mathbb{R}_+^V$, it is not hard to check that

$$\text{Schur}(\mathbb{K}) = \text{cl}(\widehat{\mathbb{M}}), \tag{25}$$

where

$$\widehat{\mathbb{M}} := \left\{ \begin{bmatrix} x_0 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{\{0\} \cup V} : X \in \mathbb{K}, x_0 \in \mathbb{R}_{++}, x_0 X \succeq_{\mathbb{K}} x x^T \right\}.$$

It is obvious that $\text{Schur}(\mathbb{K})$ is a cone. We shall prove that $\text{Schur}(\mathbb{K})$ is convex by showing that

$$\widehat{\mathbb{M}} \text{ is convex.} \tag{26}$$

Since

$$\widehat{\mathbb{M}} = \left\{ \begin{bmatrix} x_0 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{\{0\} \cup V} : X \in \mathbb{K}, x_0 \in \mathbb{R}_{++}, \langle H, x_0 X - x x^T \rangle \geq 0 \forall H \in \mathbb{K}^* \right\},$$

it suffices to show that, for each $H \in \mathbb{K}^*$,

$$\text{the function } f_H : x_0 \oplus x \in \mathbb{R}_{++} \oplus \mathbb{R}^V \mapsto \frac{x^\top H x}{x_0} \text{ is convex.} \quad (27)$$

Let $H \in \mathbb{K}^*$. The Hessian of f_H is

$$\nabla^2 f_H(x_0 \oplus x) = \frac{2}{x_0^2} \begin{bmatrix} x^\top H x / x_0 & -(Hx)^\top \\ -Hx & x_0 H \end{bmatrix}.$$

From the hypothesis that $\mathbb{K} \supseteq \mathbb{S}_+^V$ we get $\mathbb{K}^* \subseteq \mathbb{S}_+^V$ whence $H \succeq 0$, so we may write $H = \sum_{h \in \mathcal{H}} h h^\top$ for a finite subset \mathcal{H} of \mathbb{R}^V . For $u := x_0^{1/2} \oplus x_0^{-1/2} \mathbb{1} \in \mathbb{R}_{++}^{\{0\}} \oplus \mathbb{R}_{++}^V$, we have

$$\frac{x_0^2}{2} \mathcal{D}_u(\nabla^2 f_H(x_0 \oplus x)) = \sum_{h \in \mathcal{H}} \mathcal{D}_{(h,x) \oplus \mathbb{1}} \left(\begin{bmatrix} 1 & -h^\top \\ -h & h h^\top \end{bmatrix} \right) \succeq 0.$$

Thus, $\nabla^2 f_H(x_0 \oplus x) \succeq 0$, and this concludes the proofs of (27) and (26). Therefore, $\text{Schur}(\mathbb{K})$ is convex by (25).

Let

$$\hat{X} := \begin{bmatrix} x_0 & x^\top \\ x & X \end{bmatrix} \in \text{Schur}(\mathbb{K}),$$

and let $h_0 \oplus h \in \mathbb{R}_+^{\{0\}} \oplus \mathbb{R}_+^V$. The condition $\mathcal{D}_{h_0 \oplus h}(\hat{X}) \in \text{Schur}(\mathbb{K})$ is equivalent to $\mathcal{D}_h(X) \in \mathbb{K}$ and $h_0^2 x_0 \mathcal{D}_h(X) \succeq_{\mathbb{K}} h_0^2 \mathcal{D}_h(x x^\top)$, both of which follow from the diagonal scaling invariance of \mathbb{K} . It is easy to check that $\text{Schur}(\mathbb{K})$ satisfies (11a). For (11b), note that $x - (x \odot x) = \text{diag}(X - x x^\top) \geq 0$ since $\text{diag}(\mathbb{K}) \subseteq \mathbb{R}_+^V$ whence $x \subseteq [0, 1]^V$. This completes the proof that (11) holds. The remainder of the statement of the theorem follows from Proposition 1. \square

The hypothesis that $\mathbb{K} \supseteq \mathbb{S}_+^V$ holds cannot be dropped from Theorem 6. Consider the cone \mathcal{C}_V^* of completely positive matrices. Now take $V := \{1, \dots, n\}$ for some $n \geq 2$ and note that both $\mathbb{1}_{\{0,1\}} \mathbb{1}_{\{0,1\}}^\top + \mathbb{1}_{\{2\}} \mathbb{1}_{\{2\}}^\top$ and $\mathbb{1}_{\{0,2\}} \mathbb{1}_{\{0,2\}}^\top + \mathbb{1}_{\{1\}} \mathbb{1}_{\{1\}}^\top$ lie in $\text{Schur}(\mathcal{C}_V^*)$, whereas their midpoint does not.

Similar constructions of higher dimensional cones from lower dimensional cones exist in many other areas of mathematics and mathematical sciences. In addition to the obvious Schur complement connection, there are at least two other instances: one in the construction of *Siegel Domains* (see [24, 48] and the references therein to start), another in convex optimization and analysis in certain recursive quadratic reformulation of optimization problems (see [40, pp. 165–168]). In both of these constructions (which guarantee the convexity of the resulting cone), $x x^\top$ corresponds to the bilinear form. A key condition on the bilinear forms in both of these constructions, corresponds to the condition $x x^\top \in \mathbb{K}$ in our current context. Indeed, this last condition is equivalent to $\mathbb{K} \supseteq \mathbb{S}_+^V$.

PSD and Schur liftings of cones are in a sense dual to each other:

Proposition 7 *Let $\mathbb{K} \subseteq \mathbb{S}_+^V$ be a closed convex cone such that $\text{Diag}(\mathbb{R}_+^V) \subseteq \mathbb{K}$ and $\text{int}(\mathbb{K}) \neq \emptyset$. Then $[\text{Psd}(\mathbb{K})]^* = \text{Schur}(\mathbb{K}^*)$ and $[\text{Schur}(\mathbb{K}^*)]^* = \text{Psd}(\mathbb{K})$.*

Proof It suffices to prove the first equation. By definition,

$$\text{Psd}(\mathbb{K}) = \mathbb{S}_+^{\{0\} \cup V} \cap \{ \hat{X} \in \mathbb{S}^{\{0\} \cup V} : \hat{X}[V] \in \mathbb{K} \}.$$

The interiors of both cones on the RHS intersect. To see this, let $Y \in \text{int}(\mathbb{K})$ and note that, for $\varepsilon > 0$, the matrix $\varepsilon I + \begin{bmatrix} 0 & 0^\top \\ 0 & Y \end{bmatrix}$ lies in both interiors, since $Y + \varepsilon I \in \text{int}(\mathbb{K}) + \mathbb{K} \subseteq \text{int}(\mathbb{K})$. By standard convex analysis (see, e.g., [41, Corollary 16.4.2]),

$$[\text{Psd}(\mathbb{K})]^* = \mathbb{S}_+^{\{0\} \cup V} + \left\{ \begin{bmatrix} 0 & 0^\top \\ 0 & X \end{bmatrix} : X \in \mathbb{K}^* \right\} =: \widehat{\mathbb{K}}.$$

We claim that $\widehat{\mathbb{K}} = \text{Schur}(\mathbb{K}^*)$. Each summand in the definition of $\widehat{\mathbb{K}}$ is easily checked to be contained in $\text{Schur}(\mathbb{K}^*)$; since the latter cone is convex by Theorem 6, this proves that $\widehat{\mathbb{K}} \subseteq \text{Schur}(\mathbb{K}^*)$. For the reverse inclusion, let

$$\widehat{X} = \begin{bmatrix} x_0 & x^\top \\ x & X \end{bmatrix} \in \text{Schur}(\mathbb{K}^*).$$

If $x_0 = 0$, then $xx^\top \in \mathbb{K}^* \cap (-\mathbb{K}^*) = \{0\}$; the equation follows since $\text{int}(\mathbb{K}) \neq \emptyset$. Thus, $x = 0$, so $\widehat{X} \in \widehat{\mathbb{K}}$. If $x_0 > 0$, then

$$\widehat{X} = \begin{bmatrix} x_0 & x^\top \\ x & \frac{1}{x_0}xx^\top \end{bmatrix} + \begin{bmatrix} 0 & 0^\top \\ 0 & X - \frac{1}{x_0}xx^\top \end{bmatrix} \in \widehat{\mathbb{K}}.$$

This concludes the proof that $\widehat{\mathbb{K}} = \text{Schur}(\mathbb{K}^*)$. \square

In the next result, we show a containment relation between theta bodies defined using these two liftings. The relation may be regarded as a form of Weak Duality, and we shall later prove that equality, and hence a form of Strong Duality, holds.

Proposition 8 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant polyhedral cone such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Let $\mathbb{K} \subseteq \mathbb{S}_+^V$ be a diagonally scaling-invariant closed convex cone such that $\text{Diag}(\mathbb{R}_+^V) \subseteq \mathbb{K}$. Then*

$$\text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})) \subseteq \text{abl}(\text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))).$$

Proof By continuity, it suffices to show that $\langle x, y \rangle \leq 1$ if $x \in \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K}))$ and $y \in \text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))$. Let $x \in \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K}))$, and let $\widehat{X} \in \widehat{\text{TH}}(\mathbb{A}, \text{Psd}(\mathbb{K}))$ such that $x = \text{diag}(X)$ for $X := \widehat{X}[V]$. Let $y \in \text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))$, and let $\widehat{Y} \in \widehat{\text{TH}}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))$ such that $y = \text{diag}(Y)$ for $Y := \widehat{Y}[V]$. Write $Y = \text{Diag}(u) - B$ where $B \in \mathbb{A}^*$. Since $\mathbb{A}^* \subseteq \text{Null}(\text{diag})$, we have $u = y$. Then

$$\begin{aligned} 0 &\leq \langle X, Y - yy^\top \rangle = \langle X, \text{Diag}(u) - B \rangle - y^\top X y = \langle x, y \rangle - \langle X, B \rangle - y^\top X y \\ &\leq \langle x, y \rangle - y^\top (xx^\top) y = \langle x, y \rangle - \langle x, y \rangle^2. \end{aligned}$$

Hence, $\langle x, y \rangle \leq 1$. \square

We can now give an example of a natural theta body defined over the Schur lifting of a cone. Define the *weak fractional stable set polytope* of a graph $G = (V, E)$ as the polytope

$$\text{FRAC}(G) := \{x \in [0, 1]^V : x_i + x_j \leq 1 \forall ij \in E\}.$$

We shall prove that $\text{FRAC}(G)$ is a theta body over the Schur lifting of the convex cone

$$\mathbb{K}_2^V := \left\{ X \in \mathbb{S}^V : X[e] \succeq 0 \forall e \in \binom{V}{2} \right\};$$

note the similarity with the hypothesis of Corollary 2. We shall make essential use of Theorem 6 in our proof.

Theorem 9 Let $G = (V, E)$ be a graph such that $|V| \geq 2$. Then

$$\text{FRAC}(G) = \text{TH}(\mathbb{A}_{E,E}, \text{Schur}(\mathbb{K}_2^V)).$$

Proof We first prove ‘ \supseteq ’. Let $x \in \text{TH}(\mathbb{A}_{E,E}, \text{Schur}(\mathbb{K}_2^V))$, and let $\hat{X} \in \widehat{\text{TH}}(\mathbb{A}_{E,E}, \text{Schur}(\mathbb{K}_2^V))$ such that $x = \text{diag}(X)$ for $X := \hat{X}[V]$. By Theorem 6, we have $x \in [0, 1]^V$. Let $e = ij \in E$. Set $Y := X[e]$ and $y := x|_e$. Then $X \succeq_{\mathbb{K}_2^V} xx^\top$ implies $Y \succeq yy^\top$ so

$$\begin{bmatrix} 1 & x_i & x_j \\ x_i & x_i & 0 \\ x_j & 0 & x_j \end{bmatrix} \succeq 0 \implies \begin{bmatrix} 1 & -x_i & -x_j \\ -x_i & x_i & 0 \\ -x_j & 0 & x_j \end{bmatrix} \succeq 0 \implies 1 - x_i - x_j = \left\langle \begin{bmatrix} 1 & -x_i & -x_j \\ -x_i & x_i & 0 \\ -x_j & 0 & x_j \end{bmatrix}, \mathbb{1}\mathbb{1}^\top \right\rangle \geq 0.$$

Thus $x \in \text{FRAC}(G)$, and ‘ \supseteq ’ is proved.

For the reverse inclusion, it suffices by Theorem 6 to show that $\text{TH}(\mathbb{A}_{E,E}, \text{Schur}(\mathbb{K}_2^V))$ contains all the extreme points of $\text{FRAC}(G)$. So let x be an extreme point of $\text{FRAC}(G)$. By [43, Theorem 64.7], all coordinates of x lie in $\{0, \frac{1}{2}, 1\}$. Define

$$\hat{X} := \begin{bmatrix} \mathbb{1} & x^\top \\ x & X \end{bmatrix} \in \mathbb{S}^{\{0\} \cup V}$$

by setting $\text{diag}(X) := x$ and $X_{ij} := [ij \in \bar{E}] [x_i + x_j > 1] x_i x_j$ for every $ij \in \binom{V}{2}$. Note that $X \in \mathbb{A}_{E,E} \cap \mathbb{K}_2^V$ holds, and that $X \succeq_{\mathbb{K}_2^V} xx^\top$ is equivalent to

$$Y^{ij} := \begin{bmatrix} 1 & x_i & x_j \\ x_i & x_i & X_{ij} \\ x_j & X_{ij} & x_j \end{bmatrix} \in \mathbb{S}_+^{\{0\} \cup \{i,j\}},$$

for every $ij \in \binom{V}{2}$. So let $ij \in \binom{V}{2}$. If $x_i + x_j \leq 1$, then $X_{ij} = 0$ and either $0 \in \{x_i, x_j\}$ or $x_i = x_j = \frac{1}{2}$, so $Y^{ij} \succeq 0$ is easily verified. So assume $x_i + x_j > 1$. Then $ij \in \bar{E}$, so $X_{ij} = x_i x_j$. If $x_i = x_j = 1$, then $Y^{ij} = \mathbb{1}\mathbb{1}^\top \succeq 0$. If $x_i = 1$ and $x_j = \frac{1}{2}$, then

$$Y^{ij} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \mathcal{D}_{\mathbb{1}-e_2/2}(\mathbb{1}\mathbb{1}^\top + e_2 e_2^\top) \succeq 0.$$

Thus, $\hat{X} \in \text{Schur}(\mathbb{K}_2^V)$ and the proof of ‘ \subseteq ’ is complete. \square

5 Reformulations of Antiblocking Duality

We saw in Section 4.1 that the support functions of some theta bodies defined over the PSD lifting of a cone $\mathbb{K} \subseteq \mathbb{S}^V$ may be expressed as a conic optimization problem over \mathbb{K} . In this section, we shall see that something similar holds for Schur liftings. This shall be essentially a manifestation of antiblocking duality, namely, that $\text{abl}(\text{abl}(\mathcal{C})) = \mathcal{C}$ for every convex corner \mathcal{C} ; we shall make use of this fact throughout the rest of the paper. At the end of the section, we shall have an expression for ϑ_2 from (6), and we shall also introduce ϑ_1 along the way.

In the next result, we follow the rules set for [23, Eq. (9.3.6)] to interpret the quotient w_i/s_i , with $w_i, s_i \in \mathbb{R}_+$:

if $w_i = 0$, then we take the fraction w_i/s_i to be 0, even if the denominator is 0; if $w_i > 0$ but the denominator is 0, we take the fraction w_i/s_i to be $+\infty$. (28)

Proposition 10 Let $\mathcal{C} \subseteq \mathbb{R}^V$ be a convex corner. Let $w \in \mathbb{R}_+^V$. Then

$$\delta^*(w|\mathcal{C}) = \min_{s \in \text{abl}(\mathcal{C})} \max_{i \in V} \frac{w_i}{s_i}. \quad (29)$$

In particular,

$$\delta^*(w|\text{abl}(\mathcal{C})) = \min_{x \in \mathcal{C}} \max_{i \in V} \frac{w_i}{x_i}. \quad (30)$$

Moreover, all four optimization problems in (29) and (30) have optimal solutions.

Proof We may assume that $w \neq 0$. Let us prove ‘ \leq ’. Let $x \in \mathcal{C}$ and $s \in \text{abl}(\mathcal{C})$. We may assume that the max on the RHS is finite so that, by following the rules from (28), we have $W := \text{supp}(w) \subseteq \text{supp}(s) =: S$. Then

$$\langle w, x \rangle = \sum_{i \in W} w_i x_i = \sum_{i \in S} \frac{w_i}{s_i} s_i x_i \leq \left(\max_{i \in S} \frac{w_i}{s_i} \right) \sum_{i \in V} s_i x_i \leq \max_{i \in V} \frac{w_i}{s_i},$$

where (28) is only used in the rightmost term. For the reverse inequality, let $\vartheta := \delta^*(w|\mathcal{C}) > 0$. Then $s := \frac{1}{\vartheta} w \in \text{abl}(\mathcal{C})$. Since $\max_{i \in V} w_i/s_i = \vartheta$, we find that the RHS of (29) is bounded above by $\vartheta = \delta^*(w|\mathcal{C})$. This proves ‘ \geq ’ in (29), as well as attainment for its RHS. Finally, (30) follows from (29) by antiblocking duality. \square

We shall later formulate the parameter ϑ_1 (see the discussion in Subsection 1.1) essentially as the optimization problem in the RHS of (30) applied to a theta body. In a way, that formulation is unnecessary for the proof of the generalization of (4), and it may be further simplified as a line-search, i.e., by a gauge function:

Proposition 11 Let $\mathcal{C} \subseteq \mathbb{R}^V$ be a convex corner. Let $w \in \mathbb{R}_+^V$. Then

$$\min_{x \in \mathcal{C}} \max_{i \in V} \frac{w_i}{x_i} = \min\{\lambda \in \mathbb{R}_+ : w \in \lambda \mathcal{C}\}. \quad (31)$$

Moreover, the RHS is attained.

Proof We may assume that $w \neq 0$. First we show ‘ \leq ’. Let $\lambda \in \mathbb{R}_+$ such that $w \in \lambda \mathcal{C}$. Then $\lambda > 0$ since $w \neq 0$. Set $x := \frac{1}{\lambda} w \in \mathcal{C}$. Then $w_i/x_i = [w_i \neq 0]\lambda$ for every $i \in V$, according to the rules from (28), so that $\max_{i \in V} w_i/x_i = \lambda$, whence the LHS of (31) is $\leq \lambda$. This proves ‘ \leq ’.

For the reverse inequality, let $x \in \mathcal{C}$ attain the LHS of (31), and let $\lambda := \max_{i \in V} w_i/x_i$. Since $w \neq 0$, we have $\lambda > 0$. It is easy to check that $y := \frac{1}{\lambda} w$ satisfies $y \leq x$. Since $0 \leq y \leq x \in \mathcal{C}$ and \mathcal{C} is lower-comprehensive, we find that $y \in \mathcal{C}$, i.e., $w \in \lambda \mathcal{C}$. This proves ‘ \geq ’ on (31), as well as attainment in its RHS. \square

The RHS of (31) is, by definition, the gauge function $\gamma(w|\mathcal{C})$ of \mathcal{C} at w , i.e., $\gamma(x|\mathcal{C})$ is defined as

$$\gamma(x|\mathcal{C}) := \inf\{\mu : \mu \in \mathbb{R}_+, x \in \mu \mathcal{C}\} \quad \forall x \in \mathbb{R}^V. \quad (32)$$

From Propositions 10 and 11, we recover the fact that

$$\text{for a convex corner } \mathcal{C} \subseteq \mathbb{R}_+^V, \text{ we have } \delta^*(\cdot|\text{abl}(\mathcal{C})) = \gamma(\cdot|\mathcal{C}) \text{ on } \mathbb{R}_+^V; \quad (33)$$

see [41, Theorem 14.5].

A gauge function is oblivious to the upper surface of a set which is ‘almost’ a convex corner:

Proposition 12 *Let $\mathcal{C} \subseteq \mathbb{R}_+^V$ be a lower-comprehensive convex set with nonempty interior. Then*

$$\gamma(w|\mathcal{C}) = \gamma(w|\text{cl}(\mathcal{C})) \quad \forall w \in \mathbb{R}_+^V. \quad (34)$$

Proof The proof of ‘ \geq ’ is obvious. For the reverse inequality, let $w \in \mathbb{R}_+^V$ and let $\lambda \in \mathbb{R}_+$ such that $w \in \lambda \text{cl}(\mathcal{C})$. If $\lambda = 0$, then $w = 0$ and $\gamma(w|\mathcal{C}) = 0 = \gamma(w|\text{cl}(\mathcal{C}))$, so assume $\lambda > 0$. We will show that $w \in (\lambda + \varepsilon)\mathcal{C}$ for every $\varepsilon > 0$. Let $\varepsilon > 0$. Since \mathcal{C} is lower-comprehensive and has nonempty interior, there exists $M \in \mathbb{R}_{++}$ such that $\mathbb{1}/M \in \text{int}(\mathcal{C})$. Thus, for every $\mu \in \mathbb{R}$ such that $0 < \mu \leq 1$, we have $\frac{\mu}{M}\mathbb{1} + \frac{1-\mu}{\lambda}w \in \text{int}(\mathcal{C})$. For $\mu := \varepsilon/(\lambda + \varepsilon)$, this gives $\frac{\varepsilon}{M(\lambda + \varepsilon)}\mathbb{1} + \frac{1}{\lambda + \varepsilon}w \in \text{int}(\mathcal{C})$, and since \mathcal{C} is lower-comprehensive, we get $w \in (\lambda + \varepsilon)\mathcal{C}$. Since $\varepsilon > 0$ was arbitrary, this proves ‘ \leq ’ in (34). \square

We are now ready to show how an optimization problem over $\text{Schur}(\mathbb{K})$ may sometimes be reduced to an optimization problem over \mathbb{K} . We shall use the following simple fact:

$$\text{if } \mathbb{K} \supseteq \mathbb{S}_+^V, \text{ then } \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \in \text{Schur}(\mathbb{K}) \text{ if and only if } X \succeq_{\mathbb{K}} xx^\top. \quad (35)$$

Proposition 13 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ and $\widehat{\mathbb{K}} \subseteq \mathbb{S}^{\{0\} \cup V}$ be diagonally scaling-invariant closed convex cones such that (10) and (11) hold. Let $w \in \mathbb{R}_+^V$. Then*

$$\delta^*(w|\text{abl}(\text{cl}(\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})))) = \inf \left\{ \lambda \in \mathbb{R}_+ : W \in \mathbb{A}, \text{diag}(W) = \lambda \mathbb{1}, \begin{bmatrix} 1 & \sqrt{w}^\top \\ \sqrt{w} & W \end{bmatrix} \in \widehat{\mathbb{K}} \right\}. \quad (36)$$

In particular, if \mathbb{A} is polyhedral, then

$$\delta^*(w|\text{abl}(\text{cl}(\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})))) = \inf \left\{ \lambda \in \mathbb{R}_+ : Y \in -\mathbb{A} \cap \text{Null}(\text{diag}), \begin{bmatrix} 1 & \sqrt{w}^\top \\ \sqrt{w} & \lambda I - Y \end{bmatrix} \in \widehat{\mathbb{K}} \right\}, \quad (37)$$

and if $\mathbb{K} \subseteq \mathbb{S}^V$ is a diagonally scaling-invariant closed convex cone such that $\mathbb{K} \supseteq \mathbb{S}_+^V$ and $\text{diag}(\mathbb{K}) \subseteq \mathbb{R}_+^V$ then

$$\delta^*(w|\text{abl}(\text{cl}(\text{TH}(\mathbb{A}, \text{Schur}(\mathbb{K})))) = \inf \left\{ \lambda : \lambda I \succeq_{\mathbb{K}} Y + \sqrt{w} \sqrt{w}^\top, Y \in -\mathbb{A} \cap \text{Null}(\text{diag}) \right\}. \quad (38)$$

Moreover, for each of (36), (37), and (38), if the theta body on the LHS is closed, then the optimization problem on the RHS has an optimal solution.

Proof We may assume that $w \neq 0$. From Propositions 1 and 12 and from (33), we have

$$\delta^*(w|\text{abl}(\text{cl}(\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})))) = \gamma(w|\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})) = \inf \Lambda, \quad \text{where } \Lambda := \left\{ \lambda \in \mathbb{R}_{++} : w \in \lambda \text{TH}(\mathbb{A}, \widehat{\mathbb{K}}) \right\}. \quad (39)$$

Note that Λ is convex, unbounded above, and bounded away from zero since the LHS of (39) is positive by Proposition 1. Moreover, Λ is closed if $\text{TH}(\mathbb{A}, \widehat{\mathbb{K}})$ is closed by the attainment statement in Proposition 11. We may reformulate the set Λ as

$$\begin{aligned} \Lambda &= \left\{ \lambda \in \mathbb{R}_{++} : W \in \mathbb{A}, \text{diag}(W) = \frac{1}{\lambda} w, \begin{bmatrix} 1 & \frac{1}{\lambda} w^\top \\ \frac{1}{\lambda} w & W \end{bmatrix} \in \widehat{\mathbb{K}} \right\} \\ &= \left\{ \lambda \in \mathbb{R}_{++} : X \in \mathbb{A}, \text{diag}(X) = \lambda \mathbb{1}_{\text{supp}(w)}, \begin{bmatrix} 1 & \sqrt{w}^\top \\ \sqrt{w} & X \end{bmatrix} \in \widehat{\mathbb{K}} \right\} \\ &= \left\{ \lambda \in \mathbb{R}_{++} : X \in \mathbb{A}, \text{diag}(X) = \lambda \mathbb{1}, \begin{bmatrix} 1 & \sqrt{w}^\top \\ \sqrt{w} & X \end{bmatrix} \in \widehat{\mathbb{K}} \right\}, \end{aligned} \quad (40)$$

where we used the diagonal scaling invariance of \mathbb{A} and $\widehat{\mathbb{K}}$ and the change of variable

$$\hat{X} = \mathcal{D}_{1 \oplus \lambda w^{-1/2}} \left(\begin{bmatrix} 1 & \frac{1}{\lambda} w^T \\ \frac{1}{\lambda} w & W \end{bmatrix} \right), \quad \text{with } [w^{-1/2}]_i := \begin{cases} w_i^{-1/2} & \text{if } w_i > 0, \\ 1 & \text{otherwise,} \end{cases}$$

for the second equation, and the diagonal scaling invariance of \mathbb{A} and $\widehat{\mathbb{K}}$ and assumptions (10) and (11a) for the last equation. To prove (36), it now suffices to show that relaxing the constraint $\lambda \in \mathbb{R}_{++}$ to $\lambda \in \mathbb{R}_+$ in the RHS of (40) does not change the set. If it did, the relaxed set, which is convex, would contain 0, and so Λ would not be bounded away from zero, a contradiction.

Suppose that \mathbb{A} is polyhedral. It is easy to check that $\mathbb{A} \cap \text{diag}^{-1}(\lambda \mathbb{1}) = \lambda I - (-\mathbb{A} \cap \text{Null}(\text{diag}))$; the inclusion ‘ \supseteq ’ is obvious, whereas the reverse inclusion follows from Proposition 3. Thus, (37) follows. Equation (38) follows from (37) and Theorem 6, using the equivalence (35). The constraint $\lambda \in \mathbb{R}_+$ may be dropped since $\text{diag}(\mathbb{K}) \subseteq \mathbb{R}_+^V$. In all cases, attainment if the theta body is closed follows from the closedness of Λ . \square

6 A Plethora of Theta Functions

We have now introduced all formulations of the parameters ϑ_i ’s from (6) and we are ready to prove that they are all equal. Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant polyhedral cone such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Let $\mathbb{K} \subseteq \mathbb{S}_+^V$ be a diagonally scaling-invariant closed convex cone such that $\text{Diag}(\mathbb{R}_+^V) \subseteq \mathbb{K}$ and $\text{int}(\mathbb{K}) \neq \emptyset$. For each $w \in \mathbb{R}_+^V$, define:

$$\begin{aligned} \vartheta(\mathbb{A}, \mathbb{K}; w) &:= \delta^* \left(w \mid \text{abl}(\text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))) \right), \\ \vartheta_1(\mathbb{A}, \mathbb{K}; w) &:= \inf \left\{ \max_{i \in V} \frac{w_i}{x_i} : x \in \text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))) \right\}, \\ \vartheta_2(\mathbb{A}, \mathbb{K}; w) &:= \inf \left\{ \lambda : \lambda I \succeq_{\mathbb{K}^*} Y + \sqrt{w} \sqrt{w}^T, Y \in -\overline{\mathbb{A}} \cap \text{Null}(\text{diag}) \right\}, \\ \vartheta_3(\mathbb{A}, \mathbb{K}; w) &:= \sup \left\{ \sqrt{w}^T X \sqrt{w} : \text{Tr}(X) = 1, X \in \mathbb{K}, X \in \mathbb{A} \right\}, \\ \vartheta_4(\mathbb{A}, \mathbb{K}; w) &:= \delta^* \left(w \mid \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})) \right). \end{aligned}$$

Here, the objective function for $\vartheta_1(\mathbb{A}, \mathbb{K}; w)$ is evaluated according to the arithmetic rules from (28). For concreteness, we shall finally define the Lovász theta number and the variants ϑ' and ϑ^+ as special cases of the above parameters. Let $G = (V, E)$ be a graph. For each $w \in \mathbb{R}_+^V$, define

$$\begin{aligned} \vartheta(G; w) &:= \vartheta(\mathbb{A}_{E,E}, \mathbb{S}_+^V; w), \\ \vartheta'(G; w) &:= \vartheta(\mathbb{A}_{E \cup \bar{E}, E}, \mathbb{S}_+^V; w), \\ \vartheta^+(G; w) &:= \vartheta(\mathbb{A}_{\emptyset, E}, \mathbb{S}_+^V; w). \end{aligned}$$

Theorem 14 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant polyhedral cone such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Let $\mathbb{K} \subseteq \mathbb{S}_+^V$ be a diagonally scaling-invariant closed convex cone such that $\text{Diag}(\mathbb{R}_+^V) \subseteq \mathbb{K}$ and $\text{int}(\mathbb{K}) \neq \emptyset$. Let $w \in \mathbb{R}_+^V$. Then*

$$\vartheta(\mathbb{A}, \mathbb{K}; w) = \vartheta_1(\mathbb{A}, \mathbb{K}; w) = \vartheta_2(\mathbb{A}, \mathbb{K}; w) = \vartheta_3(\mathbb{A}, \mathbb{K}; w) = \vartheta_4(\mathbb{A}, \mathbb{K}; w). \quad (42)$$

Moreover, all optimization problems in (42) have optimal solutions except possibly for ϑ_2 , which has an optimal solution if $\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^))$ is closed. Furthermore,*

$$\text{abl}(\text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*)))) = \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})). \quad (43)$$

Proof The optimization problems that define $\vartheta_2(\mathbb{A}, \mathbb{K}; w)$ and $\vartheta_3(\mathbb{A}, \mathbb{K}; w)$ form a primal-dual pair of conic optimization problems; this follows from the polyhedrality of \mathbb{A} and from $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Thus, the equation $\vartheta_2(\mathbb{A}, \mathbb{K}; w) = \vartheta_3(\mathbb{A}, \mathbb{K}; w)$ follows by Conic Programming Strong Duality; see, e.g., [6] or [8, Theorem 1.1]. Although the conic formulation for $\vartheta_3(\mathbb{A}, \mathbb{K}; w)$ need not have a Slater point, the assumptions that \mathbb{A} is polyhedral and $\mathbb{K}^* \supseteq \mathbb{S}_+^V$ show that the optimization problem defining $\vartheta_2(\mathbb{A}, \mathbb{K}; w)$ has a restricted Slater point. Equation (42) follows from $\vartheta_2(\mathbb{A}, \mathbb{K}; w) = \vartheta_3(\mathbb{A}, \mathbb{K}; w)$, Propositions 10 and 13, and Theorem 5 since $\text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*)))$ is a convex corner by Theorem 6. Existence of optimal solutions follows from the corresponding statements in the previous results. Now (43) follows from conjugate duality applied to $\vartheta(\mathbb{A}, \mathbb{K}; w) = \vartheta_4(\mathbb{A}, \mathbb{K}; w)$ for every $w \in \mathbb{R}_+^V$. \square

Theorem 14 implies (4a) and (4b) using the descriptions (9) for every graph G . Note also that we could have mimicked the proof of the chain (6) as in [23] and [30]; the proof that $\vartheta_4(\mathbb{A}, \mathbb{K}; w) \leq \vartheta(\mathbb{A}, \mathbb{K}; w)$ follows from Proposition 8.

In the context of Theorem 14, the support functions of the two theta bodies that appear in (43) are gauges polar to each other; see [41, §15] and recall the definition of gauge from (32). The corresponding polar inequality (that is, the corresponding Cauchy-Schwarz inequality) for these gauges is stated next; compare with [14, Proposition 8 and Theorem 18]. For each permutation σ on V , define the linear map $P(\sigma): \mathbb{R}^V \rightarrow \mathbb{R}^V$ as the linear extension of the map $e_i \in \mathbb{R}^V \mapsto e_{\sigma(i)}$. For each $L \in \mathbb{R}^{V \times V}$, define the *congruence map* $\text{Congr}_L: \mathbb{R}^{V \times V} \rightarrow \mathbb{R}^{V \times V}$ as

$$\text{Congr}_L(X) := LXL^\top \quad \forall X \in \mathbb{R}^{V \times V}.$$

Corollary 15 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant polyhedral cone such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Let $\mathbb{K} \subseteq \mathbb{S}_+^V$ be a diagonally scaling-invariant closed convex cone such that $\text{Diag}(\mathbb{R}_+^V) \subseteq \mathbb{K}$ and $\text{int}(\mathbb{K}) \neq \emptyset$. If $w, \bar{w} \in \mathbb{R}_+^V$, then*

$$\langle w, \bar{w} \rangle \leq \delta^* \left(w \mid \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})) \right) \cdot \delta^* \left(\bar{w} \mid \text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))) \right). \quad (44)$$

Moreover, if there exists a transitive permutation group Γ on V such that

$$\{\text{Congr}_{P(\sigma)} : \sigma \in \Gamma\} \subseteq \text{Aut}(\mathbb{A}) \cap \text{Aut}(\mathbb{K}), \quad (45)$$

then

$$|V| = \delta^* \left(\mathbb{1} \mid \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})) \right) \cdot \delta^* \left(\mathbb{1} \mid \text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))) \right). \quad (46)$$

Proof By Theorem 6, we know that $\text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*)))$ is a convex corner. By (33) and Theorem 14, the gauge function $\gamma(\cdot \mid \text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))))$ is the support function $\delta^*(\cdot \mid \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})))$. Hence, the support functions $\delta^*(\cdot \mid \text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))))$ and $\delta^*(\cdot \mid \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})))$ are gauges polar to each other (when restricted to \mathbb{R}_+^V); see [41, Corollary 15.1.2]. Now (44) follows immediately.

Next, we prove that ‘ \geq ’ holds in (46) if $w = \bar{w} = \mathbb{1}$ and (45) holds. Assume the latter, and let $\widehat{\Gamma}$ denote the permutation group $\widehat{\Gamma} := \{\hat{\sigma} : \hat{\sigma}(0) = 0, \hat{\sigma}|_V \in \Gamma\}$ on $\{0\} \cup V$. It is clear that $\{\text{Congr}_{P(\hat{\sigma})} : \hat{\sigma} \in \widehat{\Gamma}\} \subseteq \text{Aut}(\widehat{\mathbb{K}})$ for each $\widehat{\mathbb{K}} \in \{\text{Psd}(\mathbb{K}), \text{Schur}(\mathbb{K}^*)\}$. Together with $\{\text{Congr}_{P(\sigma)} : \sigma \in \Gamma\} \subseteq \text{Aut}(\mathbb{A})$, this yields $\{\text{Congr}_{P(\hat{\sigma})} : \hat{\sigma} \in \widehat{\Gamma}\} \subseteq \text{Aut}(\widehat{\mathcal{C}})$ for each $\widehat{\mathcal{C}} \in \{\widehat{\text{TH}}(\mathbb{A}, \text{Psd}(\mathbb{K})), \widehat{\text{TH}}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))\}$, whence

$$\{P(\sigma) : \sigma \in \Gamma\} \subseteq \text{Aut}(\widehat{\mathcal{C}}) \quad \forall \widehat{\mathcal{C}} \in \{\widehat{\text{TH}}(\mathbb{A}, \text{Psd}(\mathbb{K})), \widehat{\text{TH}}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))\}.$$

Thus, each support function on the RHS of (46) is attained by a fixed point of the Reynolds operator

$$x \in \mathbb{R}^V \mapsto \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} P(\sigma)x.$$

Since Γ acts transitively on V , there exist $\mu, \nu \in \mathbb{R}$ such that $\mu \mathbb{1}$ attains $\delta^*(\mathbb{1} \mid \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K})))$ and $\nu \mathbb{1}$ attains $\delta^*(\mathbb{1} \mid \text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*))))$. By (43) from Theorem 14, we get $\langle \mu \mathbb{1}, \nu \mathbb{1} \rangle \leq 1$ so $\mu \nu |V| \leq 1$. Thus,

$$\delta^*(\mathbb{1} \mid \text{TH}(\mathbb{A}, \text{Psd}(\mathbb{K}))) \cdot \delta^*(\mathbb{1} \mid \text{cl}(\text{TH}(\overline{\mathbb{A}}, \text{Schur}(\mathbb{K}^*)))) = \langle \mathbb{1}, \mu \mathbb{1} \rangle \langle \mathbb{1}, \nu \mathbb{1} \rangle = \mu \nu |V|^2 \leq |V|. \quad \square$$

7 Theta Bodies over the Copositive and Completely Positive Cones

In this section, we show that the stable set polytope of a graph and one of its classical fractional relaxations are theta bodies. The key result we use to prove this is a completely positive formulation for the stability number of a graph, due to de Klerk and Pasechnik [13]. As a consequence of the antiblocker duality relation from Theorem 14, we shall derive a weighted generalization of a copositive formulation for the fractional chromatic number of a graph, due to Dukanovic and Rendl [14].

Let $G = (V, E)$ be a graph. For each $w \in \mathbb{R}_+^V$, we set

$$\alpha(G; w) := \delta^*(w \mid \text{STAB}(G)).$$

Recall that the stable set polytope $\text{STAB}(G)$ was defined as the convex hull of $\{\mathbb{1}_S : S \subseteq V \text{ stable in } G\}$, that \mathcal{C}_V denotes the cone of copositive matrices, and that \mathcal{C}_V^* is the cone of completely positive matrices. The key argument in the proof of the next result comes from [13, Theorem 2.2]:

Proposition 16 *If $G = (V, E)$ is a graph, then*

$$\text{TH}(\mathbb{A}_{E,E}, \text{Psd}(\mathcal{C}_V^*)) = \text{STAB}(G).$$

Proof To prove ‘ \supseteq ’, note that, if $S \subseteq V$ is a stable set of G , then $(1 \oplus \mathbb{1}_S)(1 \oplus \mathbb{1}_S)^\top \in \widehat{\text{TH}}(\mathbb{A}_{E,E}, \text{Psd}(\mathcal{C}_V^*))$, whence $\mathbb{1}_S \in \text{TH}(\mathbb{A}_{E,E}, \text{Psd}(\mathcal{C}_V^*))$. For the reverse inclusion it suffices by conjugate duality and Corollary 2 to show that, for $w \in \mathbb{R}_+^V$, we have $\alpha(G; w) \geq \delta^*(w \mid \text{TH}(\mathbb{A}_{E,E}, \text{Psd}(\mathcal{C}_V^*)))$. Thus, it suffices by Theorem 5 to show that, for $w \in \mathbb{R}_+^V$, we have

$$\alpha(G; w) \geq \max \left\{ \sqrt{w}^\top X \sqrt{w} : \text{Tr}(X) = 1, X \in \mathcal{C}_V^*, X \in \mathbb{A}_{E,E} \right\}. \quad (47)$$

Let $w \in \mathbb{R}_+^V$. We may assume that $w \neq 0$. The extreme rays of the cone $\mathcal{C}_V^* \cap \mathbb{A}_{E,E}$ are of the form xx^\top with $x \in \mathbb{R}_+^V$ and $\text{supp}(x)$ stable in G . So there exists an optimal solution for the RHS of (47) of the form $\bar{x}\bar{x}^\top$ for some $\bar{x} \in \mathbb{R}_+^V$ such that $\|\bar{x}\|^2 = \text{Tr}(\bar{x}\bar{x}^\top) = 1$ and $\text{supp}(\bar{x})$ is a stable set in G . In fact, for any $y \in \mathbb{R}_+^V$ such that $\|y\|^2 = 1$ and $\text{supp}(y) \subseteq \text{supp}(\bar{x})$, the point yy^\top is feasible in the RHS of (47) with objective value $\langle \sqrt{w}, y \rangle^2$ whence the RHS of (47) is equal to $\max \{ \langle \sqrt{w}, y \rangle^2 : y \in \mathbb{R}_+^V, \|y\|^2 = 1, \text{supp}(y) \subseteq \text{supp}(\bar{x}) \}$. The optimality conditions for this optimization problem (i.e., Cauchy-Schwarz) show that an optimal solution is given by $\bar{y} := \frac{\sqrt{w}}{\|\sqrt{w}\|}$ where $u := w \odot \mathbb{1}_{\text{supp}(\bar{x})}$, and its objective value is

$$\frac{\langle \sqrt{w}, \sqrt{u} \rangle^2}{\|\sqrt{u}\|^2} = \frac{\langle \sqrt{u}, \sqrt{u} \rangle^2}{\|\sqrt{u}\|^2} = \|\sqrt{u}\|^2 = \langle w, \mathbb{1}_{\text{supp}(\bar{x})} \rangle.$$

Since $\text{supp}(\bar{x})$ is stable, this concludes our proof of (47). \square

Let $G = (V, E)$ be a graph. The *fractional stable set polytope* of G is defined as

$$\text{QSTAB}(G) := \{x \in \mathbb{R}_+^V : \langle \mathbb{1}_K, x \rangle \leq 1 \text{ for every clique } K \text{ of } G\}.$$

Note that

$$\text{QSTAB}(G) = \text{abl}(\text{STAB}(\overline{G})).$$

For $w \in \mathbb{R}_+^V$, the *fractional chromatic number* of G is

$$\chi^*(G; w) := \delta^*(w | \text{QSTAB}(\overline{G})).$$

Proposition 16 yields immediately a weighted generalization of [14, Corollary 5]:

Corollary 17 *Let $G = (V, E)$ be a graph. Let $w \in \mathbb{R}_+^V$. Then*

$$\chi^*(G; w) = \min \left\{ \lambda : Y \in \mathbb{A}_{\overline{E}, \overline{E}}^\perp, \begin{bmatrix} 1 & \sqrt{w}^\top \\ \sqrt{w} & \lambda I - Y \end{bmatrix} \in \text{Psd}(\mathcal{C}_V^*) \right\}.$$

Proof By Proposition 16 and (37) from Proposition 13, we have

$$\begin{aligned} \chi^*(G; w) &= \delta^*(w | \text{QSTAB}(\overline{G})) = \delta^*(w | \text{abl}(\text{STAB}(G))) = \delta^*(w | \text{abl}(\text{TH}(\mathbb{A}_{E, E}, \text{Psd}(\mathcal{C}_V^*)))) \\ &= \min \left\{ \lambda \in \mathbb{R}_+ : Y \in -\mathbb{A}_{E, E} \cap \text{Null}(\text{diag}), \begin{bmatrix} 1 & \sqrt{w}^\top \\ \sqrt{w} & \lambda I - Y \end{bmatrix} \in \text{Psd}(\mathcal{C}_V^*) \right\} \\ &= \min \left\{ \lambda : Y \in \mathbb{A}_{\overline{E}, \overline{E}}^\perp, \begin{bmatrix} 1 & \sqrt{w}^\top \\ \sqrt{w} & \lambda I - Y \end{bmatrix} \in \text{Psd}(\mathcal{C}_V^*) \right\}. \end{aligned}$$

The constraint $\lambda \in \mathbb{R}_+$ may be dropped since $\text{diag}(\mathcal{C}_V^*) \subseteq \mathbb{R}_+$. □

By the antiblocker relation from Theorem 14, we know that $\text{QSTAB}(G)$ is the closure of a theta body. Unlike in the cases presented so far, the fact that the latter theta body is actually closed does not follow from our previous results. Thus, we proceed to prove its closedness separately. We shall use an argument from [18, Theorem 5] (more specifically, in the proof of (53) below). We denote the maximum norm by $\|\cdot\|_\infty$.

Theorem 18 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant polyhedral cone such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Then*

$$\text{TH}(\mathbb{A}, \text{Schur}(\mathcal{C}_V)) = \left\{ \text{diag}(\hat{X}[V]) : \hat{X} \in \widehat{\text{TH}}(\mathbb{A}, \text{Schur}(\mathcal{C}_V)), \|\hat{X}\|_\infty \leq 1 \right\}. \quad (48)$$

Consequently, $\text{TH}(\mathbb{A}, \text{Schur}(\mathcal{C}_V))$ is a convex corner contained in $[0, 1]^V$.

Proof The inclusion ‘ \supseteq ’ in (48) is trivial. For the reverse inclusion, let $x \in \text{TH}(\mathbb{A}, \text{Schur}(\mathcal{C}_V))$, and let $\hat{Y} \in \widehat{\text{TH}}(\mathbb{A}, \text{Schur}(\mathcal{C}_V))$ such that $x = \text{diag}(Y)$ for $Y := \hat{Y}[V]$. We shall use (35) with $\mathbb{K} = \mathcal{C}_V$ throughout the proof without further mention. Note that $Y - xx^\top \in \mathcal{C}_V$ implies that $x - (x \odot x) = \text{diag}(Y - xx^\top) \geq 0$ so

$$x \in [0, 1]^V. \quad (49)$$

Let us prove that

$$\text{we may assume that } Y \in \mathbb{S}_{\geq 0}^V \text{ and } Y = Y[\text{supp}(x)] \oplus 0. \quad (50)$$

Indeed, the principal submatrix $Y = \hat{Y}[V]$ from \hat{Y} may possibly be replaced with

$$Y - 2 \sum \left\{ [Y_{ij} < 0] Y_{ij} \text{Sym}(e_i e_j^\top) : ij \in \binom{V}{2} \right\}$$

(using the notation Sym from (13)) without affecting the relations $\hat{Y}[V] \in \mathbb{A}$ or $\hat{Y}[V] \succeq_{\mathcal{C}_V} xx^\top$, by Proposition 3 and the trivial fact that $\mathcal{C}_V + \mathbb{S}_{\geq 0}^V = \mathcal{C}_V$. Clearly, for $S := \text{supp}(x)$ and $\bar{x} := x|_S$, we have $Y[S] \succeq_{\mathcal{C}_S} \bar{x}\bar{x}^\top$. Thus, by possibly replacing $\hat{Y}[V]$ with $\hat{Y}[S] \oplus 0$ in \hat{Y} , we shall have $Y = Y[\text{supp}(x)] \oplus 0$, and the proof of (50) is complete. Thus, by possibly restricting our attention to the index set $\text{supp}(x)$,

we may assume that $\text{supp}(x) = V$.

Write $D := \text{Diag}(x)$ and $B := Y - D$. Let $G = (V, E)$ be the graph defined by $E := \{ij \in \binom{V}{2} : B_{ij} > 0\}$. Define $A \in \mathbb{A} \cap \text{Null}(\text{diag})$ by setting $A_{ij} := \frac{1}{2}[ij \in E](1/x_i + 1/x_j)$ for each $ij \in \binom{V}{2}$, where we used Proposition 3 to prove membership of A in \mathbb{A} . We claim that

$$D^{-1} + A - \mathbb{1}\mathbb{1}^\top \in \mathcal{C}_V. \quad (51)$$

We shall need to consider the following optimization problem in our proof:

$$\min \left\{ h^\top (D^{-1} + A)h : h \in \mathbb{R}_+^V, \langle \mathbb{1}, h \rangle = 1 \right\}. \quad (52)$$

Let us show that

$$\text{there exists an optimal solution } \bar{h} \text{ for (52) whose support is a stable set in } G. \quad (53)$$

Indeed, let \bar{h} be an optimal solution for (52) with minimal support. Note that an optimal solution exists by continuity and compactness. Suppose that $ij \subseteq \text{supp}(\bar{h})$ for some $ij \in E$. For each $t \in \mathbb{R}$, define $h_t := \bar{h} + t(e_i - e_j)$, and note that h_t is feasible for (52) whenever $t \in [-\bar{h}_i, \bar{h}_j]$. The objective value of h_t in (52) is, $h_t^\top (D^{-1} + A)h_t = \bar{h}^\top (D^{-1} + A)\bar{h} + 2t(e_i - e_j)^\top (D^{-1} + A)\bar{h} = \bar{h}^\top (D^{-1} + A)\bar{h}$, where the final equation follows from the optimality of $\bar{h} = h_0$. Since $h_{\bar{t}}$ is feasible in (52) for $\bar{t} := \bar{h}_j$ and $\text{supp}(h_{\bar{t}}) \subsetneq \text{supp}(\bar{h})$, the proof of (53) is complete.

It follows from (53) that $\bar{h}^\top A \bar{h} = 0$ and $\bar{h}^\top D^{-1} B D^{-1} \bar{h} = 0$. Thus, since $D^{-1} Y D^{-1} \succeq_{\mathcal{C}_V} D^{-1} x x^\top D^{-1}$ by the diagonal scaling invariance of \mathcal{C}_V , we get

$$\begin{aligned} \bar{h}^\top (D^{-1} + A)\bar{h} &= \bar{h}^\top D^{-1} \bar{h} = \bar{h}^\top (D^{-1} D D^{-1}) \bar{h} = \bar{h}^\top (D^{-1} (D + B) D^{-1}) \bar{h} \\ &\geq \bar{h}^\top D^{-1} x x^\top D^{-1} \bar{h} = \bar{h}^\top \mathbb{1} \mathbb{1}^\top \bar{h} = 1. \end{aligned}$$

Thus, $\min \{ h^\top (D^{-1} + A - \mathbb{1}\mathbb{1}^\top) h : h \in \mathbb{R}_+^V, \mathbb{1}^\top h = 1 \} \geq 0$ and (51) is proved. Set $X := \mathcal{D}_x(D^{-1} + A)$. Then (51) implies $X \succeq_{\mathcal{C}_V} \mathcal{D}_x(\mathbb{1}\mathbb{1}^\top) = x x^\top$. Moreover, $\text{diag}(X) = x$ and, for $ij \in E$, we have

$$X_{ij} = [\mathcal{D}_x(A)]_{ij} = \frac{x_i x_j}{2} \left(\frac{1}{x_i} + \frac{1}{x_j} \right) = \frac{x_j + x_i}{2} \leq 1$$

by (49). Since $X_{ij} = 0$ for $ij \in \bar{E}$, it follows that

$$\hat{X} := \begin{bmatrix} \mathbb{1} & x^\top \\ x & X \end{bmatrix} \in \widehat{\text{TH}}(\mathbb{A}, \text{Schur}(\mathcal{C}_V))$$

and $\|\hat{X}\|_\infty \leq 1$. This completes the proof of (48). It follows that the set $\text{TH}(\mathbb{A}, \text{Schur}(\mathcal{C}_V))$ is closed, since it is described by (48) as the linear image of a compact set. Thus, $\text{TH}(\mathbb{A}, \text{Schur}(\mathcal{C}_V))$ is a convex corner by Theorem 6. \square

Corollary 19 *Let $G = (V, E)$ be a graph. Then*

$$\text{QSTAB}(G) = \text{TH}(\mathbb{A}_{E,E}, \text{Schur}(\mathcal{C}_V)). \quad (54)$$

In particular, for every $w \in \mathbb{R}_+^V$, we have

$$\chi^*(G; w) = \max \left\{ \langle w, x \rangle : X \in \mathbb{A}_{\bar{E}, \bar{E}}, \text{diag}(X) = x, X \succeq_{\mathcal{C}_V} xx^\top \right\}. \quad (55)$$

Proof We know that $\text{abl}(\text{TH}(\mathbb{A}_{E,E}, \text{Schur}(\mathcal{C}_V))) = \text{TH}(\mathbb{A}_{\bar{E}, \bar{E}}, \text{Psd}(\mathcal{C}_V^*)) = \text{STAB}(\bar{G})$ by Theorems 14 and 18 and Proposition 16. Thus, (54) follows from antiblocking duality. Now (55) follows from (54) and (35) since, for each $w \in \mathbb{R}_+^V$, we have $\chi^*(G; w) = \delta^*(w \mid \text{QSTAB}(\bar{G})) = \delta^*(w \mid \text{TH}(\mathbb{A}_{\bar{E}, \bar{E}}, \text{Schur}(\mathcal{C}_V)))$. \square

8 Hoffman Bounds

The *chromatic number* of a graph $G = (V, E)$, denoted by $\chi(G)$, is the size of a smallest partition of G into stable sets. Hoffman [26] proved the following classical lower bound on $\chi(G)$:

$$\chi(G) \geq 1 - \frac{\lambda_{\max}(A_G)}{\lambda_{\min}(A_G)}. \quad (56)$$

Here, A_G denotes the adjacency matrix of G . Lovász [32] proved that the lower bound (56) on $\chi(G)$ remains valid if the adjacency matrix A_G is replaced with any matrix in $\mathbb{A}_{E,E}^\perp$, and that the tightest lower bound on $\chi(G)$ arising in this manner is precisely $\vartheta(\bar{G})$. Knuth [30, Sec. 33] defined another graph parameter, that he denoted by $\vartheta_6(G; w)$, which is in fact equal to $\vartheta(G; w)$. The parameter $\vartheta_6(G; w)$ is defined as an optimization problem, and the objective function corresponding to $\vartheta_6(\bar{G}; \mathbb{1})$ yields precisely the expression in the RHS of (56) when applied to an arbitrary matrix $A \in \mathbb{A}_{E,E}^\perp$. We shall extend our framework in this direction.

Let $\mathbb{A} \subseteq \mathbb{S}^V$ and $\mathbb{K} \subseteq \mathbb{S}^V$. Following Knuth [30, Sec. 33], we define

$$\vartheta_6(\mathbb{A}, \mathbb{K}; w) := \sup \{ \lambda_{\max}(B) : \text{diag}(B) = w, B \in \mathbb{K}, B \in \mathbb{A} \}$$

for every $w \in \mathbb{R}_+^V$. Note that the optimization problem on the RHS above is not convex. The next result relates the formulations for $\vartheta_6(\mathbb{A}, \mathbb{K}; w)$ and $\vartheta_3(\mathbb{A}, \mathbb{K}; w)$.

Theorem 20 *Let $\mathbb{M} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant closed convex cone such that (19) holds, and that $\text{Diag}(\mathbb{R}_+^V) \subseteq \mathbb{M}$. Suppose that either $\mathcal{D}_h(\mathbb{M}) \subseteq \mathbb{M}$ for every $h \in \mathbb{R}^V$ or $\mathbb{M} \subseteq \mathbb{S}_{\geq 0}^V$. Let $w \in \mathbb{R}_+^V$. Then*

$$\max \left\{ \lambda_{\max}(B) : B \in \mathbb{M}, \text{diag}(B) = w \right\} = \max \left\{ \sqrt{w}^\top X \sqrt{w} : \text{Tr}(X) = 1, X \in \mathbb{M} \right\}. \quad (57)$$

Moreover, both optimization problems have optimal solutions.

Proof Equation (57) when $w = 0$ follows from (19b). Thus, we may assume that $w \neq 0$. Then the RHS of (57) is positive, whence Lemma 4 may be applied. We start by proving ‘ \geq ’ in (57). Let X^* be an optimal solution for the RHS of (57). Define d and \bar{X} as in the statement of Lemma 4. Then $\bar{B} := \mathcal{D}_{\sqrt{w}}(\bar{X}) + \text{Diag}(w \odot \mathbb{1}_{V \setminus \text{supp}(d)})$ is feasible for the LHS and its objective value is $\lambda_{\max}(\bar{B}) \geq \lambda_{\max}(\mathcal{D}_{\sqrt{w}}(\bar{X})) = \sqrt{w}^\top X^* \sqrt{w}$ by (21c).

Next we prove ‘ \leq ’ in (57). Let \bar{B} be an optimal solution for the LHS of (57); one exists by compactness, as a consequence of (19c). Let $\lambda := \lambda_{\max}(\bar{B}) > 0$ and let $b \in \mathbb{R}^V$ be a unit vector such that $\bar{B}b = \lambda b$. Note that $\text{supp}(b) \subseteq \text{supp}(w)$ by (19b). The matrix $\tilde{X} := \text{Diag}(\sqrt{w})^\dagger \bar{B} \text{Diag}(\sqrt{w})^\dagger$ satisfies $\text{diag}(\tilde{X}) = \mathbb{1}_{\text{supp}(w)}$,

whence $\bar{X} := \mathcal{D}_b(\tilde{X})$ satisfies $\text{Tr}(\bar{X}) = \|b\|^2 = 1$. If $\mathcal{D}_h(\mathbb{M}) \subseteq \mathbb{M}$ for each $h \in \mathbb{R}^V$, then $\bar{X} \in \mathbb{M}$ follows from $\bar{B} \in \mathbb{M}$. If $\mathbb{M} \subseteq \mathbb{S}_{\geq 0}^V$, then $\bar{X} \in \mathbb{M}$ follows from $\bar{B} \in \mathbb{M}$ and by the diagonal scaling invariance of \mathbb{M} , since we may assume that $b \geq 0$ by the Perron-Frobenius Theorem; see, e.g., [27, Theorem 8.3.1]. In either case, we find that $\bar{X} \in \mathbb{M}$, whence \bar{X} is feasible in the RHS of (57). Finally, its objective value in the RHS of (57) is $\sqrt{w}^\top \bar{X} \sqrt{w} = \sqrt{w}^\top \mathcal{D}_b(\tilde{X}) \sqrt{w} = b^\top \mathcal{D}_{\sqrt{w}}(\tilde{X}) b = b^\top \bar{B} b = \lambda$, where we used (19b) to get $\bar{B} = \mathcal{D}_{\sqrt{w}}(\tilde{X})$. This completes the proof of (57). \square

Corollary 21 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant polyhedral cone such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Let $\mathbb{K} \subseteq \mathbb{S}_+^V$ be a diagonally scaling-invariant closed convex cone such that $\text{Diag}(\mathbb{R}_+^V) \subseteq \mathbb{K}$. Let $w \in \mathbb{R}_+^V$. If either $\mathcal{D}_h(\mathbb{A} \cap \mathbb{K}) \subseteq \mathbb{A} \cap \mathbb{K}$ for all $h \in \mathbb{R}^V$ or $\mathbb{A} \cap \mathbb{K} \subseteq \mathbb{S}_{\geq 0}^V$, then*

$$\vartheta_6(\mathbb{A}, \mathbb{K}; w) = \vartheta_3(\mathbb{A}, \mathbb{K}; w).$$

Proof Immediate from Theorem 20. \square

Next we shall show that, when applied to $w = \mathbb{1}$, the objective value of the LHS of (57) has the same form as the RHS of (56), and thus generalizes it:

Proposition 22 *Let $\mathbb{A}, \mathbb{K} \subseteq \mathbb{S}^V$ be diagonally scaling-invariant closed convex cones. Suppose that \mathbb{A} is polyhedral and that $I \in \mathbb{A} \cap \mathbb{K}$. Then*

$$\vartheta_6(\mathbb{A}, \mathbb{K}; \mathbb{1}) = \max \left\{ 1 - [\mu \neq 0] \frac{\lambda_{\max}(A)}{\mu} : A \in \mathbb{A} \cap \text{Null}(\text{diag}), \mu \in -\mathbb{R}_+, A \succeq_{\mathbb{K}} \mu I \right\}.$$

Proof We have

$$\begin{aligned} & \max \{ \lambda_{\max}(B) : \text{diag}(B) = \mathbb{1}, B \in \mathbb{A}, B \in \mathbb{K} \} \\ &= \max \{ \lambda_{\max}(I+A) : \text{diag}(A) = 0, I+A \in \mathbb{A}, I+A \in \mathbb{K} \} \\ &= \max \{ 1 + [\nu \neq 0] \nu \lambda_{\max}(A) : A \in \mathbb{A} \cap \text{Null}(\text{diag}), \nu \in \mathbb{R}_+, \nu A \succeq_{\mathbb{K}} -I \} \\ &= \max \left\{ 1 - [\mu \neq 0] \frac{\lambda_{\max}(A)}{\mu} : A \in \mathbb{A} \cap \text{Null}(\text{diag}), \mu \in -\mathbb{R}_+, A \succeq_{\mathbb{K}} \mu I \right\}. \end{aligned}$$

Note that we used Proposition 3 on the second equation. That equation also uses $I \in \mathbb{A}$, whereas the third one makes use of $I \in \mathbb{K}$. \square

Corollary 23 *Let $G = (V, E)$ be graph. Then*

$$\begin{aligned} \vartheta(\bar{G}; \mathbb{1}) &= \max \left\{ 1 - [A \neq 0] \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} : A \in \mathbb{A}_{E,E}^* \right\}, \\ \vartheta'(\bar{G}; \mathbb{1}) &= \max \left\{ 1 - [A \neq 0] \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} : A \in \mathbb{A}_{E \cup \bar{E}, E}^* \right\}, \\ \alpha(\bar{G}; \mathbb{1}) &= \max \left\{ 1 - [\mu \neq 0] \frac{\lambda_{\max}(A)}{\mu} : A \in \mathbb{A}_{E,E}^*, \mu \in -\mathbb{R}_+, A \succeq_{\mathcal{C}_V^*} \mu I \right\}. \end{aligned}$$

Moreover, all the optimization problems in (58) have optimal solutions.

Proof Immediate from Theorem 14, Corollary 21, and Propositions 22 and 16. \square

Finally, note that, for a graph $G = (V, E)$, we have

$$\begin{aligned} \max \left\{ 1 - [A \neq 0] \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} : A \in \mathbb{A}_{E,E}^* \right\} &= \delta^*(\mathbb{1} \mid \text{TH}(\mathbb{A}_{\bar{E},\bar{E}}, \text{Schur}(\mathbb{S}_+^V))) \quad \text{by Corollary 23,} \\ &\leq \delta^*(\mathbb{1} \mid \text{TH}(\mathbb{A}_{\bar{E},\bar{E}}, \text{Schur}(\mathcal{C}_V))) \quad \text{since } \mathbb{S}_+^V \subseteq \mathcal{C}_V, \\ &= \delta^*(\mathbb{1} \mid \text{QSTAB}(\bar{G})) \quad \text{by Corollary 19,} \\ &= \chi^*(G; \mathbb{1}) \leq \chi(G). \end{aligned}$$

This proves that the best bound from this family of lower bounds for $\chi(G)$ is $\vartheta(\bar{G})$, as was already shown by Lovász [32, Theorem 6]; see also [7].

9 Theta Bodies over the Positive Semidefinite Cone

The development of the theory makes it clear that the positive semidefinite cone plays a key role in theta bodies. For instance, \mathbb{S}_+^V delineates the range of applicability of the lifting operators in several results (e.g., Theorems 5, 6, and 14) and it provides the most symmetric antiblocking relation (43), in the form of (4a). In this section, we focus on some special properties of theta bodies defined over the semidefinite cone. Clearly, the most interesting such families are the ones defined in (9). We shall reprove two classical results about these families of theta bodies using our unifying framework, and we conclude the section and the paper with a weighted extension of the convex characterization of Luz and Schrijver [37] to all semidefinite variants of ϑ . We start by reproving that every facet of a theta body over the semidefinite cone is defined by a clique inequality. Note that, as pointed out by Shepherd [44], this implies via (4a) that every vertex of $\text{TH}(G)$ is the incidence vector of a stable set of G ; further progress on some other structural properties of the theta body, such as the existence of an exact (“NO-definite”) weak separation algorithm, may be seen in [44].

We briefly recall some basic concepts of the facial structure of convex sets. Let $\mathcal{C} \subseteq \mathbb{E}$ be a convex set. A convex subset $\mathcal{F} \subseteq \mathcal{C}$ is a *facet* of \mathcal{C} if $\dim(\mathcal{F}) = \dim(\mathcal{C}) - 1$ and $\mathcal{F} = \arg \max_{x \in \mathcal{C}} \langle c, x \rangle$ for some nonzero $c \in \mathbb{E}^*$; in this case, we say that the facet \mathcal{F} is *determined* by the inequality $\langle c, x \rangle \leq \delta^*(c \mid \mathcal{C})$.

The proof below is a slight modification of [43, Theorem 67.13]. Note how it uses the complementarity established in Proposition 8:

Theorem 24 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant polyhedral cone such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Then each facet of $\text{TH}(\mathbb{A}, \mathbb{S}_+^{\{0\} \cup V})$ is determined either by $x_i \geq 0$ for some $i \in V$, or by $\langle w, x \rangle \leq 1$ for some $w \in \text{TH}(\bar{\mathbb{A}}, \mathbb{S}_+^{\{0\} \cup V}) \cap \{0, 1\}^V$.*

Proof By Corollary 2, $\text{TH}(\mathbb{A}, \mathbb{S}_+^{\{0\} \cup V})$ is a convex corner, and its antiblocker is $\text{TH}(\bar{\mathbb{A}}, \mathbb{S}_+^{\{0\} \cup V})$ by Theorem 14. By a well-known dual characterization of facets of convex corners (see, e.g., [10, Theorem 8]), it suffices to show that, if the inequality $\langle w, x \rangle \leq 1$ determines a facet of $\text{TH}(\mathbb{A}, \mathbb{S}_+^{\{0\} \cup V})$ for some $w \in \text{TH}(\bar{\mathbb{A}}, \mathbb{S}_+^{\{0\} \cup V})$, then $w \in \{0, 1\}^V$.

So let $w \in \text{TH}(\bar{\mathbb{A}}, \mathbb{S}_+^{\{0\} \cup V})$ such that $\langle w, x \rangle \leq 1$ determines a facet \mathcal{F} of $\text{TH}(\mathbb{A}, \mathbb{S}_+^{\{0\} \cup V})$, and let $\hat{W} \in \widehat{\text{TH}}(\bar{\mathbb{A}}, \mathbb{S}_+^{\{0\} \cup V})$ such that $w = \text{diag}(W)$ for $W := \hat{W}[V]$. Write $\hat{W} = \sum_{i=1}^r (\alpha_i \oplus a_i)(\alpha_i \oplus a_i)^\top$ for linearly independent vectors $\{\alpha_i \oplus a_i : i \in [r]\} \subseteq \mathbb{R}^{\{0\} \cup V}$, where $[r] := \{1, \dots, r\}$. Let us show that,

$$\text{if } x \in \mathcal{F}, \text{ then } \langle a_i, x \rangle = \alpha_i \text{ for each } i \in [r]. \quad (59)$$

So let $x \in \mathcal{F}$ and let $\hat{X} \in \widehat{\text{TH}}(\mathbb{A}, \mathbb{S}_+^{\{0\} \cup V})$ such that $x = \text{diag}(X)$ for $X := \hat{X}[V]$. Write $W = \text{Diag}(w) - B$ for some $B \in \mathbb{A}^*$. Since $\hat{W} \in \widehat{\text{TH}}(\overline{\mathbb{A}}, \mathbb{S}_+^{\{0\} \cup V})$ and $\hat{X} \in \widehat{\text{TH}}(\mathbb{A}, \mathbb{S}_+^{\{0\} \cup V})$, we have

$$\begin{aligned} 0 &\leq \langle W, X - xx^\top \rangle = \langle \text{Diag}(w) - B, X \rangle - x^\top Wx = \langle w, x \rangle - \langle B, X \rangle - x^\top Wx \\ &\leq \langle w, x \rangle - \langle w, x \rangle^2 = 1 - 1 = 0. \end{aligned}$$

Equality throughout implies that $x^\top Wx = \langle w, x \rangle^2 = 1$ and that $\langle W, X \rangle = x^\top Wx = 1$. Thus,

$$\left\langle \begin{bmatrix} 1 & -x^\top \\ -x & X \end{bmatrix}, \begin{bmatrix} 1 & w^\top \\ w & W \end{bmatrix} \right\rangle = 0.$$

In particular,

$$(\alpha_i \oplus a_i)^\top \begin{bmatrix} 1 & -x^\top \\ -x & X \end{bmatrix} (\alpha_i \oplus a_i) = 0 \quad \forall i \in [r],$$

which implies that

$$\begin{bmatrix} 1 & -x^\top \\ -x & X \end{bmatrix} (\alpha_i \oplus a_i) = 0 \quad \forall i \in [r],$$

and, in particular, $\langle a_i, x \rangle = \alpha_i$ for all $i \in [r]$. This proves (59).

Since \mathcal{F} has n affinely independent vectors, (59) implies that $r = 1$, i.e., \hat{W} is rank-one. Thus, $\text{diag}(W) = w$ and $\hat{W} \in \mathbb{S}_+^{\{0\} \cup V}$ imply that $w \in \{0, 1\}^V$. \square

9.1 Geometric Representations from Theta Bodies

The theta bodies described in (9) all have the form $\text{TH}(\mathbb{A}_{E^+, E^-}, \mathbb{S}_+^{\{0\} \cup V})$ for some $E^+, E^- \subseteq \binom{V}{2}$. The elements of these sets arise from certain vectors which may be regarded as geometric representations of graphs (see, e.g., [33]):

Proposition 25 *Let $E^+, E^- \subseteq \binom{V}{2}$. Then $\text{TH}(\mathbb{A}_{E^+, E^-}, \mathbb{S}_+^{\{0\} \cup V})$ consists of all vectors $x \in \mathbb{R}^V$ of the form*

$$x_i = \langle u_0, u_i \rangle^2 \quad \forall i \in V \tag{60}$$

for vectors $\{u_i : i \in \{0\} \cup V\} \subseteq \mathbb{R}^{\{0\} \cup V}$ satisfying the following properties:

$$\langle u_0, u_i \rangle \geq 0 \quad \forall i \in V, \tag{61a}$$

$$\|u_i\| = 1 \quad \forall i \in \{0\} \cup V, \tag{61b}$$

$$\langle u_i, u_j \rangle \geq 0 \quad \forall ij \in E^+, \tag{61c}$$

$$\langle u_i, u_j \rangle \leq 0 \quad \forall ij \in E^-. \tag{61d}$$

Proof Denote by \mathcal{C} the set of all vectors $x \in \mathbb{R}^V$ of the form given by (60) for vectors $\{u_i : i \in \{0\} \cup V\} \subseteq \mathbb{R}^{\{0\} \cup V}$ satisfying (61).

We first verify that

$$\mathcal{C} \subseteq \text{TH}(\mathbb{A}_{E^+, E^-}, \mathbb{S}_+^{\{0\} \cup V}). \tag{62}$$

Let $\{u_i : i \in \{0\} \cup V\} \subseteq \mathbb{R}^{\{0\} \cup V}$ satisfy (61). Define $U \in \mathbb{R}^{\{0\} \cup V \times V}$ by setting $Ue_i := u_i$ for every $i \in V$. Next, set $Y := U \text{Diag}(U^\top u_0)$ and

$$\hat{X} := \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} := \begin{bmatrix} u_0^\top u_0 & u_0^\top Y \\ Y^\top u_0 & Y^\top Y \end{bmatrix} = \begin{bmatrix} u_0^\top \\ Y^\top \end{bmatrix} \begin{bmatrix} u_0 & Y \end{bmatrix} \in \mathbb{S}_+^{\{0\} \cup V},$$

where we used (61b). Let us verify that

$$X \in \mathbb{A}_{E^+, E^-}, \quad (63a)$$

$$\text{diag}(X) = x, \quad (63b)$$

$$x_i = \langle u_0, u_i \rangle^2 \quad \forall i \in V. \quad (63c)$$

We start with (63a). Note that $X = Y^\top Y = \mathcal{D}_{U^\top u_0}(U^\top U)$ and $U^\top u_0 \geq 0$ by (61a). Since \mathbb{A}_{E^+, E^-} is diagonally scaling-invariant, it suffices to show that $U^\top U \in \mathbb{A}_{E^+, E^-}$. However, this is immediate from (61c) and (61d). This proves (63a). For (63c), note that

$$x = Y^\top u_0 = [\text{Diag}(U^\top u_0) U^\top] u_0 = (U^\top u_0) \odot (U^\top u_0). \quad (64)$$

By (61b), we have $\text{diag}(U^\top U) = \mathbb{1}$. Thus, $\text{diag}(X) = \text{diag}(\mathcal{D}_{U^\top u_0}(U^\top U)) = (U^\top u_0) \odot \text{diag}(U^\top U) \odot (U^\top u_0) = (U^\top u_0) \odot (U^\top u_0) = x$ by (64), thus proving (63b). It follows that $x \in \text{TH}(\mathbb{A}_{E^+, E^-}, \mathbb{S}_+^{\{0\} \cup V})$, and the proof of (62) is complete.

Now we show that

$$\text{TH}(\mathbb{A}_{E^+, E^-}, \mathbb{S}_+^{\{0\} \cup V}) \subseteq \mathcal{C}. \quad (65)$$

Let $x \in \text{TH}(\mathbb{A}_{E^+, E^-}, \mathbb{S}_+^{\{0\} \cup V})$, and let $\hat{X} \in \widehat{\text{TH}}(\mathbb{A}_{E^+, E^-}, \mathbb{S}_+^{\{0\} \cup V})$ such that $x = \text{diag}(X)$ for $X := \hat{X}[V]$. Let $Y \in \mathbb{R}^{\{0\} \cup V \times \{0\} \cup V}$ such that $\hat{X} = Y^\top Y$. Set $y_i := Ye_i$ for each $i \in \{0\} \cup V$. Let $Z := \{i \in V : y_i = 0\}$. Define $u_i := y_i / \|y_i\|$ for each $i \in \{0\} \cup (V \setminus Z)$ and let $\{u_i : i \in Z\}$ be an orthonormal basis for a subspace of $\{u_i : i \in \{0\} \cup (V \setminus Z)\}^\perp$ of appropriate dimension.

We must show that (61) holds. Note that (61a) for $i \in V \setminus Z$ follows from $\hat{X}e_0 = \text{diag}(\hat{X}) \geq 0$, and for $i \in Z$ it holds by construction. We also know that (61b) holds by construction. Let us check (61c). Let $ij \in E^+$. If i or j is in Z , then $\langle u_i, u_j \rangle = 0$, so we may assume that $i, j \in V \setminus Z$. Then $\|y_i\| \|y_j\| \langle u_i, u_j \rangle = \langle y_i, y_j \rangle = X_{ij} \geq 0$ since $X \in \mathbb{A}_{E^+, E^-}$. This completes the proof of (61c). The proof of (61d) is analogous, so (61) holds.

Lastly, we show that x is given by (60). Let $i \in V$. Since $\hat{X}e_0 = \text{diag}(\hat{X})$, we have $x_i = [Y^\top Y]_{0i} = \langle y_0, y_i \rangle = \|y_0\| \|y_i\| \langle u_0, u_i \rangle = X_{ii}^{1/2} \langle u_0, u_i \rangle = x_i^{1/2} \langle u_0, u_i \rangle$. If $x_i > 0$, then $x_i^{1/2} = \langle u_0, u_i \rangle$. Otherwise, $u_i \perp u_0$ by construction, so $x_i = 0 = \langle u_0, u_i \rangle^2$. This proves that x is given by (60) and completes the proof of (65). \square

An *orthonormal representation* of a graph $G = (V, E)$ is a map u that sends V into the unit vectors of some Euclidean space such that $\langle u_i, u_j \rangle = 0$ whenever $ij \in \bar{E}$. If, additionally, $\langle u_i, u_j \rangle \geq 0$ whenever $ij \in E$, then u is called an *acute orthonormal representation* of G . Finally, an *obtuse representation* of G is a map u from V to the unit vectors of some Euclidean space so that $\langle u_i, u_j \rangle \leq 0$ whenever $ij \in \bar{E}$.

Proposition 25 immediately leads to the following well-known internal description of the sets in (9).

Corollary 26 *Let $G = (V, E)$ be a graph. Let $\mathcal{C} \in \{\text{TH}(G), \text{TH}^-(G), \text{TH}^+(G)\}$. Then \mathcal{C} consists of all vectors $x \in \mathbb{R}^V$ of the form $x_i = \langle u_0, u_i \rangle^2$ for every $i \in V$ for some unit vectors in $\{u_i : i \in \{0\} \cup V\} \subseteq \mathbb{R}^{\{0\} \cup V}$ such that*

- (i) $u|_V$ is an orthonormal representation of \bar{G} , if $\mathcal{C} = \text{TH}(G)$;

- (ii) $u|_V$ is an acute orthonormal representation of \bar{G} and $\langle u_0, u_i \rangle \geq 0$ for all $i \in V$, if $\mathcal{C} = \text{TH}'(G)$;
 (iii) $u|_V$ is an obtuse representation of \bar{G} and $\langle u_0, u_i \rangle \geq 0$ for all $i \in V$, if $\mathcal{C} = \text{TH}^+(G)$.

Proof Immediate from Proposition 25. When $\mathcal{C} = \text{TH}(G)$, the constraint $\langle u_0, u_i \rangle \geq 0$ may be dropped, since for each orthonormal representation u of G and $i \in V$, the map obtained from u by replacing some image u_i by $-u_i$ is also an orthonormal representation of G . \square

9.2 Luz and Schrijver's Convex Quadratic Characterization

In this subsection, we show how to generalize the convex quadratic characterization from [37] to the context of generalized theta bodies over the cone $\widehat{\mathbb{K}} = \mathbb{S}_+^{\{0\} \cup V}$. This provides a generalization of their results to all weights $w \in \mathbb{R}_+^V$ to all of the functions ϑ , ϑ' , and ϑ^+ . We remark that a convex quadratic characterization of ϑ' as well as a weighted generalization of the convex quadratic characterization of ϑ were already known to Luz; see [35, 36].

Let $C \in \mathbb{S}^V$ such that $\text{diag}(C) = 0$. For each $w \in \mathbb{R}_+^V$, define

$$v(C; w) := \max \left\{ 2\langle w, x \rangle - x^T \mathcal{D}_{\sqrt{w}}(H_C + I)x : x \in \mathbb{R}_+^V \right\}, \quad (66)$$

where

$$H_C := [C \neq 0] \frac{C}{-\lambda_{\min}(C)}.$$

Note that $\text{diag}(C) = 0$ implies that $H_C + I \succeq 0$, so the quadratic program on the RHS of (66) is convex. In particular, there is an optimal solution whenever the optimal value is finite. The necessary and sufficient conditions for optimality are:

$$x \geq 0, \quad (67a)$$

$$\mathcal{D}_{\sqrt{w}}(H_C + I)x \geq w, \quad (67b)$$

$$x^T \mathcal{D}_{\sqrt{w}}(H_C + I)x = \langle w, x \rangle = v(C; w). \quad (67c)$$

Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a polyhedral diagonally scaling-invariant cone. Define

$$v(\mathbb{A}; w) := \inf \{ v(C; w) : C \in \mathbb{A} \cap \text{Null}(\text{diag}) \}.$$

We first show that $v(\mathbb{A}; w)$ provides an upper bound for $\vartheta_1(\bar{\mathbb{A}}, \mathbb{S}_+^V)$. Note the similarity with the proof of Theorem 5.

Proposition 27 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a diagonally scaling-invariant closed convex cone such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Let $w \in \mathbb{R}_+^V$. Then*

$$\min \left\{ \max_{i \in V} \frac{w_i}{y_i} : y \in \text{TH}(\mathbb{A}, \mathbb{S}_+^{\{0\} \cup V}) \right\} \leq v(\mathbb{A}; w).$$

Proof Let $C \in \mathbb{A} \cap \text{Null}(\text{diag})$ such that $v := v(C; w) < \infty$, and let \bar{x} be an optimal solution for the corresponding optimization problem (66). We shall use the optimality conditions (67) without further mention.

We may assume that $\text{supp}(\bar{x}) \subseteq \text{supp}(w)$, so we may also assume that $\text{supp}(w) = V$ and that $v > 0$. Write $H_C + I \succeq 0$ as $H_C + I = B^T B$ for some $B \in \mathbb{R}^{V \times V}$. Set

$$\begin{aligned} c &:= v^{-1/2} B \text{Diag}(\sqrt{w}) \bar{x}, \\ z &:= B^T c, \\ \bar{B} &:= B \text{Diag}(B^T c), \\ y &:= \bar{B}^T c = \text{Diag}(z) z = z \odot z. \end{aligned}$$

Note that $\mathcal{D}_{\sqrt{w}}(H_C + I) \bar{x} \geq w$ implies that

$$z = v^{-1/2} (H_C + I) \text{Diag}(\sqrt{w}) \bar{x} \geq v^{-1/2} \sqrt{w}. \quad (68)$$

We claim that

$$\hat{Y} := \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} := \begin{bmatrix} 1 & y^T \\ y & \bar{B}^T \bar{B} \end{bmatrix} \in \text{TH}(\mathbb{A}, \mathbb{S}_+^{\{0\} \cup V}). \quad (69)$$

Positive semidefiniteness of the matrix in (69) follows from the factorization

$$\begin{bmatrix} 1 & y^T \\ y & \bar{B}^T \bar{B} \end{bmatrix} = \begin{bmatrix} c^T \\ \bar{B}^T \end{bmatrix} \begin{bmatrix} c \\ \bar{B} \end{bmatrix}.$$

Since we have $H_C + I \in \mathbb{A}$ and $z \geq 0$ follows from (68), we get $\bar{B}^T \bar{B} = \mathcal{D}_z(H_C + I) \in \mathbb{A}$. Finally

$$\text{diag}(\bar{B}^T \bar{B}) = z \odot \text{diag}(B^T B) \odot z = y$$

since $\text{diag}(C) = 0$. This concludes the proof of (69), and so we have $y \in \text{TH}(\mathbb{A}, \mathbb{S}_+^{\{0\} \cup V})$. It follows from (68) that $y_i \geq v^{-1} w_i$ for each $i \in V$, so that $\max_{i \in V} w_i / y_i \leq v$. \square

Next we show how $v(\mathbb{A}; w)$ relates to $\vartheta_2(\bar{\mathbb{A}}, \mathbb{S}_+^V)$:

Theorem 28 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a polyhedral diagonally scaling-invariant cone such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Let $w \in \mathbb{R}_+^V$. Let \bar{Y} be an optimal solution for*

$$\vartheta := \min \left\{ \lambda_{\max}(Y + \sqrt{w} \sqrt{w}^T) : Y \in -\mathbb{A} \cap \text{Null}(\text{diag}) \right\}. \quad (70)$$

Then $\vartheta \geq v(-\bar{Y}; w)$.

Proof First note that the dual of (70) has $\frac{1}{|V|} I$ as a restricted Slater point, so an optimal solution for (70) exists and $\vartheta \geq 0$, with equality only if $w = 0$. We may thus assume that $\vartheta > 0$. Set $\bar{C} := -\bar{Y}$. Clearly, $\vartheta \geq \lambda_{\max}(\bar{Y})$. If $\vartheta = \lambda_{\max}(\bar{Y})$, then the objective value of an arbitrary $x \in \mathbb{R}_+^V$ in the formulation for $v(C; w)$ is

$$\begin{aligned} 2\langle w, x \rangle - x^T \mathcal{D}_{\sqrt{w}}(H_{\bar{C}} + I)x &= 2\langle w, x \rangle - \vartheta^{-1} x^T \mathcal{D}_{\sqrt{w}}(\vartheta I - \bar{Y} - \sqrt{w} \sqrt{w}^T)x - \vartheta^{-1} \langle w, x \rangle^2 \\ &\leq 2\langle w, x \rangle - \vartheta^{-1} \langle w, x \rangle^2 \leq \vartheta \end{aligned}$$

since $(\vartheta^{1/2} - \vartheta^{-1/2} \langle w, x \rangle)^2 \geq 0$, and hence $v(-\bar{Y}; w) \leq \vartheta$. Thus, we may assume that

$$\vartheta > \lambda_{\max}(\bar{Y}). \quad (71)$$

Let \bar{B} be an optimal solution for the dual of (70), given by $\sup\{\sqrt{w}^\top B \sqrt{w} : \text{Tr}(B) = 1, B \in \bar{\mathbb{A}} \cap \mathbb{S}_+^V\}$. Note that $\vartheta I - \bar{Y}$ is nonsingular by (71), so the rank of $\vartheta I - \bar{Y} - \sqrt{w} \sqrt{w}^\top$ is $\geq |V| - 1$. Since $\langle \bar{B}, \vartheta I - \bar{Y} - \sqrt{w} \sqrt{w}^\top \rangle = 0$ by complementarity, it follows that \bar{B} has rank one. Write $\bar{B} = \vartheta^{-1} \bar{y} \bar{y}^\top$ with \bar{y} having at least one positive component, set $S := \text{supp}(\bar{y})$ and $\bar{x} := \mathbb{1}_S$. Let us show that

$$\bar{y} = \text{Diag}(\sqrt{w}) \bar{x}, \quad (72a)$$

$$S \subseteq \text{supp}(w), \quad (72b)$$

$$\langle \sqrt{w}, \bar{y} \rangle = \langle w, \bar{x} \rangle = \vartheta, \quad (72c)$$

$$\|\bar{y}\|^2 = \vartheta. \quad (72d)$$

We shall apply Lemma 4. The inclusion (72b) follows from (21a). Equation (21d) yields $\langle \sqrt{w}, \bar{y} \rangle \bar{y} = \vartheta \text{Diag}(\sqrt{w})^\dagger \text{Diag}(\bar{y}) \bar{y}$. If we multiply this equation on the left by $\text{Diag}(\sqrt{w}) \text{Diag}(\bar{y})^\dagger$ and use (72b) we get

$$\langle \sqrt{w}, \bar{y} \rangle \text{Diag}(\sqrt{w}) \bar{x} = \vartheta \bar{y}. \quad (73)$$

Thus, $\bar{y} \geq 0$ or $\bar{y} \leq 0$, so that $\bar{y} \geq 0$ since \bar{y} has at least one positive component. Then $\vartheta = \sqrt{w}^\top \bar{B} \sqrt{w}$ implies that $\langle \sqrt{w}, \bar{y} \rangle = \vartheta$, which establishes (72a) via (73) and half of (72c); the other half follows from the half already established and (72a). Finally, (72d) follows from $\text{Tr}(\bar{B}) = 1$.

We claim that

$$\bar{x} \text{ satisfies the optimality conditions (67) for } \vartheta = v(\bar{C}; w). \quad (74)$$

Clearly (67a) holds. Since complementarity yields $(\vartheta I - \bar{Y} - \sqrt{w} \sqrt{w}^\top) \bar{y} = 0$, we have $\vartheta \bar{y} = (\sqrt{w} \sqrt{w}^\top + \bar{Y}) \bar{y}$ and thus $-\bar{Y} \bar{y} = \vartheta(\sqrt{w} - \bar{y})$ using (72c). Thus,

$$\mathcal{D}_{\sqrt{w}}(H_{\bar{C}} + I) \bar{x} = \text{Diag}(\sqrt{w}) \left[\frac{-\bar{Y}}{\lambda_{\max}(\bar{Y})} + I \right] \bar{y} = \text{Diag}(\sqrt{w}) \left[\frac{\vartheta}{\lambda_{\max}(\bar{Y})} (\sqrt{w} - \bar{y}) + \bar{y} \right]. \quad (75)$$

Since $\vartheta \geq \lambda_{\max}(\bar{Y})$ and $\sqrt{w} \geq \bar{y}$, we get from (75) that $\mathcal{D}_{\sqrt{w}}(H_{\bar{C}} + I) \bar{x} \geq \text{Diag}(\sqrt{w})[(\sqrt{w} - \bar{y}) + \bar{y}] = w$, so (67b) holds. By hitting the LHS of (75) on the left with \bar{x}^\top , we get from the RHS using (72) that $\bar{x}^\top \mathcal{D}_{\sqrt{w}}(H_{\bar{C}} + I) \bar{x} = \vartheta$. Thus, (67c) follows from (72c), and the proof of (74) is complete. \square

Corollary 29 *Let $\mathbb{A} \subseteq \mathbb{S}^V$ be a polyhedral diagonally scaling-invariant cone such that $\text{Im}(\text{Diag}) \subseteq \mathbb{A}$. Let $w \in \mathbb{R}_+^V$. Then*

$$v(\mathbb{A}; w) = \vartheta(\bar{\mathbb{A}}, \mathbb{S}_+^V; w).$$

Moreover, the optimization problem for $v(\mathbb{A}; w)$ has an optimal solution.

Proof Immediate from Proposition 27 and Theorem 28. \square

10 Conclusion

We took an axiomatic viewpoint in our study of the Lovász theta function and the related theta body of graphs. We generalized the binary encoding of graphs by the notion of diagonally scaling-invariant polyhedral cones and the semidefinite cone with a more general set of convex cones. These generalization and viewpoint led to graph complementation being replaced by convex polarity and to the new notion of Schur-Lifting of cones as the dual of commonly used PSD-Lifting of cones. Our new general theory has many advantages: we are able to treat the stable set polytope and many of its convex relaxations uniformly and extend the most commonly used equivalent characterizations of Lovász theta function and the most powerful properties to these generalized theta bodies.

References

1. N. E. Aguilera, M. S. Escalante, and G. L. Nasini. A generalization of the perfect graph theorem under the disjunctive index. *Math. Oper. Res.*, 27(3):460–469, 2002. 3
2. N. Alon, K. Makarychev, Y. Makarychev, and A. Naor. Quadratic forms on graphs. *Invent. Math.*, 163(3):499–522, 2006. 2
3. C. Bachoc, G. Nebe, F. M. de Oliveira Filho, and F. Vallentin. Lower bounds for measurable chromatic numbers. *Geom. Funct. Anal.*, 19(3):645–661, 2009. 2
4. R. Bačík. *Structure of Graph Homomorphisms*. PhD thesis, Simon Fraser University, 1997. 2
5. R. Bačík and S. Mahajan. Semidefinite programming and its applications to NP problems. In *Computing and combinatorics (Xi'an, 1995)*, volume 959 of *Lecture Notes in Comput. Sci.*, pages 566–575. Springer, Berlin, 1995. 2
6. A. Ben-Tal and A. Nemirovski. *Lectures on modern convex optimization*. MPS/SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Programming Society (MPS), Philadelphia, PA, 2001. Analysis, algorithms, and engineering applications. 20
7. Y. Bilu. Tales of Hoffman: three extensions of Hoffman’s bound on the graph chromatic number. *J. Combin. Theory Ser. B*, 96(4):608–613, 2006. 26
8. M. K. de Carli Silva. *Geometric Ramifications of the Lovász Theta Function and Their Interplay with Duality*. PhD thesis, Department of Combinatorics and Optimization, University of Waterloo, 2013. <http://hdl.handle.net/10012/7812>. 20
9. M. K. de Carli Silva and L. Tunçel. Optimization problems over unit-distance representations of graphs. *Electron. J. Combin.*, 20(1):Research Paper 43, 20 pp. (electronic), 2013. 3
10. M. K. de Carli Silva and L. Tunçel. Vertices of spectrahedra arising from the ellipsope, the theta body, and their relatives. *SIAM J. Optim.*, 25(1):295–316 (electronic), 2015. 26
11. Y. H. Chan and L. C. Lau. On linear and semidefinite programming relaxations for hypergraph matching. *Math. Program.*, 135(1-2, Ser. A):123–148, 2012. 2
12. I. Csizsár, J. Körner, L. Lovász, K. Marton, and G. Simonyi. Entropy splitting for antiblocking corners and perfect graphs. *Combinatorica*, 10(1):27–40, 1990. 2
13. E. de Klerk and D. V. Pasechnik. Approximation of the stability number of a graph via copositive programming. *SIAM J. Optim.*, 12(4):875–892 (electronic), 2002. 21
14. I. Dukanovic and F. Rendl. Copositive programming motivated bounds on the stability and the chromatic numbers. *Math. Program.*, 121(2, Ser. A):249–268, 2010. 3, 20, 21, 22
15. U. Feige and L. Lovász. Two-prover one-round proof systems: their power and their problems (extended abstract). In *Proceedings of the twenty-fourth annual ACM symposium on Theory of computing*, STOC ’92, pages 733–744, New York, NY, USA, 1992. ACM. 2
16. A. Galtman. Spectral characterizations of the Lovász number and the Delsarte number of a graph. *J. Algebraic Combin.*, 12(2):131–143, 2000. 2
17. A. M. H. Gerards, G. Maróti, and A. Schrijver. A note on: “A generalization of the perfect graph theorem under the disjunctive index” [Math. Oper. Res. 27 (2002), no. 3, 460–469; mr1926653] by N. E. Aguilera, M. S. Escalante and G. L. Nasini. *Math. Oper. Res.*, 28(4):884–885, 2003. 3
18. L. E. Gibbons, D. W. Hearn, P. M. Pardalos, and M. V. Ramana. Continuous characterizations of the maximum clique problem. *Math. Oper. Res.*, 22(3):754–768, 1997. 22
19. D. Gijswijt. *Matrix Algebras and Semidefinite Programming Techniques for Codes*. PhD thesis, University of Amsterdam, 2005. 10
20. M. X. Goemans. Semidefinite programming in combinatorial optimization. *Math. Programming*, 79(1-3, Ser. B):143–161, 1997. Lectures on mathematical programming (ismp97) (Lausanne, 1997). 2
21. J. Gouveia, P. A. Parrilo, and R. R. Thomas. Theta bodies for polynomial ideals. *SIAM J. Optim.*, 20(4):2097–2118, 2010. 7
22. M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2 of *Algorithms and Combinatorics: Study and Research Texts*. Springer-Verlag, Berlin, 1988. 2
23. M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, second edition, 1993. 2, 4, 16, 20
24. O. Güler. Barrier functions in interior point methods. *Math. Oper. Res.*, 21(4):860–885, 1996. 14
25. N. Gvozdenović and M. Laurent. The operator Ψ for the chromatic number of a graph. *SIAM J. Optim.*, 19(2):572–591, 2008. 2
26. A. J. Hoffman. On eigenvalues and colorings of graphs. In *Graph Theory and its Applications (Proc. Advanced Sem., Math. Research Center, Univ. of Wisconsin, Madison, Wis., 1969)*, pages 79–91. Academic Press, New York, 1970. 2, 24
27. R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original. 5, 25
28. D. Karger, R. Motwani, and M. Sudan. Approximate graph coloring by semidefinite programming. *J. ACM*, 45(2):246–265, 1998. 2
29. J. Kleinberg and M. X. Goemans. The Lovász theta function and a semidefinite programming relaxation of vertex cover. *SIAM J. Discrete Math.*, 11(2):196–204 (electronic), 1998. 2
30. D. E. Knuth. The sandwich theorem. *Electron. J. Combin.*, 1:Article 1, approx. 48 pp. (electronic), 1994. 4, 20, 24
31. L. Lipták and L. Tunçel. Lift-and-project ranks and antiblocker duality. *Oper. Res. Lett.*, 33(1):35–41, 2005. 3
32. L. Lovász. On the Shannon capacity of a graph. *IEEE Trans. Inform. Theory*, 25(1):1–7, 1979. 2, 24, 26

33. L. Lovász. Geometric representations of graphs. <http://www.cs.elte.hu/~lovasz/geomrep.pdf>, 2009. Accessed: 2014-09-08. 27
34. L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0-1 optimization. *SIAM J. Optim.*, 1(2):166–190, 1991. 2
35. C. J. Luz. A convex quadratic characterization of the McEliece-Rodemich-Rumsey and Schrijver upper bound on the stability number of a graph. *Cadernos de Matemática CM06/I-14*, Departamento de Matemática, Universidade de Aveiro, 2006. 29
36. C. J. Luz. A characterization of the weighted Lovász number based on convex quadratic programming. *Optimization Letters* (online), pages 1–13, DOI 10.1007/s11590-015-0911-6, June 2015. 29
37. C. J. Luz and A. Schrijver. A convex quadratic characterization of the Lovász theta number. *SIAM J. Discrete Math.*, 19(2):382–387 (electronic), 2005. 2, 3, 26, 29
38. K. Marton. On the Shannon capacity of probabilistic graphs. *J. Combin. Theory Ser. B*, 57(2):183–195, 1993. 2
39. R. J. McEliece, E. R. Rodemich, and H. C. Rumsey, Jr. The Lovász bound and some generalizations. *J. Combin. Inform. System Sci.*, 3(3):134–152, 1978. 2
40. Yu. Nesterov and A. Nemirovskii. *Interior-point polynomial algorithms in convex programming*, volume 13 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994. 14
41. R. T. Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks. 5, 15, 17, 20
42. A. Schrijver. A comparison of the Delsarte and Lovász bounds. *IEEE Trans. Inform. Theory*, 25(4):425–429, 1979. 2
43. A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. B*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Matroids, trees, stable sets, Chapters 39–69. 2, 16, 26
44. F. B. Shepherd. The theta body and imperfection. In *Perfect graphs*, Wiley-Intersci. Ser. Discrete Math. Optim., pages 261–291. Wiley, Chichester, 2001. 26
45. G. Simonyi. Perfect graphs and graph entropy. An updated survey. In *Perfect graphs*, Wiley-Intersci. Ser. Discrete Math. Optim., pages 293–328. Wiley, Chichester, 2001. 2
46. M. Szegedy. A note on the theta number of Lovász and the generalized Delsarte bound. In *Proceedings of the 35th Annual IEEE Symposium on Foundations of Computer Science*, 1994. 2
47. É. Tardos. The gap between monotone and nonmonotone circuit complexity is exponential. *Combinatorica*, 8(1):141–142, 1988. 2
48. V. A. Truong and L. Tunçel. Geometry of homogeneous convex cones, duality mapping, and optimal self-concordant barriers. *Math. Program.*, 100(2):295–316, 2004. 14

Antiblocking Duality

Antiblocking duality is the form of duality most suitable for convex corners, and we used extensively the basic facts of this duality theory in this paper. Even though such facts are well known, we are not aware of any previous treatment in the literature of antiblocking duality for non-polyhedral convex corners which meets the needs of this paper. Thus, for the sake of completeness, we include a brief, self-contained description of this theory.

Recall from Section 2 that a convex corner is a compact, lower-comprehensive convex subset of the nonnegative orthant with nonempty interior, and that the antiblocker of a convex corner $\mathcal{C} \subseteq \mathbb{R}_+^n$ is $\text{abl}(\mathcal{C}) := \{y \in \mathbb{R}_+^n : \langle y, x \rangle \leq 1 \forall x \in \mathcal{C}\}$. Note that $\text{abl}(\mathcal{C})$ is also a convex corner. Closedness, convexity and inclusion in \mathbb{R}_+^n are clear, whereas lower-comprehensiveness follows from the inclusion $\mathcal{C} \subseteq \mathbb{R}_+^n$. Boundedness of $\text{abl}(\mathcal{C})$ follows from $\mathcal{C} \cap \mathbb{R}_+^n \neq \emptyset$, since $\text{int}(\mathcal{C}) \neq \emptyset$. Finally, since \mathcal{C} is bounded, $\text{abl}(\mathcal{C})$ has nonempty interior: if $M \in \mathbb{R}$ is such that $x \leq n^{-1}M\mathbb{1}$ for all $x \in \mathcal{C}$, then $M^{-1}\mathbb{1} \in \text{abl}(\mathcal{C})$, so that $\frac{1}{2}M^{-1}\mathbb{1} \in \text{int}(\text{abl}(\mathcal{C}))$ by lower-comprehensiveness of $\text{abl}(\mathcal{C})$.

The key fact of antiblocking duality is that $\text{abl}(\cdot)$ defines an involution on the class of convex corners, which we prove next.

Theorem 30 *If $\mathcal{C} \subseteq \mathbb{R}_+^n$ is a convex corner, then $\text{abl}(\text{abl}(\mathcal{C})) = \mathcal{C}$.*

Proof The inclusion ‘ \supseteq ’ is trivial. For the other inclusion, let $u \in \mathbb{R}^n \setminus \mathcal{C}$. If $u \notin \mathbb{R}_+^n$, then $u \in \mathbb{R}^n \setminus \text{abl}(\text{abl}(\mathcal{C}))$, so assume that $u \geq 0$. Let $x^* \in \mathcal{C}$ minimize $\|u - x\|^2$ over $x \in \mathcal{C}$. Set

$$y := u - x^* \quad \text{and} \quad \mu := \langle y, x^* \rangle.$$

We claim that

$$\langle y, x \rangle \leq \mu \quad \forall x \in \mathcal{C}, \quad \text{and} \quad \langle y, u \rangle > \mu. \quad (76)$$

Let $x \in \mathcal{C}$. If $t \in (0, 1]$, then $x^* + t(x - x^*) \in \mathcal{C}$, so that $\|u - x^*\|^2 \leq \|u - x^* - t(x - x^*)\|^2$, whence $\langle u - x^*, x - x^* \rangle \leq t\|x - x^*\|^2/2$. By sending $t \downarrow 0$, we get $\langle u - x^*, x - x^* \rangle \leq 0$, which implies the first half of (76). The second half is easily seen to be equivalent to $\|y\|^2 > 0$, which follows from the fact that $x^* \in \mathcal{C}$ whereas $u \notin \mathcal{C}$. This concludes the proof of (76).

Next we show that

$$y \geq 0. \quad (77)$$

Suppose that $y_i < 0$ for some $i \in \{1, \dots, n\}$. Then $x_i^* > u_i \geq 0$. Let $\varepsilon > 0$ such that $x_i^* \geq \varepsilon$ and set $\bar{x} := x^* - \varepsilon e_i \in \mathcal{C}$. By (76), we get $\mu \geq \langle y, \bar{x} \rangle = \langle y, x^* \rangle - \varepsilon \langle y, e_i \rangle = \mu - \varepsilon y_i > \mu$, a contradiction. This proves (77), which yields $\mu > 0$ when combined with (76) and $\text{int}(\mathcal{C}) \neq \emptyset$. Now (76) and (77) show that $\frac{1}{\mu}y \in \text{abl}(\mathcal{C})$, so $u \in \mathbb{R}^n \setminus \text{abl}(\text{abl}(\mathcal{C}))$ by (76). This concludes the proof of ‘ \subseteq ’. \square

We thus obtain the following optimality conditions:

Corollary 31 *Let $\mathcal{C} \subseteq \mathbb{R}_+^n$ be a convex corner. Let $w \in \mathbb{R}_+^n$, and let $\vartheta \in \mathbb{R}_+$. Then $\delta^*(w | \text{abl}(\mathcal{C})) = \vartheta$ if and only if there exists $x^* \in \text{abl}(\mathcal{C})$ and $y^* \in \mathcal{C}$ such that $\langle y^*, x^* \rangle = 1$ and $\vartheta y^* = w$.*

Proof First we prove necessity. If $w = 0$, let $x^* \in \text{abl}(\mathcal{C})$ maximize $\|x\|^2$ over $x \in \text{abl}(\mathcal{C})$, and set $y^* := x^* / \|x^*\|^2$; if $x \in \text{abl}(\mathcal{C})$, then $\langle y^*, x \rangle \leq \|x\| \|x^*\| / \|x^*\|^2 \leq 1$ by Cauchy-Schwarz, so $y^* \in \mathcal{C}$ by Theorem 30. Assume that $w \neq 0$, so $\vartheta > 0$. Let x^* be maximize $\langle w, x \rangle$ over $x \in \text{abl}(\mathcal{C})$, and set $y^* := w / \vartheta$. If $x \in \text{abl}(\mathcal{C})$, then $\langle y^*, x \rangle \leq 1$ since $\vartheta = \delta^*(w | \text{abl}(\mathcal{C}))$, so $y^* \in \mathcal{C}$ by Theorem 30. Now we prove sufficiency. If $x \in \text{abl}(\mathcal{C})$, then $\langle w, x \rangle = \vartheta \langle y^*, x \rangle \leq \vartheta$ shows that $\delta^*(w | \text{abl}(\mathcal{C})) \leq \vartheta$, whereas $\langle w, x^* \rangle = \vartheta \langle y^*, x^* \rangle = \vartheta$ shows that equality holds. \square