STRICT COMPLEMENTARITY IN SEMIDEFINITE OPTIMIZATION WITH ELLIPTOPES INCLUDING THE MAXCUT SDP

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ABSTRACT. The MaxCut approximation algorithm by Goemans and Williamson is one of the most celebrated results in semidefinite optimization, and the corresponding MaxCut SDP has many favourable properties. The feasible regions of this class of SDPs are known as elliptopes, and they have been studied extensively. One of their nicest geometric/duality properties is the fact that their vertices correspond exactly to the cuts of a graph, as proved by Laurent and Poljak in 1995. Recall that a boundary point x of a convex set $\mathscr{C}$ is called a vertex of $\mathscr{C}$ if the normal cone of $\mathscr{C}$ at x is full-dimensional. Semidefinite programs over elliptopes were also exploited by Goemans and Williamson and by Nesterov to develop approximation algorithms for the Maximum-2-Satisfiability problem and for nonconvex quadratic optimization problems, respectively.

We study how often strict complementarity holds or fails for SDPs over elliptopes when a vertex is optimal, i.e., when the SDP relaxation is tight. While strict complementarity is known to hold when the objective function is in the interior of the normal cone at any vertex, we prove that it fails generically (in a context of Hausdorff measure and Hausdorff dimension) at the boundary of such normal cones. In this regard, SDPs over elliptopes display the nastiest behavior possible for a convex optimization problem.

We also study strict complementarity with respect to two classes of objective functions. We show that, when the objective functions are sampled uniformly from a class of negative semidefinite rank-one matrices in the boundary of the normal cone at any vertex, the probability that strict complementarity holds lies in $(0, 1)$. To complete our study with a spectral graph theory based viewpoint of the data for the MaxCut SDP, we extend a construction due to Laurent and Poljak of weighted Laplacian matrices for which strict complementarity fails. Their construction works for complete graphs, and we extend it to cosums of graphs under some mild conditions.

1. INTRODUCTION

Consider a linear optimization problem (LP)

$$\max\{\, c^{\mathsf{T}}x : Ax = b,\, x \geq 0\} \tag{1}$$

and its dual $\min\{\, b^{\mathsf{T}}y : A^{\mathsf{T}}y \geq c\}$, where $A \in \mathbb{R}^{m\times n}$ is a matrix, and $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are vectors. In this context, a pair $(\bar{x}, \bar{y})$ of primal-dual feasible solutions is *complementary* if $\bar{x}_i\bar{s}_i = 0$ for each $i \in [n] := \{1, \dots, n\}$, where $\bar{s} := A^{\mathsf{T}}\bar{y} - c \geq 0$ is the slack. For each $i \in [n]$, since $\bar{s}_i$ is the slack in the dual constraint generated by the primal variable x_i, the definition of complementarity requires that at least one of the feasibility inequalities $x_i \geq 0$ (in the primal) and $(A^{\mathsf{T}}y)_i \geq c_i$ (in the dual) must be tight, i.e., they cannot both have a slack. Since slack variables allow us to express the complementarity condition more concisely, we shall always write them explicitly, and we will instead refer to a pair $(\bar{x}, \bar{y} \oplus \bar{s})$ of primal-dual solutions, in place of the more cumbersome notation $(\bar{x}, (\bar{y}, \bar{s}))$.

Complementary slackness is a fundamental optimality condition, and hence ubiquitous in optimization. It may be stated in the more general setting of continuous optimization (see [4]), and it can be expressed very conveniently in structured convex optimization (see [32]). In this paper, we are mainly concerned with semidefinite optimization problems (SDPs), in the format:

$$\max\{\, \mathrm{Tr}(CX) : \mathcal{A}(X) = b,\, X \succeq 0\}; \tag{2}$$

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here, as usual, we equip the space $\mathbb{S}^n$ of symmetric n-by-n matrices with the trace inner-product $\langle C, X\rangle := \mathrm{Tr}(CX^{\mathsf{T}}) = \sum_{i,j} C_{ij}X_{ij}$, the map $\mathcal{A}\colon \mathbb{S}^n \to \mathbb{R}^m$ is linear, and $X \succeq 0$ denotes that $X \in \mathbb{S}^n$ is positive semidefinite; most of our notation can be found in Tables 1 to 5. The dual SDP is

$$\min\{\, b^{\mathsf{T}}y : S = \mathcal{A}^*(y) - C,\, S \succeq 0\}, \tag{3}$$

where $\mathcal{A}^*\colon \mathbb{R}^m \to \mathbb{S}^n$ is the adjoint of $\mathcal{A}$, and a pair $(\bar{X}, \bar{y} \oplus \bar{S})$ of primal-dual feasible solutions is called *complementary* if $\mathrm{Tr}(\bar{X}\bar{S}) = 0$; equivalently, if $\bar{X}\bar{S} = 0$, since $\bar{X}, \bar{S} \succeq 0$.

Strict complementarity is a refinement of the notion of complementary slackness where we require *precisely one* of the feasibility conditions involved to be tight, which forces the other one to have a slack. A pair $(\bar{x}, \bar{y} \oplus \bar{s})$ of primal-dual feasible solutions for the LP in (1) and its dual is *strictly complementary* if $\bar{x}_i\bar{s}_i = 0$ and $\bar{x}_i + \bar{s}_i > 0$ for every $i \in [n]$. A pair $(\bar{X}, \bar{y} \oplus \bar{S})$ of primal-dual feasible solutions for the SDP in (2) and its dual is *strictly complementary* if $\bar{X}\bar{S} = 0$ and $\bar{X} + \bar{S} \succ 0$, i.e., $\bar{X} + \bar{S}$ is positive definite. The latter two notions can be neatly unified in the context of convex conic optimization via the concept of faces (see [33]).

Complementary slackness characterizes primal-dual optimality whenever Strong Duality holds, in both LPs and SDPs: a primal-dual pair of feasible solutions is optimal if and only if it is complementary. This is sometimes described by saying that *complementary slackness holds* for the (primal-dual pair of) programs. In the case of LPs, whenever primal and dual are both feasible, there exists a primal-dual pair of optimal solutions that is strictly complementary [15]; i.e., *strict complementarity holds* for every primal-dual pair of feasible LPs. However, there exist primal-dual pairs of SDPs (which satisfy strong regularity conditions sufficient for SDP Strong Duality) that have no strictly complementary primal-dual pair of optimal solutions (see [37]); in such cases, we say that *strict complementarity fails* for said primal-dual pair of SDPs. In fact, failure of strict complementarity is deeply related to failure of Strong Duality in the context of convex conic optimization [40].

Existence of a strictly complementary pair of optimal solutions is crucial in continuous optimization in general and in semidefinite optimization in particular. This strict complementarity property is needed or is very useful in the following scenarios:

- identifying the set of optimal solutions in primal and dual problems, detecting infeasibility and unboundedness, and efficiently recovering certificates for these [19, 29, 42];
- for establishing various optimality conditions [37];
- for establishing (finite) convergence of various SDP relaxations for polynomial optimization problems [23, 31];
- convergence of the central path to the analytic center of the optimal face [17];
- superlinear/quadratic convergence theory for interior-point algorithms [20, 21, 27, 30];
- understanding SDP instances where Strong Duality fails [40];
- establishing stability results and error bounds [2, 5, 38].

Hence, it is important to determine whether the strict complementarity property holds for a given class of SDPs.

It is known that strict complementarity holds generically for SDPs [1, 16]; for a generalization to convex optimization problems, see [32]. (We shall expand below on a more precise meaning of "generic".) However, there are some generic properties of LPs that fail in some natural, highly structured formulations arising in combinatorial optimization. For instance, whereas systems of linear inequalities are well-known to be generically nondegenerate, the natural description of many classical polytopes is degenerate (e.g., for the matching polytope, see [36, Theorem 25.4]), and "...most real-world LP problems are degenerate" according to [41]. Thus, one ought to be careful about strict complementarity when approaching combinatorial optimization problems via SDP relaxations.

In this paper, we study how often strict complementarity holds or fails for semidefinite programs over the *elliptope* $\mathscr{E}_n$, the set of n-by-n positive semidefinite matrices with all diagonal entries equal to one. Such SDPs (and their duals) may be written as:

$$\begin{array}{llcll} \max & \mathrm{Tr}(CX) & = & \min & \mathbb{1}^{\mathsf{T}}y \\ & \mathrm{diag}(X) = \mathbb{1}, & & & S = \mathrm{Diag}(y) - C, \\ & X \succeq 0, & & & S \succeq 0; \end{array} \tag{4}$$

here, $\mathrm{diag}\colon \mathbb{S}^n \to \mathbb{R}^n$ extracts the diagonal, $\mathrm{Diag}\colon \mathbb{R}^n \to \mathbb{S}^n$ is the adjoint of diag, and $\mathbb{1}$ is the vector of all-ones. Strong Duality holds for every $C \in \mathbb{S}^n$ since both SDPs have *Slater points*, i.e., feasible solutions that are positive definite.

When the objective matrix C is a weighted Laplacian matrix of a graph, the primal problem in (4) is known as the *MaxCut SDP*. Recall that the *MaxCut problem* for a given graph $G = (V, E)$ on $V = [n]$ and weight function $w\colon E \to \mathbb{R}$ can be cast as the optimization problem $\max\{\, x^\mathsf{T} Cx : x \in \{\pm 1\}^n \},$ where $C \in \mathbb{S}^n$ is defined as

$$4C \coloneqq \mathcal{L}_G(w) \coloneqq \sum_{\{i,j\}\in E} w_{\{i,j\}}(e_i - e_j)(e_i - e_j)^\mathsf{T} \tag{5}$$

and $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$. The matrix $\mathcal{L}_G(w)$ is known as (a weighted) *Laplacian* matrix of G, and it is simple to check that $\mathcal{L}_G(w) \succeq 0$ if $w \geq 0$. Hence, the MaxCut SDP is a relaxation of the MaxCut problem. This SDP was used in the celebrated approximation algorithm by Goemans and Williamson [14]. More generally, semidefinite programs over elliptopes were also exploited by Goemans and Williamson [14] to obtain an approximation algorithm for the Maximum-2-Satisfiability problem and by Nesterov [28] in designing approximation algorithms for nonconvex quadratic optimization problems.

The structure of the elliptope has been extensively studied; see [10]. An important feature of the elliptope in relation to the MaxCut problem (and other combinatorial applications) is that the vertices of $\mathscr{E}_n$ are precisely its elements that are rank-one matrices [24], i.e., matrices of the form xx^T with $x \in \{\pm 1\}^n$. Thus, they correspond precisely to the *exact* solutions of the MaxCut problem, for which the MaxCut SDP is a relaxation. The vertices of $\mathscr{E}_n$ are by definition the points of $\mathscr{E}_n$ whose normal cones are full-dimensional (we postpone the definition of normal cone to Section 2.2).

It is known [6] (see Proposition 5 below) that strict complementarity holds in (4) precisely when C lies in the relative interior of the normal cone of *some* $X \in \mathscr{E}_n$. In particular, if $\bar{X}$ is a vertex of $\mathscr{E}_n$, then strict complementarity holds for (4) whenever C is in the interior of the normal cone of $\mathscr{E}_n$ at $\bar{X}$. An imprecise though intuitive way to interpret this is that, since the interior of a full-dimensional cone makes up its "bulk", the probability of sampling C in the normal cone at $\bar{X}$ such that strict complementarity holds is 1. However, when C lies in the boundary of this normal cone, it is not clear whether strict complementarity holds, since C may or may not be in the relative interior of the normal cone of some other $X \in \mathscr{E}_n$.

A very natural question is whether strict complementarity holds for every SDP over the elliptope, that is, for every objective function. It turns out that this question was already answered by Laurent and Poljak [25] in a slightly different context. They answered in the negative by providing, for each integer $n \geq 3$, a weighted Laplacian matrix for which strict complementarity fails for MaxCut SDP when G is the complete graph K_n. Moreover, in these cases the unique optimal solution is at a vertex of the elliptope, i.e., the relaxation is exact. Hence, these SDPs are very favorable with respect to so many geometric and duality properties, yet they have a rather nasty behaviour with respect to strict complementarity. A very simple Laplacian matrix provided by Laurent and Poljak's construction is

$$4C = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix},$$

which corresponds to taking a triangle with edge weights 2, 2, and 1. (See also Section 3 below.) Note that this example has the smallest possible number of vertices with failure of strict complementarity since $\mathscr{E}_2$ is polyhedral[1]. By the discussion above, such weighted Laplacians lie in the boundary of the normal cone of a vertex of the elliptope.

In this paper, we continue the study of strict complementarity for SDPs over elliptopes with objective matrix in the boundary of the normal cone of a vertex. We prove that, when C is chosen from the boundary of the normal cone at any vertex of the elliptope $\mathscr{E}_n$ with $n \geq 3$, strict complementarity almost always fails for (4); in this regard, surprisingly, SDPs over elliptopes display the worst possible behavior for a convex optimization problem. In order to make the statement "almost always fails" rigorous, we shall make use of Hausdorff measures. However, our treatment is self-contained and it does not require prior in-depth knowledge of the theory of Hausdorff measures.

[1]It is worthwhile to point out that strict complementarity may fail even when the set of optimal solutions is not a singleton: consider the objective function $-bb^\mathsf{T} \in \mathbb{S}^4$ with $b \coloneqq [0, 2, -1, -1]^\mathsf{T}$, for which the set of optimal solutions is the line segment between the vertices $\mathbb{1}\mathbb{1}^\mathsf{T}$ and $\bar{x}\bar{x}^\mathsf{T}$ with $\bar{x} \coloneqq [-1, 1, 1, 1]^\mathsf{T}$.

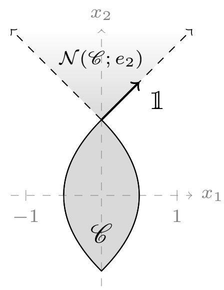

FIGURE 1. The set $\mathscr{C}$ defined in (6) and its normal cone $\mathcal{N}(\mathscr{C}; e_2)$ at e_2.

Intuitively as above, one may consider a randomized experiment where one samples an objective matrix C uniformly at the boundary of the normal cone at $\bar{X}$ and checks if strict complementarity holds for (4). Naturally, there is a problem with sampling uniformly from a cone, which is unbounded. A simple workaround would be to normalize the cone, albeit with some arbitrary normalization factor. For example, suppose we take the sample space to be the unit vectors in the boundary of the normal cone, and that the normal cone lives in an N-dimensional space. Then the sample space is the intersection of a sphere centered at the origin and the boundary of a full-dimensional cone, so we cannot use $(N-1)$-dimensional Lebesgue measure (of the sphere) to represent our "conditional" probability space. These issues are neatly solved by the elegant theory of Hausdorff measures, which allows us to compute d-dimensional measures in N-dimensional space even when $d < N$, and even on sets that are highly nonlinear and nonconvex, such as the boundaries in discussion. In fact, Hausdorff measure theory applies to any metric space, which might not have *a priori* any natural notion of (linear) dimension. The theory actually allows us to determine, for each subset of our ambient space, the most natural dimension d with which to perform the measuring as described; this is the Hausdorff dimension of such set. In this sense, we will say that a property P fails generically at a set $\mathscr{C}$ if, by taking d to be the Hausdorff dimension of $\mathscr{C}$, the set of points of $\mathscr{C}$ for which P holds has zero d-dimensional Hausdorff measure.

We also focus on two classes of objective functions for (4). We prove that, when C is sampled uniformly from (a normalization of) the negative semidefinite rank-one matrices in the normal cone at a vertex of the elliptope, the probability that strict complementarity fails for (4) is in $(0, 1)$. Naturally, we shall also use Hausdorff measures to achieve this. As Laurent and Poljak [24] proved, it is NP-hard to decide whether the optimal value of our SDP with such a rank-one matrix C is attained at a vertex. A consequence of this result is that this special class of SDPs are indeed very interesting especially in connection with strict complementarity holding or failing at a vertex of the underlying elliptope. Finally, we also extend the construction by Laurent and Poljak [25] mentioned above. Whereas their construction provides weighted Laplacian matrices that lead to failure of strict complementarity for the MaxCut SDP for complete graphs, we extend it to graphs which are cosums where one of the summands is connected and with some mild condition relating the maximum eigenvalues of their Laplacians.

It is useful to consider a low-dimensional case that helps to understand more intuitively the geometry of normal cones of elliptopes at vertices and why their boundaries do not play nicely with strict complementarity. Take, for instance, the convex set $\mathscr{C} \subseteq \mathbb{R}^2$ in Figure 1. For concreteness, an explicit description of $\mathscr{C}$ is given by

$$\mathscr{C} := \{\, x \in \mathbb{R}^2 : \|x\| + |x_1| \le 1 \,\} = \{\, x \in \mathbb{R}^2 : |x_1| \le 1/2,\ |x_2| \le \sqrt{1 - 2|x_1|} \,\}, \tag{6}$$

and it is not hard to show that $\mathscr{C}$ is the projection of the feasible region of an SDP. It is intuitive and simple to verify that $\mathbb{1}$ lies in (the boundary of) the normal cone of $\mathscr{C}$ at its vertex e_2, but $\mathbb{1}$ is not in the relative interior of any normal cone of $\mathscr{C}$. We can trace this phenomenon to the smooth, nonpolyhedral boundary of $\mathscr{C}$ around e_2. It is straightforward to extend this example to $\mathbb{R}^3$ by considering the solid of revolution obtained by rotating $\mathscr{C}$ around the e_2 axis, i.e., an American football.

The elliptope looks somewhat similar to $\mathscr{C}$ in the following sense. Let us consider the projection $\mathscr{E}'_n \subseteq \mathbb{R}^{\binom{n}{2}}$ of the elliptope $\mathscr{E}_n$ into its off-diagonal entries. For $n \ge 3$, the set $\mathscr{E}'_n$ is a compact nonpolyhedral convex set with 2^{n-1} vertices. Intuitively, $\mathscr{E}'_n$ can be thought of as being obtained from the polytope which is the convex hull of these 2^{n-1} vertices by inflating it like a balloon, while preserving the vertices fixed. (In fact, by [24, Proposition 2.9], the line segments between the 2^{n-1} vertices are also kept fixed.) In this way, $\mathscr{E}_n'$ is a round, plump convex set, whose boundary is smooth almost everywhere, and the neighborhood of $\mathscr{E}_n'$ around any vertex looks like (a generalization of) what is depicted by the set $\mathscr{C}$ from the previous paragraph. Thus, when one considers that the elliptope around a vertex "locally" looks like $\mathscr{C}$ around e_2, the poor behavior of SDPs over elliptopes that we describe in this paper makes more intuitive sense.

The order in which our results are presented is different from what we described above. Since the weighted Laplacian construction generalized from Laurent and Poljak involves only matrix analysis and spectral graph theory, and no measure theory, we start with that result. Only then we shall delve into measure theory tools to prove the other results. Hence, the rest of this paper is organized as follows. Section 2 contains some preliminaries, such as notation and background results about SDPs over elliptopes. In Section 3 we discuss failure of strict complementarity for (4) using previous results by Laurent and Poljak and we extend their Laplacian construction to cosums of graphs. In Section 4, we develop some Hausdorff measure basics and use them to prove that strict complementarity fails generically ("almost everywhere") for the SDPs over elliptopes when the objective function is in the boundary of the normal cone of a vertex of the elliptope. Finally, in Section 5, we zoom into the set of negative semidefinite rank-one matrices in the latter boundary, and prove that in this case the probability that strict complementarity holds is in $(0, 1)$.

2. Preliminaries

We refer the reader to Tables 1 to 5 for our mostly standard notation and terminology. In order to treat $\mathbb{R}^n$ and $\mathbb{S}^n$ uniformly, we adopt the language of Euclidean spaces, i.e., finite-dimensional real vectors spaces equipped with an inner product. We denote arbitrary Euclidean spaces by $\mathbb{E}$ and $\mathbb{Y}$. We adopt Minkowski's notation; for instance, $\mathscr{C} + \Lambda\mathscr{D} := \{\, x + \lambda y : x \in \mathscr{C},\, \lambda \in \Lambda,\, y \in \mathscr{D} \,\}$ for $\mathscr{C}, \mathscr{D} \subseteq \mathbb{E}$ and $\Lambda \subseteq \mathbb{R}$. Also, whenever possible we shorten singletons to their single elements, e.g., we write $\mathbb{R}_+(1 \oplus \mathscr{C})$ to denote the conic homogenization of the set $\mathscr{C}$ in one higher dimensional space.

Table 1. Notation for special sets.

$[n]$	$:=$	$\{1, \dots, n\}$ for each $n \in \mathbb{N}$
$\mathcal{P}(X)$	$:=$	the power set of X
$\mathbb{R}_+$	$:=$	$\{\, x \in \mathbb{R} : x \geq 0 \,\}$, the set of nonnegative reals
$\mathbb{R}_{++}$	$:=$	$\{\, x \in \mathbb{R} : x > 0 \,\}$, the set of positive reals
$\mathbb{R}^{n \times n}$	$:=$	the space of $n \times n$ real-valued matrices
$\mathbb{S}^n$	$:=$	$\{\, X \in \mathbb{R}^{n \times n} : X = X^\mathsf{T} \,\}$, the space of symmetric $n \times n$ matrices
$\mathbb{S}^n_+$	$:=$	$\{\, X \in \mathbb{S}^n : h^\mathsf{T} X h \geq 0 \,\forall h \in \mathbb{R}^n \,\}$, the cone of *positive semidefinite* matrices
$\mathbb{S}^n_{++}$	$:=$	$\{\, X \in \mathbb{S}^n : h^\mathsf{T} X h > 0 \,\forall h \in \mathbb{R}^n \setminus \{0\} \,\}$, the cone of *positive definite* matrices
$\mathbb{D}^n$	$:=$	the set of diagonal n-by-n matrices
$\mathscr{E}_n$	$:=$	the *elliptope*; see (9)

2.1. Uniqueness of Dual Optimal Solutions.

Delorme and Poljak [9] proved that the dual SDP in (4) has a unique optimal solution. We shall state a slightly generalized version of their result with some changes and include a proof for the sake of completeness.

Proposition 1 ([9, Theorem 2])**.** Consider the primal-dual pair of SDPs in (2) and (3), where $\mathcal{A}\colon \mathbb{S}^n \to \mathbb{R}^m$ is a surjective linear map, $C \in \mathbb{S}^n$, and $b \in \mathbb{R}^m$. Assume there exist $\mathring{X} \in \mathbb{S}^n_{++}$ and $\mathring{y} \in \mathbb{R}^m$ such that $\mathcal{A}(\mathring{X}) = b$ and $\mathcal{A}^*(\mathring{y}) \in \mathbb{S}^n_{++}$. Suppose that, for every nonzero $y \in \mathbb{R}^m$, there exists $z \in \mathbb{R}^m$ such that $b^\mathsf{T} z \neq 0$ and $\operatorname{Null}(\mathcal{A}^*(y)) \subseteq \operatorname{Null}(\mathcal{A}^*(z))$. Then (3) has a unique optimal solution.

Proof. Since $\mathring{X}$ is a Slater point for (2), there exists an optimal solution for (3). Suppose for the sake of contradiction that $y_1 \oplus S_1$ and $y_2 \oplus S_2$ are distinct optimal solutions for (3). Set $\bar{y} := \frac{1}{2}(y_1 + y_2)$ and $\bar{S} := \mathcal{A}^*(\bar{y}) - C = \frac{1}{2}(S_1 + S_2) \succeq 0$. We claim that $\bar{S} \neq 0$; otherwise from $S_1, S_2 \succeq 0$ we get $S_1 = S_2 = 0$ and

Table 2. Notation for linear algebra.

$\mathcal{A}^*$	$:=$	the *adjoint* of a linear map $\mathcal{A}$ between Euclidean spaces
$\operatorname{Tr}(X)$	$:=$	$\sum_{i=1}^n X_{ii}$, the *trace* of $X \in \mathbb{R}^{n\times n}$
I	$:=$	the identity matrix in the appropriate space
$\mathbb{1}$	$:=$	the vector of all-ones in the appropriate space
$\{e_1, \dots, e_n\}$	$:=$	the set of canonical basis vectors of $\mathbb{R}^n$
$\operatorname{Im}(A)$	$:=$	the range of $A \in \mathbb{R}^{n\times n}$
$\operatorname{Null}(A)$	$:=$	the nullspace of $A \in \mathbb{R}^{n\times n}$
$\operatorname{supp}(x)$	$:=$	$\{\, i \in [n] : x_i \neq 0\}$, the *support* of $x \in \mathbb{R}^n$
$\operatorname{diag}(X)$	$:=$	$\sum_{i=1}^n X_{ii} e_i$ for each $X \in \mathbb{R}^{n\times n}$ so $\operatorname{diag}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^n$ extracts the diagonal
$\operatorname{Diag}(x)$	$:=$	$\sum_{i=1}^n x_i e_i e_i^{\mathsf{T}} \in \mathbb{R}^{n\times n}$ for each $x \in \mathbb{R}^n$, so Diag is the adjoint of diag
$\mathscr{C}^{\perp}$	$:=$	$\{\, x \in \mathbb{E} : \langle x, s\rangle = 0\, \forall s \in \mathscr{C}\}$ for each subset $\mathscr{C}$ of an Euclidean space $\mathbb{E}$
$\oplus$	$:=$	the direct sum of two vectors or two sets of vectors
$x \perp y$	$:=$	denotes that $x, y \in \mathbb{E}$ are orthogonal, i.e., $\langle x, y\rangle = 0$
$\succeq$	$:=$	the *Löwner partial order* on $\mathbb{S}^n$, i.e., $A \succeq B \iff A - B \in \mathbb{S}^n_+$ for $A, B \in \mathbb{S}^n$
$\succ$	$:=$	the partial order on $\mathbb{S}^n$ defined as $A \succ B \iff A - B \in \mathbb{S}^n_{++}$ for $A, B \in \mathbb{S}^n$
$\lambda_{\max}(A)$	$:=$	the largest eigenvalue of $A \in \mathbb{S}^n$
$A^{\dagger}$	$:=$	the Moore-Penrose pseudoinverse of $A \in \mathbb{R}^{m\times n}$; see [3]
vec	$:=$	the map that sends a matrix in $\mathbb{R}^{n\times n}$ to a vector indexed by $[n] \times [n]$

Table 3. Notation for convex analysis on an Euclidean space $\mathbb{E}$.

$\operatorname{cl}(\mathscr{C})$	$:=$	the *closure* of $\mathscr{C} \subseteq \mathbb{E}$
$\operatorname{int}(\mathscr{C})$	$:=$	the *interior* of $\mathscr{C} \subseteq \mathbb{E}$
$\operatorname{ri}(\mathscr{C})$	$:=$	the *relative interior* of a convex set $\mathscr{C} \subseteq \mathbb{E}$
$\operatorname{bd}(\mathscr{C})$	$:=$	$\operatorname{cl}(\mathscr{C}) \setminus \operatorname{int}(\mathscr{C})$, the *boundary* of $\mathscr{C} \subseteq \mathbb{E}$
$\operatorname{rbd}(\mathscr{C})$	$:=$	$\operatorname{cl}(\mathscr{C}) \setminus \operatorname{ri}(\mathscr{C})$, the *relative boundary* of a convex set $\mathscr{C} \subseteq \mathbb{E}$
$\mathscr{F} \trianglelefteq \mathscr{C}$	$:=$	denotes that $\mathscr{F}$ is a face of a convex set $\mathscr{C} \subseteq \mathbb{E}$; see Section 4.2
$\mathscr{F} \triangleleft \mathscr{C}$	$:=$	denotes that $\mathscr{F}$ is a proper face of a convex set $\mathscr{C} \subseteq \mathbb{E}$; see Section 4.2
$\operatorname{Faces}(\mathscr{C})$	$:=$	the set of faces of a convex set $\mathscr{C} \subseteq \mathbb{E}$; see Section 4.2
$\operatorname{Normal}(\mathscr{C}; x)$	$:=$	the normal cone of a convex set $\mathscr{C} \subseteq \mathbb{E}$ at $x \in \mathscr{C}$; see (8)
$\mathbb{B}$	$:=$	the unit ball in the appropriate Euclidean space
$\mathbb{B}_{\infty}$	$:=$	the unit ball for the ∞-norm in the appropriate $\mathbb{R}^n$

Table 4. Notation for the theory of Hausdorff measures in a normed space $\mathscr{V}$.

$H_d(\mathscr{X})$	$:=$	the d-dimensional Hausdorff outer measure of $\mathscr{X} \subseteq \mathscr{V}$; see (21)
$\lambda_d(\mathscr{X})$	$:=$	the d-dimensional Lebesgue outer measure of $\mathscr{X} \subseteq \mathbb{R}^d$
$\dim_H(\mathscr{X})$	$:=$	the Hausdorff dimension of $\mathscr{X} \subseteq \mathscr{V}$; see (25)

so $y_1 = y_2$ since $\mathcal{A}$ is surjective, a contradiction. Then $\bar{y} \oplus \bar{S}$ is also optimal in (3). Let $\bar{z} \in \mathbb{R}^m$ such that $b^{\mathsf{T}}\bar{z} \neq 0$ and $\operatorname{Null}(\mathcal{A}^*(y_1 - y_2)) \subseteq \operatorname{Null}(\mathcal{A}^*(\bar{z}))$, which exists by assumption. Then

$$\operatorname{Null}(\bar{S}) \subseteq \operatorname{Null}(\mathcal{A}^*(\bar{z})); \tag{7}$$

indeed, if h lies in $\operatorname{Null}(\bar{S}) = \operatorname{Null}(S_1) \cap \operatorname{Null}(S_2)$, then we get $\mathcal{A}^*(y_1)h = Ch = \mathcal{A}^*(y_2)h$, whence $h \in \operatorname{Null}(\mathcal{A}^*(y_1 - y_2)) \subseteq \operatorname{Null}(\mathcal{A}^*(\bar{z}))$.

TABLE 5. Notation for a graph $G = (V, E)$.

$V(G)$	$:=$	the vertex set of G
$E(G)$	$:=$	the edge set of G
$\mathcal{L}_G(w)$	$:=$	the weighted Laplacian matrix of G with weights $w \in \mathbb{R}^E$; see (5)
$G \mathbin{\bar{+}} H$	$:=$	the cosum of graphs G and H; see (13)

Define

$$\beta := -\frac{b^{\mathsf{T}}\mathring{y}}{b^{\mathsf{T}}\bar{z}}, \qquad d := \mathring{y} + \beta\bar{z},$$

and note that $b^{\mathsf{T}}d = 0$. Since $\bar{S} \in \mathbb{S}^n_+$ is nonzero, it has a smallest positive eigenvalue; denote it by $\mu > 0$. Let $\|\cdot\|_2$ denote the operator 2-norm. If $\beta\|\mathcal{A}^*(\bar{z})\|_2 = 0$, set $\varepsilon := 1$; otherwise set

$$\varepsilon := \frac{\mu}{|\beta|\|\mathcal{A}^*(\bar{z})\|_2} > 0.$$

Also, set $\tilde{y} := \bar{y} + \varepsilon d$ and $\tilde{S} := \mathcal{A}^*(\tilde{y}) - C$. Let $h \in \mathbb{R}^n$. Write $h = h_1 + h_2$ with $h_1 \in \operatorname{Null}(\bar{S})$ and $h_2 \in [\operatorname{Null}(\bar{S})]^\perp$. By (7) we have

$$\begin{aligned}
h^{\mathsf{T}}\tilde{S}h &= h^{\mathsf{T}}\bar{S}h + \varepsilon h^{\mathsf{T}}\mathcal{A}^*(d)h \\
&\geq \mu\|h_2\|^2 + \varepsilon h^{\mathsf{T}}\mathcal{A}^*(\mathring{y})h + \varepsilon\beta h^{\mathsf{T}}\mathcal{A}^*(\bar{z})h \\
&\geq \mu\|h_2\|^2 + \varepsilon h^{\mathsf{T}}\mathcal{A}^*(\mathring{y})h - \varepsilon|\beta|\|\mathcal{A}^*(\bar{z})\|_2\|h_2\|^2 \\
&\geq \varepsilon h^{\mathsf{T}}\mathcal{A}^*(\mathring{y})h.
\end{aligned}$$

Thus, $\tilde{S} \succeq \varepsilon\mathcal{A}^*(\mathring{y})$ and hence $\bar{y} - \varepsilon\mathring{y}$ is feasible for (3). Its objective value is $b^{\mathsf{T}}(\bar{y} - \varepsilon\mathring{y}) < b^{\mathsf{T}}\bar{y}$ since $b^{\mathsf{T}}\mathring{y} = \langle\mathcal{A}(\mathring{X}), \mathring{y}\rangle = \langle\mathring{X}, \mathcal{A}^*(\mathring{y})\rangle > 0$ as $\mathring{X}, \mathcal{A}^*(\mathring{y}) \succ 0$. This contradiction completes the proof of the theorem. ☐

Corollary 2 ([9, Theorem 2])**.** The dual SDP in (4) has a unique optimal solution.

Proof. We shall apply Proposition 1 to (4). Let us see that the map $\mathcal{A} := \operatorname{diag}$ satisfies the required properties. Take $\mathring{X} := I$ and $\mathring{y} := \mathbb{1}$. Let $y \in \mathbb{R}^n$ be nonzero. Define $z \in \mathbb{R}^n$ as $z_i := |y_i|$ for every $i \in [n]$, and note that $\operatorname{Null}(\operatorname{Diag}(y)) = \operatorname{Null}(\operatorname{Diag}(z))$ and that $\mathbb{1}^{\mathsf{T}}z > 0$ since $y \neq 0$. ☐

2.2. **Vertices of the Elliptope.** Let $\mathscr{C}$ be a convex set in an Euclidean space $\mathbb{E}$. The *normal cone* of $\mathscr{C}$ at $\bar{x} \in \mathscr{C}$ is

$$\operatorname{Normal}(\mathscr{C}; \bar{x}) := \{\, a \in \mathbb{E} : \langle a, x\rangle \leq \langle a, \bar{x}\rangle \; \forall x \in \mathscr{C} \}, \tag{8}$$

i.e., it is the set of all normals to supporting halfspaces of $\mathscr{C}$ at $\bar{x}$. Note that we are identifying the dual space $\mathbb{E}^*$ of $\mathbb{E}$ with $\mathbb{E}$. We say that $\bar{x} \in \mathscr{C}$ is a *vertex* of $\mathscr{C}$ if $\operatorname{Normal}(\mathscr{C}; \bar{x})$ is full-dimensional. The set of vertices of the *elliptope*

$$\mathscr{E}_n := \{\, X \in \mathbb{S}^n_+ : \operatorname{diag}(X) = \mathbb{1} \} \tag{9}$$

was determined by Laurent and Poljak [24]:

Theorem 3 ([24, Theorem 2.5])**.** The set of vertices of $\mathscr{E}_n$ is $\{\, xx^{\mathsf{T}} : x \in \{\pm1\}^n \}$.

An *automorphism* of $\mathscr{E}_n$ is a nonsingular linear operator $\mathcal{T}$ on $\mathbb{S}^n$ that preserves $\mathscr{E}_n$, i.e., $\mathcal{T}(\mathscr{E}_n) = \mathscr{E}_n$. For $s \in \{\pm1\}^n$, the map $X \in \mathbb{S}^n \mapsto \operatorname{Diag}(s)X\operatorname{Diag}(s)$ is easily checked to be an automorphism of $\mathscr{E}_n$. Such maps can be seen as resigning and they correspond to the well-known switching operations on cuts; see, e.g., [24]. If $x, y \in \{\pm1\}^n$, then $y = \operatorname{Diag}(s)x$ for $s \in \{\pm1\}^n$ defined by $s_i := x_iy_i$ for each $i \in [n]$. Hence, any vertex of $\mathscr{E}_n$ can be mapped into the vertex $\mathbb{1}\mathbb{1}^{\mathsf{T}}$ by an automorphism of $\mathscr{E}_n$; i.e., the automorphism group of $\mathscr{E}_n$ acts transitively on the vertices of $\mathscr{E}_n$. This allows us to prove many linear properties about the vertices of $\mathscr{E}_n$ by just proving them for the vertex $\mathbb{1}\mathbb{1}^{\mathsf{T}}$. *We shall make extensive use of this fact without further mention.*

Laurent and Poljak [24] also provided formulas for the normal cones of the elliptope:

$$\begin{aligned}
\operatorname{Normal}(\mathscr{E}_n; X) &= \mathbb{D}^n - (\mathbb{S}^n_+ \cap \{X\}^\perp) \\
&= \mathbb{D}^n - \{\, Y \in \mathbb{S}^n_+ : \operatorname{Im}(Y) \subseteq \operatorname{Null}(X) \}
\end{aligned} \qquad \forall X \in \mathscr{E}_n. \tag{10}$$

When $\bar{X}$ is a vertex of $\mathscr{E}_n$, every element of $\mathrm{Normal}(\mathscr{E}_n;\bar{X})$ can be described in a unique way as an element of the Minkowski sum at the RHS of (10):

Lemma 4. Let $\bar{X}$ be a vertex of $\mathscr{E}_n$. Let $y_1, y_2 \in \mathbb{R}^n$ and $S_1, S_2 \in \mathbb{S}^n_+ \cap \{\bar{X}\}^\perp$ be such that $\mathrm{Diag}(y_1) - S_1 = \mathrm{Diag}(y_2) - S_2$. Then $y_1 = y_2$ and $S_1 = S_2$.

Proof. Up to resigning, we may assume that $\bar{X} = \mathbb{1}\mathbb{1}^\mathsf{T}$. Then $S_1 \in \mathbb{S}^n_+ \cap \{\mathbb{1}\mathbb{1}^\mathsf{T}\}^\perp$ implies that $S_1\mathbb{1} = 0$. Analogously, $S_2\mathbb{1} = 0$. Thus $y_1 = \mathrm{Diag}(y_1)\mathbb{1} = (\mathrm{Diag}(y_1) - S_1)\mathbb{1} = (\mathrm{Diag}(y_2) - S_2)\mathbb{1} = \mathrm{Diag}(y_2)\mathbb{1} = y_2$, so $S_1 = S_2$. $\square$

3. Failure of Strict Complementarity with Laplacian Objectives

Existence of strictly complementary optimal solutions is known to be equivalent to membership of the objective vector in the relative interior of some normal cone:

Proposition 5 ([6, Proposition 4.2])**.** If the feasible region $\mathscr{C}$ of the SDP (2) has a positive definite matrix, then strict complementarity holds for (2) and its dual if and only if $C \in \mathrm{ri}(\mathrm{Normal}(\mathscr{C}; X))$ for some $X \in \mathscr{C}$.

Hence, strict complementarity is locally generic when the objective function is chosen in the normal cone of a given feasible solution; see [6, Corollary 4.3].

By (10) and standard convex analysis,

$$\begin{aligned} \mathrm{ri}(\mathrm{Normal}(\mathscr{E}_n; X)) &= \mathbb{D}^n - \mathrm{ri}(\mathbb{S}^n_+ \cap \{X\}^\perp) \\ &= \mathbb{D}^n - \{\, Y \in \mathbb{S}^n_+ : \mathrm{Im}(Y) = \mathrm{Null}(X)\} \end{aligned} \qquad \forall X \in \mathscr{E}_n. \tag{11}$$

When $\bar{X}$ is a vertex of $\mathscr{E}_n$, we may combine (10) with (11) and Lemma 4 to conclude that

$$\begin{aligned} \mathrm{bd}(\mathrm{Normal}(\mathscr{E}_n; \bar{X})) &= \mathbb{D}^n - \mathrm{rbd}(\mathbb{S}^n_+ \cap \{\bar{X}\}^\perp) \\ &= \mathbb{D}^n - \{\, Y \in \mathbb{S}^n_+ : \mathrm{Im}(Y) \subsetneq \mathrm{Null}(\bar{X})\}. \end{aligned} \tag{12}$$

Note that Lemma 4 is used to prove the inclusion $\mathrm{bd}(\mathrm{Normal}(\mathscr{E}_n;\bar{X})) \supseteq \mathbb{D}^n - \mathrm{rbd}(\mathbb{S}^n_+ \cap \{\bar{X}\}^\perp)$ in (12), whereas the reverse inclusion is easy to prove. Indeed, if $S = D - Y$ for some $D \in \mathbb{D}^n$ and $Y \in \mathrm{rbd}(\mathbb{S}^n_+ \cap \{\bar{X}\}^\perp)$, then clearly $S \in \mathbb{D}^n - (\mathbb{S}^n_+ \cap \{X\}^\perp)$. However we also need to prove that $S \notin \mathbb{D}^n - \mathrm{ri}(\mathbb{S}^n_+ \cap \{X\}^\perp)$, which is where the uniqueness of decomposition shown in Lemma 4 comes in.

In [6], we noted that strict complementarity holds for (4) for every C of the form $C = \frac{1}{4}\mathcal{L}_G(w)$ with $w \geq 0$ provided that the polar $\mathscr{E}_n^\circ := \{\, Y \in \mathbb{S}^n : \mathrm{Tr}(YX) \leq 1 \,\forall X \in \mathscr{E}_n\}$ of the elliptope is facially exposed, and we (implicitly) asked whether the latter holds. It turns out, Laurent and Poljak [25, Example 5.10] showed, even before we raised the question, in a different context and using a slightly different terminology, that strict complementarity may fail for (4) for every $n \geq 3$, hence answering the question in the negative. For each complete graph $G = K_n$ with $n \geq 3$, they provided a weight function $w \geq 0$ for which strict complementarity fails for (4) with $C = \frac{1}{4}\mathcal{L}_G(w)$.

We generalize their construction showing that strict complementarity may fail with a weighted Laplacian objective for graphs which are cosums, with mild conditions on the (co-)summands. Recall that, if $G = (V, E)$ and $H = (U, F)$ are graphs such that $V \cap U = \varnothing$, the *cosum* of G and H is the graph

$$G \mp H := (V \cup U, E \cup F \cup \{\, \{v, u\} : (v, u) \in V \times U\}). \tag{13}$$

We shall use a characterization of positive semidefinite matrices partitioned in blocks using Schur complements and the Moore-Penrose pseudoinverse:

Lemma 6 (see [13, Theorem 4.3])**.** For $A \in \mathbb{S}^m$, $C \in \mathbb{S}^n$, and $B \in \mathbb{R}^{m\times n}$, we have

$$\begin{bmatrix} A & B \\ B^\mathsf{T} & C \end{bmatrix} \succeq 0 \iff A \succeq 0, \quad (I - AA^\dagger)B = 0, \quad \text{and } C \succeq B^\mathsf{T} A^\dagger B. \tag{14}$$

Below we only use two properties of the pseudoinverse $A^\dagger$ of $A \in \mathbb{R}^{n\times n}$. The matrix $AA^\dagger$ is the orthogonal projector onto $\mathrm{Im}(A)$. If $v \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue $\lambda \neq 0$, then v is an eigenvector of $A^\dagger$ with eigenvalue λ^{-1}.

Theorem 7. Let G and H be graphs with $n_G \geq 2$ and $n_H \geq 1$ vertices, respectively. Let $w_G\colon E(G) \to \mathbb{R}_{++}$ and $w_H\colon E(H) \to \mathbb{R}_{++}$ be weight functions, and denote the respective weighted Laplacians by $L_G \coloneqq \mathcal{L}_G(w_G)$ and $L_H \coloneqq \mathcal{L}_H(w_H)$. Set $\mu_G \coloneqq \lambda_{\max}(L_G)$ and $\mu_H \coloneqq \lambda_{\max}(L_H)$. Suppose that $n_G\mu_G > n_H\mu_H$ and that H is connected. Define $\bar{w}\colon E(G \mp H) \to \mathbb{R}_{++}$ as $\bar{w} \coloneqq w_G \oplus w_H \oplus \alpha\mathbb{1}$ where $\alpha \coloneqq \mu_G/n_H$. Then there is no strictly complementary pair of primal-dual optimal solutions for (4) when $C \coloneqq \mathcal{L}_{G\mp H}(\bar{w})$.

Proof. For enhanced clarity denote the vectors of all-ones in $\mathbb{R}^{V(G)}$ and $\mathbb{R}^{V(H)}$ by $\mathbb{1}_G$ and $\mathbb{1}_H$, respectively. We will prove that the unique pair of primal-dual optimal solutions for (4) is $(X^*, y^* \oplus S^*)$ defined by

$$\begin{aligned} X^* &\coloneqq \begin{bmatrix} -\mathbb{1}_G \\ \mathbb{1}_H \end{bmatrix} \begin{bmatrix} -\mathbb{1}_G \\ \mathbb{1}_H \end{bmatrix}^\mathsf{T}, \qquad y^* \coloneqq 2\alpha \begin{bmatrix} n_H\mathbb{1}_G \\ n_G\mathbb{1}_H \end{bmatrix}, \\ S^* &\coloneqq \operatorname{Diag}(y^*) - \mathcal{L}_{G\mp H}(\bar{w}) = \begin{bmatrix} \mu_G I - L_G & \alpha\mathbb{1}_G\mathbb{1}_H^\mathsf{T} \\ \alpha\mathbb{1}_H\mathbb{1}_G^\mathsf{T} & \alpha n_G I - L_H \end{bmatrix}. \end{aligned} \tag{15}$$

In particular, since $(X^* + S^*)(h \oplus 0) = 0$ for any μ_G-eigenvector h of L_G, there is no strictly complementary pair of primal-dual optimal solutions for (4).

It is easy to check that X^* is feasible in the primal. We have

$$S^* = 2\begin{bmatrix} \mu_G I & 0^\mathsf{T} \\ 0 & \alpha n_G I \end{bmatrix} - \begin{bmatrix} L_G + \mu_G I & -\alpha\mathbb{1}_G\mathbb{1}_H^\mathsf{T} \\ -\alpha\mathbb{1}_H\mathbb{1}_G^\mathsf{T} & L_H + \alpha n_G I \end{bmatrix} = \operatorname{Diag}(y^*) - \mathcal{L}_{G\mp H}(\bar{w}),$$

and by Lemma 6 the condition $S^* \succeq 0$ is equivalent to the conditions

$$A \coloneqq \mu_G I - L_G \succeq 0, \tag{16a}$$

$$(I - AA^\dagger)\mathbb{1}_G = 0, \tag{16b}$$

$$\alpha n_G I \succeq L_H + \alpha^2 \mathbb{1}_G^\mathsf{T} A^\dagger \mathbb{1}_G \mathbb{1}_H \mathbb{1}_H^\mathsf{T}. \tag{16c}$$

Note that (16a) holds trivially. Also $A\mathbb{1}_G = \mu_G\mathbb{1}_G$, so $\mathbb{1}_G \in \operatorname{Im}(A)$ and (16b) holds since $I - AA^\dagger$ is the orthogonal projector onto $\operatorname{Null}(A) = \operatorname{Im}(A)^\perp$. Finally, $A^\dagger\mathbb{1}_G = \mu_G^{-1}\mathbb{1}_G$ so (16c) is equivalent to $\alpha n_G I \succeq L_H + \alpha n_G \frac{1}{n_H}\mathbb{1}_H\mathbb{1}_H^\mathsf{T}$, which holds since $\alpha n_G > \mu_H$ by assumption. It follows that $y^* \oplus S^*$ is feasible in the dual. It is easy to check that $\operatorname{Tr}(X^*S^*) = 0$, so X^* and $y^* \oplus S^*$ are optimal solutions. By Corollary 2, $y^* \oplus S^*$ is the unique optimal solution for the dual.

It remains to show that X^* is the unique optimal solution for the primal. Let

$$X = \begin{bmatrix} X_G & B \\ B^\mathsf{T} & X_H \end{bmatrix}$$

be an optimal solution for the primal. Complementary slackness yields

$$0 = XS^* = \begin{bmatrix} X_G(\mu_G I - L_G) + \alpha B\mathbb{1}_H\mathbb{1}_G^\mathsf{T} & \alpha X_G\mathbb{1}_G\mathbb{1}_H^\mathsf{T} + B(\alpha n_G I - L_H) \\ B^\mathsf{T}(\mu_G I - L_G) + \alpha X_H\mathbb{1}_H\mathbb{1}_G^\mathsf{T} & \alpha B^\mathsf{T}\mathbb{1}_G\mathbb{1}_H^\mathsf{T} + X_H(\alpha n_G I - L_H) \end{bmatrix}. \tag{17}$$

If $h \perp \mathbb{1}_H$ is an eigenvector of L_H, (left-)multiplying h by the bottom right block in (17) yields $X_H h = 0$, where we used the assumption that $\alpha n_G > \mu_H$. Since H is connected, this implies that X_H is a nonnegative scalar multiple of $\mathbb{1}_H\mathbb{1}_H^\mathsf{T}$, and so

$$X_H = \mathbb{1}_H\mathbb{1}_H^\mathsf{T}.$$

Next apply $\mathbb{1}_G^\mathsf{T} \cdot \mathbb{1}_H$ and $\mathbb{1}_H^\mathsf{T} \cdot \mathbb{1}_G$ to the top right block and bottom left block of (17), respectively, to get

$$0 = n_H\mathbb{1}_G^\mathsf{T} X_G\mathbb{1}_G + n_G\mathbb{1}_G^\mathsf{T} B\mathbb{1}_H, \tag{18}$$

$$0 = n_H\mathbb{1}_H^\mathsf{T} B^\mathsf{T}\mathbb{1}_G + n_G\mathbb{1}_H^\mathsf{T} X_H\mathbb{1}_H. \tag{19}$$

Hence,

$$\frac{\mathbb{1}_G^\mathsf{T} X_G\mathbb{1}_G}{n_G^2} = \frac{\mathbb{1}_H^\mathsf{T} X_H\mathbb{1}_H}{n_H^2} \qquad \text{and} \qquad X_G = \mathbb{1}_G\mathbb{1}_G^\mathsf{T}.$$

Finally, by (18) we get $\mathbb{1}_G^\mathsf{T} B\mathbb{1}_H = -n_G n_H$, and so $B = -\mathbb{1}_G\mathbb{1}_H^\mathsf{T}$. Hence, $X = X^*$. □

Note in the comment following (15) that the dimension of the $\lambda_{\max}(\mathcal{L}_G(w))$-eigenspace controls the "degree" to which strict complementarity fails in Theorem 7. In particular, when G is the complete graph and $w_G = \mathbb{1}$, we have $\operatorname{rank}(X^*) + \operatorname{rank}(S^*) = 1 + n_H$. The construction by Laurent and Poljak [25, Example 5.10] may be recovered from Theorem 7 by taking $G = K_{n-1}$ for some $n \geq 3$, $H = K_1$, and setting $w_G := \frac{1}{n-1}\mathbb{1}$.

Theorem 7 shows that, if F is a graph which is a cosum (i.e., the complement of F is not connected) $F = G \mp H$, where G has at least one edge and H is connected, then there is a nonnegative weight function $w\colon E(F) \to \mathbb{R}_{++}$ such that strict complementarity fails for (4) with $C = \frac{1}{4}\mathcal{L}_F(w)$; one may just fix $w_H \in \mathbb{R}^{E(H)}_{++}$ arbitrarily, e.g., $w_H = \mathbb{1}$, and set $w_G := M\mathbb{1}$ for large enough M so that $n_G\mu_G > n_H\mu_H$. A natural question following from this is:

Problem 8. Characterize the set of graphs for which there exists a positive weight function on the edges such that strict complementarity fails for (4) when $4C$ is the corresponding weighted Laplacian matrix.

Positive semidefinite matrix completion problems can be phrased in terms of slices of elliptopes. In that area, the paper [39] shares the spirit of Problem 8.

4. Generic Failure of Strict Complementarity on the Boundaries of Normal Cones

In this section, we consider how often strict complementarity holds for (4) when C lies in the (relative) boundary of $\operatorname{Normal}(\mathscr{E}_n; \bar{X})$ for some vertex $\bar{X}$ of $\mathscr{E}_n$. Note that this boundary is described as a Minkowski sum in (12).

We start by considering the case $n = 3$, where (12) simplifies to

$$\operatorname{bd}(\operatorname{Normal}(\mathscr{E}_3; \bar{x}\bar{x}^{\mathsf{T}})) = \mathbb{D}^3 - \{\, zz^{\mathsf{T}} : z \in \{\bar{x}\}^{\perp} \} \tag{20}$$

for every $\bar{x} \in \{\pm 1\}^3$.

Proposition 9. Let $\bar{x} \in \{\pm 1\}^3$, and let $C = \operatorname{Diag}(\bar{y}) - \bar{z}\bar{z}^{\mathsf{T}}$ for some $\bar{y} \in \mathbb{R}^3$ and $\bar{z} \in \{\bar{x}\}^{\perp}$, so that $C \in \operatorname{bd}(\operatorname{Normal}(\mathscr{E}_3; \bar{x}\bar{x}^{\mathsf{T}}))$. Then strict complementarity holds for (4) if and only if $\bar{z}_i = 0$ for some $i \in [3]$.

Proof. Set $\bar{S} := \operatorname{Diag}(\bar{y}) - C = \bar{z}\bar{z}^{\mathsf{T}}$ and $\bar{X} := \bar{x}\bar{x}^{\mathsf{T}}$. Clearly, $\bar{y} \oplus \bar{S}$ is feasible in the dual and $\operatorname{Tr}(\bar{S}\bar{X}) = (\bar{z}^{\mathsf{T}}\bar{x})^2 = 0$, so $(\bar{X}, \bar{y} \oplus \bar{S})$ is a pair of primal-dual optimal solutions. By Corollary 2, $\bar{y} \oplus \bar{S}$ is the unique optimal solution in the dual.

Suppose that $\bar{z}_i \neq 0$ for every $i \in [3]$. We claim that $\bar{X}$ is the unique optimal solution in the primal. Indeed, let $X \in \mathscr{E}_3$ be optimal in the primal. Then $0 = \operatorname{Tr}(\bar{S}X) = \bar{z}^{\mathsf{T}}X\bar{z}$ so $X\bar{z} = 0$. Thus,

$$0 = \begin{bmatrix} 1 & X_{12} & X_{13} \\ X_{12} & 1 & X_{23} \\ X_{13} & X_{23} & 1 \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \end{bmatrix} = \begin{bmatrix} \bar{z}_1 + \bar{z}_2 X_{12} + \bar{z}_3 X_{13} \\ \bar{z}_1 X_{12} + \bar{z}_2 + \bar{z}_3 X_{23} \\ \bar{z}_1 X_{13} + \bar{z}_2 X_{23} + \bar{z}_3 \end{bmatrix},$$

so

$$\begin{bmatrix} \bar{z}_2 & \bar{z}_3 & 0 \\ \bar{z}_1 & 0 & \bar{z}_3 \\ 0 & \bar{z}_1 & \bar{z}_2 \end{bmatrix} \begin{bmatrix} X_{12} \\ X_{13} \\ X_{23} \end{bmatrix} = -\bar{z}.$$

The determinant of the matrix defining the latter linear system is $-2\bar{z}_1\bar{z}_2\bar{z}_3 \neq 0$, so the unique solution is given by the off-diagonal entries of $\bar{X}$.

Suppose now that $\bar{z}_i = 0$ for some $i \in [3]$. If $\bar{z} = 0$ then $(I, \bar{y} \oplus 0)$ satisfies strict complementarity, so assume $\bar{z} \neq 0$. Set $\tilde{x} := \operatorname{Diag}(\mathbb{1} - e_i)\bar{x}$ and $\tilde{X} := \tilde{x}\tilde{x}^{\mathsf{T}} + e_i e_i^{\mathsf{T}} \in \mathscr{E}_3$. Then $\operatorname{Tr}(\bar{S}\tilde{X}) = \bar{z}^{\mathsf{T}}(\tilde{x}\tilde{x}^{\mathsf{T}} + e_i e_i^{\mathsf{T}})\bar{z} = (\bar{z}^{\mathsf{T}}\tilde{x})^2 + \bar{z}_i^2 = 0$ since $\bar{z}^{\mathsf{T}}\tilde{x} = \bar{z}^{\mathsf{T}}\bar{x} = 0$. Hence, $(\tilde{X}, \bar{y} \oplus \bar{S})$ is a strictly complementarity pair of primal-dual optimal solutions for (4). ☐

For $n \geq 4$, characterization of strict complementarity in (4) is not as easily described. However, we can prove the following condition sufficient for the failure of strict complementarity, which will turn out to be sufficient for our purposes.

Theorem 10. Let $n \geq 3$ and let $C \in \operatorname{bd}(\operatorname{Normal}(\mathscr{E}_n; \mathbb{1}\mathbb{1}^{\mathsf{T}}))$ of the form $C = \operatorname{Diag}(\bar{y}) - \bar{S}$, where $\bar{y} \in \mathbb{R}^n$ and $\bar{S} \in \mathbb{S}^n_+$. Assume that $\operatorname{Null}(\bar{S}) = \operatorname{span}\{\mathbb{1}, h\}$ where $h \in \{\mathbb{1}\}^{\perp}$ and at least three coordinates of h have distinct values. Then strict complementarity fails for (4).

Proof. Set $\bar{X} := \mathbb{1}\mathbb{1}^\mathsf{T}$. Clearly, $\bar{y} \oplus \bar{S}$ is feasible in the dual and $\operatorname{Tr}(\bar{S}\bar{X}) = 0$, so $(\bar{X}, \bar{y} \oplus \bar{S})$ is a pair of primal-dual optimal solutions. By Corollary 2, $\bar{y} \oplus \bar{S}$ is the unique optimal solution in the dual. We shall prove that $\bar{X}$ is the unique optimal solution in the primal, which implies that strict complementarity fails as $\operatorname{rank}(\bar{X}) + \operatorname{rank}(\bar{S}) = 1 + n - 2 = n - 1 < n$.

Let $X \in \mathscr{E}_n$ be an optimal solution in the primal. By complementary slackness, $\operatorname{Tr}(X\bar{S}) = 0$, so $\bar{S}X = 0$ and $\operatorname{Im}(X) \subseteq \operatorname{Null}(\bar{S}) = \operatorname{span}\{\mathbb{1}, h\}$. Hence, $X = \alpha_1 \mathbb{1}\mathbb{1}^\mathsf{T} + \alpha_2 hh^\mathsf{T} + \alpha_3(h\mathbb{1}^\mathsf{T} + \mathbb{1}h^\mathsf{T})$ for some $\alpha \in \mathbb{R}^3$. Since $\operatorname{diag}(X) = \mathbb{1}$, we find that $\alpha_1 + \alpha_2 h_i^2 + 2\alpha_3 h_i = 1$ for every $i \in [n]$. Let $i, j, k \in [n]$ such that $|\{h_i, h_j, h_k\}| = 3$. Then

$$\begin{bmatrix} 1 & 2h_i & h_i^2 \\ 1 & 2h_j & h_j^2 \\ 1 & 2h_k & h_k^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_3 \\ \alpha_2 \end{bmatrix} = \mathbb{1}.$$

The determinant of the matrix defining this linear system is a Vandermonde determinant, and it is equal to $2^3(h_j - h_i)(h_k - h_i)(h_k - h_j) \neq 0$ by assumption. Hence, $\alpha = e_1$ is its unique solution. Thus, $X = \mathbb{1}\mathbb{1}^\mathsf{T}$. $\square$

Theorem 10 seems to indicate that strictly complementarity fails "almost everywhere" on the boundary of $\operatorname{Normal}(\mathscr{E}_n; \mathbb{1}\mathbb{1}^\mathsf{T})$, since the high rank matrices make up the bulk of the boundary (consider that the set of nonsingular matrices is open and dense) and for "most" of them the extra vector h in the nullspace has at least three coordinates with distinct values. Unfortunately, we are dealing with somewhat complicated sets (e.g., the high rank matrices in the boundary of a normal cone). In order to make our previous statements precise, we shall make use of the theory of Hausdorff measures, which we introduce below in Section 4.1.

Note also that the cone $\operatorname{Normal}(\mathscr{E}_n; \mathbb{1}\mathbb{1}^\mathsf{T})$ is not pointed, and its lineality space is $\mathbb{D}^n$. This is the case due to the constraints that all the diagonal entries are ones, so that $\mathscr{E}_n$ is not full-dimensional in $\mathbb{S}^n$. In the remainder of the paper, we shall focus on the more relevant part of $\operatorname{bd}(\operatorname{Normal}(\mathscr{E}_n; \mathbb{1}\mathbb{1}^\mathsf{T}))$, namely the term $-\operatorname{rbd}(\mathbb{S}^n_+ \cap \{\mathbb{1}\mathbb{1}^\mathsf{T}\}^\perp)$ in (12). This corresponds to modding out the diagonal perturbation in the expression (12).

4.1. Preliminaries on Hausdorff Measures.

We refer the reader to [35], though we use different notation and more standard terminology. See also [11, 32] for a somewhat similar presentation. We focus our presentation on finite-dimensional normed spaces (over the reals) but most of it could be developed for arbitrary metric spaces. Our main normed spaces are (subspaces of) $\mathbb{R}^n$ and $\mathbb{S}^n$. Since these are Euclidean spaces, they are equipped with a norm induced by their inner-products, and that is the norm that we will consider unless explicitly stated otherwise. We shall only use other norms in Section 5.

Let $\mathscr{V}$ be a finite-dimensional normed space. Let $d \in \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_{++}$. For each $\mathscr{X} \subseteq \mathscr{V}$, define

$$H_d^\varepsilon(\mathscr{X}) := \inf\left\{ \sum_{i=0}^{\infty} \left[\operatorname{diam}(\mathscr{U}_i)\right]^d : \{\mathscr{U}_i\}_{i\in\mathbb{N}} \subseteq \mathcal{P}(\mathscr{V}),\ \mathscr{X} \subseteq \bigcup_{i=0}^{\infty} \mathscr{U}_i,\ \operatorname{diam}(\mathscr{U}_i) < \varepsilon\, \forall i \in \mathbb{N} \right\},$$

where the *diameter* of $\mathscr{U} \subseteq \mathscr{V}$ is $\operatorname{diam}(\mathscr{U}) := \sup_{x,y\in\mathscr{U}} \|x - y\|$. The function $H_d \colon \mathcal{P}(\mathscr{V}) \to [0, +\infty]$ defined by

$$H_d(\mathscr{X}) := \sup_{\varepsilon>0} H_d^\varepsilon(\mathscr{X}) = \lim_{\varepsilon\downarrow 0} H_d^\varepsilon(\mathscr{X}) \qquad \forall \mathscr{X} \subseteq \mathscr{V} \tag{21}$$

is an outer measure on $\mathscr{V}$. Hence, the restriction of H_d to the H_d-measurable subsets of $\mathscr{V}$ is a complete measure on $\mathscr{V}$, called the *d-dimensional Hausdorff measure* on $\mathscr{V}$. The 0-dimensional Hausdorff measure H_0 is the cardinality of a set, H_1 is its length, H_2 is its area, and so on.

Let d be a positive integer and set $\mathscr{V} := \mathbb{R}^d$. Let $\lambda_d \colon \mathcal{P}(\mathbb{R}^d) \to [0, +\infty]$ denote the d-dimensional Lebesgue outer measure on $\mathbb{R}^d$. It can be proved [35, Theorem 30] that

$$\frac{\lambda_d(\mathscr{X})}{\lambda_d(\mathbb{B})} = \frac{H_d(\mathscr{X})}{2^d} \qquad \forall \mathscr{X} \subseteq \mathbb{R}^d. \tag{22}$$

In particular, the H_d-measurable subsets of $\mathbb{R}^d$ are the same as the λ_d-measurable sets.

Let $a, b \in \mathbb{R}_+$ with $a < b$ and let $\mathscr{X} \subseteq \mathscr{V}$. It is not hard to prove from the definition that

$$H_a(\mathscr{X}) < \infty \implies H_b(\mathscr{X}) = 0, \tag{23}$$

$$H_b(\mathscr{X}) > 0 \implies H_a(\mathscr{X}) = \infty. \tag{24}$$

Hence,

$$\sup\{\, d \in \mathbb{R}_+ : H_d(\mathscr{X}) = \infty\} = \inf\{\, d \in \mathbb{R}_+ : H_d(\mathscr{X}) = 0\}, \tag{25}$$

and the common value in (25) is the *Hausdorff dimension* of $\mathscr{X}$, denoted by $\dim_H(\mathscr{X})$, that is,

$$\dim_H(\mathscr{X}) = \sup\{\, d \in \mathbb{R}_+ : H_d(\mathscr{X}) = \infty\} = \inf\{\, d \in \mathbb{R}_+ : H_d(\mathscr{X}) = 0\}. \tag{26}$$

In particular,

$$\text{if } d \in \mathbb{R}_+ \text{ and } \mathscr{X} \subseteq \mathscr{V} \text{ satisfy } H_d(\mathscr{X}) \in (0,\infty), \text{ then } \dim_H(\mathscr{X}) = d. \tag{27}$$

Note that the widely used notion of dimension of a convex set, while consistent with the above definition (see the paragraph following the proof of Proposition 13 below), is an elementary concept. The dimension of a convex set is the dimension of the smallest affine subspace containing it (i.e., the dimension of a convex set is the dimension of its affine hull). Therefore, the notion of dimension for convex sets is no more complicated than that of linear subspaces. However, dealing with boundaries or relative boundaries of convex sets require us to understand the dimensions of these nonconvex sets. For this purpose, we employ the notion of Hausdorff dimension.

We may now define *genericity* precisely. Let $\mathscr{X}$ be a subset of a finite-dimensional normed space $\mathscr{V}$. Let P be a property that may hold or fail for points in $\mathscr{X}$, i.e., $P(x)$ is either true or false for each $x \in \mathscr{X}$. We say that P *holds generically on* $\mathscr{X}$ if $H_d(\{\, x \in \mathscr{X} : P(x) \text{ is false}\}) = 0$ for $d := \dim_H(\mathscr{X})$. We say that P *fails generically on* $\mathscr{X}$ if the negation of P holds generically on $\mathscr{X}$. In Section 4.3, we will use Theorem 10 to prove that strict complementarity fails generically at the boundary of the normal cone of any vertex of $\mathscr{E}_n$, for $n \geq 3$, modulo some qualification on the ambient space. In the remainder of this section and in the next one, we will describe a few more measure-theoretic tools that we shall use towards this goal.

Let $\mathscr{V}$ and $\mathscr{U}$ be finite-dimensional normed spaces. Let $\mathscr{X} \subseteq \mathscr{V}$. Recall that a function $\varphi\colon \mathscr{X} \to \mathscr{U}$ is *Lipschitz continuous* with Lipschitz constant $L > 0$ if

$$\|\varphi(x) - \varphi(y)\| \leq L\|x - y\| \qquad \forall x, y \in \mathscr{X}. \tag{28}$$

The following is well known and easy to prove (see, e.g., [35, Theorem 29] or [11, Proposition 2.13]):

Theorem 11. Let $\mathscr{V}$ and $\mathscr{U}$ be finite-dimensional normed spaces. Let $\mathscr{X} \subseteq \mathscr{V}$ and $d \in \mathbb{R}_+$. Let $\varphi\colon \mathscr{X} \to \mathscr{U}$ be Lipschitz continuous with Lipschitz constant L. Then

$$H_d\big(\varphi(\mathscr{X})\big) \leq L^d H_d(\mathscr{X}). \tag{29}$$

Theorem 11 is especially useful to determine some Hausdorff dimensions via bi-Lipschitz maps. We recall the definition here. Let $\mathscr{V}$ and $\mathscr{U}$ be finite-dimensional normed spaces. Let $\mathscr{X} \subseteq \mathscr{V}$, and let $\varphi\colon \mathscr{X} \to \mathscr{U}$ be a one-to-one function with range $\mathscr{Y} := \varphi(\mathscr{X})$. We say that φ is *bi-Lipschitz continuous* with Lipschitz constants $L_1 > 0$ and $L_2 > 0$ if φ is Lipschitz continuous with Lipschitz constant L_1 and $\varphi^{-1}\colon \mathscr{Y} \to \mathscr{V}$ is Lipschitz continuous with Lipschitz constant L_2.

Corollary 12. Let $\mathscr{V}$ and $\mathscr{U}$ be finite-dimensional normed spaces. Let $\mathscr{X} \subseteq \mathscr{V}$ and $d \in \mathbb{R}_+$. Let $\varphi\colon \mathscr{X} \to \mathscr{U}$ be bi-Lipschitz continuous with Lipschitz constants L_1 and L_2. Then

$$L_2^{-d} H_d(\mathscr{X}) \leq H_d(\varphi(\mathscr{X})) \leq L_1^d H_d(\mathscr{X}). \tag{30}$$

In particular, if $H_d(\mathscr{X}) \in (0,\infty)$, then $\dim_H(\varphi(\mathscr{X})) = d$.

This corollary may be used, for instance, to regard any d-dimensional Euclidean space $\mathscr{V}$ as $\mathbb{R}^d$ by considering the coordinate map $\varphi\colon \mathscr{V} \to \mathbb{R}^d$ with respect to a fixed orthonormal basis of $\mathscr{V}$. Another frequent use of Corollary 12 goes as follows. Equip the space $\mathbb{S}^n$ with the trace inner-product. If $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, the map $X \in \mathbb{S}^n \mapsto QXQ^\mathsf{T}$ preserves inner-products, and hence norms and distances; hence, the map is Lipschitz continuous with Lipschitz constant 1. Its inverse is $X \in \mathbb{S}^n \mapsto Q^\mathsf{T}XQ$ and so the map $X \in \mathbb{S}^n \mapsto QXQ^\mathsf{T}$ is bi-Lipschitz continuous with Lipschitz constants 1 and 1.

The next result is useful for determining the Hausdorff dimension of some simple unbounded sets in the σ-finite case, when (27) is not directly applicable:

Proposition 13. Let $\mathscr{X}$ be a subset of a finite-dimensional normed space $\mathscr{V}$. For each $i \in \mathbb{N}$, let $\mathscr{Y}_i$ be a subset of a finite-dimensional normed space $\mathscr{U}_i$, and let $\varphi_i\colon \mathscr{Y}_i \to \mathscr{V}$ be a Lipschitz continuous function with Lipschitz constant L_i. If $\mathscr{X} \subseteq \bigcup_{i\in\mathbb{N}} \varphi_i(\mathscr{Y}_i)$, then $\dim_H(\mathscr{X}) \leq \sup_{i\in\mathbb{N}} \dim_H(\mathscr{Y}_i)$.

Proof. Set $d := \sup_{i\in\mathbb{N}} \dim_H(\mathscr{Y}_i)$. Let $\bar{d} > d$. Then (25) yields $H_{\bar{d}}(\mathscr{Y}_i) = 0$ for each $i \in \mathbb{N}$, so by Theorem 11 we have $H_{\bar{d}}(\mathscr{X}) \leq \sum_{i\in\mathbb{N}} L_i^{\bar{d}} H_{\bar{d}}(\mathscr{Y}_i) = 0$. Hence, $\dim_H(\mathscr{X}) \leq \bar{d}$ by (26) and thus $\dim_H(\mathscr{X}) \leq d$. ☐

For instance, $\mathbb{R}^d = \bigcup_{M\in\mathbb{N}} M\mathbb{B}$ and the ball $M\mathbb{B} \subseteq \mathbb{R}^d$ with nonzero M has Hausdorff dimension d by (27) and (22), so Proposition 13 shows that $\dim_H(\mathbb{R}^d) \leq d$. Since $\mathbb{R}^d \supseteq \mathbb{B}$ shows that $H_d(\mathbb{R}^d) \geq H_d(\mathbb{B}) > 0$ by (22), we conclude by (27) that $\dim_H(\mathbb{R}^d) = d$. Together with Corollary 12, this shows that Hausdorff dimension and the usual (linear) dimension coincide on linear subspaces, and hence also for convex sets by translation invariance.

4.2. **Hausdorff Measures and the Boundary Structure of Convex Sets.** In this section we collect some results relating Hausdorff measures and the boundary structure of convex sets, including a quick review of basic facts about faces.

The following result is well known:

Theorem 14. Let $\mathbb{E}$ be an Euclidean space. If $\mathscr{C} \subseteq \mathbb{E}$ is a compact convex set with dimension $d \geq 1$, then $\dim_H(\operatorname{rbd}(\mathscr{C})) = d - 1$.

Proof. We may assume that $\dim(\mathbb{E}) = d$ so that $\mathscr{C}$ has nonempty interior. By choosing an orthonormal basis for $\mathbb{E}$, we may assume that $\mathbb{E} = \mathbb{R}^d$. We may also assume that $0 \in \operatorname{int}(\mathscr{C})$ by translation invariance of Hausdorff measure. Set $X := \operatorname{bd}(\mathbb{B}_\infty)$, and note that $H_{d-1}(X) \in (0, +\infty)$ by (22) and Corollary 12. Let $\varepsilon, M \in \mathbb{R}_{++}$ such that $2\varepsilon\mathbb{B}_\infty \subseteq \mathscr{C} \subseteq \frac{1}{2}M\mathbb{B}_\infty$. Let $p_{\mathscr{C}} \colon \mathbb{R}^d \to \mathscr{C}$ be the metric projection onto $\mathscr{C}$, i.e., $\{p_{\mathscr{C}}(x)\} = \arg\min_{y\in\mathscr{C}} \|y - x\|$ for each $x \in \mathbb{R}^d$. Then $p_{\mathscr{C}}$ is Lipschitz continuous (with Lipschitz constant 1). Theorem 11 applied to $p_{\mathscr{C}}\restriction_{MX}$, which is the restriction of $p_{\mathscr{C}}$ to the set MX, and positive homogeneity of H_{d-1} (of degree $d-1$) yield $H_{d-1}(\operatorname{bd}(\mathscr{C})) < \infty$. Similarly, applying Theorem 11 to the restriction to $\operatorname{bd}(\mathscr{C})$ of metric projection onto $\varepsilon\mathbb{B}_\infty$ yields $H_{d-1}(\operatorname{bd}(\mathscr{C})) > 0$. The theorem now follows from (27). □

Since we are dealing with convex cones, the previous result will be more useful to us when stated in a lifted form about pointed closed convex cones:

Corollary 15. Let $\mathbb{E}$ be an Euclidean space. If $\mathscr{K} \subseteq \mathbb{E}$ is a pointed closed convex cone with dimension $d \geq 1$, then $\dim_H(\operatorname{rbd}(\mathscr{K})) = d - 1$.

Proof. We may assume that $\mathbb{E} = \mathbb{R}^d$. Since $\mathscr{K}$ is pointed, after applying some rotation, which preserves Hausdorff measures by Corollary 12, we may assume that $\mathscr{K} = \mathbb{R}_+(1 \oplus \mathscr{C})$ for some compact convex set $\mathscr{C} \subseteq \mathbb{R}^{\bar{d}}$ where $\bar{d} := d - 1$. For each $N \in \mathbb{N}$, define the compact convex set $\mathscr{K}_N := \mathscr{K} \cap [N, N+1] \oplus \mathbb{R}^{\bar{d}}$. Since

$$\operatorname{rbd}(\mathscr{K}) \subseteq \bigcup_{N=0}^{\infty} \operatorname{rbd}(\mathscr{K}_N), \tag{31}$$

the result follows from Proposition 13 and Theorem 14. □

The next result refers to faces of a convex set, so before we state it we shall briefly recall the basic theory; see [34, Sec. 18]. Let $\mathbb{E}$ be an Euclidean space. Let $\mathscr{C} \subseteq \mathbb{E}$ be a convex set. A convex subset $\mathscr{F}$ of $\mathscr{C}$ is a *face* of $\mathscr{C}$ if, for each $x, y \in \mathscr{C}$ such that the open line segment $(x, y) := \{\, (1-\lambda)x + \lambda y : \lambda \in (0,1) \,\}$ between x and y meets $\mathscr{F}$, we have $x, y \in \mathscr{F}$. We use the notation $\mathscr{F} \trianglelefteq \mathscr{C}$ to denote that $\mathscr{F}$ is a face of $\mathscr{C}$, and $\mathscr{F} \triangleleft \mathscr{C}$ to denote that $\mathscr{F}$ is a *proper* face of $\mathscr{C}$, i.e., $\mathscr{F} \trianglelefteq \mathscr{C}$ and $\mathscr{F} \neq \mathscr{C}$. Denote $\operatorname{Faces}(\mathscr{C}) := \{\, \mathscr{F} : \mathscr{F} \trianglelefteq \mathscr{C} \,\}$.

Faces of closed convex sets are closed, and faces of convex cones are convex cones. An arbitrary intersection of faces of $\mathscr{C}$ is a face of $\mathscr{C}$ and, since the faces of a convex set are partially ordered by inclusion and $\mathscr{C} \trianglelefteq \mathscr{C}$, every point x of $\mathscr{C}$ lies in a unique minimal face $\mathscr{F}$ of $\mathscr{C}$; this face $\mathscr{F}$ is characterized by the property $x \in \operatorname{ri}(\mathscr{F})$. Also, it can be proved that $\{\, \operatorname{ri}(\mathscr{F}) : \varnothing \neq \mathscr{F} \trianglelefteq \mathscr{C} \,\}$ is a partition of $\mathscr{C}$. If $\mathscr{C}$ is a compact convex set, it is not hard to prove that the faces of the homogenization of $\mathscr{C}$ are described by:

$$\operatorname{Faces}\big(\mathbb{R}_+(1 \oplus \mathscr{C})\big) = \big\{\varnothing, \{0\}\big\} \cup \big\{\, \mathbb{R}_+(1 \oplus \mathscr{F}) : \varnothing \neq \mathscr{F} \trianglelefteq \mathscr{C} \,\big\}. \tag{32}$$

Theorem 16 (Larman [22])**.** Let $\mathbb{E}$ be an Euclidean space. If $\mathscr{C} \subseteq \mathbb{E}$ is a compact convex set with dimension $d \geq 1$, then

$$H_{d-1}\Big(\bigcup_{\mathscr{F} \triangleleft \mathscr{C}} \operatorname{rbd}(\mathscr{F}) \Big) = 0.$$

As before, we shall need a conic version of Larman's Theorem. We apply tools similar to the ones used to lift Theorem 14 to Corollary 15:

Theorem 17. Let $\mathbb{E}$ be an Euclidean space. If $\mathscr{K} \subseteq \mathbb{E}$ is a pointed closed convex cone with dimension $d \geq 1$, then

$$H_{d-1}\Big(\bigcup_{\mathscr{F} \lhd \mathscr{K}} \operatorname{rbd}(\mathscr{F})\Big) = 0.$$

Proof. The case $d = 1$ is easy to verify; assume that $d \geq 2$. We may assume that $\mathbb{E} = \mathbb{R} \oplus \mathbb{R}^{\bar{d}}$ for $\bar{d} \coloneqq d - 1$ and, as in the beginning of the proof of Corollary 15, we may assume that $\mathscr{K} = \mathbb{R}_+(1 \oplus \mathscr{C})$ for some compact convex set $\mathscr{C} \subseteq \mathbb{R}^{\bar{d}}$ with nonempty interior. For each $N \in \mathbb{N}$, define the compact convex set $\mathscr{K}_N \coloneqq \mathscr{K} \cap [N, N+1] \oplus \mathbb{R}^{\bar{d}}$. By elementary convex analysis,

$$\bigcup_{\mathscr{F} \lhd \mathscr{K}} \operatorname{rbd}(\mathscr{F}) \subseteq \bigcup_{N=0}^{\infty} \bigcup_{\mathscr{F}_N \lhd \mathscr{K}_N} \operatorname{rbd}(\mathscr{F}_N). \tag{33}$$

Hence,

$$H_{d-1}\left(\bigcup_{\mathscr{F} \lhd \mathscr{K}} \operatorname{rbd}(\mathscr{F})\right) \leq \sum_{N=0}^{\infty} H_{d-1}\left(\bigcup_{\mathscr{F}_N \lhd \mathscr{K}_N} \operatorname{rbd}(\mathscr{F}_N)\right) = 0,$$

where we used the fact that each summand is zero by Theorem 16. ☐

4.3. **Generic Failure of Strict Complementarity.** In this section, we prove one of our main results: strict complementarity fails generically in the relative boundary of the normal cone of the elliptope at any of its vertices.

We shall apply Theorem 17 to $\mathbb{S}^n_+$. Let us briefly recall some well-known descriptions of the faces of the positive semidefinite cone $\mathbb{S}^n_+$; see, e.g., [18]. Let $\mathfrak{L}_n$ denote the set of linear subspaces of $\mathbb{R}^n$. For each $\mathscr{L} \in \mathfrak{L}_n$, denote

$$\mathscr{F}_{\mathscr{L}} \coloneqq \{\, X \in \mathbb{S}^n_+ : \operatorname{Null}(X) \supseteq \mathscr{L} \,\} \tag{34}$$

and note that

$$\operatorname{ri}(\mathscr{F}_{\mathscr{L}}) = \{\, X \in \mathbb{S}^n_+ : \operatorname{Null}(X) = \mathscr{L} \,\}. \tag{35}$$

Then

$$\operatorname{Faces}(\mathbb{S}^n_+) = \{\varnothing\} \cup \{\, \mathscr{F}_{\mathscr{L}} : \mathscr{L} \in \mathfrak{L}_n \,\}. \tag{36}$$

Note that, for $\mathscr{L} \in \mathfrak{L}_n$ such that $\mathscr{L} \neq \mathbb{R}^n$, there is an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$\mathscr{F}_{\mathscr{L}} = \left\{ Q \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} Q^{\mathsf{T}} : U \in \mathbb{S}^r_+ \right\}, \tag{37}$$

where $r \coloneqq n - \dim(\mathscr{L})$.

Lemma 18. Let $n \geq 2$ be an integer. Then the property "$C \mapsto \operatorname{rank}(C) = n - 1$" holds generically in $\operatorname{bd}(\mathbb{S}^n_+)$.

Proof. Set $d \coloneqq \dim_H(\mathbb{S}^n_+)$. Note that $d - 1 = \dim_H(\operatorname{bd}(\mathbb{S}^n_+))$ by Corollary 15. Let $X \in \operatorname{bd}(\mathbb{S}^n_+)$ such that $\operatorname{rank}(X) = n - 1$ fails. Then $\operatorname{rank}(X) \leq n - 2$. For each nonzero $h \in \operatorname{Null}(X)$, let $\mathscr{L}$ be the linear subspace of $\mathbb{R}^n$ spanned by h and note that $X \in \operatorname{rbd}(\mathscr{F}_{\mathscr{L}})$, following the notation from (34). Hence,

$$\{\, X \in \operatorname{bd}(\mathbb{S}^n_+) : \operatorname{rank}(X) \neq n - 1 \,\} = \{\, X \in \mathbb{S}^n_+ : \operatorname{rank}(X) \leq n - 2 \,\} \subseteq \bigcup_{\mathscr{F} \lhd \mathbb{S}^n_+} \operatorname{rbd}(\mathscr{F}).$$

The $(d-1)$-dimensional Hausdorff measure of the set on the RHS above is zero by Theorem 17. ☐

We are ready to prove one of our main results:

Theorem 19. Let $n \geq 3$, and let $\bar{X}$ be a vertex of $\mathscr{E}_n$. Then the property "$C \mapsto$ strict complementarity holds for (4)" fails generically on $-\operatorname{rbd}(\mathbb{S}^n_+ \cap \{\bar{X}\}^\perp)$.

Proof. By Theorem 3 and the discussion of linear automorphisms of $\mathscr{E}_n$ from Section 2.2, we may assume that $\bar{X} = \mathbb{1}\mathbb{1}^{\mathsf{T}}$. Set

$$m \coloneqq n - 1.$$

Let $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix such that $Q^{\mathsf{T}} e_n = n^{-1/2}\mathbb{1}$ and $Q^{\mathsf{T}} e_m = 2^{-1/2}(e_1 - e_2)$. Using the map $M \in \mathbb{S}^n \mapsto QMQ^{\mathsf{T}}$ and Corollary 12, we find that $\operatorname{rbd}(\mathbb{S}^n_+ \cap \{\bar{X}\}^\perp)$ and $\operatorname{rbd}(\mathbb{S}^n_+ \cap \{e_n e_n^{\mathsf{T}}\}^\perp)$ have the same Hausdorff dimension. Since the cone $\mathbb{S}^n_+ \cap \{e_n e_n^\intercal\}^\perp$ is the image of an embedding of $\mathbb{S}^m_+$ into $\mathbb{S}^n_+$, the Hausdorff dimension of $\operatorname{rbd}(\mathbb{S}^n_+ \cap \{e_n e_n^\intercal\}^\perp)$ is $\dim_H(\mathbb{S}^m_+) - 1$ by Corollary 15. Hence,

$$d := \dim_H\big(\operatorname{rbd}\big(\mathbb{S}^n_+ \cap \{\bar{X}\}^\perp\big)\big) = \binom{n}{2} - 1. \tag{38}$$

Set $\mathscr{C} := \big\{\, C \in -\operatorname{rbd}(\mathbb{S}^n_+ \cap \{\bar{X}\}^\perp) : \text{strict complementarity holds in (4)} \big\}$. We want to show that $H_d(\mathscr{C}) = 0$. If strict complementarity holds for $C = -\bar{S}$, i.e. $C \in \mathscr{C}$, then by Theorem 10 it is not the case that $\operatorname{Null}(\bar{S}) = \operatorname{span}\{\mathbb{1}, h\}$ where $h \perp \mathbb{1}$ and at least three coordinates of h have distinct values. As $\bar{S} \in \operatorname{rbd}(\mathbb{S}^n_+ \cap \{\bar{X}\}^\perp)$, then $\operatorname{Null}(\bar{S})$ contains $\mathbb{1}$ and $\operatorname{rank}(\bar{S}) \leq n-2$. If $\operatorname{rank}(\bar{S}) \leq n-3$ then

$$\bar{S} \in \mathscr{D}_0 := \big\{\, S \in \mathbb{S}^n_+ \cap \{\bar{X}\}^\perp : \operatorname{rank}(S) \leq n-3 \big\}.$$

Otherwise, $\bar{S}$ lies in one of the sets

$$\mathscr{D}_{ij} := \big\{\, S \in \mathbb{S}^n_+ : \exists h \in \{\mathbb{1}, e_i - e_j\}^\perp,\, h \neq 0,\, \operatorname{Null}(S) = \operatorname{span}\{\mathbb{1}, h\} \big\}, \qquad \text{for each } i, j \in [n].$$

Hence,

$$-\mathscr{C} \subseteq \mathscr{D}_0 \cup \mathscr{D}_{12} \cup \mathscr{D}_{13} \cup \mathscr{D}_{23} \tag{39}$$

Clearly all the sets $\mathscr{D}_{ij}$ have the same d-dimensional Hausdorff measures, so it suffices to prove that

$$H_d(\mathscr{D}_0) = 0, \tag{40}$$

$$H_d(\mathscr{D}_{12}) = 0. \tag{41}$$

By using the map $M \in \mathbb{S}^n \mapsto QMQ^\intercal$ and Corollary 12, $\mathscr{D}_0$ and $\{\, S \in \mathbb{S}^m_+ : \operatorname{rank}(S) \leq m-2\}$ have the same d-dimensional Hausdorff measure. Hence, (40) follows from Lemma 18 and Corollary 15. Again using the map $M \in \mathbb{S}^n \mapsto QMQ^\intercal$ and Corollary 12, we find that $H_d(\mathscr{D}_{12}) = H_d(\mathscr{D}')$ where

$$\mathscr{D}' := \{\, U \in \mathbb{S}^m_+ : \operatorname{rank}(U) = m-1, e_m \in \operatorname{Im}(U)\}.$$

Hence, to prove (41) and thus the theorem, it suffices to prove that

$$H_d(\mathscr{D}') = 0. \tag{42}$$

For each $k \in [m-1]$ define the permutation matrix $P_k := \sum_{i \in [m] \setminus \{k,m\}} e_i e_i^\intercal + e_k e_m^\intercal + e_m e_k^\intercal \in \mathbb{S}^m$, so that right-multiplication by P_k permutes the columns indexed by k and m. Set $P_m := I$. For each $k \in [m]$ define the map $\varphi_k \colon \mathbb{S}^{m-1}_{++} \oplus \mathbb{R}^{m-1} \to \mathbb{S}^m$ by setting

$$\varphi_k(A \oplus c) := P_k^\intercal \begin{bmatrix} A & Ac \\ c^\intercal A & c^\intercal Ac \end{bmatrix} P_k.$$

Note that, before applying $P_k^\intercal \cdot P_k$, the map φ_k builds from $A \oplus c \in \mathbb{S}^{m-1}_{++} \oplus \mathbb{R}^{m-1}$ a flat extension (see [26, Sec. 1.3.3]) of A in $\mathbb{S}^m_+$ whose nullspace is spanned by $-c \oplus 1$. Then the application of $P_k^\intercal \cdot P_k$ symmetrically permutes the last row/column with the kth row/column. It is easy to verify that the images of these maps decompose the rank-$(m-1)$ matrices in $\mathbb{S}^m_+$:

$$\{\, U \in \mathbb{S}^m_+ : \operatorname{rank}(U) = m-1\} = \bigcup_{k \in [m]} \varphi_k(\mathbb{S}^{m-1}_{++} \oplus \mathbb{R}^{m-1}). \tag{43}$$

Also, the nullspace of each shuffled flat extension is easy to compute:

$$\operatorname{Null}(\varphi_k(A \oplus c)) = P_k \operatorname{span}\{-c \oplus 1\} \qquad \forall A \oplus c \in \mathbb{S}^{m-1}_{++} \oplus \mathbb{R}^{m-1}. \tag{44}$$

Let $U \in \mathbb{S}^m_+$ with $\operatorname{rank}(U) = m-1$, and let $k \in [m]$ and $A \oplus c \in \mathbb{S}^{m-1}_{++} \oplus \mathbb{R}^{m-1}$ such that $U = \varphi_k(A \oplus c)$. Then $e_m \in \operatorname{Im}(U)$ is equivalent to $e_m \perp P_k(-c \oplus 1)$, which is equivalent to $k \in [m-1]$ and $c \perp e_k$. Hence,

$$\mathscr{D}' = \bigcup_{k \in [m-1]} \varphi_k(\mathbb{S}^{m-1}_{++} \oplus \{e_k\}^\perp). \tag{45}$$

Let $k \in [m-1]$. Since each entry of $\varphi_k(A \oplus c)$ is (component-wise) polynomial function of the input, the map φ_k is Lipschitz continuous on any compact subset of the domain. It follows from Proposition 13 that

$$\dim_H(\varphi_k(\mathbb{S}^{m-1}_{++} \oplus \{e_k\}^\perp)) \leq \binom{m}{2} + m - 2 = d - 1; \tag{46}$$

note that the subspace $\{e_k\}^\perp$ in the LHS is $(m-2)$-dimensional, as this subspace is the set of vectors in $\mathbb{R}^{m-1}$ orthogonal to e_k. Now (42) follows from (45) and (46), since $\dim_H(\mathscr{D}') \leq d-1$ implies that $H_d(\mathscr{D}') = 0$. $\square$

Note that in the statement of Theorem 19, the objective function matrices C vary over the relative boundary of $-\mathbb{S}^n_+ \cap \{\bar{X}\}^\perp$, where one might have expected that it would vary over the boundary of the normal cone at $\bar{X}$. As mentioned in our comment preceding Section 4.1, here we are modding out the trivial part, the lineality space of the normal cone (equivalent to taking the diagonal perturbation as fixed).

5. Failure of Strict Complementarity for Rank-One Objectives

In Section 4, we zoomed into the boundary of the normal cone of an arbitrary vertex of the elliptope and proved that strict complementarity fails generically there. *Informally*, we might say that with zero "probability" a "uniformly chosen" objective function in the boundary of such normal cone yields an SDP that satisfies strict complementarity. Recall that Theorem 19 is based on Theorem 10, which provides a sufficient condition for failure of strict complementarity based only on the *highest rank* matrices in the boundary of the normal cone. This may seen counterintuitive since we are choosing objectives with corank 2, and so any primal optimal solution with rank 2 would already be sufficient for strict complementarity to hold.

In this section, we shall zoom in even further in the boundary of the normal cone, into the set of negative semidefinite rank-one objectives, and consider again how often strict complementarity holds. In a sense, we are now proceeding in the opposite direction of Theorem 10, by looking at the *lowest possible rank* in the boundary of the normal cone (excluding the zero matrix). For such objectives, strict complementarity would require existence of extremely high-rank primal optimal solutions for strict complementarity to hold, namely corank 1. Even though the requirements seem even harder to achieve, we shall see that in this "conditional" probability space, strict complementarity holds with positive probability. In fact, we will prove that such probability lies strictly between 0 and 1.

We will state and prove a self-contained result in Theorem 23 below. However, in order to motivate the objects of the construction and the intermediate results, we start with an informal discussion. Assume throughout this discussion that $n \geq 4$. We will normalize the "sample space" so that we can have a probability space. Without loss of generality, let us focus our attention on the vertex $\mathbb{1}\mathbb{1}^\mathsf{T}$ of $\mathscr{E}_n$ and consider the sample space to be

$$\begin{aligned} \Omega &:= \{\, C \in -\operatorname{rbd}(\mathbb{S}^n_+ \cap \{\mathbb{1}\mathbb{1}^\mathsf{T}\}^\perp) : \operatorname{rank}(C) = 1,\ \|\operatorname{vec}(C)\|_\infty = 1\} \\ &= \{\, -bb^\mathsf{T} : b \in \mathbb{R}^n,\ \|b\|_\infty = 1,\ b \perp \mathbb{1}\}. \end{aligned} \tag{47}$$

Accordingly, equip $\mathbb{S}^n$ with the norm $X \in \mathbb{S}^n \mapsto \|\operatorname{vec}(X)\|_\infty$. Set $d := \dim_H(\Omega)$. In order to obtain a probability space on Ω, we will define a probability measure

$$\mathbb{P}(\mathscr{A}) := \frac{H_d(\mathscr{A})}{H_d(\Omega)} \tag{48}$$

over all H_d-measurable subsets $\mathscr{A}$ of Ω; we shall prove that $H_{n-2}(\Omega) \in (0, \infty)$, so that (48) is properly defined and $d = n-2$. Our goal is to prove that the probability of the event

$$\mathscr{G} := \{\, C \in \Omega : \text{strict complementarity holds for (4) with } C\} \tag{49}$$

lies in $(0,1)$.

In order to achieve this, we shall reduce the problem to the space of vectors that generate the rank-one tensors in Ω and $\mathscr{G}$, which lie in the matrix space. In order to carry results back and forth between these spaces, we rely on Corollary 12. For each $s \in \{\pm 1\}^n$, define

$$\mathbb{R}^n_s := \operatorname{Diag}(s)\mathbb{R}^n_+, \tag{50}$$

$$\varphi_s \colon b \in \mathbb{R}^n_s \cap \operatorname{bd}(\mathbb{B}_\infty) \mapsto -bb^\mathsf{T}. \tag{51}$$

Equip $\mathbb{R}^n$ with the norm $x \in \mathbb{R}^n \mapsto \|x\|_\infty$. We shall split our analysis to each of the 2^n bi-Lipschitz maps φ_s, one for each chamber/orthant of $\mathbb{R}^n$, according to their sign vectors:

Theorem 20. Let $s \in \{\pm 1\}^n$. Then the map φ_s defined in (51) is bi-Lipschitz continuous with Lipschitz constants 2 and 1, where we equip the domain with the ∞-norm, and we equip the range with the norm $\|\operatorname{vec}(\cdot)\|_\infty$.

Proof. To see that φ_s is Lipschitz continuous with Lipschitz constant 2, let $x, y \in \mathbb{R}^n_s \cap \mathrm{bd}(\mathbb{B}_\infty)$ and note that

$$\|2\,\mathrm{vec}(xx^\top - yy^\top)\|_\infty = \|\,\mathrm{vec}[(x-y)(x+y)^\top + (x+y)(x-y)^\top]\|_\infty \leq 2\|x+y\|_\infty\|x-y\|_\infty \leq 4\|x-y\|_\infty.$$

The proof that φ_s^{-1} is Lipschitz continuous with Lipschitz constant 1 is also simple but it involves case analysis. Set $A := xx^\top - yy^\top$. Let $k \in [n]$ such that $|x_k| = 1$, so $x_k = s_k$. Similarly, let $\ell \in [n]$ such that $|y_\ell| = 1$, so $y_\ell = s_\ell$. Let $j \in [n]$. We shall make use of the following facts:

$$\alpha_k := \frac{y_k}{s_k} \in [0,1], \qquad \beta_\ell := \frac{x_\ell}{s_\ell} \in [0,1], \qquad |A_{kj}| = |x_j - \alpha_k y_j|, \qquad |A_{\ell j}| = |\beta_\ell x_j - y_j|.$$

We consider 4 cases, according to which of x_j or y_j is largest, and according to their signs; note that both x_j and y_j have the same sign.

We have

$$\begin{aligned}
x_j \geq y_j \geq 0 &\implies 0 \leq |x_j - y_j| = x_j - y_j \leq x_j - \alpha_k y_j = |A_{kj}|;\\
y_j \geq x_j \geq 0 &\implies 0 \leq |x_j - y_j| = y_j - x_j \leq y_j - \beta_\ell x_j = |A_{\ell j}|;\\
0 \geq x_j \geq y_j &\implies 0 \leq |x_j - y_j| = x_j - y_j \leq \beta_\ell x_j - y_j = |A_{\ell j}|;\\
0 \geq y_j \geq x_j &\implies 0 \leq |x_j - y_j| = y_j - x_j \leq \alpha_k y_j - x_j = |A_{kj}|.
\end{aligned}$$

Hence, $\|x-y\|_\infty \leq \|\,\mathrm{vec}(xx^\top - yy^\top)\|_\infty$. ☐

Note that restricting the domain of φ_s in Theorem 20 to chambers of $\mathbb{R}^n$ is necessary. Indeed, consider $x := (1,-1,\varepsilon)^\top$ and $y := (-1,1,0)^\top$, for an arbitrary $\varepsilon \in (0,1)$. Then $\|x-y\|_\infty = 2$ but $\|\,\mathrm{vec}(xx^\top - yy^\top)\|_\infty = \varepsilon$.

Finally, we need to relate $\mathscr{G}$ with the vectors that appear in the rank-one tensors. A vector $b \in \mathbb{R}^n$ is *strictly balanced* if $|b_i| < \sum_{j\in[n]\setminus\{i\}}|b_j|$ for every $i \in [n]$. It is easy to verify that,

$$\text{if } b \in \mathbb{R}^n \text{ and } i \in [n] \text{ is such that } |b_i| = \|b\|_\infty, \text{ then } b \text{ is strictly balanced } \iff |b_i| < \textstyle\sum_{j\in[n]\setminus\{i\}}|b_j|. \tag{52}$$

We shall rely on yet another result by Laurent and Poljak:

Theorem 21 ([25, Theorem 2.6])**.** Let $b \in \mathbb{R}^n$ such that $b \perp \mathbb{1}$ and $\mathrm{supp}(b) = [n]$. Then there exists $X \in \mathscr{E}_n$ such that $\mathrm{Null}(X) = \mathrm{span}\{b\}$ if and only if b is strictly balanced.

Theorem 21 provides a neat characterization of strict complementarity for full support matrices in Ω in terms of strict balancedness:

Proposition 22. Let $b \in \mathbb{R}^n$ such that $b \perp \mathbb{1}$ and $\mathrm{supp}(b) = [n]$. Then strict complementarity holds for (4) with $C = -bb^\top$ if and only if b is strictly balanced.

Proof. Note that $\mathbb{1}\mathbb{1}^\top$ is an optimal solution for (4) if $C = -bb^\top$. By Proposition 5, we must show that existence of $X \in \mathscr{E}_n$ such that $-bb^\top \in \mathrm{ri}(\mathrm{Normal}(\mathscr{E}_n; X))$ is equivalent to strict balancedness of b. We will show that, for each $X \in \mathscr{E}_n$,

$$-\,bb^\top \in \mathrm{ri}(\mathrm{Normal}(\mathscr{E}_n; X)) \iff bb^\top \in \{\, Z \in \mathbb{S}^n_+ : \mathrm{Im}(Z) = \mathrm{Null}(X)\}. \tag{53}$$

Since existence of $X \in \mathscr{E}_n$ such that the RHS of (53) holds is equivalent to b being strictly balanced by Theorem 21, the result will follow.

The proof of sufficiency in (53) follows from (11) and $\mathrm{ri}(\mathbb{S}^n_+ \cap \{X\}^\perp) = \{\, Z \in \mathbb{S}^n_+ : \mathrm{Im}(Z) = \mathrm{Null}(X)\}$. For the proof of necessity, recall (11) and suppose that there exists $X \in \mathscr{E}_n$ such that $-bb^\top = \mathrm{Diag}(y) - S$ for some $y \in \mathbb{R}^n$ and $S \in \mathrm{ri}(\mathbb{S}^n_+ \cap \{X\}^\perp)$. Then $0 = -bb^\top\mathbb{1} = (\mathrm{Diag}(y) - S)\mathbb{1} = y - S\mathbb{1}$ shows that

$$y = S\mathbb{1}. \tag{54}$$

Since X and $\mathbb{1}\mathbb{1}^\top$ are optimal solutions for (4), we find that $0 = \mathrm{Tr}(-bb^\top\mathbb{1}\mathbb{1}^\top) = \mathrm{Tr}(-bb^\top X) = y^\top\mathrm{diag}(X) - \mathrm{Tr}(SX)$ so $\mathbb{1}^\top y = \mathrm{Tr}(SX) = 0$. By (54), $\mathbb{1}^\top S\mathbb{1} = \mathbb{1}^\top y = 0$, so $\mathbb{1} \in \mathrm{Null}(S)$ and $y = 0$. ☐

We are now in position to present the main result of this section:

Theorem 23. Let $n \geq 4$ be an integer. Equip $\mathbb{S}^n$ with the norm $\|\,\mathrm{vec}(\cdot)\|_\infty$. Set

$$\Omega := \{\, -bb^\top : b \in \mathbb{R}^n,\ \|b\|_\infty = 1,\ b \perp \mathbb{1}\} \subseteq \mathbb{S}^n,$$

$$\mathscr{G} := \{\, C \in \Omega : \text{strict complementarity holds for (4) with } C\}.$$

Set $d := \dim_H(\Omega)$. Let Σ_d be the σ-algebra of H_d-measurable subsets of $\mathbb{S}^n$ and set $\Sigma := \{\, \mathscr{A} \in \Sigma_d : \mathscr{A} \subseteq \Omega\}$. Then

(i) $\Omega \in \Sigma_d$,
(ii) $H_{n-2}(\Omega) \in (0,\infty)$, so $d = n-2$,
(iii) $\mathscr{G} \in \Sigma$,
(iv) $H_d(\mathscr{G}) > 0$ and $H_d(\overline{\mathscr{G}}) > 0$, where $\overline{\mathscr{G}} := \Omega \setminus \mathscr{G}$.

In particular, if we set

$$\mathbb{P}(\mathscr{A}) := \frac{H_d(\mathscr{A})}{H_d(\Omega)} \qquad \forall \mathscr{A} \in \Sigma, \tag{55}$$

then $(\Omega, \Sigma, \mathbb{P})$ is a probability space and the event $\mathscr{G}$ satisfies $\mathbb{P}(\mathscr{G}) \in (0,1)$.

Proof. Define φ_s as in (51) for each $s \in \{\pm 1\}^n$.

(i): Trivial since Ω is compact.

(ii): We will prove that $H_{n-2}(\Omega) \in (0,\infty)$, from which it will follow via (27) that $d = n-2$. We have

$$\Omega \supseteq \{\, -bb^{\mathsf{T}} : b = -1 \oplus c,\, c \in \mathbb{R}^{n-1}_{+},\, \mathbb{1}^{\mathsf{T}} c = 1\} \implies H_{n-2}(\Omega) > 0.$$

For each $s \in \{\pm 1\}^n$ and $i \in [n]$, the polytope $\mathscr{P}_{s,i} := \{\, b \in \mathbb{R}^n_s : b \perp \mathbb{1},\, -\mathbb{1} \leq b \leq \mathbb{1},\, b_i = s_i\}$ has dimension less than or equal to $n-2$. Since

$$\Omega = \bigcup_{s \in \{\pm 1\}^n} \bigcup_{i \in [n]} \varphi_s(\mathscr{P}_{s,i})$$

and each $\varphi(\mathscr{P}_{s,i})$ has finite d-dimensional Hausdorff measure by Corollary 12 and Theorem 20, the proof of (ii) is complete.

(iii): In the remainder of the proof we shall use subsets of $\mathbb{R}^n$ with constraints on the coordinates that are zero:

$$\mathscr{Z}_i := \{\, b \in \mathbb{R}^n : b_i = 0\} \quad \forall i \in [n], \qquad \text{and} \qquad \mathscr{Z}_\varnothing := \mathbb{R}^n \setminus \bigcup_{i \in [n]} \mathscr{Z}_i = \{\, b \in \mathbb{R}^n : \operatorname{supp}(b) = [n]\}.$$

We will transfer measures of sets in a space of vectors to sets in a space of matrices. To distinguish them, we shall decorate the sets in spaces of vectors with a prime:

$$\begin{aligned} \Omega' &:= \{\, b \in \mathbb{R}^n : b \perp \mathbb{1},\, \|b\|_\infty = 1\}, \\ \mathscr{G}' &:= \{\, b \in \Omega' : -bb^{\mathsf{T}} \in \mathscr{G}\}, \\ \overline{\mathscr{G}'} &:= \Omega \setminus \mathscr{G}', \\ \mathscr{B}_{\text{bal}} &:= \{\, b \in \Omega' : b \text{ is strictly balanced}\}, \\ \overline{\mathscr{B}_{\text{bal}}} &:= \Omega' \setminus \mathscr{B}_{\text{bal}}. \end{aligned}$$

Proposition 22 implies that

$$\mathscr{G}' \cap \mathscr{Z}_\varnothing = \mathscr{B}_{\text{bal}} \cap \mathscr{Z}_\varnothing, \tag{56}$$

$$\overline{\mathscr{G}'} \cap \mathscr{Z}_\varnothing = \overline{\mathscr{B}_{\text{bal}}} \cap \mathscr{Z}_\varnothing. \tag{57}$$

For each $i \in [n]$, we have $\mathscr{G}' \cap \mathscr{Z}_i \subseteq \Omega' \cap \mathscr{Z}_i$ and the set on the RHS has zero d-dimensional Hausdorff measure. Hence,

$$H_d(\mathscr{G}' \cap \mathscr{Z}_i) = 0 \qquad \forall i \in [n]. \tag{58}$$

Define φ_s as in (51) for each $s \in \{\pm 1\}^n$. By putting together (58) and (56), we find that

$$\mathscr{G} = \mathscr{N} \cup \bigcup_{s \in \{\pm 1\}^n} \varphi_s(\mathscr{G}' \cap \mathscr{Z}_\varnothing \cap \mathbb{R}^n_s) = \mathscr{N} \cup \bigcup_{s \in \{\pm 1\}^n} \varphi_s(\mathscr{B}_{\text{bal}} \cap \mathscr{Z}_\varnothing \cap \mathbb{R}^n_s) \tag{59}$$

for some subset $\mathscr{N} \subseteq \Omega$ such that $H_d(\mathscr{N}) = 0$. This set $\mathscr{N}$ arises as the image under the various maps φ_s of the sets $\mathscr{G}' \cap \mathscr{Z}_i$; again we are relying on Corollary 12 and Theorem 20.

Let us prove that

$$\mathscr{G} \in \Sigma. \tag{60}$$

By standard Hausdorff measure theory, Σ_d contains every Borel set of $\mathbb{S}^n$; see, e.g., [35, Theorem 27]. Recall that the Borel sets of $\mathbb{S}^n$ are the elements of the smallest σ-algebra on $\mathbb{S}^n$ that contains all the open subsets of $\mathbb{S}^n$. For each $m \in \mathbb{N} \setminus \{0\}$, define

$$\mathscr{B}_{\mathrm{bal},m} := \Big\{ b \in \mathbb{R}^n : b \perp \mathbb{1},\, \|b\|_\infty = 1,\, |b_i| \geq \tfrac{1}{m}\, \forall i \in [n],\, |b_i| + \tfrac{1}{m} \leq \sum_{j \in [n]\setminus\{i\}} |b_j|\, \forall i \in [n] \Big\}.$$

Clearly, $\mathscr{B}_{\mathrm{bal}} = \bigcup_{m=1}^{\infty} \mathscr{B}_{\mathrm{bal},m}$. Hence, by (59),

$$\mathscr{G} = \mathscr{N} \cup \bigcup_{m=1}^{\infty} \bigcup_{s \in \{\pm 1\}^n} \varphi_s(\mathscr{B}_{\mathrm{bal},m} \cap \mathscr{Z}_{\varnothing} \cap \mathbb{R}^n_s). \tag{61}$$

Since each $\varphi_s(\mathscr{B}_{\mathrm{bal},m} \cap \mathscr{Z}_{\varnothing} \cap \mathbb{R}^n_s)$ is compact, it follows that $\mathscr{G}$ is the union of a null set with an F_σ, i.e., a countable union of closed sets, and hence a Borel set. Thus, $\mathscr{G} \in \Sigma_d$, and the proof of (iii) is complete.

(iv): Set

$$\mathring{x} := 1 \oplus \frac{1}{n-1} \oplus \frac{-n}{(n-1)(n-2)} \mathbb{1} \in \mathbb{R}^n, \quad \varepsilon := \frac{3}{4(n-1)(n-2)}, \quad \text{and} \quad s(x) := 1 \oplus 1 \oplus -\mathbb{1} \in \{\pm 1\}^n.$$

It is not hard to verify that

$$\mathring{x} + \varepsilon(\mathbb{B}_\infty \cap \{e_1, \mathbb{1}\}^\perp) \subseteq \mathscr{B}_{\mathrm{bal}} \cap \mathscr{Z}_{\varnothing} \cap \mathbb{R}^n_{s(x)}. \tag{62}$$

Since the set in the LHS of (62) has positive d-dimensional measure, so does the set in the RHS of (62), whence

$$H_d(\mathscr{G}) > 0 \tag{63}$$

by Corollary 12, Theorem 20, and (59).

Set

$$\mathring{y} := 1 \oplus -\frac{1}{n-1} \mathbb{1} \in \mathbb{R}^n, \quad \delta := \frac{1}{2(n-1)}, \quad \text{and} \quad s(y) := 1 \oplus -\mathbb{1} \in \{\pm 1\}^n.$$

It is not hard to verify that

$$\mathring{y} + \delta(\mathbb{B}_\infty \cap \{e_1, \mathbb{1}\}^\perp) \subseteq \overline{\mathscr{B}_{\mathrm{bal}}} \cap \mathscr{Z}_{\varnothing} \cap \mathbb{R}^n_{s(y)}. \tag{64}$$

Hence,

$$\overline{\mathscr{G}} \supseteq \varphi(\overline{\mathscr{G}'} \cap \mathscr{Z}_{\varnothing}) = \varphi(\overline{\mathscr{B}_{\mathrm{bal}}} \cap \mathscr{Z}_{\varnothing}) \supseteq \varphi_{s(y)}(\overline{\mathscr{B}_{\mathrm{bal}}} \cap \mathscr{Z}_{\varnothing} \cap \mathbb{R}^n_{s(y)}) \supseteq \varphi_{s(y)}(\mathring{y} + \delta(\mathbb{B}_\infty \cap \{e_1, \mathbb{1}\}^\perp)).$$

Thus,

$$H_d(\overline{\mathscr{G}}) > 0$$

by Corollary 12 and Theorem 20. □

We note that, in the above proof, it is possible to compute the volume of the set $\mathscr{B}_{\mathrm{bal}}$ exactly, since it can be expressed as a combinatorial function of certain slabs of hypercubes in an Euclidean space. Some of the underlying volume formulae go back at least to works of Laplace as well as Pólya (see [7]), and they are related to Ehrhart Theory (see [8, 12]). We opted to present the above high-level elegant proof instead of long computations and analysis for at least two reasons: (i) the above proof is more clearly adaptable to similar situations in convex optimization; (ii) the usage of the bi-Lipschitz map with constants 2 and 1 from Theorem 20 together with Corollary 12, degrades the probabilities by a factor of $2^{\Omega(n)}$. Therefore, to present an exact value (or near exact value) for this probability would take us far afield.

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