Homogeneous Cone Complementarity Problems and P Properties *

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Abstract

We consider existence and uniqueness properties of a solution to homogeneous cone complementarity problem (HCCP). Employing the T-algebraic characterization of homogeneous cones, we generalize the P, P_0, R_0 properties for a nonlinear function associated with the standard nonlinear complementarity problem to the setting of HCCP. We prove that if a continuous function has either the order-P_0 and R_0, or the P_0 and R_0 properties then all the associated HCCPs have solutions. In particular, if a continuous monotone function has the trace-P property then the associated HCCP has a unique solution (if any); if it has the uniform-trace-P property then the associated HCCP has the global uniqueness (of the solution) property (GUS). Moreover, we establish a global error bound for the HCCP with the uniform-trace-P property under some conditions for homogeneous cone linear complementarity problem.

Keywords: Homogeneous cone complementarity problem, P property, existence of a solution, globally uniquely solvability property.

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1 Introduction

In this paper, we are interested in the *homogeneous cone complementarity problem* (HCCP(F, q) for short) which is to find a vector $x \in K$ such that

$$x \in K,\ y \in K^*,\ \langle x, y\rangle = 0, \quad y = F(x) + q, \tag{1}$$

where K is a homogeneous cone (the automorphism group of the cone acts transitively on the interior of the cone, see Section 2 for the details) in a finite-dimensional inner product space $\mathbb{H}$ over $\mathbb{R}$ with its dual K^* given by $K^* := \{y \in \mathbb{H} : \langle x, y\rangle \geq 0, \forall x \in K\}$, $F : \mathbb{H} \to \mathbb{H}$ is a continuous function and $q \in \mathbb{H}$. If $F(x) = L(x)$ is linear, we call problem (1) the *homogeneous cone linear complementarity problem* (HCLCP(L, q)). When K is a symmetric cone in a Euclidean Jordan algebra, it is the *symmetric cone complementarity problem* (SCCP), which includes the

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so-called *nonlinear complementarity problem* (NCP, where $\mathbb{H} = \mathbb{R}^n$, the space of n-dimensional real column vectors, and $K = \mathbb{R}^n_+$, the nonnegative orthant) and *semidefinite complementarity problem* (SDCP, where $\mathbb{H} = \mathbb{S}^n$, the space of $n \times n$ real symmetric matrices, and $K = \mathbb{S}^n_+$, the cone of positive semidefinite symmetric matrices) as special cases.

In the NCP context, a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a *P function (has the P property)* if the following implication holds

$$(x - y) \circ [f(x) - f(y)] \leq 0 \quad \Rightarrow \quad x = y,$$

where "$\circ$" denotes the Hadamard (componentwise) product and $z \leq 0$ means that all components of z are nonpositive. There are many applications of P functions in engineering, economics, management science, as well as several other fields, see, e.g., [7, 23]. We say f is a P_0 function if $f + \varepsilon I$ is a P function for any $\varepsilon > 0$ where I is the identity transformation. This is a generalization of P-matrices. It is known that if f is a P_0 function and satisfies the so-called R_0 condition then the NCP(f, q) has a solution for every $q \in \mathbb{R}^n$. (Here, we only consider the R_0 condition as described in Definition 3.2 below.) In the setting of a Cartesian product of sets in $\mathbb{R}^n$, Facchinei and Pang [7] proposed the notions of P and P_0 functions and studied some of their properties. In the setting of Euclidean Jordan algebras, Gowda, Sznajder and Tao [12] studied some P and P_0 properties for linear transformations; Tao and Gowda [24] introduced P and P_0 functions and established the existence result for SCCP. Moreover, Gowda and Sznajder [11] studied the automorphism invariance of P and *globally uniquely solvability* (GUS) properties for linear transformations on Euclidean Jordan algebras. a necessary condition for F Symmetric cones are *homogeneous* and *self-dual*, see [8, 19]. A natural next step in this line of generalizations is to drop the requirement that K is self-dual. While there is a finite number of non-isomorphic symmetric cones of each dimension, the number is uncountable for homogeneous cones when the dimension $n > 11$, see [27]. There has been some increase in the interest and activity in the area of homogeneous cones and optimization problems over homogeneous cones, see, e.g., [1, 3, 4, 5, 6, 9, 13, 14, 18, 21, 22, 25, 26, 27, 28]. These papers deal with either certain theoretical properties of homogeneous cones, primal-dual interior-point methods for linear programming over homogeneous cones or their applications. In this paper, we work on the P properties for nonlinear transformations in the setting of HCCP. The aim of our work is to establish the existence and uniqueness results of a solution to HCCP.

With the help of the T-algebraic characterization of homogeneous cones, we first study the *metric projection* onto homogeneous cone K and its properties related to HCCP. Based on them, we introduce P, order-P, trace-P, uniform-trace-P, trace-P_0, order-P_0, P_0 and R_0 properties for a function $F : \mathbb{H} \to \mathbb{H}$ in the setting of HCCP. Then, we show that if F has either the order-P_0 and R_0, or the P_0 and R_0 properties then the HCCP(F, q) has a solution for every $q \in \mathbb{H}$ by applying degree theory. We give some sufficient conditions under which the HCCP has GUS property. For instance, if F is monotone and has the trace-P property then the associated HCCP(F, q) has a unique solution (if not empty); if F is monotone and has the uniform-trace-P property then the associated HCCP(F, q) has GUS property. Moreover, we establish a global error bound for the HCCP with F having the uniform-trace-P property.

This paper is organized as follows. In Section 2, we briefly review some basic concepts and results on T-algebras, and describe some fundamental results on metric projection onto homogenous cones. In Section 3, we introduce various P properties and show our existence result for HCCP. In Section 4, we study the GUS property and give an error bound for HCCP. In Section 5, we include some concluding remarks. While most of our focus is on Homogeneous cones, many of our results apply more generally (in the setting of arbitrary convex cones).

2 Preliminaries

We first briefly review some basic concepts and results on homogeneous cones and T-algebras from [4, 27, 28], and then provide some fundamental results on metric projection onto homogeneous cones.

2.1 Homogeneous cones and T-algebras

Definition 2.1 *A closed, convex cone K with nonempty interior is* homogeneous *if the group of automorphisms of K acts transitively on the interior of K.*

Note that a cone K is homogeneous then so is its dual K^*. Vinberg [27] introduced a constructive way to build homogeneous cones by employing the so-called *T-algebra* which connects homogeneous cones to abstract matrices whose elements are vectors. We first review the following concept of *matrix algebra*.

Definition 2.2 *A* matrix algebra *$\mathcal{A}$ is a bi-graded algebra $\bigoplus_{i,j=1}^{r} \mathcal{A}_{ij}$ over the reals with a bilinear product of $a_{ij} \in \mathcal{A}_{ij}$ and $a_{kl} \in \mathcal{A}_{kl}$ $(1 \leq i, j, k, l \leq r)$ satisfying*

$$a_{ij}a_{kl} \in \begin{cases} \mathcal{A}_{il} & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

where $\mathcal{A}_{ij}$ is a n_{ij}-dimensional vector space. The positive integer r is called the rank *of the matrix algebra $\mathcal{A}$.*

Every element $a \in \mathcal{A}$ is a generalized matrix with its component in $\mathcal{A}_{ij}$ being an n_{ij}-dimensional generalized element of the matrix a_{ij}, i.e., a_{ij} is the projection of a onto $\mathcal{A}_{ij}$. The multiplication of two elements $a, b \in \mathcal{A}$ is analogous to the multiplication of matrices,

$$(ab)_{ij} = \sum_{k=1}^{r} a_{ik}b_{kj}.$$

Assume that for every i, $\mathcal{A}_{ii}$ is isomorphic to $\mathbb{R}$, and let ρ_i be the isomorphism and let e_i denote the representation of the *unit element of $\mathcal{A}_{ii}$ in $\mathcal{A}$*. We define the *trace* of an element a as

$$\mathrm{Tr}(a) = \sum_{i=1}^{r} \rho_i(a_{ii}).$$

The following notion generalizes the classical (conjugate) transpose. An *involution* $*$ of the matrix algebra $\mathcal{A}$ of rank r is a linear automorphism on $\mathcal{A}$ that satisfies

(i) $(a^*)^* = a$ for all $a \in \mathcal{A}$ (involutory).

(ii) $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$ (anti-homomorphic).

(iii) $(a^*)_{ij} = (a_{ji})^*$ for $1 \leq i, j \leq r$.

(iv) $\mathcal{A}_{ij}^* = \mathcal{A}_{ji}$ for $1 \leq i, j \leq r$.

We are ready to state the following definition of T-algebra, which was originally introduced by Vinberg [27].

Definition 2.3 *A* T-algebra *of rank r is a matrix algebra $\mathcal{A}$ of rank r with involution $*$ satisfying the following axioms:*

I. For each $1 \leq i \leq r$, the subalgebra $\mathcal{A}_{ii}$ is isomorphic to the reals.

II. For each $a \in \mathcal{A}$ and each $1 \leq i, j \leq r$,

$$a_{ji}e_i = a_{ji} \quad \text{and} \quad e_i a_{ij} = a_{ij}.$$

III. For each $a, b \in \mathcal{A}$, $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$.
IV. For each $a, b, c \in \mathcal{A}$, $\mathrm{Tr}((ab)c) = \mathrm{Tr}(a(bc))$.
V. For each $a \in \mathcal{A}$, $\mathrm{Tr}(a^*a) \geq 0$, with equality if and only if $a = 0$.
VI. For each $a, b, c \in \mathcal{A}$ and each $i, j, k, l \in \{1, 2, \cdots, r\}$ with $i \leq j \leq k \leq l$,

$$a_{ij}(b_{jk}c_{kl}) = (a_{ij}b_{jk})c_{kl}.$$

VII. For each $a, b \in \mathcal{A}$ and each $i, j, k, l \in \{1, 2, \cdots, r\}$ with $i \leq j \leq k$ and $l \leq k$,

$$a_{ij}(b_{jk}b_{lk}^*) = (a_{ij}b_{jk})b_{lk}^*.$$

Thus, the *cone associated with a T-algebra $\mathcal{A}$* of rank r, denoted by $\mathrm{int}(K(\mathcal{A}))$, is given by

$$\mathrm{int}(K(\mathcal{A})) := \{tt^* : t \in \mathcal{A}, t_{ij} = 0, \ \forall 1 \leq j < i \leq r, \text{ and } t_{ii} > 0, \ \forall 1 \leq i \leq r\}.$$

It is easy to see from Axiom II that $e := \sum_{i=1}^r e_i$ is the unit element in $\mathcal{A}$, i.e., $ea = ae$ for all $a \in \mathcal{A}$. It is necessary to note that multiplication in a T-algebra is neither commutative nor associative. Define the subalgebra of upper triangular elements of $\mathcal{A}$ and the subspace of "Hermitian" elements, respectively, as

$$\mathcal{T} := \bigoplus_{i \leq j, i,j=1}^{r} \mathcal{A}_{ij}, \qquad \mathcal{H} := \{a \in \mathcal{A} : a = a^*\}.$$

Clearly, Axiom VI is equivalent to $t(uw) = (tu)w$ for all $t, u, w \in \mathcal{T}$. Taking involution, we get another equivalent statement: $t(uw) = (tu)w$ for all $t, u, w \in \mathcal{T}^*$. Similarly, Axiom VII is equivalent to $t(uu^*) = (tu)u^*$ for all $t, u \in \mathcal{T}$; or equivalently, $(u^*u)t = u^*(ut)$ for all $t, u \in \mathcal{T}^*$.

Let $\mathcal{T}_*$ (resp., $\mathcal{T}_+$ and $\mathcal{T}_{++}$) denote the set of elements of $\mathcal{T}$ with nonzero (resp., nonnegative and positive) diagonal components. It is easy to see that $\mathrm{int}(K(\mathcal{A})) = \{tt^* : t \in \mathcal{T}_{++}\}$ and $\mathcal{H}$ is the linear span of $\mathrm{int}(K(\mathcal{A}))$.

We recall below a fundamental characterization of homogeneous cones established by Vinberg [27].

Theorem 2.4 *(T-algebraic representation of homogeneous cones) A cone K is homogeneous if and only if $\mathrm{int}(K)$ is isomorphic to the cone $\mathrm{int}(K(\mathcal{A}))$ associated with some T-algebra $\mathcal{A}$. Moreover, given $\mathrm{int}(K(\mathcal{A}))$, the representation of an element from $\mathrm{int}(K)$ in the form tt^* is unique. Finally, the dual cone $\mathrm{int}(K^*)$ can be represented as $\{t^*t : t \in \mathcal{T}_{++}\}$.*

Remark 2.1 This theorem is analogous to the representation of a symmetric cone as the set of squares over a Jordan algebra. However, for an arbitrary T-algebra and its associated cone it is not true that for a given $a \in \mathcal{A}$ we have $aa^* \in \mathrm{int}(K(\mathcal{A}))$. In general, the closures of the cone $\mathrm{int}(K(\mathcal{A}))$ and its dual are specified by

$$K(\mathcal{A}) = \{tt^* : t \in \mathcal{T}_+\}, \quad K^*(\mathcal{A}) = \{t^*t : t \in \mathcal{T}_+\}. \tag{2}$$

Thus, $K(\mathcal{A})$ is a closed convex cone in the inner product space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is given by $\langle a, b \rangle := \mathrm{Tr}(a^*b) = \mathrm{Tr}(ab)$ for all $a, b \in \mathbb{H}$. It is worth noting that for all $a, b, c \in \mathcal{A}$,

$$\langle ab, c \rangle = \langle b^*a^*, c^* \rangle = \langle a, cb^* \rangle = \langle b, a^*c \rangle. \tag{3}$$

We define the *norm* induced by the inner product as $\|a\| := \sqrt{\langle a, a\rangle}$. In what follows, we simply write K and $\mathbb{H}$ for $K(\mathcal{A})$ and $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, respectively.

Example 2.1 Consider the following five-dimensional closed convex cone with nonempty interior (Vinberg [27]):

$$K := \left\{ x \in \mathbb{R}^5 : \begin{pmatrix} x_1 & x_2 & x_4 \\ x_2 & x_3 & 0 \\ x_4 & 0 & x_5 \end{pmatrix} \in \mathbb{S}^3_+ \right\}.$$

This cone is homogeneous; but, it is not a symmetric cone since there does not exist any inner product on $\mathbb{R}^5$ under which $K = K^*$. Let us choose the inner product implied by the trace inner product on $\mathbb{S}^3$; that is, $\forall x, y \in \mathbb{R}^5$,

$$\langle x, y\rangle := x_1y_1 + x_3y_3 + x_5y_5 + 2x_2y_2 + 2x_4y_4.$$

Then,

$$K^* = \left\{ y \in \mathbb{R}^5 : \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \in \mathbb{S}^2_+, \begin{pmatrix} y_1 & y_4 \\ y_4 & y_5 \end{pmatrix} \in \mathbb{S}^2_+ \right\}.$$

With this (natural) choice of the inner product, $K \subseteq K^*$. Moreover, it is straightforward to verify that $a^2 \in K + K^*$ for all $a \in \mathbb{H}$.

Remark 2.2 Note that every $z \in \mathbb{H}$ may be rewritten as $z = t + t^*$ with $t \in \mathcal{T}$ and $\langle z^2, e_i\rangle \geq 0$ for all $i \in \{1, 2, \cdots, r\}$. Then $z^2 = tt^* + t^*t + t^2 + (t^*)^2$. Motivated by the fact that a symmetric cone is the set of squares over a Jordan algebra, we propose the following question:

> for every homogeneous cone K, does there exist
>
> an inner product on $\mathbb{H}$ such that $z^2 \in K + K^*, \forall z \in \mathbb{H}$?

As we stated in the Introduction, we are generalizing the underlying theorems and in many cases their existing proofs from the symmetric cone setting to the more general, homogeneous cone setting. The existing results for the symmetric cone setting essentially fix an inner product under which $K = K^*$ and treat both K and K^* in the same finite dimensional Euclidean space. This allows operations like $x + y$ for $x \in K, y \in K^*$ and the related metric projections onto the cones K, K^* to be treated in the same space. Since much of the related theory is based on metric projections, one is required to fix an inner product in our more general setting as well. We remind the reader that the choice of the inner product is up to the goals of the user of the theory (for example, to recover the existing results for the special case of symmetric cones, we would pick the inner product so that $K = K^*$).

2.2 Metric Projection

Let $\Pi_K(x)$ denote the *metric projection* of x onto K, i.e.,

$$\Pi_K(x) := \operatorname{argmin}\left\{ \frac{1}{2}\|x - z\|^2 : z \in K \right\}.$$

In other words, $y = \Pi_K(x)$ if and only if $y \in K$ and

$$\|x - y\| \leq \|x - z\|, \ \forall z \in K,$$

or equivalently, the so-called obtuse angle property (or the Kolmogorov criterion) holds:

$$\langle z - \Pi_K(x), x - \Pi_K(x)\rangle \leq 0, \ \forall z \in K.$$

It is well-known [29] that the metric projector Π_K is unique and contractive, i.e.,

$$\|\Pi_K(x) - \Pi_K(y)\| \leq \|x - y\|, \ \forall x, y \in \mathbb{H}.$$

Utilizing the Moreau decomposition, any $x \in \mathbb{H}$ can be written as

$$x = \Pi_K(x) - \Pi_{K^*}(-x) \ \text{ with } \langle \Pi_K(x), \Pi_{K^*}(-x)\rangle = 0. \tag{4}$$

Based on the metric projection operator, we define the following operations for any $x, y \in \mathbb{H}$,

$$x \wedge_K y \ := \ x - \Pi_K(x - y), \quad x \vee_K y \ := \ y + \Pi_K(x - y). \tag{5}$$

Then, by direct calculation, we obtain

$$x \wedge_K y = y \wedge_{K^*} x, \quad (-x) \wedge_K (-y) = -(x \vee_{K^*} y). \tag{6}$$

Summarizing the above arguments, we have the following proposition.

Proposition 2.5 *Let K be a closed convex cone in $\mathbb{H}$ with its dual K^*. Then the following statements hold for all $x, y \in \mathbb{H}$.*

(a) *We have* $x = \Pi_K(x) - \Pi_{K^*}(-x)$ with $\langle \Pi_K(x), \Pi_{K^*}(-x)\rangle = 0$. *This decomposition is unique in the sense that if $x = x_1 - x_2$ with $x_1 \in K, x_2 \in K^*$ and $\langle x_1, x_2\rangle = 0$ then $x_1 = \Pi_K(x)$ and $x_2 = \Pi_{K^*}(-x)$.*

(b) $x \wedge_K y = y \wedge_{K^*} x, \quad x \vee_K y = y \vee_{K^*} x.$

(c) $(-x) \wedge_K (-y) = -(x \vee_{K^*} y)$ *and* $(-x) \wedge_{K^*} (-y) = -(x \vee_K y)$.

In particular, $\Pi_K(x) \wedge_K \Pi_{K^*}(-x) = 0$, *and* $\Pi_K(x) \vee_K \Pi_{K^*}(-x) = \Pi_K(x) + \Pi_{K^*}(-x)$.

Considering the characterization of homogeneous cones, from (4) and (2), we obtain that any $x \in \mathbb{H}$ can be expressed as

$$x = uu^* - v^*v \ \text{ with } \ \langle uu^*, v^*v\rangle = 0,$$

where $u, v \in \mathcal{T}_+$. Observe that

$$\begin{aligned} \langle uu^*, v^*v\rangle &= \langle (uu^*)v^*, v^*\rangle \\ &= \langle u(u^*v^*), v^*\rangle \\ &= \langle (vu)^*, u^*v^*\rangle \\ &= \langle (vu)^*, (vu)^*\rangle \\ &= \|vu\|^2, \end{aligned}$$

where the first equality holds by (3), the second equality holds by Axiom VII, the third holds by (3) and the fact that $*$ is anti-homomorphic. Then, Axiom V implies that $\langle uu^*, v^*v\rangle = 0$ if and only if $vu = 0$. We have actually proved the following.

Theorem 2.6 *Let K be a homogeneous cone in $\mathbb{H}$ with its dual K^*. Then, every $x \in \mathbb{H}$ can be uniquely expressed as*

$$x = uu^* - v^*v \quad \text{with} \quad vu = 0, \quad u, v \in \mathcal{T}_+. \tag{7}$$

Moreover, we have $\Pi_K(x) = uu^*, \quad \Pi_{K^*}(-x) = v^*v$.

Applying Proposition 2.5 and the above theorem, we obtain the following equivalent statements related to HCCP (1).

Proposition 2.7 *Let K be a homogeneous cone in $\mathbb{H}$ with its dual K^*. Then the following statements are equivalent:*

(a) $x \wedge_K y = 0$;

(b) $y \wedge_{K^*} x = 0$;

(c) $x \in K, \ y \in K^*, \ \langle x, y\rangle = 0$;

(d) $x \in K, \ y \in K^*, \ \langle xy, e_i\rangle = \langle yx, e_i\rangle = 0, \ \forall i \in \{1, 2, \cdots, r\}$;

(e) There exist $u, v \in \mathcal{T}_+$ such that $x = uu^*, \ y = v^*v, \quad vu = 0$;

(f) $x \in K, \ y \in K^*, \ (xy)_{lj} = 0, \ \forall \ l, j \in \{1, 2, \cdots, r\}$ *such that $l \geq j$.*

In particular, if $xy = yx$, then (f) becomes

(f′) $x \in K, \ y \in K^*, \ xy = 0$.

Proof. By Proposition 2.5 and Theorem 2.6, $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (e)$. Clearly, $(f) \Rightarrow (d) \Rightarrow (c)$. Therefore, we need only to show that $(e) \Rightarrow (f)$. Choose any $w^* \in \mathcal{T}^*$. Applying arguments similar to those before Theorem 2.6, we obtain

$$\begin{aligned} \langle xy, w^*\rangle &= \langle (uu^*)(v^*v), w^*\rangle \\ &= \langle uu^*, w^*(v^*v)\rangle \\ &= \langle uu^*, (w^*v^*)v\rangle \\ &= \langle (uu^*)v^*, w^*v^*\rangle \\ &= \langle u(vu)^*, w^*v^*\rangle. \end{aligned}$$

Then, the desired conclusion of equivalence follows. (f') is self-evident. □

We next address the following result which will be used to establish the GUS-property for HCCP.

Proposition 2.8 *Let K be a homogeneous cone in $\mathbb{H}$ with its dual K^*. Then, for every $x, y \in \mathbb{H}$, the following statements hold:*

(i) if $x \in K, \ y \in K^*$, and $x = \sum_{i=1}^r x_i e_i$, then $\langle xy, e_i\rangle \geq 0, \ \forall i \in \{1, 2, \cdots, r\}$;

(ii) $\langle (x \wedge_K y)(x \vee_K y), e_i\rangle = \langle xy, e_i\rangle, \ \forall i \in \{1, 2, \cdots, r\}$.

Proof. We first prove (i). Since $x \in K, \ y \in K^*$, noting that $e_i \in K \cap K^*$, we have $\langle x, e_i\rangle \geq 0, \langle y, e_i\rangle \geq 0$ for all $i \in \{1, 2, \cdots, r\}$. Using $x = \sum_{i=1}^r x_i e_i, \ x_i = \langle x, e_i\rangle \geq 0$. Thus, for every $i \in \{1, 2, \cdots, r\}$,

$$\langle xy, e_i\rangle = \langle y, xe_i\rangle = x_i\langle y, e_i\rangle \geq 0.$$

Therefore, the desired conclusion (i) holds.

For part (ii), direct calculation yields

$$\begin{aligned}\langle (x \wedge_K y)(x \vee_K y), e_i\rangle &= \langle [x - \Pi_K(x-y)][y + \Pi_K(x-y)], e_i\rangle \\ &= \langle xy, e_i\rangle + \langle x\Pi_K(x-y), e_i\rangle - \langle \Pi_K(x-y)y, e_i\rangle - \langle \Pi_K(x-y)\Pi_K(x-y), e_i\rangle \\ &= \langle xy, e_i\rangle + \langle x, \Pi_K(x-y)e_i\rangle - \langle y, \Pi_K(x-y)e_i\rangle - \langle \Pi_K(x-y), \Pi_K(x-y)e_i\rangle \\ &= \langle xy, e_i\rangle + \langle (x-y) - \Pi_K(x-y), \Pi_K(x-y)e_i\rangle \\ &= \langle xy, e_i\rangle,\end{aligned}$$

where the last equality follows from the fact $\langle (x-y) - \Pi_K(x-y), \Pi_K(x-y)e_i\rangle = \langle \Pi_{K^*}(y-x)\Pi_K(x-y), e_i\rangle = 0$ by (4). Thus, we proved (ii). □

Note that in the above proposition, the condition $x = \sum_{i=1}^r x_i e_i$ is necessary, which is illustrated by the following example.

Example 2.1 Let K be the homogeneous cone in $\mathbb{R}^5$ as in Example 2.1. Clearly, it is easy to verify that

$$x = \begin{pmatrix} 5 & -2 & -2 \\ -2 & 1 & 0 \\ -2 & 0 & 5 \end{pmatrix} \in K, \quad y = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix} \in K^*, \quad \text{and} \quad xy = \begin{pmatrix} -3 & 2 & 2 \\ 0 & 0 & 0 \\ 8 & 0 & 16 \end{pmatrix}.$$

Obviously, $\langle xy, e_1\rangle = -3$. Therefore, without some additional condition, one can not conclude $\langle xy, e_i\rangle \geq 0$ for all $i \in \{1, 2, 3\}$.

We end this section with the following property of the elements in K.

Proposition 2.9 *Let K be a homogeneous cone in $\mathbb{H}$ and $x \in K$ with $x = \sum_{i,j=1}^r x_{ij}$. If $x_{kk} = 0$ for some $k \in \{1, 2, \cdots, r\}$, then*

$$\sum_{j=1}^r x_{kj} + \sum_{i=1}^r x_{ik} = 0.$$

In particular, if $x \in K \cap \left(\bigoplus_{i \neq j} \mathcal{A}_{ij}\right)$, then $x = 0$.

Proof. Since $x \in K$, there exists $u \in \mathcal{T}_+$ such that $x = uu^*$. Set $u = \sum_{i,j=1}^r u_{ij}$ with $u_{ij} = 0$ for every $i, j \in \{1, 2, \cdots, r\}$ such that $i > j$. Direct calculation yields

$$x_{kk} = \sum_{j=k}^r u_{kj} u_{kj}^*.$$

Since $\langle e_k, u_{kj}u_{kj}^*\rangle = \langle u_{kj}, u_{kj}\rangle = \|u_{kj}\|^2$, by the assumption $x_{kk} = 0$, we obtain

$$0 = \langle e_k, x_{kk}\rangle = \sum_{j=k}^r \|u_{kj}\|^2.$$

This along with $u \in \mathcal{T}_+$ leads to $u_{kl} = 0$ for every $l \in \{1, 2, \cdots, r\}$. Therefore, we obtain

$$x_{ik} = \sum_{l=1}^r u_{il} u_{kl}^* = 0, \quad x_{kj} = \sum_{l=1}^r u_{kl} u_{jl}^* = 0, \quad \forall i, j \in \{1, 2, \cdots, r\},$$

as desired. □

3 P and R_0 properties

We first give the definitions of various $P(P_0)$ and R_0 properties.

Definition 3.1 *For a continuous function $F : \mathbb{H} \to \mathbb{H}$, we say that it has*

(i) the order-P property *if for any pair $x, y \in \mathbb{H}$,*

$$(x-y) \wedge_K (F(x)-F(y)) \in -(K \cap K^*) \text{ and } (x-y) \vee_K (F(x)-F(y)) \in (K+K^*) \;\Rightarrow\; x=y;$$

(ii) the order-P_0 property *if $F(x)+\varepsilon B(x)$ has the order-P property for any $\varepsilon > 0$ where $B : \mathbb{H} \to \mathbb{H}$ is a given linear function and satisfies $\langle x, B(x)\rangle > 0, \langle xB(x), e_i\rangle \geq 0,\ \forall x \neq 0,\ \forall i \in \{1, 2, \cdots, r\}$;*

(iii) the P property *if for any pair $x, y \in \mathbb{H}$,*

$$\sum_{l \geq j} \left([(x-y)(F(x)-F(y))]_{lj} + [(x-y)(F(x)-F(y))]_{lj}^* \right) - \sum_{i=1}^{r} [(x-y)(F(x)-F(y))]_{ii} \;\in\; -(K+K^*) \;\Rightarrow\; x=y;$$

(iv) the P_0 property *if $F(x)+\varepsilon B(x)$ has the P property for any $\varepsilon > 0$;*

(v) the trace-P property *if for any pair $x, y \in \mathbb{H}$ with $x \neq y$,*

$$\max_i \langle (x-y)(F(x)-F(y)), e_i\rangle > 0;$$

(vi) the trace-P_0 property *if $F(x)+\varepsilon B(x)$ has the trace P property for any $\varepsilon > 0$;*

(vii) the uniform-trace-P property *if there is an $\alpha > 0$ such that for any pair $x, y \in \mathbb{H}$,*

$$\max_i \langle (x-y)(F(x)-F(y)), e_i\rangle \geq \alpha \|x-y\|.$$

In general, we may choose the above B as the identity transformation.

Remark 3.1 By Proposition 2.5 (b), the implication condition of the order-P property is equivalent to the following: for any pair $x, y \in \mathbb{H}$,

$$(x-y) \wedge_K (F(x)-F(y)) \in -(K \cap K^*) \text{ and } (F(x)-F(y)) \vee_{K^*} (x-y) \in (K+K^*) \;\Rightarrow\; x=y.$$

Note that when K is self-dual, $K \cap K^* = K$ and $K + K^* = K$. It is easy to see that all the above order-P and P properties become the order-P and Jordan P properties given by Tao and Gowda [24] in the setting of SCCP, respectively. In particular, when $\mathbb{H} = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$, they are all the same as the P function (see Introduction).

Remark 3.2 Using the related definitions, we can easily verify the following one-way implications of the properties for nonlinear transformation F:

$$\begin{aligned}
\text{Strong monotonicity} \Rightarrow\ \text{uniform-trace-}P \quad &\Rightarrow \quad \text{trace-}P \Rightarrow \text{trace-}P_0,\\
\text{Strong monotonicity} \Rightarrow \text{strict monotonicity} \quad &\Rightarrow \quad \text{trace-}P \Rightarrow P,\\
\text{Monotonicity} \quad &\Rightarrow \quad \text{trace-}P_0 \Rightarrow P_0.
\end{aligned}$$

Here, we say that F is *monotone* if $\langle x-y, F(x)-F(y)\rangle \geq 0,\ \ \forall\, x, y \in \mathbb{H}$; F is *strictly monotone* if $\langle x-y, F(x)-F(y)\rangle > 0,\ \ \forall\, x \neq y, x, y \in \mathbb{H}$; and F is *strongly monotone with modulus* $\mu > 0$ if $\langle x-y, F(x)-F(y)\rangle \geq \mu\|x-y\|^2,\ \ \forall\, x, y \in \mathbb{H}$.

Remark 3.3 Observe that there are very many possible generalizations of the definition of the order-P property from symmetric cones to homogeneous cones. For instance, for any pair of sets $\hat{K}$ and $\check{K}$ such that $K \cap K^* \subseteq \hat{K}, \check{K} \subseteq K + K^*$, we say F has the *order-P property with respect to $\hat{K}$ and $\check{K}$* if

$$(x - y) \wedge_K (F(x) - F(y)) \in -\hat{K} \text{ and } (x - y) \vee_K (F(x) - F(y)) \in \check{K} \;\Rightarrow\; x = y;$$

F has the *order-P_0 property with respect to $\hat{K}$ and $\check{K}$* if $F(x) + \varepsilon B(x)$ has the order-P property for any $\varepsilon > 0$. In order to establish the existence result of a solution to HCCP (see Theorem 3.7), we set $\check{K} := K + K^*$ from the equation (10) in the proof of Lemma 3.6.

Moreover, we may define the *order-P property* of F by the implication

$$(x - y) \wedge_{K^*} (F(x) - F(y)) \in -(K \cap K^*) \text{ and } (x - y) \vee_{K^*} (F(x) - F(y)) \in (K + K^*) \;\Rightarrow\; x = y;$$

and the corresponding order-P_0 property. However, in this case, we cannot guarantee the existence result of a solution to HCCP (see Theorem 3.7).

Remark 3.4 Similarly, there are very many possible generalizations of the definition of the P property to homogeneous cones. For instance, for every $\tilde{K}$ such that $K \cap K^* \subseteq \tilde{K} \subseteq K + K^*$, we say F has the *P property with respect to $\tilde{K}$* if

$$\sum_{l \geq j} \left([(x - y)(F(x) - F(y))]_{lj} + [(x - y)(F(x) - F(y))]^*_{lj}\right) - \sum_{i=1}^{r} [(x - y)(F(x) - F(y))]_{ii} \in -\tilde{K} \Rightarrow x = y;$$

and F has the *P_0 property with respect to $\tilde{K}$* if $F(x) + \varepsilon B(x)$ has the P property for any $\varepsilon > 0$. Clearly, from the proofs of Lemma 3.6 and Theorem 3.7, we obtain that for any $\tilde{K}$ if F has the P_0 property with respect to $\tilde{K}$ and R_0 properties then the associated HCCPs have solutions. If the answer to our question in Remark 2.2 is "yes," with the choice of $\tilde{K} = K + K^*$, then there is some hope for proving $P \Rightarrow P_0$.

Remark 3.5 In the case of SCCP, the trace-P property implies the order-P property, and the order-P property implies the P (Jordan P) property. However, it is not clear whether this is valid for HCCP.

Definition 3.2 *A continuous function $F : \mathbb{H} \to \mathbb{H}$ is said to have the R_0 property if the following condition holds: for every sequence $\{x^{(k)}\} \subset \mathbb{H}$ with*

$$\|x^{(k)}\| \to \infty, \quad \liminf_{k \to \infty} \frac{x^{(k)}}{\|x^{(k)}\|} \in K, \quad \liminf_{k \to \infty} \frac{F(x^{(k)})}{\|x^{(k)}\|} \in K^*,$$

we have $\liminf_{k \to \infty} \frac{\max_i \langle x^{(k)} F(x^{(k)}), e_i \rangle}{\|x^{(k)}\|^2} > 0$.

The above definition is motivated by Definition 3.2 of Tao and Gowda [24], which was originally introduced for NCP by Chen and Harker [2]. In the setting of $\mathbb{R}^n$, it becomes that a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ has the R_0 property if for every sequence $\{x^{(k)}\} \subset \mathbb{R}^n$ with

$$\|x^{(k)}\| \to \infty, \quad \liminf_{k \to \infty} \frac{\min_i x_i^{(k)}}{\|x^{(k)}\|} \geq 0, \quad \liminf_{k \to \infty} \frac{\min_i f_i(x^{(k)})}{\|x^{(k)}\|} \geq 0,$$

we have $\liminf_{k \to \infty} \frac{\max_i x_i^{(k)} f_i(x^{(k)})}{\|x^{(k)}\|^2} > 0$. Clearly, when f is linear the above condition becomes equivalent to the statement that the standard linear complementarity problem $LCP(f, 0)$ has a unique solution, namely, zero.

Applying the related definitions we can easily derive two conditions under which the R_0 property holds, which is a generalization of Proposition 3.2 in [24].

Proposition 3.3 *Let $F : \mathbb{H} \to \mathbb{H}$ be a continuous function. If F has either the uniform-trace-P property or satisfies the following implication: for every sequence $\{x^{(k)}\} \subset \mathbb{H}$ with*

$$\|x^{(k)}\| \to \infty, \quad \liminf_{k\to\infty} \frac{x^{(k)}}{\|x^{(k)}\|} \in K, \quad \liminf_{k\to\infty} \frac{F(x^{(k)})}{\|x^{(k)}\|} \in K^*,$$

we have $\liminf_{k\to\infty} \frac{\langle x^{(k)}, F(x^{(k)})\rangle}{\|x^{(k)}\|^2} > 0$, then F has the R_0 property.

It is well-known that the notion of R_0 property of a function is closely related its coercivity, which plays a central role in describing the boundedness of the solution set to NCP, see, e.g. [7]. In the case of HCCP, we have a similar result.

Proposition 3.4 *Let $F : \mathbb{H} \to \mathbb{H}$ be a continuous function. If F has the R_0 property, then for every $\delta > 0$, the set $\{x \in \mathbb{H} : \; x \text{ solves } HCCP(F, q), \; \|q\| \le \delta\}$ is bounded.*

Proof. Suppose the set $\{x \in \mathbb{H} : \; x \text{ solves } HCCP(F, q), \; \|q\| \le \delta\}$ is unbounded. Then, there exist sequences $\{q^{(k)}\}$ with $\|q^{(k)}\| \le \delta$ and $\{x^{(k)}\}$ with $\|x^{(k)}\| \to \infty$ such that

$$x^{(k)} \in K, \; y^{(k)} = F(x^{(k)}) + q^{(k)} \in K^*, \; \langle x^{(k)}, y^{(k)}\rangle = 0, \; \forall k.$$

Since $\{x^{(k)}\} \subset K$, $\{y^{(k)}\} \subset K^*$ and K, K^* are closed, $\liminf_{k\to\infty} \frac{x^{(k)}}{\|x^{(k)}\|} \in K$ and $\liminf_{k\to\infty} \frac{y^{(k)}}{\|x^{(k)}\|} \in K^*$. Since $q^{(k)}$ is bounded, $\liminf_{k\to\infty} \frac{q^{(k)}}{\|x^{(k)}\|} = 0$. Thus,

$$\liminf_{k\to\infty} \frac{F(x^{(k)})}{\|x^{(k)}\|} = \liminf_{k\to\infty} \frac{F(x^{(k)}) + q^{(k)}}{\|x^{(k)}\|} = \liminf_{k\to\infty} \frac{y^{(k)}}{\|x^{(k)}\|} \in K^*.$$

This together with the R_0 property of F gives

$$\liminf_{k\to\infty} \frac{\max_i \langle x^{(k)} F(x^{(k)}), e_i\rangle}{\|x^{(k)}\|^2} > 0.$$

However, noting that $\langle x^{(k)}, y^{(k)}\rangle = 0$ and the boundedness $q^{(k)}$, by Proposition 2.7, we obtain that for every $i \in \{1, 2, \cdots, r\}$,

$$\frac{\langle x^{(k)} F(x^{(k)}), e_i\rangle}{\|x^{(k)}\|^2} = \frac{\langle x^{(k)} y^{(k)}, e_i\rangle}{\|x^{(k)}\|^2} - \frac{\langle x^{(k)} q^{(k)}, e_i\rangle}{\|x^{(k)}\|^2} = -\frac{\langle x^{(k)} q^{(k)}, e_i\rangle}{\|x^{(k)}\|^2} \to 0 \text{ as } k \to \infty.$$

This is a contradiction and hence the desired conclusion follows. □

Before stating our main result in this section, we recall below a useful result from degree theory. The topological degree technique plays an important role in the study of complementarity problems and variational inequality problems, see, e.g, [10, 15, 16, 17, 24, 30, 31]. Let Ω be a bounded open set in $\mathbb{H}$ with its closure $\mathrm{cl}(\Omega)$ and boundary $\partial\Omega$. For a continuous function $\Phi : \mathrm{cl}(\Omega) \to \mathbb{H}$ and $p \notin \Phi(\partial\Omega)$, we denote $\deg(\Phi, \Omega, p)$ the (topological) *degree* of Φ with respect to Ω at p, see Lloyd [20] for the details.

Lemma 3.5 *(Theorem 2.1.2, [20]) (1) Suppose that $\Phi, \varphi : \mathrm{cl}(\Omega) \to \mathbb{H}$ are continuous and $p \notin \Phi(\partial\Omega)$. If $\sup_{x\in \mathrm{cl}(\Omega)} \|\Phi(x) - \varphi(x)\| < \mathrm{dist}(p, \Phi(\partial\Omega))$, then $\deg(\varphi, \Omega, p)$ is defined and*

$$\deg(\varphi, \Omega, p) = \deg(\Phi, \Omega, p).$$

(2) If $g_t(x)$ is a homotopy and $p \notin g_t(\partial\Omega)$ for $0 \le t \le 1$, then $\deg(g_t, \Omega, p)$ is independent of $t \in [0, 1]$.

The next lemma relies heavily on the invariance of degree under suitable homotopies and generalizes Theorem 3.1 of [24] and its proof.

Lemma 3.6 *Let $F : \mathbb{H} \to \mathbb{H}$ be a continuous function, and for every $\delta > 0$ the set*

$$\{x \in \mathbb{H} : \ x \text{ solves } HCCP(F, q), \ \|q\| \le \delta\} \tag{8}$$

is bounded. If F has either the order-P_0 property, or the P_0 property, then for every $q \in \mathbb{H}$, the solution set of HCCP(F,q) is nonempty and bounded.

Proof. Choose any $q \in \mathbb{H}$. Consider the function

$$\Phi(x) := x \wedge_K (F(x) + q).$$

Define the homotopy

$$G_1(x, t) := x \wedge_K [F(x) + tq + (t-1)F(0)], \ t \in [0, 1].$$

Clearly, $G_1(x, 0) = x \wedge_K [F(x) - F(0)]$ and $G_1(x, 1) = \Phi(x)$ for all x. By the assumption, the sets $\{x \in \mathbb{H} : G_1(x, t) = 0\}$ $(t \in [0, 1])$ are uniformly bounded. Thus, we may take a bounded open set $\Omega \in \mathbb{H}$ such that

$$\bigcup_{t\in[0,1]} \{x \in \mathbb{H} : G_1(x, t) = 0\} \subseteq \Omega.$$

Then, $0 \in \Omega$ and $0 \notin G_1(\partial\Omega, 0)$ since $G_1(0, 0) = 0$. Therefore, by Lemma 3.5 (2),

$$\deg(G_1(\cdot, 0), \Omega, 0) = \deg(G_1(\cdot, 1), \Omega, 0) = \deg(\Phi, \Omega, 0).$$

Define $\varphi_\varepsilon(x) := x \wedge_K [F(x) + \varepsilon B(x) - F(0)]$ for any $\varepsilon > 0$, where B is linear and strictly monotone. Note that

$$\begin{aligned}
\|\varphi_\varepsilon(x) - G_1(x, 0)\| &= \|x \wedge_K [F(x) + \varepsilon B(x) - F(0)] - x \wedge_K [F(x) - F(0)]\| \\
&= \|\Pi_K[x - (F(x) + \varepsilon B(x) - F(0))] - \Pi_K[x - (F(x) - F(0))]\| \\
&\le \|[x - (F(x) + \varepsilon B(x) - F(0))] - [x - (F(x) - F(0))]\| \\
&= \|\varepsilon B(x)\|,
\end{aligned}$$

where the inequality follows from (4). Since $\operatorname{dist}(0, G_1(\partial\Omega, 0)) > 0$ by $0 \notin G_1(\partial\Omega, 0)$, we pick $\varepsilon_0 > 0$ such that

$$\sup_{x\in \mathrm{cl}(\Omega)} \|\varphi_\varepsilon(x) - G_1(x, 0)\| < \operatorname{dist}(0, G_1(\partial\Omega, 0)).$$

Then, by Lemma 3.5 (1), $\deg(G_1(\cdot, 0), \Omega, 0) = \deg(\varphi_\varepsilon, \Omega, 0)$. So, we obtain

$$\deg(\varphi_\varepsilon, \Omega, 0) = \deg(\Phi, \Omega, 0). \tag{9}$$

For small $\varepsilon > 0$, we define the homotopy

$$G_2(x, t) := x \wedge_K [t(F(x) - F(0) + \varepsilon B(x)) + (1 - t)x], \ t \in [0, 1].$$

Clearly, $G_2(x, 0) = x \wedge_K x = x$ and $G_2(x, 1) = \varphi_\varepsilon(x)$ for all x. We now show that $0 \notin G_2(\partial\Omega, t)$ for any $t \in [0, 1]$. Suppose not, then there exist $t_0 \in [0, 1]$ and $x_0 \in \partial\Omega$ such that $G_2(x_0, t_0) = 0$.

If $t_0 = 0$, then $G_2(x_0, 0) = 0$ means that $x_0 = 0$, which contradicts $0 \in \Omega$. We may assume $t_0 \in (0, 1]$. By Proposition 2.7, $G_2(x_0, t_0) = 0$ is equivalent to the following

$$x_0 \in K,\ F(x_0) - F(0) + \varepsilon B(x_0) + \frac{1 - t_0}{t_0} x_0 \in K^*,\ \left\langle x_0, F(x_0) - F(0) + \varepsilon B(x_0) + \frac{1 - t_0}{t_0} x_0 \right\rangle = 0.$$

Letting $\widetilde{F}(x) := F(x) + \varepsilon B(x) + (\frac{1}{t_0} - 1)x$, the above can be written as

$$x_0 \in K,\ \widetilde{F}(x_0) - \widetilde{F}(0) \in K^*,\ \left\langle x_0, \widetilde{F}(x_0) - \widetilde{F}(0) \right\rangle = 0.$$

Thus, by Propositions 2.5 and 2.7, we obtain

$$\begin{aligned} (x_0 - 0) \wedge_K (\widetilde{F}(x_0) - \widetilde{F}(0)) &= 0 \in -(K \cap K^*), \\ (x_0 - 0) \vee_K (\widetilde{F}(x_0) - \widetilde{F}(0)) &= x_0 + (\widetilde{F}(x_0) - \widetilde{F}(0)) \in K + K^*, \end{aligned} \tag{10}$$

and

$$[(x_0 - 0)(\widetilde{F}(x_0) - \widetilde{F}(0))]_{lj} = 0,\ \forall l, j \in \{1, 2, \cdots, r\} \text{ such that } l \geq j. \tag{11}$$

If F has the order-P_0 property, then $\widetilde{F}$ has the order-P property. Hence, by (10), $x_0 = 0$, a contradiction. If F has the P_0 property, then $\widetilde{F}$ has the P property and hence, by (11), $x_0 = 0$. This is also a contradiction.

Thus, $0 \notin G_2(\partial\Omega, t)$. Again, by Lemma 3.5 (2),

$$\deg(G_2(\cdot, 0), \Omega, 0) = \deg(G_2(\cdot, 1), \Omega, 0) = \deg(\varphi_\varepsilon, \Omega, 0). \tag{12}$$

Notice that $\deg(G_2(\cdot, 0), \Omega, 0) = 1$. This together with (9) and (12) yields $\deg(\Phi, \Omega, 0) = 1$, which says that $\Phi(x) = 0$ has a solution. By Proposition 2.7, we proved that HCCP(F, q) has a solution. The desired conclusion follows from the assumption (8). □

We state below our main result in this section.

Theorem 3.7 *Let $F : \mathbb{H} \to \mathbb{H}$ be a continuous function. Suppose that F has either the order-P_0 and R_0 properties, or the P_0 and R_0 properties. Then for every $q \in \mathbb{H}$, the solution set of HCCP(F,q) is nonempty and bounded.*

Proof. It follows immediately from Proposition 3.4 and Lemma 3.6. □

As a direct consequence of the above theorem, we have the following.

Corollary 3.8 *Let $F : \mathbb{H} \to \mathbb{H}$ be a continuous function. Suppose that F has either the order-P_0 and R_0 properties, or the P_0 and R_0 properties. Then, there exists $\bar{x} \in \mathbb{H}$ such that*

$$\bar{x} \in \text{int}(K), \quad F(\bar{x}) \in \text{int}(K^*).$$

Proof. It follows from Theorem 3.7 that for every $q \in \mathbb{H}$, the solution set of HCCP(F, q) is nonempty and bounded. Take $-q_0 \in \text{int}(K^*)$. Let $\hat{x}$ be a solution to HCCP(F, q_0). Then, we have $\hat{x} \in K, \hat{y} := F(\hat{x}) + q_0 \in K^*$. Therefore, $F(\hat{x}) = -q_0 + \hat{y} \in \text{int}(K^*)$ and the desired conclusion follows from the continuity of F. □

4 GUS Property

We say that a continuous transformation $F : \mathbb{H} \to \mathbb{H}$ has the *GUS property* if for every $q \in \mathbb{H}$, HCCP(F, q) has a unique solution. In this section, we show that the trace-P property is sufficient for F having the GUS property in the setting of HCCP. We also establish an error bound for the HCCP with the uniform-trace-P property.

4.1 Sufficient conditions for the GUS property

Next, we show that if F has the trace-P property together with monotonicity or some other suitable property, then the associated HCCP(F, q) has the GUS property.

Theorem 4.1 *Let $F : \mathbb{H} \to \mathbb{H}$ be a monotone continuous function. Suppose that F has the trace-P property. Then for any $q \in \mathbb{H}$, the solution set of HCCP(F, q) is a singleton, if it is nonempty.*

Proof. Assume that there exist two solutions x, y to HCCP(F, q). Then, by Proposition 2.7,

$$x, y \in K,\ F(x) + q, F(y) + q \in K^*,\ \langle x(F(x) + q), e_i\rangle = \langle y(F(y) + q), e_i\rangle = 0,\ \forall i \in \{1, 2, \cdots, r\}.$$

Then,

$$\langle x - y, F(x) - F(y)\rangle = \langle x - y, (F(x) + q) - (F(y) + q)\rangle = -\langle x, F(y) + q\rangle - \langle y, F(x) + q\rangle \leq 0.$$

Also, since F is monotone, $\langle x - y, F(x) - F(y)\rangle \geq 0$. So, $\langle x - y, F(x) - F(y)\rangle = 0$ and $\langle x, F(y) + q\rangle = \langle y, F(x) + q\rangle = 0$. Similarly, by Proposition 2.7, $\langle x(F(y) + q), e_i\rangle = \langle y(F(x) + q), e_i\rangle = 0$. Thus, we obtain that

$$\langle (x - y)(F(x) - F(y)), e_i\rangle = \langle (x - y)[(F(x) + q) - (F(y) + q)], e_i\rangle = 0.$$

Therefore, the trace-P property of F yields $x = y$, as desired. □

As a direct application of the above Theorems 4.1 and 3.7 and the connection between various P properties, we have that under the monotonicity assumption, the uniform-trace-P property implies the GUS property.

Corollary 4.2 *Let $F : \mathbb{H} \to \mathbb{H}$ be a monotone continuous function. If F has the uniform-trace-P property then it has the GUS property.*

Note that in the special case of SCCP, if $\bar{x}$ is a solution to SCCP(F, q), then $\bar{x}, F(\bar{x}) + q$ share a common Jordan frame. This motivates us to give the following result.

Theorem 4.3 *Let $F : \mathbb{H} \to \mathbb{H}$ be a continuous function. Suppose that F has the trace-P property. For every $q \in \mathbb{H}$, if HCCP(F, q) has a solution x such that $x = \sum_{i=1} x_i e_i, F(x) + q = \sum_{i=1} y_i e_i$, then it has a unique solution.*

Proof. As in the proof of Theorem 4.1, assume that there exist two solutions x, y to HCCP(F, q). Then, by Proposition 2.7,

$$x, y \in K,\ F(x) + q, F(y) + q \in K^*,\ \langle x(F(x) + q), e_i\rangle = \langle y(F(y) + q), e_i\rangle = 0,\ \forall i \in \{1, 2, \cdots, r\}.$$

Thus, we obtain that

$$\begin{aligned} \langle (x - y)(F(x) - F(y)), e_i\rangle &= \langle (x - y)[(F(x) + q) - (F(y) + q)], e_i\rangle \\ &= -\langle x(F(y) + q), e_i\rangle - \langle y(F(x) + q), e_i\rangle \\ &\leq 0, \end{aligned}$$

where the second equality holds by Proposition 2.7 and the inequality holds by Proposition 2.8. Therefore, the trace-P property of F yields $x = y$, as desired. □

Corollary 4.4 *Let $F : \mathbb{H} \to \mathbb{H}$ be a continuous function. If F has the uniform-trace-P property and for any $q \in \mathbb{H}$, HCCP(F, q) has a solution x such that $x = \sum_{i=1} x_i e_i, F(x) + q = \sum_{i=1} y_i e_i$, then x is the unique solution to HCCP(F, q).*

4.2 Error Bound

We give an error bound for the HCCP with the uniform-trace-P property.

Theorem 4.5 *Suppose F has the uniform-trace-P property with modulus $\alpha > 0$ and is Lipschitz continuous with constant $\kappa > 0$. Let x^* be the unique solution of problem HCCP(F, q) such that $x^* = \sum_{i=1} x_i^* e_i, F(x^*) + q = \sum_{i=1} y_i^* e_i$. Then*

$$\frac{1}{2+\kappa}\|x \wedge_K (F(x) + q)\| \le \|x - x^*\| \le \frac{1+\kappa}{\alpha}\|x \wedge_K (F(x) + q)\|, \ \forall x \in \mathbb{H}. \tag{13}$$

Proof. Notice that $F(x)+q-x\wedge_K(F(x)+q) = \Pi_{K^*}(F(x)+q-x) \in K^*$, $x-x\wedge_K(F(x)+q) = \Pi_K(x - F(x) - q) \in K$, and $x^* \wedge_K (F(x^*) + q) = 0$. By Propositions 2.7 and 2.8 and the fact that x^* solves HCCP(F, q) such that $x^* = \sum_{i=1} x_i^* e_i, F(x^*) + q = \sum_{i=1} y_i^* e_i$, we obtain

$$\begin{aligned}
\langle [F(x) + q - x \wedge_K (F(x) + q)][x^* - x^* \wedge_K (F(x^*) + q)], e_i\rangle &\ge 0,\\
\langle [x - x \wedge_K (F(x) + q)][F(x^*) + q - x^* \wedge_K (F(x^*) + q)], e_i\rangle &\ge 0,\\
\langle [F(x^*) + q - x^* \wedge_K (F(x^*) + q)][x^* - x^* \wedge_K (F(x^*) + q))], e_i\rangle &= 0.
\end{aligned}$$

Thus, by direction calculation, we obtain that for every $i \in \{1, 2, \cdots, r\}$,

$$\begin{aligned}
&\langle [F(x) + q - x \wedge_K (F(x) + q)][x - x \wedge_K (F(x) + q)], e_i\rangle\\
\ge\ &\langle ([F(x) + q - x \wedge_K (F(x) + q)] - [F(x^*) + q - x^* \wedge_K (F(x^*) + q)])\\
&([x - x \wedge_K (F(x) + q)] - [x^* - x^* \wedge_K (F(x^*) + q)], e_i\rangle\\
=\ &\langle [F(x) - F(x^*) - x \wedge_K (F(x) + q)][x - x^* - x \wedge_K (F(x) + q)], e_i\rangle\\
\ge\ &\langle [F(x) - F(x^*)](x - x^*), e_i\rangle - \langle x \wedge_K (F(x) + q)[F(x) - F(x^*) + x - x^*], e_i\rangle\\
\ge\ &\langle (F(x) - F(x^*))(x - x^*), e_i\rangle - \|x \wedge_K (F(x) + q)\| \cdot (1 + \kappa)\|x - x^*\|,
\end{aligned}$$

where the second inequality holds by the fact $\langle a^2, e_i\rangle = \langle aa^*, e_i\rangle \ge 0$ for all $a \in \mathbb{H}$ because $a = a^*$; the last inequality follows from the Lipschitz continuity of F and $\|ab\| \le \|a\|\|b\|$ for any $a, b \in \mathcal{A}$ by Theorem 2 of [4]. By Propositions 2.5 and 2.7, $\langle [F(x) + q - x \wedge_K (F(x) + q)][x - x \wedge_K (F(x) + q)], e_i\rangle = 0$. Thus, we conclude from the above inequality and the uniform-trace-P property of F,

$$(1 + \kappa)\|x \wedge_K (F(x) + q)\|\|x - x^*\| \ \ge\ \max_i \langle (F(x) - F(x^*))(x - x^*), e_i\rangle \ge \alpha\|x - x^*\|^2.$$

This leads to the right-hand side of inequality (13).

Note that $\|\Pi_K(y) - \Pi_K(z)\| \le \|y - z\|$ for every $y, z \in \mathbb{H}$. From the Lipschitz continuity of F, we obtain by direct manipulation that

$$\begin{aligned}
\|x \wedge_K (F(x) + q)\| &= \|[x - \Pi_K(x - F(x) - q)] - [x^* - \Pi_K(x^* - F(x^*) - q)]\|\\
&= \|[x - x^*] - [\Pi_K(x - F(x) - q) - \Pi_K(x^* - F(x^*) - q)]\|\\
&\le \|x - x^*\| + \|\Pi_K(x - F(x) - q) - \Pi_K(x^* - F(x^*) - q)\|
\end{aligned}$$

$$\begin{aligned} &\leq \|x - x^*\| + \|(x - F(x) - q) - (x^* - F(x^*) - q)\| \\ &\leq 2\|x - x^*\| + \|F(x) - F(x^*)\| \\ &\leq (2 + \kappa)\|x - x^*\|. \end{aligned}$$

This means that the left-hand side of inequality (13) holds. □

Note that in the SCCP context, the condition $x^* = \sum_{i=1} x_i^* e_i, F(x^*) + q = \sum_{i=1} y_i^* e_i$ is naturally satisfied. Then, the above error bound result becomes a generalization of the corresponding one for NCP presented by Chen and Harker [2]. It is not clear whether the above error bound would hold for every monotone, continuous function F with the uniform-trace-P property.

5 Final Remarks

In this paper, by employing the T-algebraic characterization of homogeneous cones we prove that if a continuous function has either the order-P_0 and R_0, or the P_0 and R_0 properties then all the associated HCCPs have solutions. We give sufficient conditions under which the associated HCCP has the GUS property. Moreover, we establish a global error bound for HCCP under some conditions.

Many of our results apply to the more general setting of arbitrary convex cones (the order-P property, order-P_0 property, trace-P property, and R_0 property). Further generalizations of similar results and the theory to arbitrary convex cone setting represent a good direction for future research. The design of algorithms for HCCP (and beyond) and a study of their mathematical and computational properties provide other interesting future research avenues.

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