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STABLE SET POLYTOPES WITH RANK $|V(G)|/3$ FOR THE LOVÁSZ–SCHRIJVER SDP OPERATOR

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Abstract. We study the lift-and-project rank of the stable set polytope of graphs with respect to the Lovász–Schrijver SDP operator LS_+ applied to the fractional stable set polytope. In particular, we show that for every positive integer ℓ, the smallest possible graph with LS_+-rank ℓ contains 3ℓ vertices. This result is sharp and settles a conjecture posed by Lipták and the second author in 2003, as well as answers a generalization of a problem posed by Knuth in 1994. We also show that for every positive integer ℓ there exists a vertex-transitive graph on $4\ell + 12$ vertices with LS_+-rank at least ℓ.

1. Introduction

In discrete optimization, a common and very successful approach for tackling a given problem is to model it as an integer program and analyze it using convex optimization techniques. More precisely, suppose we are interested in solving the integer program

$$\max\left\{c^\top x : x \in P \cap \{0,1\}^n\right\}, \tag{1}$$

where $c \in \mathbb{R}^n$ and $P \subseteq [0,1]^n$ are given. Notice that we can replace $P \cap \{0,1\}^n$ by

$$P_I := \operatorname{conv}\left\{P \cap \{0,1\}^n\right\},$$

the *integer hull* of P, as the feasible region in (1) to obtain a convex optimization problem. However, for a general given P (given as the solution set of a system of linear inequalities), it is $\mathcal{NP}$-hard to efficiently obtain a description (such as a list of its facet-inducing inequalities) of P_I. Now if the given set P is tractable (i.e., it admits a polynomial-time separation oracle), we could also choose to simply optimize $c^\top x$ over P and at least obtain an approximate solution and an upper bound on the optimal value of (1) in polynomial time. However, as many sets P can share the same integer hull, the quality of the approximate solution obtained under this approach very much depends on whether P is a "tight" or "loose" relaxation of P_I.

One way to systematically tighten a given relaxation is via *lift-and-project methods*. While a number of operators fall under this approach, most notably those devised in [SA90, LS91, BCC93, Las01, BZ04, GPT10, AT16], in this work we focus on the operator LS_+ (also known as N_+ in the literature) devised by Lovász and Schrijver [LS91].

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Before we define LS_+, we need some notation. Given a set $P \subseteq [0,1]^n$, we define the *homogenized cone* of P to be

$$\operatorname{cone}(P) := \left\{ \begin{bmatrix} \lambda \\ \lambda x \end{bmatrix} : \lambda \geq 0, x \in P \right\}.$$

Notice that $\operatorname{cone}(P) \subseteq \mathbb{R}^{n+1}$, and we will index the new coordinate by 0. Also, given a vector x (which, by default, is a column vector) and index i, we let x_i or $[x]_i$ denote the i-entry in x. Next, we let e_i be the unit vector whose i-entry is 1, with all other entries being 0. We let $\mathbb{S}_+^n$ denote the set of $n \times n$ real symmetric positive semidefinite matrices. To express that a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is positive semidefinite, we may write $M \in \mathbb{S}_+^n$, or alternatively use the notation $M \succeq 0$. We also let $\operatorname{diag}(M)$ denote the vector made up of the diagonal entries of M. Also, given a positive integer n, we let $[n] := \{1, 2, \ldots, n\}$.

The LS_+ operator can be defined as follows. Given $P \subseteq [0,1]^n$, let

$$\widehat{\mathrm{LS}}_+(P) := \left\{ Y \in \mathbb{S}_+^{n+1} : Ye_0 = \operatorname{diag}(Y), Ye_i, Y(e_0 - e_i) \in \operatorname{cone}(P)\ \forall i \in [n] \right\}.$$

Then we define

$$\mathrm{LS}_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \widehat{\mathrm{LS}}_+(P), Ye_0 = \begin{bmatrix} 1 \\ x \end{bmatrix} \right\}.$$

Intuitively, LS_+ *lifts* P to a set of $(n+1) \times (n+1)$ matrices and imposes some constraints in the lifted space to obtain $\widehat{\mathrm{LS}}_+(P)$, and then *projects* it back down to $\mathbb{R}^n$ to obtain the tightened relaxation $\mathrm{LS}_+(P)$. Then one can show that $P_I \subseteq \mathrm{LS}_+(P) \subseteq P$ (see, for instance, [AT24a, Lemma 3] for a proof).

Moreover, we can apply LS_+ successively to a set P to obtain yet tighter relaxations. Define $\mathrm{LS}_+^0(P) := P$, and for every positive integer $k \geq 1$ define $\mathrm{LS}_+^k(P) := \mathrm{LS}_+\left(\mathrm{LS}_+^{k-1}(P)\right)$. Then, for every set P, LS_+ generates a hierarchy of nested convex relaxations which satisfy

$$P \supseteq \mathrm{LS}_+(P) \supseteq \mathrm{LS}_+^2(P) \supseteq \cdots \supseteq \mathrm{LS}_+^n(P) = P_I.$$

Thus, instead of optimizing over P, one can optimize over the tightened relaxation $\mathrm{LS}_+^k(P)$ for a chosen k and obtain a potentially better approximate solution. Furthermore, if P is tractable and $k = O(1)$, then $\mathrm{LS}_+^k(P)$ is also tractable. Thus, the LS_+-relaxations offer a "generic" polynomial-time approximation algorithm for a broad range of 0, 1 integer programs — and as an immediate extension, many hard discrete optimization problems. We define the LS_+*-rank* of a set P to be the smallest integer k where $\mathrm{LS}_+^k(P) = P_I$. Since the n-th relaxation generated by LS_+ is guaranteed to be equal to P_I [LS91], every $P \subseteq [0,1]^n$ has LS_+-rank at most n.

In this manuscript, we are particularly interested in studying the LS_+-relaxations for the stable set problem of graphs. Given a simple, undirected graph $G = (V(G), E(G))$, we say that a set of vertices $S \subseteq V(G)$ is a *stable set* in G if no two vertices in S are joined by an edge in G. The *(maximum) stable set problem*, which aims to find the stable set of the largest cardinality in a given graph, is one of the most well-studied problems in combinatorial optimization and is well-known to be $\mathcal{NP}$-hard.

Given a graph G, we define its *fractional stable set polytope* to be

$$\mathrm{FRAC}(G) := \left\{ x \in [0,1]^{V(G)} : x_i + x_j \leq 1\ \forall \{i,j\} \in E(G) \right\},$$

and its *stable set polytope* to be $\mathrm{STAB}(G) := \mathrm{FRAC}(G)_I$. Notice that $x \in \{0,1\}^{V(G)}$ belongs to $\mathrm{FRAC}(G)$ if and only if it is the incidence vector of a stable set in G, and that $\mathrm{STAB}(G)$ is precisely the convex hull of the incidence vectors of all stable sets in G. It is well-known that $\mathrm{FRAC}(G) = \mathrm{STAB}(G)$ if and only if the given graph is bipartite. In other cases, we can then apply LS_+ to $\mathrm{FRAC}(G)$ to obtain a hierarchy of convex relaxations that approximate

STAB(G). Furthermore, the LS_+-rank of FRAC(G) (which we will simply call the LS_+-rank of G and denote by $r_+(G)$) gives a measure of the level of complexity of the stable set problem on G in the perspective of LS_+ and FRAC(G). For instance, Lovász and Schrijver [LS91] showed that many well-known families of graphs, including perfect graphs, odd cycles, odd antiholes, and odd wheels, have LS_+-rank 1. In the last decade, there has been significant progress (see, for instance, [BENT13, BENT17, Wag22, BENW23]) on obtaining a combinatorial characterization of graphs with LS_+-rank 1, which are commonly known as *LS_+-perfect graphs* in the literature.

While LS_+ can compute STAB(G) in polynomial time for many graphs G, this also raises the natural question of which graphs give the worst-case instances for LS_+. While there are simple polytopes in $[0,1]^n$ which have the highest possible LS_+-rank of n (see, for instance, [GT01, AT18]), Lipták and the second author proved the following [LT03, Theorem 39].

Theorem 1. *For every graph G, $r_+(G) \leq \frac{|V(G)|}{3}$.*

Given an integer $\ell \geq 1$, let $n_+(\ell)$ denote the smallest number of vertices on which there exists a graph with LS_+-rank ℓ. Then Theorem 1 readily implies that $n_+(\ell) \geq 3\ell$ for every positive integer ℓ. Thus, we say that a graph G is *ℓ-minimal* if $r_+(G) = \ell$ and $|V(G)| = 3\ell$.

Well, for which ℓ do ℓ-minimal graphs exist? Again, since FRAC(G) = STAB(G) if and only if G is bipartite, it is easy to see that $n_+(1) = 3$, attained by the 3-cycle. Lipták and the second author [LT03] showed that $G_{2,1}$ from Figure 1 is 2-minimal, and went on to conjecture that ℓ-minimal graphs exist for every positive integer ℓ. Subsequently, Escalante, Montelar, and Nasini [EMN06] showed that there is only one other 2-minimal graph ($G_{2,2}$ from Figure 1, also see [AT24a, discussion following Proposition 21]), as well as discovered the first known 3-minimal graph ($G_{3,1}$ from Figure 1). Then, after nearly two decades of relative lack of new progress on this front, the authors [AT24a] recently discovered the first known 4-minimal graph ($G_{4,1}$ from Figure 1), which implies the existence of several other new 3- and 4-minimal graphs.

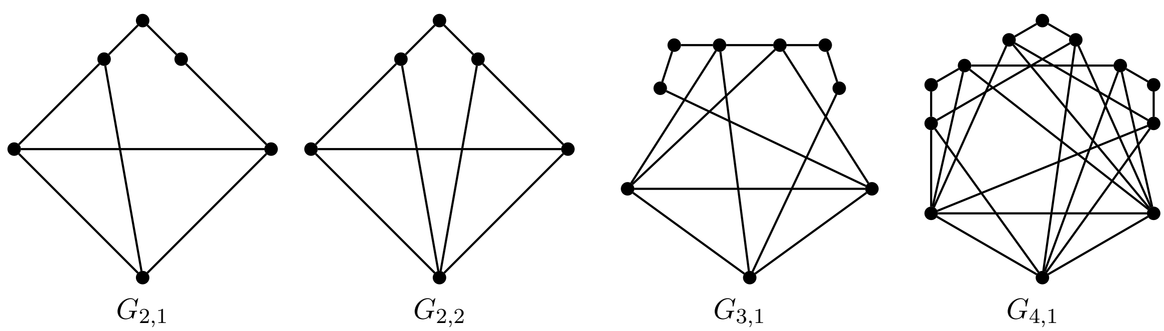


FIGURE 1. Several known ℓ-minimal graphs due to [LT03, EMN06, AT24a]

As for the asymptotic behaviour of $n_+(\ell)$, Stephen and the second author [ST99] showed that the line graph of the complete graph on $2\ell + 1$ vertices has LS_+-rank ℓ, which implies that $n_+(\ell) \leq 2\ell^2 + \ell$ in general. Recently, the authors [AT24b] discovered a family of graphs which showed that $n_+(\ell) \leq 16\ell$ for every positive integer ℓ, thus, implying that $n_+(\ell) = \Theta(\ell)$ asymptotically.

In this work, we show that, indeed, $n_+(\ell) = 3\ell$ for every $\ell \geq 1$, which settles the aforementioned conjecture in [LT03]. The ℓ-minimal graphs we present herein are all *stretched cliques* — graphs which can be obtained by starting with a complete graph, and applying a number of vertex-stretching operations, which we define in Section 2. We remark that our vertex-stretching operation is a slight variant of that defined in [AT24a], and is closely related to similar operations studied earlier in [LT03, AEF14, BENT17]. More generally, a number of stretched cliques have also been studied as instances of interest for other lift-and-project hierarchies [PnVZ07, DV15, LV23, Var23].

Understanding the behaviour of the LS_+ operator, and in particular, the behaviour of the LS_+-rank under graph operations was a natural research direction following the seminal paper of Lovász and Schrijver [LS91]. Related questions about the behaviour of the LS_+-rank under basic graph operations were also raised by Goemans and the second author [GT01]. There has been some very nice work establishing connections between lift-and-project operator ranks (for some operators related to LS_+) and graph minor operations especially on the maximum cut problem (see [Lau02, Lau03a, Lau03b]). However, as it was shown in [LT03], LS_+-rank behaves rather erraticaly with respect to many established graph operations. Thus, deeper investigations on the LS_+-rank of graphs was justified. A key piece of such investigations is understanding the combinatorial structure of minimal obstructions to effectiveness of such convex relaxations of the stable set problem obtained via the LS_+ operator. Importance of such questions were raised by many others. Notably, in a very well-known survey "The Sandwich Theorem" [Knu94] about the Lovász theta function, Knuth poses six "perplexing questions." Two of these questions involve the effectiveness of the LS_+ operator on the stable set problem. One of the questions asked about what we call here 2-minimal graphs (answered in [LT03]). One of the main results of this paper answers a more general version of Knuth's question by giving constructions of $2^{\ell-1}$ non-isomorphic ℓ-minimal graphs for every positive integer ℓ.

The manuscript is organized as follows: In Section 2, we define the aforementioned vertex-stretching operation, and mention some properties of the stable set polytopes of stretched cliques. Then we return to analyzing LS_+-relaxations in Section 3, as we establish the necessary facts — some regarding general LS_+-relaxations and some specifically applicable to the graphs of our interest — and build up to our main results (Theorem 19 and Corollary 20). We then explore the implications of these results in Section 4. In particular, we offer an explicit construction of an ℓ-minimal graph for every positive integer ℓ (Proposition 21 and Figure 4), and go on to show that there are in fact at least $2^{\ell-1}$ distinct ℓ-minimal graphs for every positive integer ℓ (Theorem 23). As a consequence of our construction, we also obtain a family of vertex-transitive graphs on $4\ell + 8$ vertices with LS_+-rank at least ℓ for every odd $\ell \geq 1$ (Proposition 26 and Figure 9). We conclude our manuscript by discussing some relevant remaining open questions in Section 5.

2. Stretched cliques and their properties

In this section, we revisit the vertex-stretching operation introduced in [AT24a] and investigate some graphs that can be obtained by iteratively applying this operation to a complete graph.

First, we need some graph theoretical notation. Given a positive integer n, let K_n denote the *complete graph* on n vertices, with $V(K_n) := [n]$ and $E(K_n) := \{\{i,j\} : 1 \leq i < j \leq n\}$. Also, given a graph G and vertex $v \in V(G)$, we define the *(open) neighborhood* of v to be

$$\Gamma_G(v) := \{u \in V(G) : \{u,v\} \in E(G)\}.$$

Next, given a set of vertices $S \subseteq V(G)$, we let $G - S$ denote the subgraph of G induced by the vertices $V(G) \setminus S$, and call this the graph obtained from G by the *deletion* of S. (When $S = \{v\}$, we will simply write $G - v$ instead of $G - \{v\}$.) Then we define

$$G \ominus v := G - (\Gamma_G(v) \cup \{v\})$$

to be the graph obtained from G by the *destruction* of v. We also let $\alpha(G)$ denote the *stability number* of a graph G, which is defined to be the cardinality of the largest stable set in G.

Next, given a graph G, vertex $v \in V(G)$, and (possibly empty) sets $A_1, \ldots, A_k \subseteq \Gamma_G(v)$ where $\bigcup_{i=1}^k A_i = \Gamma_G(v)$, we define the *stretching* of v to be the following transformation to G:

- Replace v by $k+1$ vertices: $v_0, v_1, \ldots, v_k$;
- For every $j \in [k]$, add an edge between v_j to every vertex in $\{v_0\} \cup A_j$.

We remark that the above definition of vertex stretching is a slight variant of that defined in [AT24a], which has an additional requirement that $A_1, \ldots, A_k$ must each be a non-empty and proper subset of $\Gamma_G(v)$. Herein we will say that a vertex-stretching operation is *proper* if it satisfies this more restrictive definition. We will also call the operation *k-stretching* when we need to specify k. For example, in Figure 2, $G_1 = K_6$, G_2 is obtained from 2-stretching vertex 5 in G_1, and G_3 is obtained from 2-stretching vertex 6 in G_2.

The following is a key property of the vertex-stretching operation.

Lemma 2. *Let G' be a graph obtained from G by stretching a vertex in G. Then $r_+(G') \geq r_+(G)$.*

Proof. The case for when the vertex stretching is proper was shown in [AT24a, Proposition 14], so it remains to prove our claim for when the operation is not proper. Suppose G' is obtained by k-stretching the vertex $v \in V(G)$ with $A_1, \ldots, A_k$ satisfying $\bigcup_{i=1}^k A_i = \Gamma_G(v)$. The stretching not being proper implies that $A_i = \Gamma_G(v)$ for some $i \in [k]$, and/or $A_i = \emptyset$ for some $i \in [k]$.

Define $S \coloneqq \{i \in [k] : A_i \neq \emptyset\}$, and G'' be the subgraph of G' induced by $(V(G) \setminus \{v\}) \cup \{v_0\} \cup \{v_i : i \in S\}$. Since G'' is an induced subgraph of G', we have $r_+(G') \geq r_+(G'')$. We next show that $r_+(G'') \geq r_+(G)$, which implies our claim. If $A_i \neq \Gamma_G(v)$ for every $i \in S$, then G'' can be obtained from G by a proper vertex-stretching operation, and so $r_+(G'') \geq r_+(G)$. Otherwise, there exists $j \in S$ where $A_j = \Gamma_G(v)$. In that case, the subgraph of G'' induced by the vertices $(V(G) \setminus \{v\}) \cup \{v_j\}$ is isomorphic to G, which implies $r_+(G'') \geq r_+(G)$. Thus, our claim follows. □

The relationship between vertex-stretching operations and the LS$_+$-rank of a graph was first studied in [LT03]. Among other results, it was shown therein that applying a type-1 stretching operation (which is a proper 2-stretching of a vertex where A_1 and A_2 are disjoint) cannot decrease the LS$_+$-rank of a graph. Similar vertex-stretching operations and their impact on the LS$_+$-rank of a graph were also analyzed in [AEF14, BENT17]. More recently, the authors showed that applying a proper 2-stretching operation to a complete graph on at least 4 vertices always increases its LS$_+$-rank from 1 to 2. The key breakthrough of this work is that we are now able to prove that, for every $n \geq 4$, it is possible to 2-stretch $n-3$ vertices in K_n to increase its LS$_+$-rank from 1 to $n-2$, which produces an $(n-2)$-minimal graph. These results are detailed in Section 3.

Thus, we will restrict our discussion to 2-stretching below. In particular, given integers $n \geq 3$ and $d \geq 0$, let $\mathcal{K}_{n,d}$ denote the set of graphs that can be obtained from 2-stretching d of the n vertices from K_n. We also introduce some terminology that will ease our subsequent discussion of graphs in $\mathcal{K}_{n,d}$. Given $G \in \mathcal{K}_{n,d}$, let $D(G) \subseteq [n]$ be the set of vertices of K_n which were stretched to obtain G. For each $i \in D(G)$, we call i_0 a *hub* vertex, and i_1, i_2 *wing* vertices. We also call each $i \in [n] \setminus D(G)$ an *unstretched* vertex in G. Finally, given an index $i \in [n]$, we define the vertices *associated* with i to be i_0, i_1, i_2 if $i \in D(G)$, and just the unstretched vertex i otherwise. Notice that every vertex in G is associated with a unique $i \in [n]$. Also, observe that, given $G \in \mathcal{K}_{n,d}$ and distinct $i, j \in [n]$, it follows from the definition of vertex stretching that there must be at least one edge in G joining a vertex associated with i and a vertex associated with j.

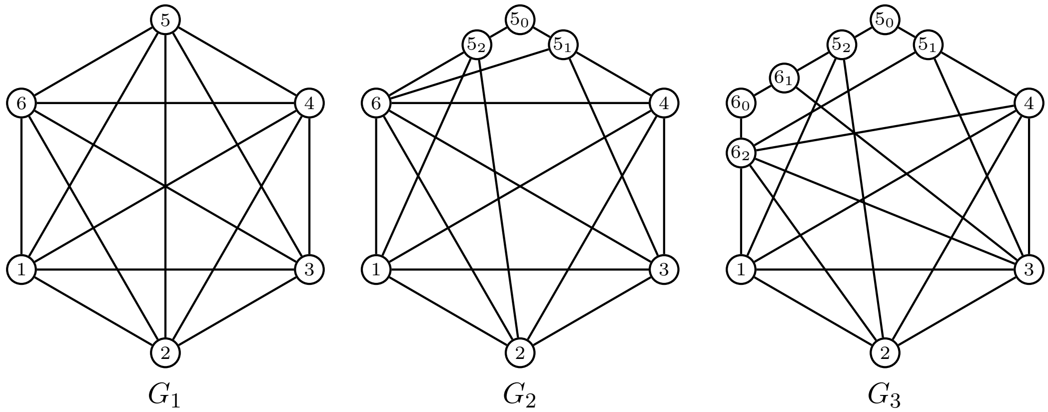


FIGURE 2. Illustrating the vertex-stretching operation

Example 3. Consider the graphs in Figure 2. First, we have $G_1 \in \mathcal{K}_{6,0}$, $G_2 \in \mathcal{K}_{6,1}$, and $G_3 \in \mathcal{K}_{6,2}$. Also, notice that $D(G_3) = \{5, 6\}$, and that the wing vertex 6_2 is adjacent to a vertex associated with i for every $i \in \{1, 2, 3, 4, 5\}$ — we will revisit this point later on in Example 5.

The following lemma gives some basic properties of graphs in $\mathcal{K}_{n,d}$.

Lemma 4. *Let $G \in \mathcal{K}_{n,d}$ where $n \geq 3$ and $d \geq 0$. Then*

(i) *for every $i \in D(G)$, $G \ominus i_0 \in \mathcal{K}_{n-1,d-1}$;*
(ii) *for every $i \in [n] \setminus D(G)$, $G - i \in \mathcal{K}_{n-1,d}$;*
(iii) $\alpha(G) = d + 1$.

Proof. Both (i) and (ii) follow directly from the definition of $\mathcal{K}_{n,d}$ and our definition of vertex stretching. (iii) follows from the proof of [AT24a, Lemma 20] (which does not require that the vertex stretching is proper in its argument). □

Let $\bar{e}$ denote the vector of all-ones (whose dimension will be clear from the context). Given a graph G, the inequality $\bar{e}^\top x \leq \alpha(G)$ is often known as the *rank inequality* of G, and is always valid for STAB(G). We next describe the graphs $G \in \mathcal{K}_{n,d}$ for which the rank inequality is in fact a facet-inducing inequality for STAB(G). Given $G \in \mathcal{K}_{n,d}$, $i \in D(G)$, and $\ell \in \{1, 2\}$, define $\tilde{\Gamma}_G(i_\ell)$ to be the set of indices $j \in [n] \setminus \{i\}$ where there is an edge between i_ℓ and a vertex associated with j. We define $\tilde{\mathcal{K}}_{n,d} \subseteq \mathcal{K}_{n,d}$ to be the set of graphs G where

$$\tilde{\Gamma}_G(i_\ell) \subset [n] \setminus \{i\}$$

for every $i \in D(G)$ and for every $\ell \in \{1, 2\}$.

Example 5. Let us revisit the graphs from Figure 2. First, $G_1 \in \tilde{\mathcal{K}}_{6,0}$ as $D(G_1) = \emptyset$. More generally, we have $\tilde{\mathcal{K}}_{n,0} = \mathcal{K}_{n,0}$ for every $n \geq 3$.

For G_2, we have $D(G_2) = \{5\}$, and

$$\tilde{\Gamma}_{G_2}(5_1) = \{3, 4, 6\}, \quad \tilde{\Gamma}_{G_2}(5_2) = \{1, 2, 6\},$$

both of which are proper subsets of $[6] \setminus \{5\}$. Thus, $G_2 \in \tilde{\mathcal{K}}_{6,1}$. On the other hand (and as mentioned in Example 3), notice that for G_3 we have $\tilde{\Gamma}_{G_3}(6_2) = \{1, 2, 3, 4, 5\} = [6] \setminus \{6\}$. Thus, $G_3 \in \mathcal{K}_{6,2} \setminus \tilde{\mathcal{K}}_{6,2}$.

Notice that to obtain a graph $G \in \tilde{\mathcal{K}}_{n,d}$ from K_n, every vertex-stretching operation must be proper. Thus, [AT24a, Proposition 24] applies to graphs in $\tilde{\mathcal{K}}_{n,d}$.

Lemma 6. *For every graph $G \in \tilde{\mathcal{K}}_{n,d}$, $\bar{e}^\top x \leq d+1$ is a facet-inducing inequality for* $\text{STAB}(G)$.

Next, we take a closer look at the graphs in $\mathcal{K}_{n,d} \setminus \tilde{\mathcal{K}}_{n,d}$. The next result helps show that the rank inequality is indeed not a facet inducing inequality for the stable set polytope for these graphs.

Lemma 7. *Let $G \in \mathcal{K}_{n,d}$ where $n \geq 3$ and $d \geq 1$. If $G \not\in \tilde{\mathcal{K}}_{n,d}$, then there exists an edge $\{u,v\} \in E(G)$ consisting of a hub vertex and a wing vertex where $G - \{u,v\} \in \mathcal{K}_{n,d-1}$.*

Proof. Given $G \not\in \tilde{\mathcal{K}}_{n,d}$, there exists $i \in D(G)$ and $\ell \in \{1,2\}$ where the wing vertex i_ℓ satisfies $\tilde{\Gamma}_G(i_\ell) = [n] \setminus \{i\}$. Then notice that $G' \coloneqq G - \{i_0, i_{3-\ell}\} \in \mathcal{K}_{n,d-1}$, with the vertex $i_\ell \in V(G')$ now taking on the role of an unstretched vertex. □

Example 8. Recall that, as shown in Example 5, G_3 from Figure 2 is in $\mathcal{K}_{6,2} \setminus \tilde{\mathcal{K}}_{6,2}$. Since $\tilde{\Gamma}_G(6_2) = [6] \setminus \{6\}$, we can remove the vertices $6_0, 6_1$ to obtain an induced subgraph in $\mathcal{K}_{6,1}$. In fact, observe that $G_3 - \{6_0, 6_1\} \in \tilde{\mathcal{K}}_{6,1}$.

Thus, applying Lemma 7 iteratively, we see that for every graph $G \in \mathcal{K}_{n,d}$, there exists a partition of $V(G)$ $\{C_0, C_1, \ldots, C_k\}$ where

- C_0 induces a graph in $\tilde{\mathcal{K}}_{n,d-k}$;
- $C_1, \ldots, C_k \in E(G)$, with each C_j consisting of a hub vertex and a wing vertex.

We call such a partition $\{C_0, \ldots, C_k\}$ a *stretched-clique decomposition* of G, and the subgraph of G induced by C_0 a *core stretched clique* of G. Notice that the rank inequality of G is exactly the sum of the rank inequality of the core stretched clique induced by C_0, as well as the k edge inequalities corresponding to $C_1, \ldots, C_k$. Moreover, since every hub vertex in G has degree 2 (and thus does not belong to any clique of size at least 3), each of the edge inequalities for the edges $C_1, \ldots, C_k$ is facet inducing for $\text{STAB}(G)$. We next show that the rank inequality corresponding to C_0 is also a facet-inducing inequality for $\text{STAB}(G)$.

Lemma 9. *Let $G \in \mathcal{K}_{n,d}$, and let $\{C_0, \ldots, C_k\}$ be a stretched-clique decomposition of G. Then*

$$\sum_{v \in C_0} x_v \leq d - k + 1 \tag{2}$$

is a facet-inducing inequality for $\text{STAB}(G)$.

Proof. Let G' be the subgraph of G induced by C_0. Since G' is an induced subgraph of G, (2) is valid for $\text{STAB}(G)$. Next, we show that there are indeed $|V(G)|$ affinely independent incidence vectors of stable sets in G that satisfy (2) with equality. First, if $k = 0$, then $G' = G$ and the claim follows from Lemma 6. Next, suppose $k \geq 1$, and so G' is a proper subgraph of G. Then there is a collection of stable sets $\mathcal{S}$ of G' where $|\mathcal{S}| = |V(G')|$ and the vectors $\{\chi_S : S \in \mathcal{S}\}$ are affinely independent and all satisfy (2) with equality. Moreover, since the facet-inducing inequality (2) has full support (in the perspective of G'), we know that for every $v \in V(G')$, $v \in S$ for at least one $S \in \mathcal{S}$, and $v \not\in S$ for at least one $S \in \mathcal{S}$.

Now consider a vertex $v \in V(G) \setminus V(G')$. If $v = i_0$ for some $i \in D(G)$ (i.e., v is a hub vertex), then by the construction in Lemma 7 we know that exactly one of i_1, i_2 is in $V(G')$. Suppose without loss of generality that it is i_2 (and so $i_1 \in V(G) \setminus V(G')$). Choose $S \in \mathcal{S}$ where $i_2 \not\in S$, and define $T_v \coloneqq S \cup \{v\}$. Then T_v is a stable set in G and χ_{T_v} satisfies (2) with equality.

Next, suppose $v \in V(G) \setminus V(G')$ is a wing vertex, and so without loss of generality let $v = i_1$ for some $i \in D(G)$. By the construction in Lemma 7 again, we know that $i_2 \in V(G')$, and $\tilde{\Gamma}_G(i_2) = [n] \setminus \{i\}$. Then choose $S \in \mathcal{S}$ such that $i_2 \in S$.

Next, notice that for every $j \in D(G) \setminus \{i\}$, S cannot contain both wing vertices j_1 and j_2 since $i_2 \in S$ and i_2 is adjacent to at least one of them (due to $\tilde{\Gamma}_G(i_2) = [n] \setminus \{i\}$). Now if $j_1 \in S$, then $\{i_2, j_1\}$ is not an edge, and so $i \notin \tilde{\Gamma}_{G'}(j_1)$, and so $j_2 \in V(G')$. Now both $j_1, j_2 \in V(G')$ implies that $j_0 \in V(G')$. Thus, if we define S' to be the set obtained from S by replacing every wing vertex $j_\ell \in S$ that is not adjacent to i_2 by the hub vertex j_0, then S' must be a stable set in $V(G')$. Also, $|S'| = |S|$, and so $\chi_{S'}$ must also satisfy (2) with equality. Finally, by the construction of S', it must not contain any vertex that is adjacent to $v = i_1$. Hence, if we define $T_v := S' \cup \{v\}$ in this case, then we obtain a stable set in G whose incidence vector satisfies (2) with equality.

Applying this process for all $v \in V(G) \setminus V(G')$, we see that

$$\mathcal{S} \cup \left\{ T_v : v \in V(G) \setminus V(G') \right\} \tag{3}$$

gives a set of $|V(G)|$ stable sets whose incidence vectors satisfy (2) with equality. Also, observe that each $v \in V(G) \setminus V(G')$ appears in T_v but not any other stable set (3), and thus we see that their incidence vectors must also be affinely independent. This finishes the proof. □

Next, given $G \in \mathcal{K}_{n,d}$, we define the *deficiency* of G to be the minimum k for which there exists a stretched-clique decomposition of G with k edges. For example, our discussion in Example 8 shows that G_3 from Figure 2 has deficiency 1. In general, graphs in $\tilde{\mathcal{K}}_{n,d}$ have deficiency 0, and Lemma 7 shows that every graph in $\mathcal{K}_{n,d}$ has deficiency at most d.

Given a graph G, let $\omega(G)$ denote the size of the largest clique in G (the *clique number* of G). The next result relates the deficiency and clique number of stretched cliques.

Lemma 10. *Let $G \in \mathcal{K}_{n,d}$. Then the deficiency of G is at most* $\max\{0, \omega(G) - n + d\}$.

Proof. First, suppose $\omega(G) \leq n - d$. We show that $G \in \tilde{\mathcal{K}}_{n,d}$ (and thus has deficiency 0) in this case. Consider any wing vertex $i_\ell \in V(G)$ where $i \in D(G)$ and $\ell \in \{1, 2\}$. If i_ℓ were adjacent to every unstretched vertex in $[n] \setminus D(G)$, then $\{i_\ell\} \cup ([n] \setminus D(G))$ would induce a clique of size $n - d + 1$ in G, a contradiction. Thus, there exists $j \in [n] \setminus D(G)$ where $\{i, j\}$ is not an edge, and thus, $j \notin \tilde{\Gamma}_G(i_\ell)$. This shows that $\tilde{\Gamma}_G(i_\ell) \subset [n] \setminus \{i\}$ for every wing vertex, and thus $G \in \tilde{\mathcal{K}}_{n,d}$.

Next, suppose $\omega(G) \geq n - d$. We prove our claim by induction on $\omega(G) - n + d$. The case $\omega(G) - n + d = 0$ has already been verified above. Next, given $G \in \mathcal{K}_{n,d}$, either $G \in \tilde{\mathcal{K}}_{n,d}$ (in which case G has deficiency 0 and the claim follows), or by Lemma 7 there exists $G' \in \mathcal{K}_{n,d-1}$ where G' is an induced subgraph of G and that the deficiency of G' is that of G minus one. By the inductive hypothesis we know that G' has deficiency at most $\omega(G') - n + d - 1$. Since $\omega(G') \leq \omega(G)$, it follows that G has deficiency at most $\omega(G) - n + d$. □

3. Structural results for LS_+-relaxations

In this section we establish several results for analyzing LS_+-relaxations that we will use for establishing the existence of ℓ-minimal graphs. We will begin with results that apply to analyzing general LS_+-relaxation, and then zero in on observations more specifically for the LS_+-relaxations of stretched cliques towards the end of this section.

First, we prove a general convexity result which will allow us more freedom in finding points in the relaxations $\text{LS}_+^k(P)$.

Lemma 11. *Let $P \subseteq [0,1]^n$ be a closed convex set, and let k be a non-negative integer. Suppose that P_I is full-dimensional and let $F := \left\{ x \in P_I : c^\top x = c_0 \right\}$ be a facet of P_I where the facet-defining inequality $c^\top x \leq c_0$ is not valid for $\text{LS}_+^k(P)$. Then, the relative interior of F is a strict subset of the interior of $\text{LS}_+^k(P)$.*

Proof. Suppose the assumptions of the lemma hold. Let $\bar{x}$ be an optimal solution of

$$\max\left\{c^\top x \,:\, x \in \mathrm{LS}_+^k(P)\right\}.$$

This maximum is attained as by assumptions and the properties of LS_+, $\mathrm{LS}_+^k(P)$ is a non-empty compact set and the objective function is continuous. Since $c^\top x \leq c_0$ is not a valid inequality for $\mathrm{LS}_+^k(P)$, and it is a facet-inducing inequality for P_I, we have $\bar{x} \notin P_I$. Let $Q := \mathrm{conv}\left(P_I \cup \{\bar{x}\}\right)$. Note that since P_I is full-dimensional, $\mathrm{relint}(F) \subset \mathrm{int}(Q) \subseteq \mathrm{int}(\mathrm{LS}_+^k(P))$, where the last inclusion follows from the facts that $\bar{x} \in \mathrm{LS}_+^k(P)$, $P_I \subset \mathrm{LS}_+^k(P)$, and $\mathrm{LS}_+^k(P)$ is a convex set. □

Next, given a set $P \subseteq [0,1]^n$ and a linear equation $c^\top x = c_0$, we define the vector

$$u_{c^\top x = c_0}(P) := \sum_{\substack{z \in \{0,1\}^n \cap P, \\ c^\top z = c_0}} \begin{bmatrix} 1 \\ z \end{bmatrix},$$

when the intersection is not empty. Observe that, for every choice of equation $c^\top x = c_0$, $u_{c^\top x = c_0}(P) \in \mathrm{cone}(P_I)$. We also extend the notation to allow multiple equalities — e.g., $u_{c^\top x = c_0, c'^\top x = c'_0}(P) = \sum_{\substack{z \in \{0,1\}^n \cap P, \\ c^\top z = c_0, c'^\top z = c'_0}} \begin{bmatrix} 1 \\ z \end{bmatrix}$ (again, when the intersection is non-empty). Also, given a set $P \subseteq \mathbb{R}^n$, we say that P is *lower-comprehensive* if for every $x \in P$ and for every y where $x \geq y \geq 0$, it must be the case that $y \in P$ as well. Observe that $\mathrm{FRAC}(G)$ is lower-comprehensive for every graph G. Since LS_+ preserves lower-comprehensiveness, it follows that $\mathrm{LS}_+^k(G)$ is lower-comprehensive for every non-negative integer k (see, for instance, [GT01]).

The following is a key lemma to our main result, as it will be used in the inductive step of our argument that the vertex-stretching operation does increase the LS_+-rank of a graph under some circumstances.

Lemma 12. *Let $P \subseteq [0,1]^n$ be a lower-comprehensive closed convex set and let k be a non-negative integer. Suppose P_I is full-dimensional and let $c^\top x \leq c_0$ be a facet-inducing inequality for P_I. If there exists a set of indices $D \subseteq [n]$ where*

- *$c_0 > c^\top \chi_D > 0$;*
- *for every $i \in D$, there exists $\epsilon > 0$ where $u_{x_i = 1, c^\top x = c_0}(P) - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix} \in \mathrm{cone}(\mathrm{LS}_+^k(P))$;*
- *for every $i \in [n] \setminus D$, there exists $\epsilon > 0$ where $u_{x_i = 0, c^\top x = c_0}(P) - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix} \in \mathrm{cone}(\mathrm{LS}_+^k(P))$.*

Then, $u_{c^\top x = c_0}(P) - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix} \in \mathrm{cone}(\mathrm{LS}_+^{k+1}(P))$ for some $\epsilon > 0$, and the LS_+-rank of $c^\top x \leq c_0$ is at least $k+2$.

Proof. Define $\mathcal{T} := \left\{x \in \{0,1\}^n \cap P : c^\top x = c_0\right\}$ (i.e., $\mathcal{T}$ consists of the integral points in P which lie on the facet of P_I defined by the inequality $c^\top x \leq c_0$), and let

$$Y_0 := \sum_{T \in \mathcal{T}} \begin{bmatrix} 1 \\ \chi_T \end{bmatrix} \begin{bmatrix} 1 \\ \chi_T \end{bmatrix}^\top.$$

Then observe that, for every $i \in [n]$, $Y_0 e_i = u_{x_i = 1, c^\top x = c_0}(P)$ and $Y_0(e_0 - e_i) = u_{x_i = 0, c^\top x = c_0}(P)$, and thus $Y_0 e_i, Y_0(e_0 - e_i) \in \mathrm{cone}(P_I)$ for every $i \in [n]$. Also, since Y_0 is defined to be a sum of symmetric positive semidefinite matrices, we have $Y_0 \succeq 0$.

Next, we claim that the null space of Y_0 has dimension 1 and is spanned by the vector $c' := \begin{bmatrix} -c_0 \\ c \end{bmatrix}$. Notice that if vector x satisfies $Y_0 x = 0$, then $x^\top Y_0 x = 0$, and so $x^\top \begin{bmatrix} 1 \\ \chi_T \end{bmatrix} = 0$ for every $T \in \mathcal{T}$. Since $c^\top x \leq c_0$ is a facet-inducing inequality for P_I, this implies that x must be a multiple of c'.

Next, we define the matrix $Y_1 \in \mathbb{R}^{(n+1)\times(n+1)}$ where

$$Y_1 e_i := \begin{cases} \begin{bmatrix} \frac{-c^\top \chi_D}{c_0}\epsilon(1-\epsilon) \\ -\epsilon\chi_D \end{bmatrix} & \text{if } i = 0; \\ \begin{bmatrix} -\epsilon \\ -\epsilon\chi_D \end{bmatrix} & \text{if } i \in D; \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $Y_1 = Y_1^\top$, and $\operatorname{diag}(Y_1) = Y_1 e_0$.

We show that there exists $\epsilon > 0$ where $Y := Y_0 + Y_1 \in \widehat{\mathrm{LS}}_+^{k+1}(P)$. First, it is apparent that Y is symmetric and satisfies $Ye_0 = \operatorname{diag}(Y)$ (as both Y_0 and Y_1 satisfy these properties). Now observe that

$$\begin{aligned} c'^\top Y c' = c'^\top Y_1 c' &= \begin{bmatrix} -c_0 & c^\top \end{bmatrix} \begin{bmatrix} \frac{-c^\top \chi_D}{c_0}\epsilon(1-\epsilon) & -\epsilon\chi_D^\top \\ -\epsilon\chi_D & -\epsilon\chi_D\chi_D^\top \end{bmatrix} \begin{bmatrix} -c_0 \\ c \end{bmatrix} \\ &= \epsilon\left((1+\epsilon)c^\top\chi_D c_0 - \left(c^\top\chi_D\right)^2\right) \\ &= \epsilon c^\top \chi_D \left((1+\epsilon)c_0 - c^\top\chi_D\right), \end{aligned}$$

which is positive for all $\epsilon > 0$ (due to the assumption that $c_0 > c^\top\chi_D > 0$). Thus, $Y \succeq 0$ for all sufficiently small $\epsilon > 0$.

Next, we show that $Ye_i \in \operatorname{cone}(\mathrm{LS}_+^k(P))$ for every $i \in [n]$. If $i \in [n] \setminus D$, then $Y_1 e_i = 0$, and so

$$Ye_i = Y_0 e_i = u_{x_i=1, c^\top x = c_0}(P) \in \operatorname{cone}(P_I) \subseteq \operatorname{cone}(\mathrm{LS}_+^k(P)).$$

For $i \in D$, observe that $Y_1 e_i = -\epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix}$, and so $Ye_i \in \operatorname{cone}(\mathrm{LS}_+^k(P))$ follows from the second assumption.

Next, we show that $Y(e_0 - e_i) \in \operatorname{cone}(\mathrm{LS}_+^k(P))$ for every $i \in [n]$. Observe that

$$Y(e_0 - e_i) = \begin{cases} u_{x_i=0, c^\top x = c_0}(P) + \epsilon \begin{bmatrix} 1 - (1-\epsilon)\frac{c^\top\chi_D}{c_0} \\ 0 \end{bmatrix} & \text{if } i \in D; \\ u_{x_i=0, c^\top x = c_0}(P) - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix} + \epsilon \begin{bmatrix} 1 - (1-\epsilon)\frac{c^\top\chi_D}{c_0} \\ 0 \end{bmatrix} & \text{if } i \in [n] \setminus D. \end{cases}$$

Notice that $u_{x_i=0, c^\top x = c_0}(P) \in \operatorname{cone}(P_I) \subseteq \operatorname{cone}(\mathrm{LS}_+^k(P))$ for every $i \in D$, and $u_{x_i=0, c^\top x = c_0}(P) - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix} \in \operatorname{cone}(\mathrm{LS}_+^k(P))$ for every $i \in [n] \setminus D$ by the third assumption. Also, we have $0 \leq (1-\epsilon)\frac{c^\top\chi_D}{c_0} < 1$ for all small $\epsilon > 0$. Since P is assumed to be lower-comprehensive, $0 \in P_I$, and so $\begin{bmatrix} 1 - (1-\epsilon)\frac{c^\top\chi_D}{c_0} \\ 0 \end{bmatrix} \in \operatorname{cone}(P_I) \subseteq \operatorname{cone}(\mathrm{LS}_+^k(P))$. Since $\operatorname{cone}(\mathrm{LS}_+^k(P))$ is closed under vector addition, we have $Y(e_0 - e_i) \in \operatorname{cone}(\mathrm{LS}_+^k(P))$ for every $i \in [n]$.

Thus, it follows that $Ye_0 \in \mathrm{cone}(\mathrm{LS}_+^{k+1}(P))$ for some $\epsilon > 0$. Now

$$c'^{\top}(Ye_0) = c'^{\top}(Y_1 e_0) = \begin{bmatrix} -c_0 & c^{\top} \end{bmatrix} \begin{bmatrix} \frac{-c^{\top}\chi_D}{c_0}\epsilon(1-\epsilon) \\ -\epsilon\chi_D \end{bmatrix} = -\epsilon^2 c^{\top}\chi_D < 0$$

for every $\epsilon > 0$. Thus, $Ye_0 \not\in \mathrm{cone}(P_I)$, which implies that the facet-inducing inequality $c^{\top}x \leq c_0$ is not valid for $\mathrm{LS}_+^{k+1}(P)$.

Finally, given $\epsilon \geq 0$, define $\bar{x} \in \mathbb{R}^n$ where $\bar{x}_i := \frac{Y_0[i,0]}{Y_0[0,0]}$ for every $i \in [n]$. (Note that it is necessary that $Y_0[0,0] > 0$, or otherwise no integral point in P satisfies $c^{\top}x = c_0$. Then the second and/or third assumption would imply that there exist a point in $\mathrm{cone}(P)$ with negative entries, contradicting $P \subseteq [0,1]^n$.) By the construction of Y_0 and that $c^{\top}x \leq c_0$ is a facet-inducing inequality for P_I, $\bar{x}$ is in the relative interior of $\left\{x \in P_I : c^{\top}x = c_0\right\}$. Thus, Lemma 11 implies that $\bar{x}$ is in the interior of $\mathrm{LS}_+^{k+1}(P)$, which in turn implies that Y_0e_0 is in the interior of $\mathrm{cone}(\mathrm{LS}_+^{k+1}(P))$. Thus, it follows that $Y_0e_0 - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix} \in \mathrm{cone}(\mathrm{LS}_+^{k+1}(P))$ for some $\epsilon > 0$, and our claim follows. □

Before we apply Lemma 12 to prove LS_+-rank bounds for stretched cliques, we need more notation. Given $G \in \mathcal{K}_{n,d}$ and $\epsilon > 0$, let $D_0 := \{i_0 : i \in D(G)\}$ (i.e., D_0 is the set of hub vertices in G), and define the vector

$$v(G,\epsilon) := u_{\bar{e}^{\top}x = d+1}(\mathrm{FRAC}(G)) - \epsilon \begin{bmatrix} 1 \\ \chi_{D_0} \end{bmatrix}.$$

Notice that $|D_0| = d$, and thus $\begin{bmatrix} -(d+1) \\ \bar{e} \end{bmatrix}^{\top} v(G,\epsilon) = \epsilon$. Therefore, if we manage to show that $v(G,\epsilon) \in \mathrm{cone}(\mathrm{LS}_+^k(G))$ for some $\epsilon > 0$ and a given non-negative integer k, then it would follow that $\bar{e}^{\top}x \leq d+1$ is not valid for $\mathrm{LS}_+^k(G)$, which would imply that G has LS_+-rank at least $k+1$.

Lemma 13. *Let $G \in \tilde{\mathcal{K}}_{n,d}$ where $n \geq 3$ and $d \geq 1$, and let $k \geq 0$ be an integer. Suppose that*

- *for every $i \in D(G)$, $v(G \ominus i_0, \epsilon) \in \mathrm{cone}(\mathrm{LS}_+^k(G \ominus i_0))$ for some $\epsilon > 0$;*
- *for every $i \in [n] \setminus D(G)$, $v(G - i, \epsilon) \in \mathrm{cone}(\mathrm{LS}_+^k(G - i))$ for some $\epsilon > 0$.*

Then $v(G,\epsilon) \in \mathrm{cone}(\mathrm{LS}_+^{k+1}(G))$ for some $\epsilon > 0$.

Proof. We prove the result using Lemma 12 with $P := \mathrm{FRAC}(G)$, which is indeed lower-comprehensive and convex; moreover, $P_I = \mathrm{STAB}(G)$ which is full-dimensional. First, given $G \in \tilde{\mathcal{K}}_{n,d}$, it follows from Lemma 6 that $\bar{e}^{\top}x \leq d+1$ is a facet-inducing inequality for $\mathrm{STAB}(G)$. Now let D be the set of hub vertices in G. Then $|D| = d \geq 1$, and so $0 < \bar{e}^{\top}\chi_D < d+1$.

Next, let $i \in D(G)$. Notice that given $S \subseteq V(G)$ where $i_0 \in S$, S is a stable set of G of size $d+1$ if and only if $S \setminus \{i_0\}$ is a stable set of $G \ominus i_0$ of size d. Therefore,

$$\begin{aligned} & u_{x_{i_0}=1, \bar{e}^{\top}x = d+1}(\mathrm{FRAC}(G)) - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix} \in \mathrm{cone}(\mathrm{LS}_+^k(G)) \\ \iff\ & u_{\bar{e}^{\top}x = d}(\mathrm{FRAC}(G \ominus i_0)) - \epsilon \begin{bmatrix} 1 \\ \chi_{D \setminus \{i_0\}} \end{bmatrix} \in \mathrm{cone}(\mathrm{LS}_+^k(G \ominus i_0)) \\ \iff\ & v(G \ominus i_0, \epsilon) \in \mathrm{cone}(\mathrm{LS}_+^k(G \ominus i_0)). \end{aligned}$$

Thus, the first assumption here exactly fulfills the second condition in Lemma 12. Similarly, we see that

$$u_{x_i=0,\bar{e}^\top x=d+1}(\text{FRAC}(G)) - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix} \in \text{cone}(\text{LS}_+^k(G)) \iff v(G-i,\epsilon) \in \text{cone}(\text{LS}_+^k(G-i))$$

for every vertex $i \notin D$. Thus, the second assumption here fulfills the third assumption in Lemma 12 for the cases when i is an unstretched vertex, and it only remains to establish that $v(G-i,\epsilon) \in \text{cone}(\text{LS}_+^k(G-i))$ when i is a wing vertex.

Without loss of generality, suppose $i = j_1$ where $j \in D(G)$. Let $S \subseteq V(G)$ be a stable set in G where $|S| = d+1$ and $j_1 \notin S$. Then S either contains j_0 or it does not. Thus, we have

$$\begin{aligned} & u_{x_{j_1}=0,\bar{e}^\top x=d+1}(\text{FRAC}(G)) - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix} \\ = \; & \underbrace{u_{x_{j_1}=0,x_{j_0}=1,\bar{e}^\top x=d+1}(\text{FRAC}(G)) - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix}}_{v^{(1)}} + \underbrace{u_{x_{j_1}=0,x_{j_0}=0,\bar{e}^\top x=d+1}(\text{FRAC}(G))}_{v^{(2)}}. \end{aligned}$$

Notice that since $\{j_0, j_1\} \in E(G)$, if x is an incidence vector of a stable set in G with $x_{j_0} = 1$, then it must follow that $x_{j_1} = 0$. Thus, $v^{(1)} = u_{x_{j_0}=1,\bar{e}^\top x=d+1}(\text{FRAC}(G)) - \epsilon \begin{bmatrix} 1 \\ \chi_D \end{bmatrix}$, which we already showed above is in $\text{cone}(\text{LS}_+^k(G))$ due to our first assumption for hub vertices in G. Also, $v^{(2)} \in \text{cone}(\text{STAB}(G)) \subseteq \text{cone}(\text{LS}_+^k(G))$. Thus, it follows that $v^{(1)} + v^{(2)} \in \text{cone}(\text{LS}_+^k(G))$ (which is closed under vector addition). This in turn implies that $v(G-j_1,\epsilon) \in \text{cone}(\text{LS}_+^k(G-j_1))$. Thus, Lemma 12 applies, and we conclude that $v(G,\epsilon) \in \text{cone}(\text{LS}_+^{k+1}(G))$ in this case. ☐

Lemma 13 provides a framework for us to prove LS_+-rank lower bounds for stretched cliques in $\tilde{\mathcal{K}}_{n,d}$. However, given $G \in \tilde{\mathcal{K}}_{n,d}$ and a hub vertex $i_0 \in V(G)$, while it follows from Lemma 4 that $G \ominus i_0 \in \mathcal{K}_{n-1,d-1}$, it is possible that $G \ominus i_0 \notin \tilde{\mathcal{K}}_{n-1,d-1}$. Likewise, given an unstretched vertex j, $G - j$ does not necessarily belong to $\tilde{\mathcal{K}}_{n-1,d}$. Thus, we cannot simply apply induction on n banking on the subgraphs $G \ominus i_0$ and $G - j$ satisfying the inductive hypothesis. The next few lemmas provide the additional intermediate results we need to bridge this gap.

Lemma 14. *Let $G \in \mathcal{K}_{n,d}$ where $n \geq 3$ and d is non-negative. Let $\epsilon > 0$ be a real number, and let k be a non-negative integer. Also let $G' \in \mathcal{K}_{n,d+1}$ be a graph obtained from G by stretching an unstretched vertex in $[n] \setminus D(G)$. If $v(G,\epsilon) \in \text{cone}(\text{LS}_+^k(G))$, then $v(G',\epsilon) \in \text{cone}(\text{LS}_+^k(G'))$.*

Proof. Let $i \in [n] \setminus D(G)$ be the vertex in G that is stretched to obtain G'. (Thus, $D(G') = D(G) \cup \{i\}$.) For convenience, we also let D_0 and D'_0 respectively denote the set of hub vertices in G and G'. Next, we define $v' \in \mathbb{R}^{\{0\} \cup V(G')}$, where

$$v'_j = \begin{cases} (v(G,\epsilon))_j & \text{if } j = 0 \text{ or } j \in V(G) \setminus \{i\}; \\ (v(G,\epsilon))_i & \text{if } j = i_1 \text{ or } j = i_2; \\ (v(G,\epsilon))_0 - (v(G,\epsilon))_i & \text{if } j = i_0. \end{cases}$$

Since the vertex-stretching operation (regardless if it is proper or not) is a star-homomorphism (as defined in [AT24a, Section 3]), it follows from [AT24a, Proposition 11] that $v(G,\epsilon) \in \text{cone}(\text{LS}_+^k(G)) \Rightarrow v' \in \text{cone}(\text{LS}_+^k(G'))$. Now recall that

$$v(G,\epsilon) = u_{\bar{e}^\top x=d+1}(\text{FRAC}(G)) - \epsilon \begin{bmatrix} 1 \\ \chi_{D_0} \end{bmatrix}.$$

Then, by the construction of v', we have

$$v' = u_{x_{i_0}=1,\bar{e}^\top x=d+2}(\text{FRAC}(G')) + u_{x_{i_1}=1,x_{i_2}=1,\bar{e}^\top x=d+2}(\text{FRAC}(G')) - \epsilon \begin{bmatrix} 1 \\ \chi_{D_0'} \end{bmatrix}.$$

Now observe that every stable set of size $d+2$ in G' must either contain none of i_1, i_2 (in which case it must contain i_0), both of i_1 and i_2, or exactly one of i_1 and i_2. Thus, we can write

$$\begin{aligned} v(G',\epsilon) &= u_{\bar{e}^\top x=d+2}(\text{FRAC}(G')) - \epsilon \begin{bmatrix} 1 \\ \chi_{D_0'} \end{bmatrix} \\ &= u_{x_{i_0}=1,\bar{e}^\top x=d+2}(\text{FRAC}(G')) + u_{x_{i_1}=1,x_{i_2}=1,\bar{e}^\top x=d+2}(\text{FRAC}(G')) + \\ &\quad u_{x_{i_1}+x_{i_2}=1,\bar{e}^\top x=d+2}(\text{FRAC}(G')) - \epsilon \begin{bmatrix} 1 \\ \chi_{D_0'} \end{bmatrix} \\ &= v' + u_{x_{i_1}+x_{i_2}=1,\bar{e}^\top x=d+2}(\text{FRAC}(G')). \end{aligned}$$

Since cone($\text{LS}_+^k(G')$ is closed under vector addition, and that

$$u_{x_{i_1}+x_{i_2}=1,\bar{e}^\top x=d+2}(\text{FRAC}(G')) \in \text{cone}(\text{STAB}(G')) \subseteq \text{cone}(\text{LS}_+^k(G')),$$

we conclude that $v(G',\epsilon) \in \text{cone}(\text{LS}_+^k(G'))$. □

Next, given integers $n \geq 3$ and $d \geq 0$, we define $\hat{\mathcal{K}}_{n,d} \subseteq \mathcal{K}_{n,d}$ to be the set of stretched cliques G where

$$|\{\{i_1,j_1\},\{i_1,j_2\},\{i_2,j_1\},\{i_2,j_2\}\} \cap E(G)| = 1, \quad \forall i,j \in D(G), i \neq j. \tag{4}$$

In other words, given $G \in \mathcal{K}_{n,d}$, if there is exactly one edge in G that joins a vertex associated with i and a vertex associated with j for every pair of distinct indices $i,j \in D(G)$, then $G \in \hat{\mathcal{K}}_{n,d}$.

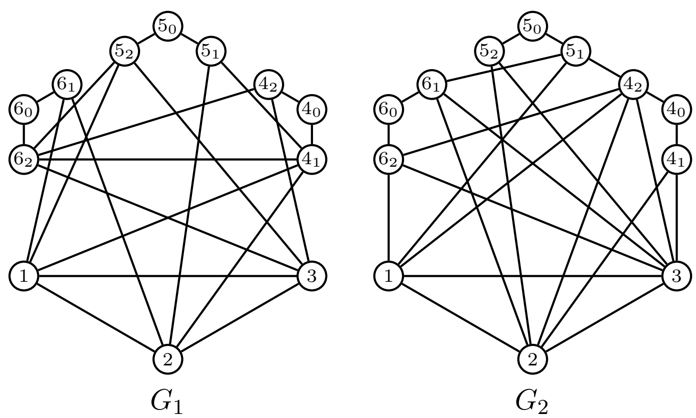


FIGURE 3. Illustrating the definition of the set of graphs $\hat{\mathcal{K}}_{n,d}$ (see Example 15)

Example 15. Consider the graphs in Figure 3. First, notice that $G_1 \in \tilde{\mathcal{K}}_{6,3}$. However, since there are two edges ($\{4_1, 6_2\}$ and $\{4_2, 6_2\}$) joining vertices associated with $i = 4$ and $j = 6$, G_1 violates (4), and thus $G_1 \notin \hat{\mathcal{K}}_{6,3}$.

On the other hand, notice that $G_2 \in \hat{\mathcal{K}}_{6,3}$ as there is exactly one edge joining vertices associated i and j for every $(i,j) \in \{(4,5),(4,6),(5,6)\}$. However, $\tilde{\Gamma}_{G_2}(4_2) = \{1,2,3,5,6\}$, and so $G_2 \notin \tilde{\mathcal{K}}_{6,3}$. Using the procedure outlined in Lemma 7, we can obtain that $G' := G_2 - \{4_0, 4_1\}$ is a core stretched clique of G_2.

Now notice that $\Gamma_{G_2}(4_1) \subseteq \Gamma_{G_2}(4_2)$. Thus, we can start with G' and apply a (improper) vertex-stretching operation to 4_2 to obtain G_2. As we show in Lemma 16, this is not a coincidence for graphs in $\hat{\mathcal{K}}_{n,d} \setminus \tilde{\mathcal{K}}_{n,d}$.

We remark that every known ℓ-minimal graph at the time of this writing (i.e., the 3-cycle, the graphs $G_{2,1}$, $G_{2,2}$, $G_{3,1}$, and $G_{4,1}$ from Figure 1, as well as all other 3-minimal and 4-minimal graphs found in [AT24a]) belongs to $\hat{\mathcal{K}}_{\ell+2,\ell-1}$. We shall show by the end of this section that many more graphs in $\hat{\mathcal{K}}_{\ell+2,\ell-1}$ are ℓ-minimal. (We will also show in the Section 4 that ℓ-minimal graphs also exist outside of $\hat{\mathcal{K}}_{\ell+2,\ell-1}$.)

First, the following lemma makes concrete a property of the stretched cliques in $\hat{\mathcal{K}}_{n,d} \setminus \tilde{\mathcal{K}}_{n,d}$ we mentioned in Example 15.

Lemma 16. *Let $G \in \hat{\mathcal{K}}_{n,d}$ where $n \geq 3$ and d is non-negative. Suppose G has deficiency $k \geq 1$. Then G contains a core stretched clique $G' \in \hat{\mathcal{K}}_{n,d-k}$. Moreover, G can be obtained from G' by 2-stretching k vertices.*

Proof. Given $G \in \hat{\mathcal{K}}_{n,d}$ with deficiency $k \geq 1$, there exists a stretched-clique decomposition $\{C_0, \ldots, C_k\}$ of G. Let G' be the core stretched clique of G induced by C_0. Since G' is an induced subgraph of G, it cannot violate (4). Hence, we have $G' \in \hat{\mathcal{K}}_{n,d-k}$.

It remains to show that G can be obtained from G' by 2-stretching k vertices in G'. Consider again the stretched-clique decomposition of G, and focus on the edge $C_1 \in E(G)$. Without loss of generality, suppose $C_1 = \{i_0, i_1\}$ for some $i \in D(G)$. Since $i_2 \in V(G')$, we know that $\tilde{\Gamma}_G(i_2) = [n] \setminus \{i\}$. Thus, for every $j \in D(G) \setminus \{i\}$, there is an edge between i_2 and one of j_1, j_2. Since $G \in \hat{\mathcal{K}}_{n,d}$ and thus satisfies (4), there is no edge between i_1 and a vertex associated with j. This implies that $\Gamma_G(i_1) \subseteq \Gamma_G(i_2)$. Thus, we can 2-stretch $i_2 \in V(G')$ to obtain the subgraph of G induced by $V(G') \cup C_1$. Applying this procedure iteratively to $C_2, \ldots, C_k$ finishes the proof. □

Lemmas 14 and 16 combine to imply the following.

Lemma 17. *Let $G \in \hat{\mathcal{K}}_{n,d}$ where $n \geq 3$ and d is non-negative. Let $\epsilon > 0$ be a real number, let k be a non-negative integer, and let G' be a core stretched clique of G. If $v(G', \epsilon) \in \operatorname{cone}(\mathrm{LS}_+^k(G'))$, then $v(G, \epsilon) \in \operatorname{cone}(\mathrm{LS}_+^k(G))$.*

We are now finally ready to use Lemma 13 to show that, for many stretched cliques, the point $v(G, \epsilon)$ (which does not belong to $\mathrm{STAB}(G)$ for all sufficiently small $\epsilon > 0$) survives many iterations of LS_+.

Proposition 18. *Let $G \in \hat{\mathcal{K}}_{n,d} \cap \tilde{\mathcal{K}}_{n,d}$, where $n \geq 3$ and d is non-negative. Also, let $k := \max\{3, \omega(G)\}$. Then $v(G, \epsilon) \in \mathrm{LS}_+^{n-k}(G)$ for some $\epsilon > 0$.*

Proof. We prove our claim by induction on n. If $k = 3$ (which implies $\omega(G) \in \{2, 3\}$), then the base case is $n = 3$. Here, we have $d \in \{0, 1, 2, 3\}$, and the only graph in $\hat{\mathcal{K}}_{3,d} \cap \tilde{\mathcal{K}}_{3,d}$ is the $(2d+1)$-cycle. One can check that $v(G, \epsilon) \in \mathrm{FRAC}(G) = \mathrm{LS}_+^0(G)$ for all $\epsilon \in [0, 1]$ in all four cases. If $k \geq 4$, then the base case is when $n = k$, which implies that $G = K_n$. In this case, we have $v(K_n, \epsilon) \in \mathrm{FRAC}(K_n)$ for every $\epsilon \in [0, n-2]$.

Next, we prove the inductive step by applying the framework outlined in Lemma 13. Let $G \in \hat{\mathcal{K}}_{n,d} \cap \tilde{\mathcal{K}}_{n,d}$ where $n \geq k+1$ (which implies $d \geq 1$). Given $i \in D(G)$, observe that $G \ominus i_0 \in \hat{\mathcal{K}}_{n-1,d}$. Then $G \ominus i_0$ contains a core stretched clique $G' \in \hat{\mathcal{K}}_{n-1,d'} \cap \tilde{\mathcal{K}}_{n-1,d'}$ for some $d' \leq d$. Also, since G' is a subgraph of G, we have $\omega(G') \leq \omega(G) \leq k$. Thus, by the inductive hypothesis, we have $v(G', \epsilon) \in \text{cone}(\text{LS}_+^{n-k-1}(G'))$ for some $\epsilon > 0$. Then Lemma 17 implies that $v(G \ominus i_0, \epsilon) \in \text{cone}(\text{LS}_+^{n-k-1}(G \ominus i_0))$ for some $\epsilon > 0$.

Next, given $i \in [n] \setminus D(G)$, we have $G - i \in \hat{\mathcal{K}}_{n-1,d-1}$. Using a similar argument as in the preceding paragraph, observe that $G - i$ contains a core stretched clique $G'' \in \hat{\mathcal{K}}_{n-1,d''} \cap \tilde{\mathcal{K}}_{n-1,d''}$ for some $d'' \leq d - 1$, with $\omega(G'') \leq \omega(G) \leq k$. Thus, the inductive hypothesis implies that $v(G'', \epsilon) \in \text{cone}(\text{LS}_+^{n-k-1}(G''))$ for some $\epsilon > 0$, which (due to Lemma 17 again) implies that $v(G - i, \epsilon) \in \text{cone}(\text{LS}_+^{n-k-1}(G - i))$ for some $\epsilon > 0$. Thus, it follows from Lemma 13 that $v(G, \epsilon) \in \text{cone}(\text{LS}_+^{n-k}(G))$ for some $\epsilon > 0$. □

Proposition 18 readily implies the following, which provides a LS_+-rank lower bound to many stretched cliques that depends on the clique number of the graph.

Theorem 19. *Let $G \in \hat{\mathcal{K}}_{n,d}$ where $n \geq 3$ and d is non-negative. Let $k := \max\{3, \omega(G)\}$. Then $v(G, \epsilon) \in \text{LS}_+^{n-k}(G)$, and $r_+(G) \geq n - k + 1$.*

Proof. Given $G \in \hat{\mathcal{K}}_{n,d}$, it must contain a core stretched clique $G' \in \hat{\mathcal{K}}_{n,d'} \cap \tilde{\mathcal{K}}_{n,d'}$ for some $d' \leq d$. Since $\omega(G') \leq \omega(G)$, Proposition 18 implies that $v(G', \epsilon) \in \text{cone}(\text{LS}_+^{n-k}(G'))$ for some $\epsilon > 0$. Then Lemma 17 implies that $v(G, \epsilon) \in \text{cone}(\text{LS}_+^{n-k}(G))$ for some $\epsilon > 0$. Since $v(G, \epsilon)$ violates $\bar{e}^\top x \leq d + 1$ (which is valid for $\text{STAB}(G)$), it follows that $r_+(G) \geq n - k + 1$. □

4. ℓ-minimal graphs and implications

Recall that a graph G is ℓ-minimal if $r_+(G) = \ell$ and $|V(G)| = 3\ell$. In this section, we describe a number of ℓ-minimal graphs, as well as other implications of the structural results we developed in Section 3.

First, notice that every graph $G \in \mathcal{K}_{\ell+2,\ell-1}$ contains exactly 3ℓ vertices. Thus, Theorem 19 implies the following.

Corollary 20. *Let ℓ be a positive integer, and let $G \in \hat{\mathcal{K}}_{\ell+2,\ell-1}$ where $\omega(G) \leq 3$. Then G is ℓ-minimal.*

We next construct a family of ℓ-minimal graphs using Corollary 20. Given an integer $k \geq 3$, define the graph $\mathcal{A}_k$ where

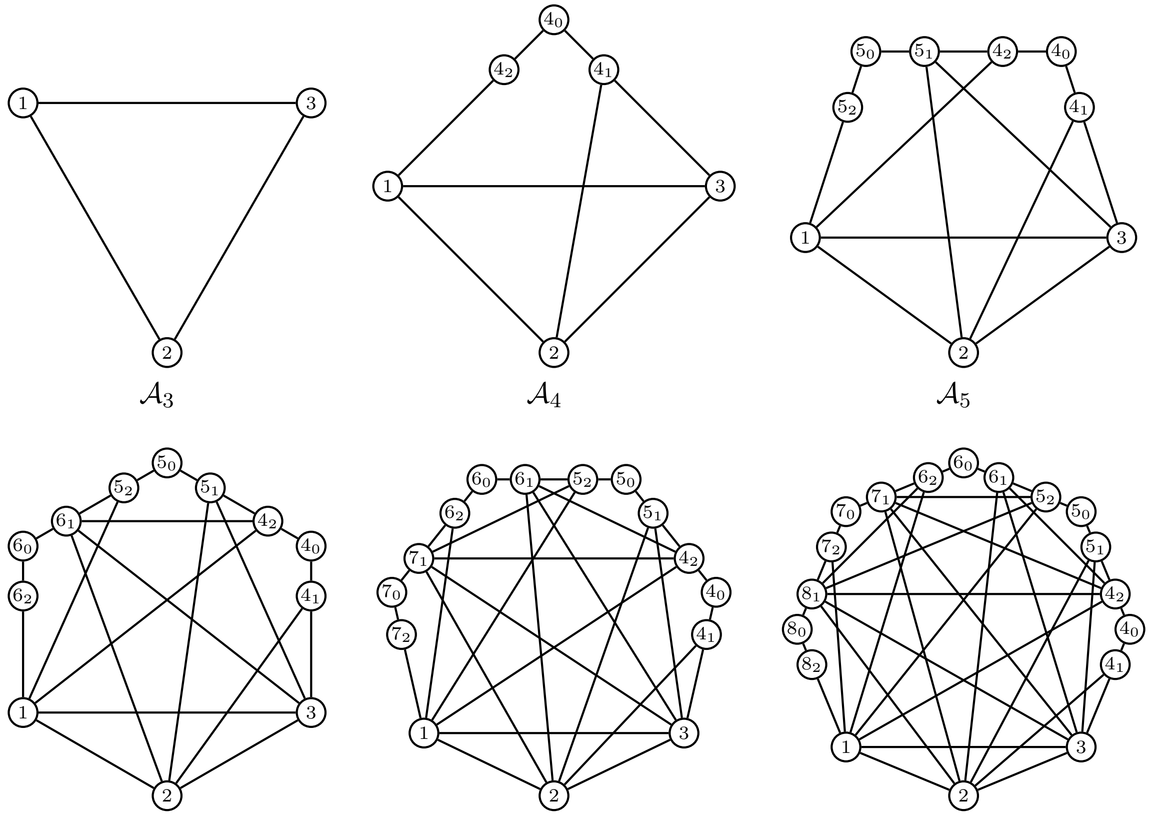
$$\begin{aligned} V(\mathcal{A}_k) &:= \{1,2,3\} \cup \{i_0, i_1, i_2 : 4 \leq i \leq k\}, \\ E(\mathcal{A}_k) &:= \{\{1,2\},\{2,3\},\{1,3\}\} \cup \\ &\quad \{\{i_0,i_1\},\{i_0,i_2\},\{i_1,2\},\{i_1,3\},\{i_2,1\} : 4 \leq i \leq k\} \cup \\ &\quad \{\{i_2,j_1\} : 4 \leq i < j \leq k\}. \end{aligned}$$

Figure 4 gives the drawings of $\mathcal{A}_k$ for $3 \leq k \leq 8$. Then we have the following:

Proposition 21. *For every $k \geq 3$, $\mathcal{A}_k$ is $(k-2)$-minimal.*

Proof. We prove the result using Corollary 20. First, as suggested by the vertex labels, we see that $\mathcal{A}_k \in \mathcal{K}_{k,k-3}$ for every $k \geq 3$, with $D(\mathcal{A}_k) = \{4, \ldots, k\}$. Now for every $i, j \in D(\mathcal{A}_k)$ where $i < j$, the only edge between vertices associated with i and j is $\{i_2, j_1\}$, and thus $\mathcal{A}_k \in \hat{\mathcal{K}}_{k,k-3}$.

Next, we show that $\mathcal{A}_k$ does not contain K_4 as an induced subgraph. Observe that each hub vertex has degree 2 and thus cannot be contained in a K_4. Also, $\Gamma_{\mathcal{A}_k}(1) = \{2, 3, 4_2, \ldots, k_2\}$. Since $\{4_2, \ldots, k_2\}$ is a stable set, 1 cannot be contained in a K_4 either. However, among the remaining vertices (the unstretched vertices 2, 3, and the wing vertices), the only 3-cycles are induced by $\{2, 3, i_1\}$ for some $i \in \{4, \ldots, k\}$. Thus, if follows from Corollary 20 that $\mathcal{A}_k$ is $(k-2)$-minimal. □



$\mathcal{A}_3$ $\mathcal{A}_4$ $\mathcal{A}_5$ $\mathcal{A}_6$ $\mathcal{A}_7$ $\mathcal{A}_8$

FIGURE 4. Several members of the family of graphs $\mathcal{A}_k$

Thus, we now know that ℓ-minimal graphs do exist for every positive integer ℓ. In fact, we show that the number of ℓ-minimal graphs grows (at least) exponentially as a function of ℓ. Given an integer $k \geq 3$ and a subset $S \subseteq \{4, \ldots, k\}$, we define the graph $\mathcal{A}_{k,S}$ where

$$V(\mathcal{A}_{k,S}) := V(\mathcal{A}_k),$$
$$E(\mathcal{A}_{k,S}) := E(\mathcal{A}_k) \cup \{\{i_2, 2\} : i \in S\}.$$

Using the same ideas from the proof of Proposition 21, one can show that $\mathcal{A}_{k,S} \in \hat{\mathcal{K}}_{k,k-3}$ and $\omega(\mathcal{A}_{k,S}) = 3$ for all possible choices of S, and so $r_+(\mathcal{A}_{k,S}) = k - 2$ for every possible choice of S. The next result shows that distinct choices of S indeed produce non-isomorphic graphs.

Lemma 22. *Let $k \geq 3$ be an integer, and let $S, S' \subseteq \{4, \ldots, k\}$. If $S \neq S'$, then $\mathcal{A}_{k,S}$ and $\mathcal{A}_{k,S'}$ are not isomorphic to each other.*

Proof. First, notice that the degrees of vertices in $\{1, 3\} \cup \{i_0, i_1 : 4 \leq i \leq k\}$ are invariant under the choice of S. For the other vertices, we have $\deg(2) = k - 1 + |S|$, and

$$\deg(i_2) = \begin{cases} k - i + 2 & \text{if } i \notin S; \\ k - i + 3 & \text{if } i \in S. \end{cases}$$

Thus, given the list of vertex degrees of $\mathcal{A}_{k,S}$, we can remove the entries that we know correspond to the vertex degrees of $\{1, 3\} \cup \{i_0, i_1 : 4 \leq i \leq k\}$, and then uniquely recover the set S from the remaining vertex degrees. Therefore, we see that $\mathcal{A}_{k,S}$ and $\mathcal{A}_{k,S'}$ have distinct lists of vertex degrees whenever $S \neq S'$, and so the two graphs cannot possibly be isomorphic to each other. □

Therefore, for every $k \geq 3$, $\{\mathcal{A}_{k,S} : S \subseteq \{4, \ldots, k\}\}$ gives a set of 2^{k-3} non-isomorphic $(k-2)$-minimal graphs, and we have the following.

Theorem 23. *There are at least* $2^{\ell-1}$ *non-isomorphic* ℓ*-minimal graphs for every positive integer* ℓ*.*

Theorem 23 is tight for $\ell = 1$ and $\ell = 2$, as all 1-minimal and 2-minimal graphs are known. For $\ell = 3$, an exhaustive computational search found 13 non-isomorphic graphs that satisfy the conditions in Corollary 20. We also computed the optimal value of $\max\left\{\bar{e}^\top x : x \in \text{LS}_+^2(G)\right\}$ for each of these graphs using CVX, a package for specifying and solving convex programs [GB14, GB08] with the SeDuMi solver [Stu99]. All 13 graphs have an optimal value of at least 3.004, which aligns with our analytical findings that they are all 3-minimal. These graphs and their corresponding optimal values are listed in Figure 5. We also performed a similar search for $\ell = 4$, and found 588 non-isomorphic graphs in $\hat{\mathcal{K}}_{6,3}$ with clique number at most 3. This suggests that the number of ℓ-minimal graphs grows rather rapidly as a function of ℓ.

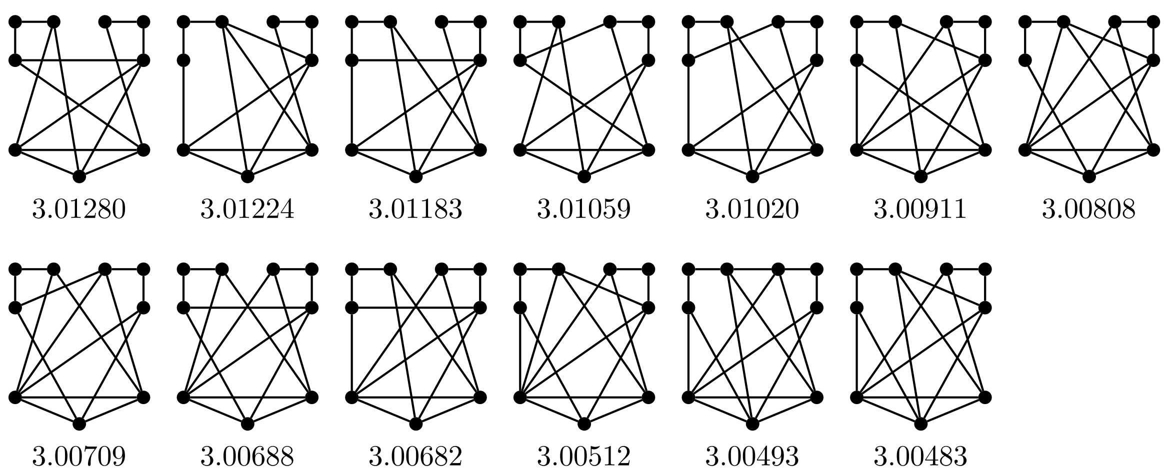


FIGURE 5. The 13 graphs $G \in \hat{\mathcal{K}}_{5,2}$ with $\omega(G) \leq 3$, and their corresponding optimal values of $\max\left\{\bar{e}^\top x : x \in \text{LS}_+^2(G)\right\}$ according to CVX

Next, given a graph G, we define the *edge density* of G to be $d(G) := \frac{|E(G)|}{\binom{|V(G)|}{2}}$. It is known that the graphs on the two extremes of the edge density spectrum (i.e., the empty graph and the complete graph) both have low LS_+-rank. This raises the natural question of finding the possible range of edge densities among ℓ-minimal graphs (see [AT24a, Problem 34]). Given an integer $\ell \geq 1$, let $d^+(\ell)$ (resp. $d^-(\ell)$) be the maximum (resp. minimum) edge density among ℓ-minimal graphs. Our analysis of the graphs $\mathcal{A}_{k,S}$ above implies the following.

Proposition 24. *For every positive integer* ℓ*,*

$$d^-(\ell) \leq \frac{\ell^2 + 7\ell - 2}{9\ell^2 - 3\ell} \quad \textit{and} \quad d^+(\ell) \geq \frac{\ell^2 + 9\ell - 4}{9\ell^2 - 3\ell}.$$

Proof. Observe that $\mathcal{A}_{k,\emptyset}$ has $\frac{1}{2}(k^2+3k-12)$ edges, and so

$$d^-(\ell) \leq \frac{\frac{1}{2}((\ell+2)^2+3(\ell+2)-12)}{\binom{3\ell}{2}} = \frac{\ell^2+7\ell-2}{9\ell^2-3\ell}.$$

Likewise, the bound for $d^+(\ell)$ follows from the fact that $\mathcal{A}_{k,\{4,\ldots,k\}}$ contains $\frac{1}{2}(k^2+5k-18)$ edges. □

The bounds of $d^-(\ell), d^+(\ell)$ from Proposition 24 are tight for $\ell=1$ and $\ell=2$ (again, due to all 1- and 2-minimal graphs being known). The bound for $d^-(3)$ is also tight [AT24a, Proposition 28]. However, the bound for $d^+(\ell)$ above (which is based an ℓ-minimal graph in $\hat{\mathcal{K}}_{\ell+2,\ell-1}$) is likely not tight in general, as we show below that there does exist ℓ-minimal graphs outside of $\hat{\mathcal{K}}_{\ell+2,\ell-1}$.

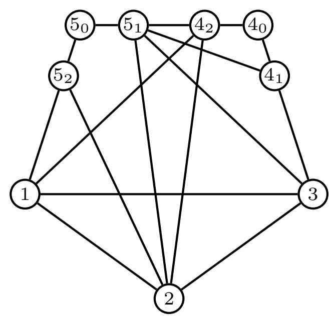


FIGURE 6. A 3-minimal graph which does not belong to $\hat{\mathcal{K}}_{5,2}$

Proposition 25. *The graph in Figure 6 is* 3*-minimal.*

Proof. Let G be the graph in Figure 6, and we prove our claim using Lemma 13. Notice that $G \in \tilde{\mathcal{K}}_{5,2}$ with $D(G) = \{4,5\}$. Next, $G \ominus 4_0$ is isomorphic to the 2-minimal graph $G_{2,2}$ from Figure 1. Since $G_{2,2} \in \hat{\mathcal{K}}_{4,1}$ and $\omega(G_{2,2}) = 3$, it follows from Proposition 19 that $v(G_{2,2},\epsilon) \in \text{cone}(\text{LS}_+(G_{2,2}))$ for some $\epsilon > 0$. Thus, we know that $v(G\ominus 4_0,\epsilon) \in \text{cone}(\text{LS}_+(G\ominus 4_0))$ for some $\epsilon > 0$. Likewise, $G \ominus 5_0$ is isomorphic to $G_{2,1}$, and the same argument shows that $v(G\ominus 5_0,\epsilon) \in \text{cone}(\text{LS}_+(G\ominus 5_0))$ for some $\epsilon > 0$.

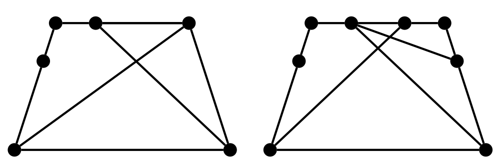


FIGURE 7. Illustrating the proof of Proposition 25

Next, observe that we can 2-stretch a vertex in $G_{2,1}$ to obtain a graph isomorphic to $G-2$ (see Figure 7). Thus, Lemma 14 implies that $v(G-2,\epsilon) \in \text{cone}(\text{LS}_+(G-2))$ for some $\epsilon > 0$. Similarly, observe that $G-1$ and $G-3$ can be obtained from stretching a vertex in $G_{2,1}$ and $G_{2,2}$, respectively. Thus, using the same rationale as above, we conclude that there exists $\epsilon > 0$ where $v(G-i,\epsilon) \in \text{cone}(\text{LS}_+(G-i))$ for every $i \in [n] \setminus D(G)$.

Hence, Lemma 13 applies, and we conclude that $v(G, \epsilon) \in \text{cone}(\text{LS}_+^2(G))$ for some $\epsilon > 0$, which implies that $r_+(G) \geq 3$. Finally, $r_+(G) \leq 3$ follows from $|V(G)| = 9$ and Theorem 1, and this finishes the proof. $\square$

The graph in Figure 6 provides what we believe is the first example of an ℓ-minimal graph that does not belong to $\hat{\mathcal{K}}_{\ell+2,\ell-1}$. Moreover, it is very likely not the only such graph. Figure 8 lists the 25 non-isomorphic graphs in $\mathcal{K}_{5,2} \setminus \hat{\mathcal{K}}_{5,2}$ with clique number at most 3, as well as their corresponding optimal value of $\max\left\{\bar{e}^\top x : x \in \text{LS}_+^2(G)\right\}$ computed in CVX. The computational results suggest that 18 of these graphs are indeed 3-minimal. On the other hand, the remaining 7 graphs have an optimal value that is very close to 3, which seems to indicate that the facet-inducing inequality $\bar{e}^\top x \leq 3$ has LS_+-rank 2 in those cases.

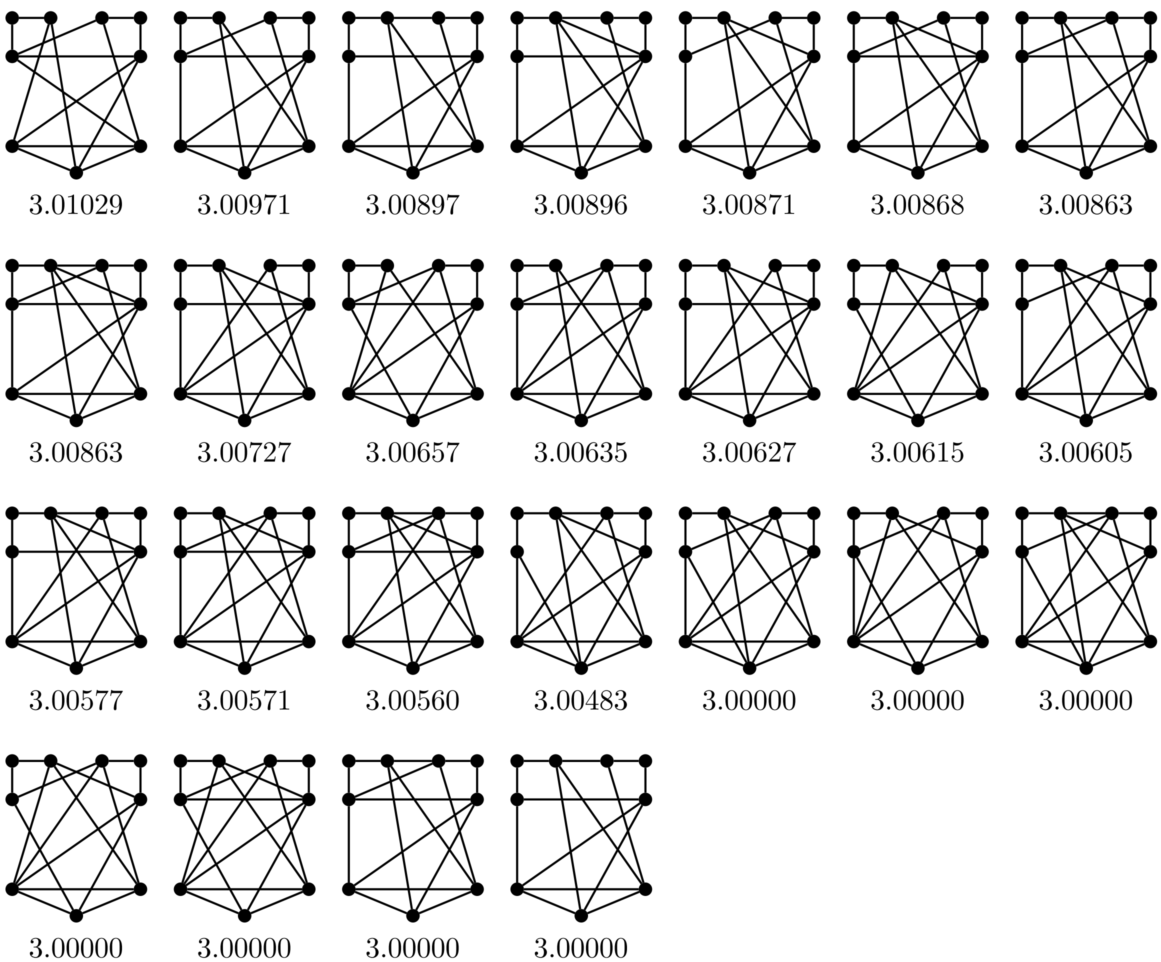


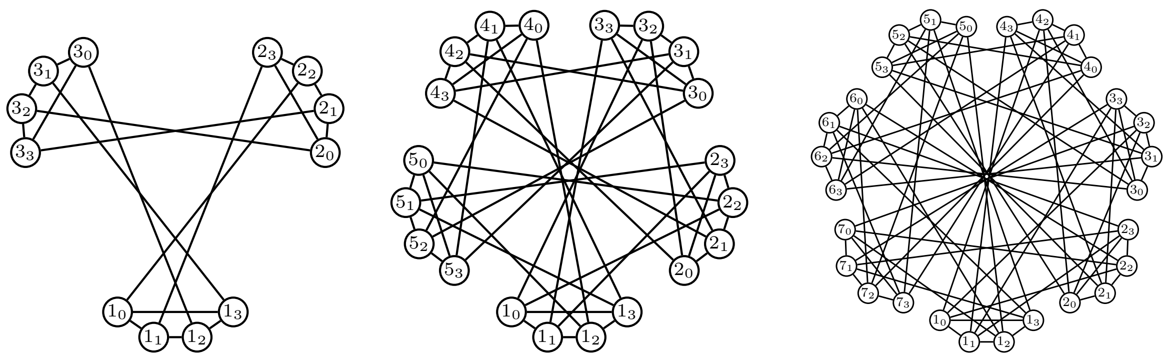
FIGURE 8. The 25 graphs $G \in \mathcal{K}_{5,2} \setminus \hat{\mathcal{K}}_{5,2}$ with $\omega(G) \leq 3$, and their corresponding optimal values of $\max\left\{\bar{e}^\top x : x \in \text{LS}_+^2(G)\right\}$ according to CVX

Finally, we finish this section by describing a family of relatively small vertex-transitive graphs with high LS_+-rank. Given integers a, b and $n \geq 1$, we define $a +_n b$ to be the unique integer $c \in [n]$ where $a + b - c$ is a multiple of n. (I.e., $a +_n b$ works similarly to addition modulo-n, except the operation outputs n instead of 0 when $a + b$ is divisible by n.) We also define $a -_n b$

analogously. Then, given an odd integer $k \geq 3$, we define the graph $\mathcal{B}_k$ where

$$\begin{aligned} V(\mathcal{B}_k) &:= \{i_0, i_1, i_2, i_3 : i \in [k]\} \\ E(\mathcal{B}_k) &:= \{\{i_0, i_1\}, \{i_1, i_2\}, \{i_2, i_3\}, \{i_3, i_0\} : i \in [k]\} \cup \\ &\left\{ \{i_0, j_2\}, \{i_1, j_3\} : (j -_k i) \in \left\{1, 2, \ldots, \frac{k-1}{2}\right\}\right\}. \end{aligned}$$

Figure 9 illustrates the first few members of the family of graphs $\mathcal{B}_k$. Then we have the following.

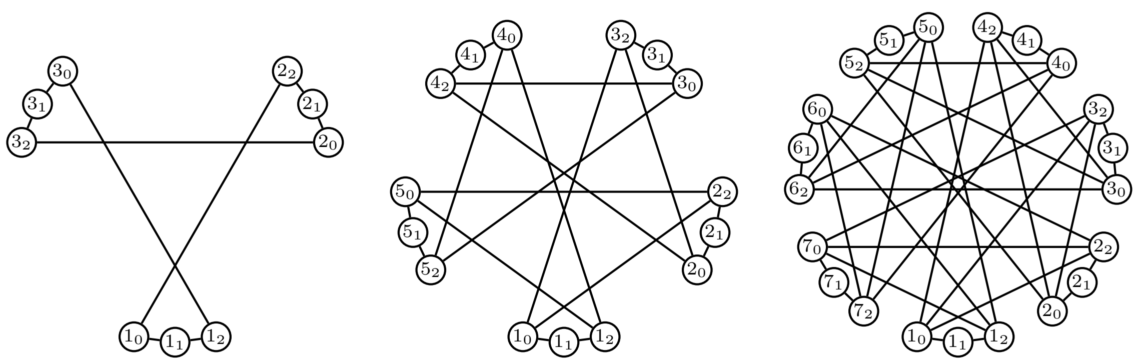


$\mathcal{B}_3$ $\mathcal{B}_5$ $\mathcal{B}_7$

FIGURE 9. Illustrating the graphs $\mathcal{B}_k$

Proposition 26. *For every odd integer $k \geq 3$, $\mathcal{B}_k$ is vertex-transitive and $r_+(\mathcal{B}_k) \geq k-2$.*

Proof. First, let $\mathcal{B}'_k$ denote the graph obtained from $\mathcal{B}_k$ by removing the vertices $\{i_3 : i \in [k]\}$. We see that $\mathcal{B}'_k \in \hat{\mathcal{K}}_{k,k}$, with $\omega(\mathcal{B}'_k) = 2$. (In fact, for every $j \in \{0, 1, 2\}$, removing all vertices in $\{i_j : i \in [k]\}$ from $\mathcal{B}_k$ also results in a graph isomorphic to $\mathcal{B}'_k$.) Hence, it follows from Theorem 19 that $r_+(\mathcal{B}'_k) \geq k-2$. Since $\mathcal{B}'_k$ is an induced subgraph of $\mathcal{B}_k$, it follows that $r_+(\mathcal{B}_k) \geq k-2$.



$\mathcal{B}'_3$ $\mathcal{B}'_5$ $\mathcal{B}'_7$

FIGURE 10. Illustrating the graphs $\mathcal{B}'_k$ for the proof of Proposition 26

It remains to show that $\mathcal{B}_k$ is vertex-transitive, and we do so via describing three automorphisms of $\mathcal{B}_k$. For every $i \in [k]$, define the functions $f_1, f_2, f_3 : V(\mathcal{B}_k) \to V(\mathcal{B}_k)$ as follows:

j	0	1	2	3
$f_1(i_j)$	i_1	i_0	i_3	i_2
$f_2(i_j)$	$(i +_k 1)_0$	$(i +_k 1)_1$	$(i +_k 1)_2$	$(i +_k 1)_3$
$f_3(i_j)$	$(2 -_k i)_3$	$(2 -_k i)_2$	$(2 -_k i)_1$	$(2 -_k i)_0$

In all cases, one can show that if $\{u, v\}$ is an edge in $\mathcal{B}_k$, then so are $\{f_1(u), f_1(v)\}$, $\{f_2(u), f_2(v)\}$, and $\{f_3(u), f_3(v)\}$. Furthermore, for every distinct $u, v \in V(\mathcal{B}_k)$, there exists a composition of f_1, f_2, and f_3 which maps u to v. Hence, we conclude that $\mathcal{B}_k$ is vertex-transitive. □

Proposition 26 readily implies the following:

Theorem 27. *For every positive integer ℓ, there exists a vertex-transitive graph G where $|V(G)| \leq 4\ell + 12$ and $r_+(G) \geq \ell$.*

Proof. If ℓ is odd, then $\mathcal{B}_{\ell+2}$ (which has $4\ell + 8$ vertices) would do; and if ℓ is even $\mathcal{B}_{\ell+3}$ (with $4\ell + 12$ vertices) would satisfy $r_+(G) \geq \ell$. □

4.1. CG**-rank of stretched cliques.** Next, we comment on the hardness of $\text{STAB}(G)$ for some stretched cliques G with respect to another well-studied cutting-plane procedure, which is due to Chvátal [Chv73] and Gomory [Gom58]. Given a set $P \subseteq [0, 1]^n$ and a valid inequality $a^\top x \leq \beta$ for P, where $a \in \mathbb{Z}^n$, we say that $a^\top x \leq \lfloor \beta \rfloor$ is a *Chvátal–Gomory cut* of P. Observe that every Chvátal–Gomory cut of P is valid for P_I. Thus, if we define $\text{CG}(P)$ to be the set of points which satisfy all Chvátal–Gomory cuts for P, then we have $P_I \subseteq \text{CG}(P) \subseteq P$. The set $\text{CG}(P)$ is known as the *Chvátal–Gomory closure* of P, and is a closed convex set for every P.

As with LS_+, we can also apply this cutting-plane procedure iteratively. Given an integer $\ell \geq 2$, we recursively define $\text{CG}^\ell(P) \coloneqq \text{CG}\left(\text{CG}^{\ell-1}(P)\right)$. We can then define the CG*-rank* of a valid inequality of P_I (relative to P) to be the smallest integer ℓ for which it is valid for $\text{CG}^\ell(P)$, and let the CG-rank of a set P to be the smallest integer ℓ where $\text{CG}^\ell(P) = P_I$. For convenience and for consistency with our discussion on LS_+, given a graph G, we will also write $\text{CG}^\ell(G)$ instead of $\text{CG}^\ell(\text{FRAC}(G))$, and refer to the CG-rank of $\text{FRAC}(G)$ simply as the CG-rank of G. A notable distinction between the procedures LS_+ and CG is that optimizing a linear function over $\text{CG}^\ell(P)$ is $\mathcal{NP}$-hard in general, even for $\ell = O(1)$.

After establishing that stretched cliques can be the worst-case instances for LS_+, we now show that there are families of stretched cliques with unbounded CG-rank. To do so, let us consider a special family of graphs. Given an integer $k \geq 3$, we define the graph H'_k where

$$\begin{aligned} V(H'_k) &\coloneqq \{1, 2\} \cup \{i_0, i_1, i_2 : i \in \{3, \ldots, k\}\} \\ E(H'_k) &\coloneqq \{\{i_0, i_1\}, \{i_0, i_2\}, \{i_1, 2\}, \{i_2, 1\} : i \in \{3, \ldots, k\}\} \cup \\ &\quad\; \{\{i_1, j_2\} : i, j \in \{3, \ldots, k\}, i \neq j\}. \end{aligned}$$

Figure 11 gives the drawings of H'_k for $k \in \{3, 4, 5\}$. The authors recently studied the LS_+-relaxations of H'_k in [AT24a] (where the graphs had slightly different vertex labels). These graphs are also closely related to the graphs H_k, which is the first known family of graphs G where $r_+(G)$ is asymptotically a linear function of $|V(G)|$ [AT24b].

Next, observe that $H'_k \in \tilde{\mathcal{K}}_{k,k-2}$ for every $k \geq 3$, and so it follows from Lemma 6 that $\bar{e}^\top x \leq k - 1$ is a facet-inducing inequality for $\text{STAB}(H'_k)$. The following result is a consequence of [AT24a, Proposition 29] and [AT24b, Theorem 29].

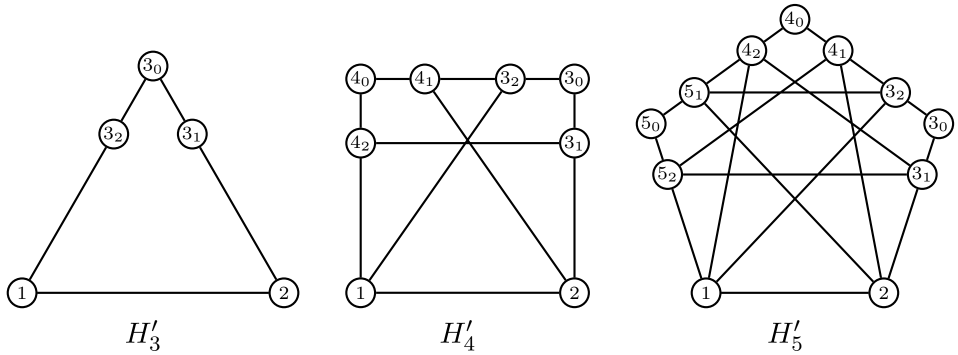


FIGURE 11. Several graphs in the family H'_k

Proposition 28. *For every $k \geq 3$, the facet-inducing inequality $\bar{e}^\top x \leq k-1$ for* STAB(H'_k) *has* LS$_+$*-rank at least $\frac{3k}{16}$, and has* CG*-rank at least* $\log_4\left(\frac{3k-7}{2}\right)$.

Next, we use the CG-rank bound on H'_k above to prove a CG-rank lower bound on some stretched cliques.

Proposition 29. *Let $G \in \mathcal{K}_{n,d}$ where $n \geq d+2$ and $d \geq 1$. Furthermore, suppose that $\{i_1, j_1\}, \{i_2, j_2\} \notin E(G)$ for all distinct $i, j \in D(G)$. Then the valid inequality $\bar{e}^\top x \leq d+1$ of* STAB(G) *has* CG*-rank at least* $\log_4\left(\frac{3d-1}{2}\right)$.

Proof. We first prove the claim for the case when $n = d+2$. Given that G does not contain the edges $\{i_1, j_1\}$ and $\{i_2, j_2\}$ for every distinct $i, j \in D(G)$, then $E(G) \subseteq E(H'_{d+2})$. Hence, FRAC$(G)$ is defined by a subset of the inequalities defining FRAC(H'_{d+2}), and as a result CG$^\ell(G)$ is defined by a subset of the inequalities defining CG$^\ell(H'_{d+2})$ for every $\ell \geq 1$. This implies that CG$^\ell(H'_{d+2}) \subseteq$ CG$^\ell(G)$ for every $\ell \in \mathbb{N}$, and thus the CG-rank bound of the inequality $\bar{e}^\top x \leq d+1$ for H'_{d+2} in Proposition 28 applies for G.

Next, suppose $n \geq d+3$. Then we can delete all but two unstretched vertices from G to obtain a subgraph $G' \in \mathcal{K}_{d+2,d}$ where $E(G') \subseteq E(H'_{d+2})$. Now the argument from the preceding paragraph applies, and we obtain that the CG-rank of the inequality $\sum_{i \in V(G'} x_i \leq d+1$ is at least that of H'_{d+2}. Furthermore, since FRAC(G') is a projection of FRAC(G), if $\sum_{i \in V(G')} x_i \leq d+1$ is not valid for CG$^\ell(G')$ for a given $\ell \in \mathbb{N}$, then $\sum_{i \in V(G)} x_i \leq d+1$ cannot be valid for CG$^\ell(G)$. Hence, the CG-rank of $\sum_{i \in V(G)} x_i \leq d+1$ for STAB(G) is at least that of $\sum_{i \in V(G')} x_i \leq d+1$ for STAB(G'). Thus, our claim follows in this case as well. □

Proposition 29 shows that one can construct families of stretched cliques with arbitrarily high CG-rank. One such family is the graphs $\mathcal{A}_{k,S}$ (for any choice of S). Therefore, we see that the stable set polytopes of these graphs are not only challenging instances for LS$_+$, but are also computationally costly for CG.

5. FUTURE RESEARCH DIRECTIONS

We conclude the manuscript by mentioning a few natural questions raised by our work herein.

Problem 30. Obtain a combinatorial characterization of all ℓ-minimal graphs.

This problem is related to Conjecture 40 of [LT03]. The conjecture has two parts. The first part is the existence of ℓ-minimal graphs for every positive integer ℓ. The second part of the conjecture stated "Moreover, the equality is attained by a subdivision of the clique $K_{\ell+2}$." (We used ℓ in place of k used in [LT03].) Escalante, Montelar and Nasini [EMN06] proved that if the word "subdivision" is interpreted as "only replacing edges with paths of length at least one" that second part of the conjecture is true for $\ell = 3$, but false for all $\ell \geq 4$. In the paper [LT03] other, more general subdivision operations were discussed (including certain versions of *stretching*). Note that here, we proved that with this interpretation of subdivision of a clique (which includes stretching), the second part of Conjecture 40 of [LT03] also holds.

Given a positive integer ℓ, we showed (Corollary 20) that $G \in \hat{\mathcal{K}}_{\ell+2,\ell-1}$ with $\omega(G) \leq 3$ is sufficient for a given graph G to be ℓ-minimal. However, as shown in Proposition 25 and suggested by the numerical evidence presented in Figure 8, being in $\hat{\mathcal{K}}_{\ell+2,\ell-1}$ is not a necessary condition for ℓ-minimal graphs. In fact, we believe that it is not necessary for ℓ-minimal graphs to belong to $\mathcal{K}_{\ell+2,\ell-1}$, as we will present numerical evidence in a forthcoming manuscript that there are at least 18 3-minimal graphs which do not belong to $\mathcal{K}_{5,2}$.

Thus, there are still plenty about ℓ-minimal graphs that we have yet to understand. What are some other interesting properties of these graphs? More ambitiously, can we obtain a combinatorial characterization of exactly when a given graph is ℓ-minimal?

Problem 31. Let $\bar{n}_+(\ell)$ be the smallest possible number of vertices needed for a vertex-transitive graph G to have $r_+(G) \geq \ell$. What is $\lim_{\ell\to\infty} \frac{\bar{n}_+(\ell)}{\ell}$?

It follows immediately from Theorem 27 that $\bar{n}_+(\ell) \leq 4\ell + 12$, and so $\lim_{\ell\to\infty} \frac{\bar{n}_+(\ell)}{\ell} \leq 4$. On the other hand, it is obvious that $\bar{n}_+(\ell) \geq n_+(\ell) = 3\ell$ for all $\ell \geq 1$, and so $\lim_{\ell\to\infty} \frac{\bar{n}_+(\ell)}{\ell} \geq 3$. Can we find out what the true value of the limit (or even a closed-form formula for $\bar{n}_+(\ell)$), or at least prove tighter bounds?

(As an aside, we remark that the problem could be rather different if we defined $\bar{n}_+(\ell)$ to be the smallest possible number of vertices needed for a vertex-transitive graph G to have $r_+(G)$ being *equal* to ℓ. In this case, since the line graph of odd cliques are vertex-transitive, we know that $\bar{n}_+(\ell) \leq 2\ell^2 + \ell$ [ST99]. However, it is not immediately clear to us that $\bar{n}_+(\ell)$ must be an increasing function of ℓ in this case, and so there is a chance that the limit $\lim_{\ell\to\infty} \frac{\bar{n}_+(\ell)}{\ell}$ may not exist in this case.)

Problem 32. For each pair of positive integers (n, ℓ) with $n \geq \ell$, characterize the family of graphs G on n vertices which maximize the integrality ratio:

$$\frac{\alpha_{\text{LS}_+^\ell}(G)}{\alpha(G)},$$

where $\alpha_{\text{LS}_+^\ell}(G) := \max \left\{ \bar{e}^\top x \, : \, x \in \text{LS}_+^\ell(G) \right\}$.

In this manuscript, we showed that ℓ-minimal graphs exist for every $\ell \geq 1$, establishing graphs which are worst-case scenarios for LS_+ in the sense of needing the maximum possible number of iterations of LS_+ to "compute" the stable set polytope. In addition to the LS_+-rank, another measure of the hardness of a graph is the *integrality gap* for the relaxation $\text{LS}_+(G)$. Progress in this direction would provide new understanding about the LS_+-relaxations of the stable set polytope of graphs from a different angle. Note that for random graphs $G_{n,1/2}$ we understand such integrality ratios to some extent. $\alpha(G_{n,1/2})$ is almost surely around $2\log_2(n)$. Feige and

Krauthgamer [FK03] showed (in providing an answer to the other question of Knuth about LS_+ in [Knu94]) that $\alpha_{\mathrm{LS}_+^\ell}(G)$ is almost surely around $\sqrt{n/2^\ell}$ for $\ell = o(\log(n))$.

Problem 33. Are there other applications for Lemma 12 and the ideas used in its proof?

The definition of LS_+^k naturally lends itself to inductive arguments when it comes to establishing rank lower bounds for a family of instances. Previous examples of this type of argument includes the aforementioned result by Stephen and the second author on the line graphs of odd cliques [ST99], as well as for the family of graphs H_k in [AT24b]. For our main result in this manuscript, a key insight was to build our proof around certifying the membership of the vector $v(G, \epsilon)$, which behaves well under deletion and destruction of vertices, even when the underlying graphs in $\mathcal{K}_{n,d}$ do not exhibit nearly as much symmetry as the two previous families of examples.

In particular, the foundation of our argument is Lemma 12, a noteworthy feature of which is that it allows us to establish LS_+-rank lower bounds without having to construct and verify specific numerical certificates. We intentionally stated this lemma as a result for LS_+-relaxations in general, and it would be interesting to see if this result and its insights can lead to breakthroughs in the analysis of other convex relaxations.

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