Polynomial optimization with a focus on hyperbolic polynomials

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A problem of minimizing (or maximizing) a multivariate polynomial over a subset of an Euclidean Space defined by the solution set of finitely many polynomial equations and inequalities is called a *Polynomial Optimization Problem* (PoP). PoPs describe a class of optimization problems with a nontrivial amount of geometric, algebraic and analytic properties. At the same time, POPs are general enough to capture as a special case, a very wide swath of finite dimensional optimization problems and even some semi-infinite optimization problems. We can reformulate PoPs in many equivalent forms. For example, by introducing a new variable (increasing the dimension of the space by one), we can push the objective function's multivariate polynomial into the constraints (hence, without loss of generality, we may assume that the objective function is always linear).

In this setting, an interesting, nicely structured, and powerful class of convex PoPs is Semidefinite Programming (SDP) problems. Objective functions of SDPs are linear, and their feasible solution sets are defined as the intersection of the convex cone of n-by-n symmetric positive semidefinite matrices (all real entries), denoted $\mathbb{S}^n_+$ with an affine subspace. Such convex sets are called *spectrahedra.* In SDP problems we may also use additional auxiliary variables that effectively get projected away due to our carefully picked choices for the objective function of the SDP. This observation shows that SDPs can also deal with convex sets that are orthogonal projections of spectrahedra. These latter convex sets are called *spectrahedral shadows.* Indeed, spectrahedral shadows yield a strict superset of spectrahedra. However, except for utilization of facial exposedness property (of spectrahedra), we do not have many elegant, useful certificates helping us distinguish these two families of convex sets precisely (see [25, 8, 23, 20, 3, 16]).

We consider PoPs from a convex optimization viewpoint (see for instance [9, 11, 19]). Then a central question is "when is the feasible region of a PoP convex?" This leads us to *hyperbolic polynomials* (a.k.a. *stable polynomials*, under a suitable transformation) which naturally define convex domains. For the sake of convenience, we work with homogeneous hyperbolic polynomials so that the underlying convex domains become convex cones called *hyperbolicity cones.* Let p be a homogeneous polynomial (this is without loss of generality in our current context) of degree d in n variables, and let $e \in \mathbb{R}^n$. p is said to be *hyperbolic in direction e* if $p(e) > 0$ and, for all $x \in \mathbb{R}^n$, the scalar polynomial $\lambda \mapsto p(x - \lambda e)$ has only real roots. Studies of hyperbolic polynomials go back at least to the work of Petrovsky (from the 1930s). Considerable amount of work has been done by Gårding, Atiyah, Bott and Gårding as well as Hörmander. Since the early 1990's there has been an amazing amount of activity allowing the subject to branch into systems and control theory, operator theory (see Marcus-Spielman-Srivastava [13] solution of Kadison-Singer problem) interior-point methods (see, for instance, [5, 20, 15] and the references therein), discrete optimization and combinatorics (see, for instance, Gurvits [4], Wagner [27] and references therein) semidefinite programming and semidefinite representations, matrix theory as well as theoretical computer science.

Fix a direction e and a polynomial p hyperbolic in direction e. We call the roots of $\lambda \mapsto p(x - \lambda e)$ the *eigenvalues of* x. Let Λ_{++} denote the set of points that have only positive eigenvalues and let Λ_+ denote its closure. Λ_+ is called the *hyperbolicity cone of* p *in direction* e. It is a convex cone. A very nice example is $\mathbb{S}^n_+$ associated with the hyperbolic polynomial $p(x) := \det(x)$ and the direction e given by the n-by-n identity matrix. Helton-Vinnikov Theorem [26, 7, 12] implies that all three dimensional hyperbolicity cones are spectrahedra and every hyperbolic polynomial giving rise to a 3-dimensional hyperbolicity cone admits a very strong determinantal representation. Using Helton-Vinnikov Theorem, one can prove: some general facts about all hyperbolicity cones, some general facts about all hyperbolic polynomials, and generalizations of many theorems from matrix analysis to "hyperbolicity cone optimization" setting. There are many generalizations of Helton-Vinnikov theorem (see [21] and the references therein), counter examples to certain proposed generalizations of Helton-Vinnikov Theorem (see Brändén [2] and the references therein), various spectrahedral and spectrahedral-shadow representations for interesting hyperbolicity cones (see Netzer and Sanyal [17] and the references therein, in the light of [18]).

If $K = \Lambda_+(p)$, then $F : \mathbb{R}^n \to \mathbb{R}$, $\quad F(x) := \begin{cases} -\ln(p(x)), & \text{if } x \in \Lambda_{++}(p); \\ +\infty, & \text{otherwise.} \end{cases}$
has very useful properties for modern interior-point methods (see [5, 15]). Let F be a normal barrier (see [15] for a definition) for the regular cone K. We say that F has *negative curvature* if for every $x \in \text{int}(K)$ and $h \in K$ we have $\nabla^3 F(x)[h]$ negative semidefinite. Negation of logarithms of hyperbolic polynomials have negative curvature [10, 5]. While the dual cone of a hyperbolicity cone is not necessarily hyperbolic [3], the *dual barrier function* $F_*(s) := \max_{x \in \text{int}(K)} \{-\langle s, x\rangle - F(x)\}$ is always a normal barrier for the dual cone K^*. F_* does not necessarily have negative curvature.

Open Problems: 1. Does there exist an algebraic convex cone (defined as the solution set of homogeneous multivariate polynomial inequalities) which admits a normal barrier with negative curvature but it is not a hyperbolicity cone? **2.** [15] Characterize the set of convex cones which admit normal barriers with negative curvature. **3.** (*Generalized Lax Conjecture*) Every hyperbolicity cone is a spectrahedron.

Conjecture 1: [24] Every hyperbolicity cone is a spectrahedral shadow.

A few months before this writing, Scheiderer [22] answered a related question of Nemirovski [14] by disproving the Helton-Nie conjecture [6] (Helton-Nie conjecture is a stronger version of our Conjecture 1 above).

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