Perturbed Sums of Squares Theorem for Polynomial Optimization and its Applications

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March 2, 2015

Abstract

We consider a property of positive polynomials on a compact set with a small perturbation. When applied to a Polynomial Optimization Problem (POP), the property implies that the optimal value of the corresponding SemiDefinite Programming (SDP) relaxation with sufficiently large relaxation order is bounded from below by $(f^* - \epsilon)$ and from above by $f^* + \epsilon(n+1)$, where f^* is the optimal value of the POP. We propose new SDP relaxations for POP based on modifications of existing sums-of-squares representation theorems. An advantage of our SDP relaxations is that in many cases they are of considerably smaller dimension than those originally proposed by Lasserre. We present some applications and the results of our computational experiments.

1. Introduction

1.1. Lasserre's SDP relaxation for POP

We consider the POP:

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad f_i(x) \geq 0 \ (i = 1, \ldots, m), \tag{1}$$

where $f, f_1, \ldots, f_m : \mathrm{R}^n \to \mathrm{R}$ are polynomials. The feasible region is denoted by $K = \{ x \in \mathrm{R}^n : f_j(x) \geq 0 \ (j = 1, \ldots, m) \}$. Then it is easy to see that the optimal value f^* can be represented as

$$f^* = \sup \{ \rho : f(x) - \rho \geq 0 \ (\forall x \in K) \} .$$

First, we briefly describe the framework of the SDP relaxation method for the POP (1) proposed by Lasserre [17]. See also research conducted by Parrilo [25]. We denote the set of polynomials and sums of squares by $\mathrm{R}[x]$ and Σ, respectively. $\mathrm{R}[x]_r$ is the set of polynomials whose degree is less than or equal to r. We let $\Sigma_r = \Sigma \cap \mathrm{R}[x]_{2r}$. We define the quadratic module generated by $f_1, \ldots, f_m$ as

$$M(f_1, \ldots, f_m) = \left\{ \sigma_0 + \sum_{j=1}^{m} \sigma_j f_j : \sigma_0, \ldots, \sigma_m \in \Sigma \right\}.$$

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The truncated quadratic module whose degree is less than or equal to $2r$ is defined by

$$M_r(f_1,\ldots,f_m) = \left\{ \sigma_0 + \sum_{i=1}^{m} \sigma_j f_j \;:\; \sigma_0 \in \Sigma_r, \sigma_j \in \Sigma_{r_j}(j=1,\ldots,m) \right\},$$

where $r_j = r - \lceil \deg f_j/2 \rceil$ for $j = 1,\ldots,m$.

Replacing the condition that $f(x) - \rho$ is nonnegative by a relaxed condition that the polynomial is contained in $M_r(f_1,\ldots,f_m)$, we obtain the following SOS relaxation:

$$\rho_r = \sup \{ \rho \;:\; f(x) - \rho \in M_r(f_1,\ldots,f_m) \}. \tag{2}$$

Lasserre[17] showed that $\rho_r \to f^*$ as $r \to \infty$ if $M(f_1,\ldots,f_m)$ is archimedean. See [22, 26] for a definition of archimedean. An easy way to ensure that $M(f_1,\ldots,f_m)$ is archimedean is to make sure that $M(f_1,\ldots,f_m)$ contains a representation of a ball of finite (but possibly very large) radius. In particular, we point out that when $M(f_1,\ldots,f_m)$ is archimedean, K is compact.

Problem (2) can be encoded as an SDP problem. Note that we can express the sum of squares $\sigma \in \Sigma_r$ by using a positive semidefinite matrix $X \in \mathbb{S}^{s(r)}_+$ as $\sigma(x) = u_r(x)^T X u_r(x)$, where $s(r) = \binom{n+r}{n}$ and $u_r(x)$ is the monomial vector that contains all the monomials in n variables up to and including degree r with an appropriate order. By using this relation, the containment by $M_r(f_1,\ldots,f_m)$ constraints in (2), *i.e.*,

$$f - \rho = \sigma_0 + \sum_{j=1}^{m} \sigma_j f_j,$$

can be transformed to linear equations involving semidefinite matrix variables corresponding to σ_0 and σ_j's.

Note that, in this paper, we neither assume that K is compact nor that $M(f_1,\ldots,f_m)$ is archimedean. Still, the framework of Lasserre's SDP relaxation described above can be applied to (1), although some of the good theoretical convergence properties may be lost.

1.2. Problems in SDP relaxation for POP

Since POP is NP-hard, solving POP in practice is sometimes extremely difficult. The SDP relaxation method described above also has some difficulties. A major difficulty arises from the size of the SDP relaxation problem (2). In fact, (2) contains $\binom{n+2r}{n}$ variables and $s(r) \times s(r)$ matrix. When n and/or r become larger, solving (2) can become totally impossible.

To overcome this difficulty, several techniques using sparsity of polynomials are proposed, e.g., [15, 19, 22, 23, 29]. Based on the fact that most of the practical POPs are sparse in some sense, these techniques exploit special sparsity structure of POPs to reduce the number of variables and the size of the matrix variable in the SDP (2). Recent work in this direction, e.g., [6, 7] also exploits the special structure of POPs to solve larger problems. Nie and Wang [24] propose a regularization method for solving SDP relaxation problems instead of primal-dual interior-point methods.

Another problem with SDP relaxation is that (2) is often ill-posed. Strange behaviors of SDP solvers have been reported [11, 31, 33]. One of them is that an SDP solver returns an 'optimal' value of (2) which is significantly different from the true optimal value without reporting any numerical errors. Even more strange is that the returned value by the SDP solver is nothing but the real optimal value of the POP (1). We refer to this as the 'super-accurate' property of the SDP relaxation for POP.

1.3. Contribution of this paper

POP contains very hard problems as well as some easier ones. Our approach will exploit the structure in the easier instances of POP. In the context of this paper, the notion of "easiness" will be based on sums of squares certificate and sparsity. Based on Theorems 2.1, 2.2 and their variants, we propose new SDP relaxations. We call it *Adaptive SOS relaxation* in this paper. Adaptive SOS relaxations can be interpreted as relaxations of those originally proposed by Lasserre. As a result, the bounds generated by our approach cannot be superior to those generated by Lasserre's approach for the same order relaxations. However, Adaptive SOS relaxations are of significantly smaller dimensions (compared to Lasserre's SDP relaxations) and as the computational experiments in Section 3 indicate, we obtain very significant speed-up factors and are able to solve larger instances of POPs and higher-order SDP relaxations. Moreover, in most cases, the amount of loss in the quality of bounds is small, even for the same order SDP relaxations.

The rest of this paper is organized as follows. Section 2 gives our main results and Adaptive SOS relaxation based on Theorem 2.1. In Section 3, we present the results of some numerical experiments. We give a proof of Theorem 2.1, some of its extensions, and present some of the related work in Section 4.

2. Adaptive SOS relaxation

2.1. Main results

We assume that there exists an optimal solution x^* of (1). Let

$$\begin{aligned} b &= \max\left(1, \max\{\, |x_i^*| \;:\; i = 1, \dots, n \,\}\right) \\ B &= [-b, b]^n. \end{aligned}$$

Obviously $x^* \in B$. We define:

$$\begin{aligned} \bar{K} &= B \cap K \\ R_j &= \max\{\, |f_j(x)| \;:\; x \in B \,\} \;\; (j = 1, \dots, m). \end{aligned}$$

We also define for a positive integer r,

$$\begin{aligned} \psi_r(x) &= -\sum_{j=1}^{m} f_j(x) \left(1 - \frac{f_j(x)}{R_j}\right)^{2r}, \\ \Theta_r(x) &= 1 + \sum_{i=1}^{n} x_i^{2r}, \\ \Theta_{r,b}(x) &= 1 + \sum_{i=1}^{n} \left(\frac{x_i}{b}\right)^{2r}. \end{aligned}$$

We start with the following theorem.

Theorem 2.1 *Suppose that for $\rho \in \mathrm{R}$, $f(x) - \rho > 0$ for every $x \in \bar{K}$, i.e., ρ is a lower bound of f^*.*

i Then, there exists $\tilde{r} \in \mathrm{N}$ such that for all $r \geq \tilde{r}$, $f - \rho + \psi_r$ is positive over B.
ii In addition, for every $\epsilon > 0$, there exists a positive integer $\hat{r}$ such that, for every $r \geq \hat{r}$,

$$f - \rho + \epsilon\Theta_{r,b} + \psi_{\tilde{r}} \in \Sigma.$$

Theorem 2.1 will be proved in Section 4 as a corollary of Theorem 4.2. Note that $\hat{r}$ depends on ρ and ϵ while $\tilde{r}$ depends on ρ, but not ϵ. The implication of this theorem is twofold.

First, it elucidates the super-accurate property of the SDP relaxation for POPs. Note that by construction, $-\psi_{\tilde{r}}(x) \in M_{\bar{r}}(f_1, \ldots, f_m)$ where $\bar{r} = \tilde{r}\max_j(\deg(f_j))$. Now assume that in (2), $r \geq \bar{r}$. Then, for any lower bound $\bar{\rho}$ of f^*, Theorem 2.1 means that $f - \bar{\rho} + \epsilon\Theta_{r,b} \in M_r(f_1, \ldots, f_m)$ for arbitrarily small $\epsilon > 0$ and sufficiently large r. Let us discuss this in more detail. Define Π to be the set of the polynomials such that the absolute value of each coefficient is less than or equal to 1. Suppose that $\bar{\rho}$ is a "close" lower bound of f^* such that the system $f - \bar{\rho} + \psi_{\tilde{r}} \in \Sigma$ is infeasible. Let us *admit* an error ϵ in the above system, i.e., consider

$$f - \bar{\rho} + \epsilon h + \psi_{\tilde{r}} \in \Sigma, \ h \in \Pi. \tag{3}$$

The system (3) restricts the amount of the infinity norm error in the equality condition of the SDP relaxation problem to be less than or equal to ϵ. Since we can decompose $h = h_+ - h_-$ where $h_+, h_- \in \Sigma \cap \Pi$, now the system (3) is equivalent to:

$$f - \bar{\rho} + \epsilon h_+ + \psi_{\tilde{r}} \in \Sigma, \ h_+ \in \Pi \cap \Sigma. \tag{4}$$

This observation shows that $-h_-$ is not the direction of errors. Furthermore, because $\Theta_{r,b} \in \Pi \cap \Sigma$, the system (4) is feasible due to part ii of Theorem 2.1. Therefore, if we admit an error ϵ, the system $f - \bar{\rho} + \psi_{\tilde{r}} \in \Sigma$ is considered to be feasible, and $\bar{\rho}$ is recognized as a lower bound for f^*. As a result, we may obtain f^* due to the numerical errors. On the other hand, when we do not admit an error, but are given a direction of the error h implicitly by floating-point arithmetic, it does not necessarily satisfy the left inclusion of (3). However, some numerical experiments show that this is true in most cases (e.g., [31]). The reason is not clear.

Second, we can use the result to construct some new sparse SDP relaxations for POP (1). Our SDP relaxation is weaker than Lasserre's, but the size of our SDP relaxation can become smaller than Lasserre's. As a result, for some large-scale and middle-scale POPs, our SDP relaxation can often obtain a lower bound while Lasserre's cannot.

A naive idea is to use (1) as is. Note that $-\psi_{\tilde{r}}(x)$ contains only monomials whose exponents are contained in

$$\bigcup_{j=1}^{m} \left(\mathcal{F}_j + \underbrace{\tilde{\mathcal{F}}_j + \cdots + \tilde{\mathcal{F}}_j}_{2\tilde{r}} \right),$$

where $\mathcal{F}_j$ is the *support* of the polynomial f_j, *i.e.*, the set of exponents of monomials with nonzero coefficients in f_j, and $\tilde{\mathcal{F}}_j = \mathcal{F}_j \cup \{0\}$. To state the idea more precisely, we introduce some notation. For a finite set $\mathcal{F} \subseteq \mathrm{N}^n$ and a positive integer r, we denote

$r\mathcal{F} = \underbrace{\mathcal{F} + \cdots + \mathcal{F}}_{r}$ and

$$\Sigma(\mathcal{F}) = \left\{ \sum_{k=1}^{q} g_k(x)^2 \ : \ \mathrm{supp}(g_k) \subseteq \mathcal{F} \right\},$$

where $\mathrm{supp}(g_k)$ is the support of g_k. Note that $\Sigma(\mathcal{F})$ is the set of sums of squares of polynomials whose supports are contained in $\mathcal{F}$.

Now, fix an admissible error $\epsilon > 0$ and $\tilde{r}$ as in Theorem 2.1, and consider:

$$\hat{\rho}(\epsilon, \tilde{r}, r) = \sup \left\{ \rho \ : \ f - \rho + \epsilon\Theta_{r,b} - \sum_{j=1}^{m} f_j\sigma_j = \sigma_0, \sigma_0 \in \Sigma_r, \sigma_j \in \Sigma(\tilde{r}\tilde{\mathcal{F}}_j) \right\} \qquad (5)$$

for some $r \geq \tilde{r}$. Due to Theorem 2.1, (5) has a feasible solution for all sufficiently large r.

THEOREM 2.2 *For every $\epsilon > 0$, there exist $\tilde{r}, r \in \mathrm{N}$ such that $f^* - \epsilon \leq \hat{\rho}(\epsilon, \tilde{r}, r) \leq f^* + \epsilon(n+1)$.*

Proof: We apply Theorem 2.1 to POP (1) with $\rho = f^* - \epsilon$. Then for any $\epsilon > 0$, there exist $\hat{r}, \tilde{r} \in \mathrm{N}$ such that for every $r \geq \hat{r}$, $f - (f^* - \epsilon) + \epsilon\Theta_{r,b} + \psi_{\tilde{r}} \in \Sigma$. Choose a positive integer $r \geq \hat{r}$ which satisfies

$$r \geq \max\{\lceil \deg(f)/2 \rceil, \lceil (\tilde{r} + 1/2)\deg(f_1) \rceil, \ldots, \lceil (\tilde{r} + 1/2)\deg(f_m) \rceil\}. \qquad (6)$$

Then there exists $\tilde{\sigma}_0 \in \Sigma_r$ such that $f - (f^* - \epsilon) + \epsilon\Theta_{r,b} + \psi_{\tilde{r}} = \tilde{\sigma}_0$ because the degree of the polynomial in the left hand side is equal to $2r$. We denote $\tilde{\sigma}_j := (1 - f_j/R_j)^{2\tilde{r}}$ for all j. The triplet $(f^* - \epsilon, \tilde{\sigma}_0, \tilde{\sigma}_j)$ is feasible in (5) because $(1 - f_j/R_j)^{2\tilde{r}} \in \Sigma(\tilde{r}\tilde{\mathcal{F}}_j)$. Therefore, we have $f^* - \epsilon \leq \hat{\rho}(\epsilon, \tilde{r}, r)$.

We prove that $\hat{\rho}(\epsilon, \tilde{r}, r) \leq f^* + \epsilon(n+1)$. We choose r as in (6) and consider the following POP:

$$\tilde{f} := \inf_{x \in \mathrm{R}^n} \left\{ f(x) + \epsilon\Theta_{r,b}(x) : f_1(x) \geq 0, \ldots, f_m(x) \geq 0 \right\}. \qquad (7)$$

Applying Lasserre's SDP relaxation with relaxation order r to (7), we obtain the following SOS relaxation problem:

$$\hat{\rho}(\epsilon, r) := \sup \left\{ \rho \ : \ f - \rho + \epsilon\Theta_{r,b} = \sigma_0 + \sum_{j=1}^{m} f_j\sigma_j, \sigma_0 \in \Sigma_r, \sigma_j \in \Sigma_{r_j} \right\}, \qquad (8)$$

where $r_j := r - \lceil \deg(f_j)/2 \rceil$ for $j = 1, \ldots, m$. Then we have $\hat{\rho}(\epsilon, r) \geq \hat{\rho}(\epsilon, \tilde{r}, r)$ because $\Sigma(\tilde{r}\tilde{\mathcal{F}}_j) \subseteq \Sigma_{r_j}$ for all j. Indeed, it follows from (6) and the definition of r_j that $r_j \geq \tilde{r}\deg(f_j)$, and thus $\Sigma(\tilde{r}\tilde{\mathcal{F}}_j) \subseteq \Sigma_{r_j}$.

Every optimal solution x^* of POP (1) is feasible for (7) and its objective value is $f^* + \Theta_{r,b}(x^*)$. We have $f^* + \Theta_{r,b}(x^*) \geq \hat{\rho}(\epsilon, r)$ because (8) is the relaxation problem of (7). In addition, it follows from $x^* \in B$ that $n + 1 \geq \Theta_{r,b}(x^*)$, and thus $\hat{\rho}(\epsilon, \tilde{r}, r) \leq \hat{\rho}(\epsilon, r) \leq f^* + \epsilon(n+1)$. □

Lasserre [17] proved the convergence of his SDP relaxation under the assumption that the quadratic module $M(f_1, \ldots, f_m)$ associated with POP (1) is archimedean. In contrast, Theorem 2.2 does not require such an assumption and ensures that we can obtain a sufficiently close approximation to the optimal value f^* of POP (1) by solving (5).

We delete the perturbed part $\epsilon\Theta_{r,b}(x)$ from the above sparse relaxation (5) in our computations, because it may be implicitly introduced in the computation due to the usage of floating-point arithmetic. In the above sparse relaxation (5), we have to consider only those positive semidefinite matrices whose rows and columns correspond to $\tilde{r}\tilde{\mathcal{F}}_j$ for f_j. In contrast, in Lasserre's SDP relaxation, we have to consider the whole set of monomials whose degree is less than or equal to r_j for each polynomial f_j. Only σ_0 is large; it contains the set of all monomials whose degree is less than or equal to r. However, since the other polynomials do not contain most of the monomials of σ_0, such monomials can safely be eliminated to reduce the size of σ_0 [15]. As a result, our sparse relaxation reduces the size of the matrix significantly if each $|\mathcal{F}_j|$ is small enough. We note that in many of the practical cases, this is true. We call this new relaxation *Adaptive SOS relaxation.*

2.2. *Proposed approach: Adaptive SOS relaxation*

An SOS relaxation (5) for POP (1) has been introduced. However, this relaxation has some weak points. In particular, we do not know the value $\tilde{r}$ in advance. Also, introducing a small perturbation ϵ intentionally may lead to numerical difficulties in solving SDP.

To overcome these difficulties, we ignore the perturbation part $\epsilon\Theta_{r,b}(x)$ in (5) because the perturbation part may be implicitly introduced by floating-point arithmetic. In addition, we choose a positive integer r and find $\tilde{r}$ by increasing r. Furthermore, we replace $\sigma_j \in \Sigma(\tilde{r}\tilde{\mathcal{F}}_j)$ by $\sigma_j \in \Sigma(\tilde{r}_j\tilde{\mathcal{F}}_j)$ in (5), where $\tilde{r}_j$ is defined for a given integer r as

$$\tilde{r}_j = \left\lfloor \frac{r}{\deg(f_j)} - \frac{1}{2} \right\rfloor,$$

to have $\deg(f_j\sigma_j) \leq 2r$ for all $j = 1, \ldots, m$. Then, we obtain the following SOS problem:

$$\rho^*(r) := \sup_{\rho\in\mathbb{R}, \sigma_0\in\Sigma_r, \sigma_j\in\Sigma(\tilde{r}_j\tilde{\mathcal{F}}_j)} \left\{ \rho : f - \rho - \sum_{j=1}^{m} f_j\sigma_j = \sigma_0 \right\}. \tag{9}$$

We call (9) *Adaptive SOS relaxation* for POP (1). Note that we try to use numerical errors in a positive way; even though Adaptive SOS relaxation has a different optimal value from that of POP, we hope that the contaminated computation produces the correct optimal value of POP.

In general, we have $\Sigma(\tilde{r}_j\tilde{\mathcal{F}}_j) \subseteq \Sigma_{r_j}$ because of $\tilde{r}_j \deg(f_j) \leq r_j$. Recall that $r_j = r - \lceil \deg(f_j)/2 \rceil$ and is used in Lasserre's SDP relaxation (2). This implies that Adaptive SOS relaxation is no stronger than Lasserre's SDP relaxation, *i.e.*, the optimal value $\rho^*(r)$ is lower than or equal to the optimal value $\rho(r)$ of Lasserre's SDP relaxation for POP (1) for all r. We further remark that $\rho^*(r)$ may not converge to the optimal value f^* of POP (1). However, we can hope for the convergence of $\rho^*(r)$ to f^* from Theorem 2.1 and some numerical results in [11, 31, 33].

In the rest of this subsection, we provide a property of Adaptive SOS relaxation for the quadratic optimization problem

$$\inf_{x\in\mathbb{R}^n} \left\{ f(x) := x^TP_0x + c_0^Tx : f_j(x) := x^TP_jx + c_j^Tx + \gamma_j \geq 0 \ (j = 1, \ldots, m) \right\}. \tag{10}$$

The next proposition implies that we do not need to compute $\rho^*(r)$ for even r.

Proposition 2.3 *Assume that the degree* $\deg(f_j) = 2$ *for all* $j = 1, \ldots, m$ *for QOP (10). Then, the optimal value* $\rho^*(r)$ *of Adaptive SOS relaxation is equal to* $\rho^*(r-1)$ *if* r *is even.*

Proof : It follows from definition of $\tilde{r}_j$ that we have

$$\tilde{r}_j = \left\lfloor \frac{r-1}{2} \right\rfloor = \begin{cases} \frac{r-1}{2} & \text{if } r \text{ is odd,} \\ \frac{r}{2} - 1 & \text{if } r \text{ is even.} \end{cases}$$

We assume that r is even and give Adaptive SOS relaxation problems with relaxation order r and $r-1$:

$$\rho^*(r) = \sup \left\{ \rho : \begin{array}{l} f - \rho - \sum_{j=1}^m f_j \sigma_j = \sigma_0, \rho \in \mathrm{R}, \sigma_0 \in \Sigma_r, \\ \sigma_j \in \Sigma\left(\left(\frac{r}{2} - 1\right) \tilde{\mathcal{F}}_j\right) \end{array} \right\}, \tag{11}$$

$$\rho^*(r-1) = \sup \left\{ \rho : \begin{array}{l} f - \rho - \sum_{j=1}^m f_j \sigma_j = \sigma_0, \rho \in \mathrm{R}, \sigma_0 \in \Sigma_{r-1}, \\ \sigma_j \in \Sigma\left(\left(\frac{r}{2} - 1\right) \tilde{\mathcal{F}}_j\right) \end{array} \right\}. \tag{12}$$

We have $\rho^*(r) \geq \rho^*(r-1)$ for (11) and (12). All feasible solutions $(\rho, \sigma_0, \sigma_j)$ of (11) satisfy the following identity:

$$f_0 - \rho = \sigma_0 + \sum_{j=1}^m \sigma_j f_j.$$

Since r is even, the degrees of $\sum_{j=1}^m \sigma_j(x) f_j(x)$ and $f_0(x) - \rho$ are less than or equal to $2r-2$ and 2 respectively, and thus the degree of σ_0 is less than or equal to $2r-2$. Indeed, we can write $\sigma_0(x) = \sum_{k=1}^{\ell} (g_k(x) + h_k(x))^2$ where $\deg(g_k) \leq r-1$ and h_k is a homogenous polynomial with degree r. Then we obtain $0 = \sum_{k=1}^{\ell} h_k^2(x)$, which implies $h_k = 0$ for all $k = 1, \ldots, \ell$. Therefore, all feasible solutions $(\rho, \sigma_0, \sigma_j)$ in the SDP relaxation problem (11) are also feasible in the SDP relaxation problem (12), and we have $\rho^*(r) = \rho^*(r-1)$ if r is even. □

3. Numerical Experiments

In this section, we compare Adaptive SOS relaxation with Lasserre's SDP relaxation and the sparse SDP relaxation using correlative sparsity proposed in [29]. To this end, we perform some numerical experiments. We observe from the results of our computational experiments that (i) although Adaptive SOS relaxation is often strictly weaker than Lasserre's, *i.e.*, the value obtained by Adaptive SOS relaxation is less than that obtained by Lasserre's, and the difference is small in many cases, (ii) Adaptive SOS relaxation solves at least 10 times faster than Lasserre's in middle- to large-scale problems. Therefore, we conclude that Adaptive SOS relaxation can be more effective than Lasserre's for large- and middle-scale POPs. We will also observe a similar relationship against the sparse relaxation in [29]; Adaptive SOS relaxation is weaker but much faster than the sparse one.

We use a computer with Intel (R) Xeon (R) 3.10 GHz cpus and 128GB memory, and MATLAB R2014a. To construct Lasserre's [17], sparse [29] and Adaptive SOS problems, we use SparsePOP 3.00 [30]. To solve the resulting SDP relaxation problems, we use SeDuMi 1.3 [27] and SDPT3 4.0 [28] with the default parameters. The default tolerances for the stopping criterion for SeDuMi and SDPT3 are 1.0e-9 and 1.0e-8, respectively.

To determine whether the optimal value of an SDP relaxation problem is the exact optimal value of a given POP, we use the following two criteria ϵ_{obj} and ϵ_{feas}. Let $\hat{x}$ be a candidate of an optimal solution of the POP obtained from the SDP relaxations. We apply a projection of the dual solution of the SDP relaxation problem onto R^n for obtaining $\hat{x}$ in this section. See [29] for the details. We define:

$$
\begin{aligned}
\epsilon_{\text{obj}} &:= \frac{|\text{optimal value of SDP relaxation} - f(\hat{x})|}{\max\{1, |f(\hat{x})|\}}, \\
\epsilon_{\text{feas}} &:= \min_{k=1,\ldots,m} \{f_k(\hat{x})\}.
\end{aligned}
$$

If $\epsilon_{\text{feas}} \geq 0$, then $\hat{x}$ is feasible for the POP. In addition, if $\epsilon_{\text{obj}} = 0$, then $\hat{x}$ is an optimal solution of the POP and $f(\hat{x})$ is the optimal value of the POP.

We introduce the following value to indicate the closeness between the obtained values of Lasserre's, sparse and Adaptive SOS relaxations.

$$
\text{Ratio} := \frac{(\text{obj. val. of Lasserre's or sparse SDP relax. })}{(\text{obj. val. of Adaptive SOS relax.})} = \frac{\rho_r^*}{\rho^*(r)}. \tag{13}
$$

If the signs of both optimal values are the same and Ratio is sufficiently close to 1, then the optimal value of Adaptive SOS relaxation is close to the optimal value of Lasserre's and sparse SDP relaxations. In general, this value is meaningless for measuring the closeness if those signs are different or either of values is zero. Fortunately, those values are not zero and those signs are the same in most cases of the numerical experiments in this section.

To reduce the size of the resulting SDP relaxation problems, SparsePOP has functions based on the methods proposed in [15, 34]. These methods are closely related to a facial reduction algorithm proposed by Borwein and Wolkowicz [1, 2], and thus we can expect that the numerical stability of the primal-dual interior-point methods for the SDP relaxations may be improved. In this section, except for Subsection 3.1, we apply the method proposed in [34].

It should be noted that we solved randomly generated problems in Sections 3.3 to 3.5. To produce such problems, we tested both the uniform distribution and normal distribution, and observed that the computational results are very similar. The only notable difference is that instances generated by the normal distribution are slightly harder for both algorithms (Lasserre and Adaptive SOS) in terms of the number of instances successfully solved. Therefore, we only present those results using the normal distribution in this paper.

For POPs that have lower and upper bounds on variables, we can strengthen the SDP relaxations by adding valid inequalities based on these bound constraints. In this section, we add them as in [29]. See Subsection 5.5 in [29] for the details.

Table 1 shows the notation used in the description of numerical experiments in the following subsections.

Table 1. Notation

iter.	number of iterations in SeDuMi and SDPT3
rowA, colA	size of coefficient matrix A in the SeDuMi input format
nnzA	number of nonzero elements in coefficient matrix A in the SeDuMi input format
SDPobj	objective value obtained by SeDuMi for the resulting SDP relaxation problem
POPobj	value of f at a solution $\hat{x}$ retrieved by SparsePOP
#solved	number of the POPs which are solved by SDP relaxation in 30 problems. If ϵ_{obj} is smaller than 1.0e-7 and ϵ_{feas} is greater than -1.0e-7, we regard that the SDP relaxation attains the optimal value of the POP.
minR	minimum value of Ratio defined in (13) in 30 problems
aveR	average of Ratio defined in (13) in 30 problems
maxR	maximum value of Ratio defined in (13) in 30 problems
sec	CPU time consumed by SeDuMi or SDPT3 in seconds
min.t	minimum CPU time consumed by SeDuMi or SDPT3 in seconds among 30 resulting SDP relaxations
ave.t	average CPU time consumed by SeDuMi or SDPT3 in seconds among 30 resulting SDP relaxations
max.t	maximum CPU time consumed by SeDuMi or SDPT3 in seconds among 30 resulting SDP relaxations
Lasserre	Results of Lasserre's SDP relaxation proposed in [17]
Sparse	Results of Sparse SDP relaxations proposed in [29]

3.1. Numerical results for POP whose quadratic module is non-archimedean

In this subsection, we give the following POP and apply Adaptive SOS relaxation:

$$\inf_{x,y\in\mathbb{R}} \left\{ -x-y : \begin{array}{c} f_1(x,y) := x-0.5 \geq 0, \\ f_2(x,y) := y-0.5 \geq 0, \\ f_3(x,y) := 0.5-xy \geq 0 \end{array} \right\}. \tag{14}$$

The optimal value is -1.5 and the solutions are $(0.5, 1)$ and $(1, 0.5)$. It has been proved in [26, 33] that the quadratic module associated with POP (14) is non-archimedean and that all the resulting SDP relaxation problems are weakly infeasible. However, the convergence of computed values of Lasserre's SDP relaxation for POP (14) has been observed in [33].

In [33], it was shown that Lasserre's SDP relaxation (2) for (14) is weakly infeasible. Since Adaptive SOS relaxation for (14) has fewer monomials for representing σ_j's than that of Lasserre's, the resulting SDP relaxation problems are necessarily infeasible. However, we expect from Thorem 2.2 that Adaptive SOS relaxation attains the optimal value of -1.5. Table 2 provides numerical results for Adaptive SOS relaxation based on (9). In fact, we observe from Table 2 that $\rho^*(r)$ obtained by SeDuMi is equal to -1.5 at $r = 7, 8, 9, 10$. By SDPT3, we observe similar results.

3.2. Difference between Lasserre's and Adaptive SOS relaxations

In this subsection, we present a POP where Adaptive SOS relaxation converges to the optimal value strictly slower than Lasserre's computationally. This POP called "st_e08.gms" is available at [8].

$$\inf_{x,y\in\mathbb{R}} \left\{ 2x+y : \begin{array}{ll} f_1(x,y) := xy-1/16 \geq 0, & f_2(x,y) := x^2+y^2-1/4 \geq 0, \\ f_3(x,y) := x \geq 0, & f_4(x,y) := 1-x \geq 0, \\ f_5(x,y) := y \geq 0, & f_6(x,y) := 1-y \geq 0. \end{array} \right\}. \tag{15}$$

The optimal value is $(3\sqrt{6}-\sqrt{2})/8 \approx 0.741781958247055$ and the solution is $(x^*, y^*) = ((\sqrt{6}-\sqrt{2})/8, (\sqrt{6}+\sqrt{2})/8)$.

Table 3 shows the numerical results of SDP relaxations for POP (15) by SeDuMi and SDPT3. We observe that Lasserre's SDP relaxation attains the optimal value of (15) by relaxation order $r = 3$ while Adaptive SOS relaxation attains it only at the relaxation order $r = 6$.

Table 2. Approximate optimal value, cpu time, the number of iterations by SeDuMi and SDPT3

r	Software	Iter.	SDPobj	[sec]
1	SeDuMi	45	-5.8354275e+07	0.59
	SDPT3	37	-1.8924840e+06	1.39
2	SeDuMi	38	-6.9021505e+02	0.27
	SDPT3	71	-8.2193715e+00	1.16
3	SeDuMi	32	-4.2408472e+01	0.22
	SDPT3	77	-2.0928888e+00	1.31
4	SeDuMi	35	-1.2522882e+01	0.28
	SDPT3	79	-1.8195861e+00	1.66
5	SeDuMi	33	-3.7345516e+00	0.50
	SDPT3	84	-1.6015288e+00	2.23
6	SeDuMi	33	-1.8814407e+00	0.60
	SDPT3	89	-1.5025613e+00	2.99
7	SeDuMi	18	-1.5000027e+00	0.40
	SDPT3	21	-1.5000022e+00	1.04
8	SeDuMi	16	-1.5000031e+00	0.33
	SDPT3	25	-1.5000003e+00	1.93
9	SeDuMi	16	-1.5000014e+00	0.58
	SDPT3	17	-1.5001049e+00	1.76
10	SeDuMi	16	-1.5000003e+00	1.28
	SDPT3	21	-1.4999676e+00	2.71

Table 3. Numerical results on SDP relaxation problems in Subsection 3.2

<table>
<tr><td></td><td></td><td>Lasserre</td><td>Adaptive SOS</td></tr>
<tr><td>r</td><td>Software</td><td>(SDPobj, POPobj| $\epsilon_{\rm obj}, \epsilon_{\rm feas}$ | [sec])</td><td>(SDPobj, POPobj| $\epsilon_{\rm obj}, \epsilon_{\rm feas}$ | [sec])</td></tr>
<tr><td>1</td><td>SeDuMi</td><td>(0.00000e+00, 5.69253e-20, | 5.7e-20, -1.0e+00 | 0.06)</td><td>(0.00000e+00, 5.69253e-20, | 5.7e-20, -1.0e+00 | 0.02)</td></tr>
<tr><td></td><td>SDPT3</td><td>(-1.33858e-11, 1.16414e-11, | 2.5e-11, -1.0e+00 | 0.14)</td><td>(-1.33858e-11, 1.16414e-11, | 2.5e-11, -1.0e+00 | 0.07)</td></tr>
<tr><td>2</td><td>SeDuMi</td><td>(3.12500e-01, 3.12500e-01, | -9.5e-10, -8.4e-01 | 0.11)</td><td>(2.69356e-01, 2.69356e-01, | -1.7e-10, -9.3e-01 | 0.10)</td></tr>
<tr><td></td><td>SDPT3</td><td>(3.12500e-01, 3.12500e-01, | 9.7e-11, -8.4e-01 | 0.25)</td><td>(2.69356e-01, 2.69356e-01, | 1.1e-09, -9.3e-01 | 0.22)</td></tr>
<tr><td>3</td><td>SeDuMi</td><td>(7.41782e-01, 7.41782e-01, | -2.0e-11, -1.1e-09 | 0.16)</td><td>(3.06312e-01, 3.06312e-01, | -1.1e-09, -8.3e-01 | 0.15)</td></tr>
<tr><td></td><td>SDPT3</td><td>(7.41782e-01, 7.41782e-01, | 1.1e-09, 0.0e+00 | 0.29)</td><td>(3.06312e-01, 3.06312e-01, | 1.3e-10, -8.3e-01 | 0.25)</td></tr>
<tr><td>4</td><td>SeDuMi</td><td>(7.41782e-01, 7.41782e-01, | 1.1e-10, -1.5e-09 | 0.16)</td><td>(7.29855e-01, 7.29855e-01, | -1.2e-07, -4.9e-02 | 0.26)</td></tr>
<tr><td></td><td>SDPT3</td><td>(7.41782e-01, 7.41782e-01, | 1.3e-10, 0.0e+00 | 0.33)</td><td>(7.29855e-01, 7.29855e-01, | 2.9e-08, -4.9e-02 | 0.32)</td></tr>
<tr><td>5</td><td>SeDuMi</td><td>(7.41782e-01, 7.41782e-01, | 8.3e-11, -4.5e-10 | 0.20)</td><td>(7.36195e-01, 7.36194e-01, | -9.5e-07, -4.2e-02 | 0.34)</td></tr>
<tr><td></td><td>SDPT3</td><td>(7.41782e-01, 7.41782e-01, | 5.2e-10, 0.0e+00 | 0.57)</td><td>(7.36195e-01, 7.36195e-01, | 4.6e-08, -4.2e-02 | 0.40)</td></tr>
<tr><td>6</td><td>SeDuMi</td><td>(7.41782e-01, 7.41782e-01, | 2.3e-11, -6.1e-11 | 0.27)</td><td>(7.41782e-01, 7.41782e-01, | -1.0e-09, -6.6e-09 | 0.21)</td></tr>
<tr><td></td><td>SDPT3</td><td>(7.41782e-01, 7.41782e-01, | 1.5e-10, 0.0e+00 | 0.80)</td><td>(7.41782e-01, 7.41782e-01, | 3.0e-10, 0.0e+00 | 0.59)</td></tr>
</table>

3.3. Numerical results for nonconvex quadratic programs over the simplex

In this subsection, we consider the following nonconvex quadratic program over the simplex in $\mathbb{R}^n$:

$$\inf_{x \in \mathbb{R}^n} \left\{ x^T A x : f_i(x) := x_i \geq 0 \ (i = 1, \ldots, n), f_{n+1}(x) := 1 - \sum_{i=1}^n x_i = 0 \right\}, \tag{16}$$

where A is a symmetric matrix. The symmetric matrix A is said to be *copositive* if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n_+$. The problem (16) can be regarded as a problem of detecting copositivity; if the optimal value of this problem is nonnegative, then A is copositive.

We provide two numerical experiments on (16). In the former one, each A_{ij} is taken from the normal distribution $N(0, 1)$.

In the latter, we focus on checking copositivity. Note that if any of the diagonal element of A is negative, then A cannot be copositive. We deduce that all the matrices produced in the former experiment are not copositive, and this is in fact true. To produce nearly copositive matrices, we set all the coefficients of diagonal of A to be $\sqrt{n}/2$ and the other coefficients are chosen from $N(0, 1)$. In addition, since the positive semidefiniteness implies the copositivity and it is easy to check copositivity in this case, we filtered out positive semidefinite matrices and chose only non-positive semidefinite matrices.

In each experiment, we solve 30 problems generated randomly for every n, and apply

Lasserre's and Adaptive SOS relaxations with relaxation order $r = 2$.

Table 4. Information on SDP relaxation problems in Subsection 3.3 – General Case –

		Lasserre	Adaptive SOS	
n	Software	(#solved | min.t, ave.t, max.t)	(#solved | min.t, ave.t, max.t)	(minR, aveR, maxR)
5	SeDuMi	(30 | 0.08 0.11 0.16)	(30 | 0.07 0.10 0.14)	(1.0, 1.0, 1.0)
	SDPT3	(30 | 0.28 0.32 0.42)	(30 | 0.26 0.30 0.42)	(1.0, 1.0, 1.0)
10	SeDuMi	(28 | 0.19 0.26 0.38)	(27 | 0.13 0.17 0.24)	(1.0, 1.0, 1.0)
	SDPT3	(30 | 0.49 0.56 0.67)	(30 | 0.46 0.52 0.58)	(1.0, 1.0, 1.0)
15	SeDuMi	(28 | 0.66 0.97 1.36)	(23 | 0.34 0.50 0.64)	(1.0, 1.0, 1.0)
	SDPT3	(30 | 1.59 1.86 2.76)	(29 | 0.93 1.11 1.60)	(1.0, 1.0, 1.0)
20	SeDuMi	(24 | 4.33 5.74 7.40)	(20 | 2.21 2.97 3.92)	(1.0, 1.0, 1.0)
	SDPT3	(29 | 8.41 9.66 11.80)	(30 | 3.41 4.16 5.67)	(1.0, 1.0, 1.0)
25	SeDuMi	(17 | 31.53 44.80 59.16)	(18 | 12.91 17.69 23.78)	(1.0, 1.0, 1.0)
	SDPT3	(24 | 31.08 36.78 47.54)	(30 | 12.84 14.60 21.92)	(1.0, 1.0, 1.0)
30	SeDuMi	(14 | 258.23 355.42 458.14)	(9 | 130.48 174.64 220.04)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 79.23 92.97 141.83)	(29 | 34.72 47.19 58.35)	(1.0, 1.0, 1.0)

Table 5. Information on SDP relaxation problems in Subsection 3.3 – Detecting Copositivity –

		Lasserre	Adaptive SOS	
n	Software	(#solved | min.t, ave.t, max.t)	(#solved | min.t, ave.t, max.t)	(minR, aveR, maxR)
5	SeDuMi	(30 | 0.09 0.12 0.15)	(30 | 0.10 0.12 0.16)	(1.0, 1.0, 1.0)
	SDPT3	(30 | 0.29 0.33 0.43)	(30 | 0.27 0.31 0.40)	(1.0, 1.0, 1.0)
10	SeDuMi	(29 | 0.23 0.31 0.39)	(29 | 0.17 0.22 0.34)	(1.0, 1.0, 1.0)
	SDPT3	(29 | 0.53 0.68 0.86)	(29 | 0.42 0.57 0.81)	(1.0, 1.0, 1.0)
15	SeDuMi	(26 | 1.02 1.27 1.63)	(26 | 0.47 0.62 0.94)	(1.0, 1.0, 1.0)
	SDPT3	(30 | 1.65 2.53 3.01)	(29 | 1.08 1.43 1.86)	(1.0, 1.0, 1.0)
20	SeDuMi	(17 | 6.80 8.23 9.58)	(20 | 2.98 3.65 4.58)	(1.0, 1.0, 1.0)
	SDPT3	(30 | 8.61 11.08 13.05)	(29 | 3.80 5.59 7.34)	(1.0, 1.0, 1.0)
25	SeDuMi	(19 | 48.62 55.24 72.31)	(20 | 17.35 21.13 27.71)	(1.0, 1.0, 1.0)
	SDPT3	(19 | 30.71 38.17 47.01)	(28 | 16.47 20.10 24.73)	(1.0, 1.0, 1.0)
30	SeDuMi	(13 | 336.37 420.32 481.13)	(10 | 165.04 218.49 265.37)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 87.12 116.18 277.85)	(1 | 43.33 59.50 84.97)	(1.0, 1.0, 1.0)

Tables 4 and 5 show the numerical results by SeDuMi and SDPT3 for (16), respectively. We observe the following.

- SeDuMi and SDPT3 solve Adaptive SOS relaxation problems faster than Lasserre's because the sizes of Adaptive SOS relaxation problems are smaller than those of Lasserre's.
- SDPT3 cannot solve any problems with $n = 30$ by Lasserre's and Adaptive SOS relaxation although it terminates faster than SeDuMi. In particular, for almost all SDP relaxation problems, SDPT3 returns the message "stop: progress is bad" or "stop: progress is slow" and terminates. This means that it is difficult for SDPT3 to solve those SDP relaxation problems numerically. In contrast, SeDuMi terminates normally with no messages. In fact, it is not difficult to prove that the resulting Lasserre's and Adaptive SOS relaxation problems are not strictly feasible.
- The results of the two experiments are more or less similar to each other except the largest case $n = 30$ for SDPT3. In the general case, SDPT3 successfully solves 29 problems out of 30 for Adaptive SOS relaxation, while it cannot solve any problems for Lasserre's SDP relaxation.

3.4. Numerical results for higher order POP

We solve the following polynomial optimization problems:

$$\inf_{x\in\mathbb{R}^n} \{p(x) : x_i x_{i+1} \geq 0.5 \ (i = 1, \ldots, n-1), 0 \leq x_i \leq 1 \ (i = 1, \ldots, n)\}. \qquad (17)$$

Here, p is a homogeneous polynomial of degree three. All coefficients of p are chosen from the standard normal distribution $N(0, 1)$. By density of the objective function, we mean the ratio of nonzero coefficients in p. In this experiment, we vary the density between 0.2 and 0.8. We compare Adaptive SOS relaxation based on Theorem 4.2 with the sparse SDP relaxation [29] instead of Lasserre's. Indeed, when the density is small, (17) has correlatively sparse structure, and thus the sparse SDP relaxation is more effective than Lasserre's.

Table 6. Information on SDP relaxation problems in Subsection 3.4 with density 0.4

		Sparse	Adaptive SOS	
n	Software	(#solved | min.t, ave.t, max.t)	(#solved | min.t, ave.t, max.t)	(minR, aveR, maxR)
5	SeDuMi	(7 | 0.14 0.21 0.31)	(0 | 0.07 0.15 0.22)	(-0.3, 0.3, 0.7)
	SDPT3	(8 | 0.22 0.27 0.37)	(0 | 0.13 0.19 0.24)	(-0.3, 0.3, 0.7)
10	SeDuMi	(2 | 0.46 0.71 1.06)	(0 | 0.16 0.33 1.16)	(-0.1, 0.2, 0.6)
	SDPT3	(2 | 0.46 0.57 0.73)	(0 | 0.24 0.27 0.32)	(-0.1, 0.2, 0.6)
15	SeDuMi	(0 | 1.90 2.72 3.93)	(0 | 0.26 0.45 0.94)	(0.1, 0.2, 0.4)
	SDPT3	(0 | 1.35 1.68 2.28)	(0 | 0.27 0.33 0.40)	(0.1, 0.2, 0.4)
20	SeDuMi	(0 | 11.18 15.49 19.71)	(0 | 0.51 0.76 1.63)	(0.1, 0.2, 0.3)
	SDPT3	(0 | 5.07 6.37 7.64)	(0 | 0.34 0.41 0.55)	(0.1, 0.2, 0.3)
25	SeDuMi	(0 | 60.32 84.37 127.69)	(0 | 0.78 1.05 1.64)	(0.0, 0.1, 0.2)
	SDPT3	(0 | 16.22 19.52 33.66)	(0 | 0.54 0.67 0.79)	(0.0, 0.1, 0.2)
30	SeDuMi	(0 | 403.71 469.59 748.23)	(0 | 1.18 1.56 2.73)	(0.0, 0.1, 0.2)
	SDPT3	(0 | 87.15 96.84 124.65)	(0 | 0.90 1.08 1.30)	(0.0, 0.1, 0.2)

Table 7. Information on SDP relaxation problems in Subsection 3.4 with density 0.8

		Sparse	Adaptive SOS	
n	Software	(#solved | min.t, ave.t, max.t)	(#solved | min.t, ave.t, max.t)	(minR, aveR, maxR)
5	SeDuMi	(14 | 0.13 0.20 0.29)	(0 | 0.08 0.16 0.22)	(-0.0, 0.4, 0.8)
	SDPT3	(14 | 0.23 0.26 0.38)	(0 | 0.15 0.19 0.23)	(-0.0, 0.4, 0.8)
10	SeDuMi	(2 | 0.47 0.68 0.98)	(0 | 0.26 0.33 0.53)	(-0.1, 0.2, 0.4)
	SDPT3	(2 | 0.46 0.58 0.80)	(0 | 0.23 0.29 0.39)	(-0.1, 0.2, 0.4)
15	SeDuMi	(0 | 1.95 2.85 5.69)	(0 | 0.32 0.46 0.72)	(0.1, 0.2, 0.3)
	SDPT3	(0 | 1.44 1.84 2.66)	(0 | 0.30 0.35 0.51)	(0.1, 0.2, 0.3)
20	SeDuMi	(0 | 12.21 14.89 19.93)	(0 | 0.54 0.72 0.91)	(0.0, 0.1, 0.2)
	SDPT3	(0 | 5.61 6.42 7.99)	(0 | 0.40 0.45 0.54)	(0.0, 0.1, 0.2)
25	SeDuMi	(0 | 76.51 100.60 159.44)	(0 | 0.93 1.17 1.71)	(0.1, 0.1, 0.2)
	SDPT3	(0 | 17.23 19.38 22.94)	(0 | 0.63 0.74 0.93)	(0.1, 0.1, 0.2)
30	SeDuMi	(0 | 449.74 518.16 689.11)	(0 | 1.33 1.79 4.72)	(0.0, 0.1, 0.1)
	SDPT3	(0 | 85.79 98.59 115.40)	(0 | 1.00 1.19 1.37)	(0.0, 0.1, 0.1)

We observe the following from Tables 6 and 7.

- For small cases, $n = 5$ and 10, we observe that some ratios are negative. This is because the optimal values of Adaptive SOS relaxation for some problems are negative, while those of sparse SDP relaxation are positive. In contrast, for $n \geq 15$, the ratios are positive but not close to one.
- The sparse relaxation is much stronger than Adaptive SOS relaxation in this case. Adaptive SOS relaxation cannot solve any of the problems while the sparse relaxation solves several problems when $n = 5$. The ratio of the objective values obtained by the two relaxations is as low as 0.1 on average when n is large and the density is high.
- The computation time of Adaptive SOS relaxation is much smaller than that of the sparse relaxation. The difference is especially clear for larger n. The density of the objective function does not contribute to CPU times as the number of variables n does.

From this experiment, we can say that there are classes of POPs where Adaptive SOS relaxation can be applied to much larger instances than the sparse relaxation for computing reasonable bounds.

3.5. Numerical results for Bilinear matrix inequality eigenvalue problems

In this subsection, we solve bilinear matrix inequality eigenvalue problems.

$$\inf_{s\in \mathrm{R}, x\in \mathrm{R}^n, y\in \mathrm{R}^m} \left\{ s \,:\, sI_k - B_k(x,y) \in \mathrm{S}^k_+, x\in[0,1]^n, y\in[0,1]^m \right\}, \tag{18}$$

where we define for $k \in \mathrm{N}$, $x \in \mathrm{R}^n$ and $y \in \mathrm{R}^m$:

$$B_k(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} B_{ij}x_iy_j + \sum_{i=1}^{n} B_{i0}x_i + \sum_{j=1}^{m} B_{0j}y_j + B_{00},$$

where $B_{ij} (i = 0,\dots,n, j = 0,\dots,m)$ are $k \times k$ symmetric matrices. In this numerical experiment, each element of B_{ij} is chosen from the standard normal distribution $N(0,1)$. (18) is the problem of minimizing the maximum eigenvalue of $B_k(x,y)$ keeping $B_k(x,y)$ positive semidefinite.

We apply Lasserre's and Adaptive SOS relaxations with relaxation order $r = 1,2,3$. Tables 8 to 10 show the numerical results for BMIEP (18) with $k = 5,10,15$ by SeDuMi and SDPT3, respectively.

Table 8. Information on SDP relaxation problems in Subsection 3.5 with relaxation order 1

		Lasserre	Adaptive SOS	
(n,m,k)	Software	(#solved | min.t, ave.t, max.t)	(#solved | min.t, ave.t, max.t)	(minR, aveR, maxR)
(1, 1, 5)	SeDuMi	(16 | 0.04 0.06 0.09)	(16 | 0.04 0.06 0.09)	(1.0, 1.0, 1.0)
	SDPT3	(16 | 0.13 0.16 0.21)	(16 | 0.13 0.16 0.23)	(1.0, 1.0, 1.0)
(1, 1, 10)	SeDuMi	(10 | 0.05 0.07 0.09)	(10 | 0.05 0.07 0.09)	(1.0, 1.0, 1.0)
	SDPT3	(10 | 0.14 0.18 0.23)	(10 | 0.14 0.17 0.20)	(1.0, 1.0, 1.0)
(1, 1, 15)	SeDuMi	(15 | 0.06 0.08 0.11)	(15 | 0.06 0.08 0.11)	(1.0, 1.0, 1.0)
	SDPT3	(15 | 0.15 0.18 0.20)	(15 | 0.15 0.17 0.20)	(1.0, 1.0, 1.0)
(3, 3, 5)	SeDuMi	(0 | 0.06 0.08 0.09)	(0 | 0.07 0.08 0.10)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 0.17 0.21 0.26)	(0 | 0.17 0.20 0.28)	(1.0, 1.0, 1.0)
(3, 3, 10)	SeDuMi	(0 | 0.07 0.08 0.09)	(0 | 0.07 0.08 0.09)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 0.19 0.20 0.22)	(0 | 0.19 0.20 0.22)	(1.0, 1.0, 1.0)
(3, 3, 15)	SeDuMi	(0 | 0.07 0.10 0.12)	(0 | 0.07 0.10 0.12)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 0.21 0.23 0.27)	(0 | 0.21 0.23 0.27)	(1.0, 1.0, 1.0)
(5, 5, 5)	SeDuMi	(0 | 0.06 0.08 0.10)	(0 | 0.06 0.08 0.10)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 0.23 0.28 0.36)	(0 | 0.23 0.28 0.36)	(1.0, 1.0, 1.0)
(5, 5, 10)	SeDuMi	(0 | 0.08 0.10 0.13)	(0 | 0.08 0.11 0.15)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 0.27 0.32 0.37)	(0 | 0.27 0.31 0.37)	(1.0, 1.0, 1.0)
(5, 5, 15)	SeDuMi	(0 | 0.09 0.11 0.13)	(0 | 0.09 0.11 0.13)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 0.26 0.32 0.38)	(0 | 0.25 0.33 0.38)	(1.0, 1.0, 1.0)

We observe the following:

- When the relaxation order is 1, almost no difference exists between Lasserre's and Adaptive SOS relaxations. In fact, the SDP problems obtained by Adaptive SOS relaxation are the same as those of Lasserre's.
- When the relaxation order is 2, Lasserre's relaxation solves a few more problems than Adaptive SOS, consuming slightly more CPU time. When the relaxation order is 3, the difference between the two relaxations is now clear; when BMIEPs (18) are large, Lasserre's relaxation consumes significantly more time than Adaptive SOS, while Adaptive SOS gives as good bounds as Lasserre's relaxation does.
- It is apparent that SeDuMi is slower than SDPT3 at relaxation order $r = 3$ for large problems.

Table 9. Information on SDP relaxation problems in Subsection 3.5 with relaxation order 2

		Lasserre	Adaptive SOS	
(n, m, k)	Software	(#solved | min.t, ave.t, max.t)	(#solved | min.t, ave.t, max.t)	(minR, aveR, maxR)
(1, 1, 5)	SeDuMi	(20 | 0.07 0.12 0.17)	(21 | 0.08 0.11 0.14)	(1.0, 1.0, 1.1)
	SDPT3	(21 | 0.19 0.25 0.36)	(21 | 0.17 0.20 0.27)	(1.0, 1.0, 1.1)
(1, 1, 10)	SeDuMi	(14 | 0.10 0.14 0.18)	(12 | 0.10 0.12 0.16)	(1.0, 1.0, 1.0)
	SDPT3	(14 | 0.21 0.27 0.36)	(12 | 0.18 0.22 0.26)	(1.0, 1.0, 1.0)
(1, 1, 15)	SeDuMi	(20 | 0.11 0.14 0.19)	(19 | 0.10 0.13 0.18)	(1.0, 1.0, 1.0)
	SDPT3	(20 | 0.23 0.27 0.35)	(19 | 0.20 0.23 0.28)	(1.0, 1.0, 1.0)
(3, 3, 5)	SeDuMi	(0 | 0.34 0.41 0.53)	(0 | 0.17 0.21 0.24)	(0.6, 1.0, 1.4)
	SDPT3	(0 | 0.30 0.39 0.53)	(0 | 0.20 0.26 0.38)	(0.6, 1.0, 1.4)
(3, 3, 10)	SeDuMi	(2 | 0.36 0.44 0.54)	(0 | 0.19 0.22 0.28)	(1.0, 1.0, 1.1)
	SDPT3	(2 | 0.34 0.44 0.57)	(0 | 0.22 0.28 0.34)	(1.0, 1.0, 1.1)
(3, 3, 15)	SeDuMi	(1 | 0.36 0.44 0.52)	(1 | 0.18 0.24 0.37)	(1.0, 1.0, 1.0)
	SDPT3	(1 | 0.34 0.45 0.59)	(1 | 0.25 0.28 0.39)	(1.0, 1.0, 1.0)
(5, 5, 5)	SeDuMi	(0 | 0.63 0.79 1.13)	(0 | 0.31 0.37 0.46)	(-1.2, 1.6, 19.7)
	SDPT3	(0 | 0.65 0.88 1.14)	(0 | 0.25 0.29 0.33)	(-1.2, 1.6, 19.7)
(5, 5, 10)	SeDuMi	(0 | 0.66 0.82 1.03)	(0 | 0.32 0.37 0.43)	(1.0, 1.0, 1.1)
	SDPT3	(0 | 0.73 0.91 1.21)	(0 | 0.29 0.31 0.37)	(1.0, 1.0, 1.1)
(5, 5, 15)	SeDuMi	(0 | 0.73 0.86 1.22)	(0 | 0.35 0.40 0.52)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 0.80 1.04 1.27)	(0 | 0.31 0.36 0.43)	(1.0, 1.0, 1.0)

Table 10. Information on SDP relaxation problems in Subsection 3.5 with relaxation order 3

		Lasserre	Adaptive SOS	
(n, m, k)	Software	(#solved | min.t, ave.t, max.t)	(#solved | min.t, ave.t, max.t)	(minR, aveR, maxR)
(1, 1, 5)	SeDuMi	(22 | 0.09 0.14 0.28)	(21 | 0.08 0.15 0.22)	(1.0, 1.0, 1.0)
	SDPT3	(22 | 0.26 0.34 0.49)	(20 | 0.24 0.30 0.40)	(1.0, 1.0, 1.0)
(1, 1, 10)	SeDuMi	(14 | 0.10 0.16 0.23)	(12 | 0.12 0.17 0.28)	(1.0, 1.0, 1.0)
	SDPT3	(14 | 0.28 0.38 0.48)	(12 | 0.24 0.34 0.45)	(1.0, 1.0, 1.0)
(1, 1, 15)	SeDuMi	(20 | 0.11 0.17 0.28)	(19 | 0.10 0.17 0.26)	(1.0, 1.0, 1.0)
	SDPT3	(20 | 0.30 0.35 0.49)	(19 | 0.26 0.33 0.46)	(1.0, 1.0, 1.0)
(3, 3, 5)	SeDuMi	(2 | 2.36 3.35 5.01)	(0 | 0.55 0.73 0.99)	(0.9, 1.0, 1.1)
	SDPT3	(1 | 2.60 3.71 5.08)	(0 | 0.51 0.75 0.98)	(0.9, 1.0, 1.1)
(3, 3, 10)	SeDuMi	(2 | 2.82 3.73 4.61)	(0 | 0.57 0.74 0.91)	(1.0, 1.0, 1.0)
	SDPT3	(2 | 3.17 4.12 5.11)	(1 | 0.55 0.75 0.95)	(1.0, 1.0, 1.0)
(3, 3, 15)	SeDuMi	(1 | 2.29 3.88 4.81)	(1 | 0.41 0.74 0.93)	(1.0, 1.0, 1.0)
	SDPT3	(1 | 3.05 4.45 5.80)	(1 | 0.54 0.78 0.94)	(1.0, 1.0, 1.0)
(5, 5, 5)	SeDuMi	(0 | 141.74 226.92 289.11)	(0 | 7.00 9.77 12.77)	(0.1, 1.0, 1.7)
	SDPT3	(0 | 118.91 167.82 210.19)	(0 | 2.71 3.86 4.73)	(0.1, 1.0, 1.7)
(5, 5, 10)	SeDuMi	(0 | 172.47 286.46 357.09)	(0 | 7.88 9.90 12.78)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 130.71 195.31 233.59)	(0 | 3.47 3.99 4.68)	(1.0, 1.0, 1.0)
(5, 5, 15)	SeDuMi	(0 | 250.93 305.07 376.59)	(0 | 7.59 10.16 13.27)	(1.0, 1.0, 1.0)
	SDPT3	(0 | 162.89 206.92 247.36)	(0 | 3.39 4.17 5.33)	(1.0, 1.0, 1.0)

4. Extensions

In this section, we give three extensions of Theorem 2.1 and present some related work to Theorem 2.1.

4.1. Sums of squares of rational polynomials

We can extend part i. of Theorem 2.1 with sums of squares of rational polynomials. We assume that for all $j = 1, \ldots, m$, there exists $g_j \in \mathrm{R}[x]$ such that $|f_j(x)| \leq g_j(x)$ and $g_j(x) \neq 0$ for all $x \in B$. We define

$$\tilde{\psi}_r(x) = -\sum_{j=1}^{m} f_j(x) \left(1 - \frac{f_j(x)}{g_j(x)}\right)^{2r}$$

for all $r \in \mathrm{N}$. Then, we can prove the following corollary by using almost the same arguments as those used for Theorem 2.1.

Corollary 4.1 *Suppose that for $\rho \in \mathbb{R}$, $f(x) - \rho > 0$ for every $x \in \bar{K}$, i.e., ρ is a lower bound of f^*. Then there exists $\tilde{r} \in \mathbb{N}$ such that for all $r \geq \tilde{r}$, $f - \rho + \tilde{\psi}_r$ is positive over B.*

It is difficult to apply Corollary 4.1 to the framework of SDP relaxations because we deal with rational polynomials in $\tilde{\psi}_r$. However, we may be able to reduce the degrees of sums of squares in $\tilde{\psi}_r$ by using Corollary 4.1. For instance, we consider $f_1(x) = 1 - x^4$ and $B = [-1, 1]$. Choose $g_1(x) = 2(1+x^2)$. Then g_1 dominates $|f_1|$ over B, *i.e.*, $|f_1(x)| \leq g_1(x)$ for all $x \in B$. We have

$$\tilde{\psi}_r(x) = -(1-x^4)\left(1 - \frac{1-x^4}{2(1+x^2)}\right)^{2r} = -(1-x^4)\left(1 - \frac{1-x^2}{2}\right)^{2r},$$

and the degree of $\tilde{\psi}$ in Corollary 4.1 is $4r$, while the degree of ψ in Theorem 2.1 is $8r$.

4.2. Extension to POP with correlative sparsity

In [29], the authors introduced the notion of correlative sparsity for POP (1), and proposed a sparse SDP relaxation that exploits the correlative sparsity. They then demonstrated that the sparse SDP relaxation outperforms Lasserre's SDP relaxation. The sparse SDP relaxation is implemented in [30] and its source code is freely available.

We give some of the definitions of the correlative sparsity for POP (1). For this, we use an $n \times n$ symbolic symmetric matrix R, whose elements are either 0 or $\star$ representing a nonzero value. We assign either 0 or $\star$ as follows:

$$R_{k,\ell} = \begin{cases} \star \text{ if } k = \ell, \\ \star \text{ if } \alpha_k \geq 1 \text{ and } \alpha_\ell \geq 1 \text{ for some } \alpha \in \mathcal{F}, \\ \star \text{ if } x_k \text{ and } x_\ell \text{ are involved in the polynomial } f_j \text{ for some } j = 1, \dots, m, \\ 0 \text{ o.w.} \end{cases}$$

POP (1) is said to be *correlatively sparse* if the matrix R is sparse.

We give some of the details of the sparse SDP relaxation proposed in [29] for the sake of completeness. We construct an undirected graph $G = (V, E)$ from R. Here $V := \{1, \dots, n\}$ and $E := \{(k, \ell) : R_{k,\ell} = \star\}$. After applying the chordal extension to $G = (V, E)$, we generate all maximal cliques $C_1, \dots, C_p$ of the extension $G = (V, \tilde{E})$ with $E \subseteq \tilde{E}$. See [5, 29] and references therein for the details of the construction of the chordal extension. For a finite set $C \subseteq \mathbb{N}$, x_C denotes the subvector that consists of x_i $(i \in C)$. For all $f_1, \dots, f_m$ in POP (1), F_j denotes the set of indices whose variables are involved in f_j, *i.e.*, $F_j := \{i \in \{1, \dots, n\} : \alpha_i \geq 1 \text{ for some } \alpha \in \mathcal{F}_j\}$. For a finite set $C \subseteq \mathbb{N}$, the sets $\Sigma_{r,C}$ and $\Sigma_{\infty,C}$ denote the subsets of Σ_r as follows:

$$\Sigma_{r,C} := \left\{\sum_{k=1}^{q} g_k(x)^2 : \forall k = 1, \dots, q, g_k \in \mathbb{R}[x_C]_r\right\},$$
$$\Sigma_{\infty,C} := \bigcup_{r \geq 0} \Sigma_{r,C}.$$

Note that if $C = \{1, \dots, n\}$, then we have $\Sigma_{r,C} = \Sigma_r$ and $\Sigma_{\infty,C} = \Sigma$. The sparse SDP relaxation problem with relaxation order r for POP (1) is obtained from the following

SOS relaxation problem:

$$\rho_r^{\text{sparse}} := \sup \left\{ \rho : \begin{array}{l} f - \rho = \sum_{h=1}^p \sigma_{0,h} + \sum_{j=1}^m \sigma_j f_j, \\ \sigma_{0,h} \in \Sigma_{r,C_h} \ (h = 1, \ldots, p), \sigma_j \in \Sigma_{r_j,D_j} \ (j = 1, \ldots, m) \end{array} \right\}, \tag{19}$$

where D_j is the union of some of the maximal cliques $C_1, \ldots, C_p$ such that $F_j \subseteq C_h$ and $r_j = r - \lceil \deg(f_j)/2 \rceil$ for $j = 1, \ldots, m$.

It should be noted that other sparse SDP relaxations have been proposed in [9, 19, 22] and the asymptotic convergence is proved. In contrast, the convergence of the sparse SDP relaxation (19) is not shown in [29].

We give an extension of Theorem 2.1 to POP with correlative sparsity. If $C_1, \ldots, C_p \subseteq \{1, \ldots, n\}$ satisfy the following property, we refer this property as *the running intersection property* (RIP):

$$\forall h \in \{1, \ldots, p-1\}, \exists t \in \{1, \ldots, p\} \text{ such that } C_{h+1} \cap (C_1 \cup \cdots \cup C_h) \subsetneq C_t.$$

For $C_1, \ldots, C_p \subseteq \{1, \ldots, n\}$, we define sets $J_1, \ldots, J_p$ as follows:

$$J_h := \{ j \in \{1, \ldots, m\} : f_j \in \mathrm{R}[x_{C_h}] \}.$$

Clearly, we have $\cup_{h=1}^p J_h = \{1, \ldots, m\}$. In addition, we define

$$\psi_{r,h}(x) := - \sum_{j \in J_h} f_j(x) \left(1 - \frac{f_j(x)}{R_j} \right)^{2r},$$

$$\Theta_{r,h,b}(x) := 1 + \sum_{i \in C_h} \left(\frac{x_i}{b} \right)^{2r}$$

for $h = 1, \ldots, p$.

Using a proof similar to the one for the theorem on convergence of the sparse SDP relaxation given in [9], we can establish the correlatively sparse case of Theorem 2.1. Indeed, we can obtain the next theorem by using [9, Lemma 4] and Theorem 2.1.

Theorem 4.2 *Assume that nonempty sets $C_1, \ldots, C_p \subseteq \{1, \ldots, n\}$ satisfy (RIP) and we can decompose f into $f = \hat{f}_1 + \cdots + \hat{f}_p$ with $\hat{f}_h \in \mathrm{R}[x_{C_h}]$ $(h = 1, \ldots, p)$. Under the assumptions of Theorem 2.1, there exists $\tilde{r} \in \mathrm{N}$ such that for all $r \geq \tilde{r}$, $f - \rho + \sum_{h=1}^p \psi_{r,h}$ is positive over $B = [-b, b]^n$. In addition, for every $\epsilon > 0$, there exists $\hat{r} \in \mathrm{N}$ such that for all $r \geq \hat{r}$,*

$$f - \rho + \epsilon \sum_{h=1}^p \Theta_{r,h,b} + \sum_{h=1}^p \psi_{\tilde{r},h} \in \Sigma_{\infty,C_1} + \cdots + \Sigma_{\infty,C_p}. \tag{20}$$

Note that if $p = 1$, *i.e.*, $C_1 = \{1, \ldots, n\}$, then we have $\psi_{r,1} = \psi_r$ and $\Theta_{r,1,b} = \Theta_{r,b}$, and thus Theorem 4.2 is reduced to Theorem 2.1. Therefore, we will concentrate our effort to prove Theorem 4.2 in the following. In addition, it would follow from [9, Theorem 5] that (20) holds without the polynomial $\epsilon \sum_{h=1}^p \Theta_{r,h,b}$ if we assume that all quadratic modules generated by f_j $(j \in C_h)$ for all $h = 1, \ldots, p$ are archimedean.

To prove Theorem 4.2, we use Lemma 4 in [9] and Corollary 3.3 of [21].

Lemma 4.3 *(modified version of [9, Lemma 4]) Assume that we decompose f into $f = \hat{f}_1 + \cdots + \hat{f}_p$ with $\hat{f}_h \in \mathrm{R}[x_{C_h}]$ and $f > 0$ on K. Then, for any bounded set $B \subseteq \mathrm{R}^n$,* *there exist* $\tilde{r} \in \mathrm{N}$ *and* $g_h \in \mathrm{R}[x_{C_h}]$ *with* $g_h > 0$ *on* B *such that for every* $r \geq \tilde{r}$*,*

$$f = -\sum_{h=1}^{p} \psi_{r,h} + \sum_{h=1}^{p} g_h.$$

Remark 1 The original statement in [9, Lemma 4] is slightly different from Lemma 4.3. In the original statement, it is proved that there exists $\lambda \in (0,1]$, $\tilde{r} \in \mathrm{N}$ and $g_h \in \mathrm{R}[x_{C_h}]$ with $g_h > 0$ on B such that

$$f = \sum_{h=1}^{p} \sum_{j \in J_h} (1 - \lambda f_j)^{2\tilde{r}} f_j + \sum_{h=1}^{p} g_h.$$

In Appendix A, we establish the correctness of Lemma 4.3 by using [9, Lemma 4].

Lemma 4.4 *(Corollary 3.3 of [21]) Let* $f \in \mathrm{R}[x]$ *be a polynomial nonnegative on* $[-1,1]^n$*. For arbitrary* $\epsilon > 0$*, there exists some* $\hat{r}$ *such that for every* $r \geq \hat{r}$*, the polynomial* $(f + \epsilon\Theta_r)$ *is a SOS.*

Proof of Theorem 4.2 : We may choose $[-b,b]^n$ as B in Lemma 4.3. It follows from the assumption in Theorem 4.2 that we can decompose $f - \rho$ into $(\hat{f}_1 - \rho) + \hat{f}_2 + \cdots + \hat{f}_p$. Since $\hat{f}_1 - \rho \in \mathrm{R}[x_{C_1}]$, it follows from Lemma 4.3 that there exist $\tilde{r} \in \mathrm{N}$ and $g_h \in \mathrm{R}[x_{C_h}]$ with $g_h > 0$ on B such that for every $r \geq \tilde{r}$,

$$f - \rho = (\hat{f}_1 - \rho) + \hat{f}_2 + \cdots + \hat{f}_p = -\sum_{h=1}^{p} \psi_{r,h} + \sum_{h=1}^{p} g_h.$$

Therefore, the polynomial $f - \rho + \sum_{h=1}^{p} \psi_{r,h}$ is positive on B for all $r \geq \tilde{r}$.

For simplicity, we fix h and define $C_h = \{c_1, \ldots, c_k\}$. Then, g_h consists of the k variables $x_{c_1}, \ldots, x_{c_k}$. Since $g_h > 0$ on B, it is also positive on $B' := \{(x_{c_1}, \ldots, x_{c_k}) : -b \leq x_{c_j} \leq b\ (j = 1, \ldots, k)\}$. We define $\hat{g}_h(y) = g_h(by)$. Since g_h is positive on B', $\hat{g}_h \in \mathrm{R}[y_{c_1}, \ldots, y_{c_k}]$ is also positive on the set $\{(y_{c_1}, \ldots, y_{c_k}) : -1 \leq y_{c_j} \leq 1\ (j = 1, \ldots, k)\}$. Applying Lemma 4.4 to $\hat{g}_h$, for all $\epsilon > 0$, there exists $\hat{r}_h \in \mathrm{N}$ such that for every $r \geq \hat{r}_h$,

$$\hat{g}_h(y_{c_1}, \ldots, y_{c_k}) + \epsilon \sum_{i=1}^{k} y_{c_i}^{2r} = \sigma_h(y_{c_1}, \ldots, y_{c_k})$$

for some $\sigma_h \in \Sigma_{\infty, C_h}$. Substituting $x_{c_1} = by_{c_1}, \ldots, x_{c_k} = by_{c_k}$, we obtain

$$g_h + \epsilon\Theta_{r,h,b} \in \Sigma_{\infty, C_h}.$$

We fix $\epsilon > 0$. Applying the above discussion to all $h = 1, \ldots, p$, we obtain the numbers $\hat{r}_1, \ldots, \hat{r}_p$. We denote the maximum over $\hat{r}_1, \ldots, \hat{r}_p$ by $\hat{r}$. Then, we have

$$f - \rho + \epsilon \sum_{h=1}^{p} \Theta_{r,h,b} + \sum_{h=1}^{p} \psi_{\tilde{r},h} \in \Sigma_{\infty, C_1} + \cdots + \Sigma_{\infty, C_p}$$

for every $r \geq \hat{r}$. □

4.3. Extension to POP with symmetric cones

In this subsection, we extend Theorem 2.1 to POP over symmetric cones, *i.e.*,

$$f^* := \inf_{x \in \mathrm{R}^n} \{f(x) : G(x) \in \mathcal{E}_+\}, \tag{21}$$

where $f \in \mathrm{R}[x]$, $\mathcal{E}_+$ is a symmetric cone associated with an N-dimensional Euclidean Jordan algebra $\mathcal{E}$, and G is $\mathcal{E}$-valued polynomial in x. The feasible region K of POP (21) is $\{x \in \mathrm{R}^n : G(x) \in \mathcal{E}_+\}$. Note that if $\mathcal{E}$ is R^m and $\mathcal{E}_+$ is the nonnegative orthant R^m_+, then (21) is identical to (1). In addition, S^n_+, the cone of $n \times n$ symmetric positive semidefinite matrices, is a symmetric cone, the bilinear matrix inequalities can be formulated as (21).

To construct ψ_r for (21), we introduce some notation and symbols. The Jordan product and inner product of $x, y \in \mathcal{E}$ are denoted by $x \circ y$ and $x \bullet y$, respectively. Let e be the identity element in the Jordan algebra $\mathcal{E}$. For any $x \in \mathcal{E}$, we have $e \circ x = x \circ e = x$. We can define eigenvalues for all elements in the Jordan algebra $\mathcal{E}$, generalizing those for Hermitian matrices. See [4] for the details. We construct ψ_r for (21) as follows:

$$\begin{aligned} M &:= \sup\left\{\text{maximum absolute eigenvalue of } G(x) : x \in \bar{K}\right\}, \\ \psi_r(x) &:= -G(x) \bullet \left(e - \frac{G(x)}{M}\right)^{2r}, \end{aligned} \tag{22}$$

where we define $x^k := x^{k-1} \circ x$ for $k \in \mathrm{N}$ and $x \in \mathcal{E}$.

Lemma 4 in [16] shows that ψ_r defined in (22) has the same properties as ψ_r in Theorem 2.1.

THEOREM 4.5 *For a given ρ, suppose that $f(x) - \rho > 0$ for every $x \in \bar{K}$. Then, there exists $\tilde{r} \in \mathrm{N}$ such that for all $r \geq \tilde{r}$, $f - \rho + \psi_r$ is positive over B. Moreover, for any $\epsilon > 0$, there exists $\hat{r} \in \mathrm{N}$ such that for every $r \geq \hat{r}$,*

$$f - \rho + \epsilon\Theta_{r,b} + \psi_{\tilde{r}} \in \Sigma.$$

4.4. Another perturbed sums of squares theorem

In this subsection, we present another perturbed sums of squares theorem for POP (1) which is obtained by combining some of the results in [14, 18].

To use the result in [14], we introduce some notation and symbols. We assume that $K \subseteq B := [-b, b]^n$. We choose $\gamma \geq 1$ such that for all $j = 0, 1, \ldots, m$,

$$\begin{aligned} |f_j(x)/\gamma| &\leq 1 \text{ if } \|x\|_\infty \leq \sqrt{2}b, \\ |f_j(x)/\gamma| &\leq \|x/b\|_\infty^d \text{ if } \|x\|_\infty \geq \sqrt{2}b, \end{aligned}$$

where f_0 denotes the objective function f in POP (1), and $d = \max\{\deg(f), \deg(f_1), \ldots, \deg(f_m)\}$. For $r \in \mathrm{N}$, we define

$$\begin{aligned} \psi_r(x) &:= -\sum_{j=1}^m \left(1 - \frac{f_j(x)}{\gamma}\right)^{2r} f_j(x), \\ \phi_{r,b}(x) &:= -\frac{(m+2)\gamma}{b^2} \sum_{i=1}^n \left(\frac{x_i}{b}\right)^{2d(r+1)} (b^2 - x_i^2). \end{aligned}$$

From (a), (b) and (c) of Lemma 3.2 in [14], we obtain the following result:

Proposition 4.6 *Assume that the feasible region K of POP (1) is contained in $B = [-b, b]^n$. In addition, we assume that for $\rho \in \mathbb{R}$, we have $f - \rho > 0$ over K. Then there exists $\tilde{r} \in \mathbb{N}$ such that for all $r \geq \tilde{r}$, $(f - \rho + \psi_r + \phi_{r,b})$ is positive over $\mathbb{R}^n$.*

We do not need to impose the assumption on the compactness of K in Proposition 4.6. Indeed, we can drop it by replacing K by $\bar{K}$ defined in Subsection 2.1 as in Theorem 2.1.

Next, we describe a result from [18] which is useful in deriving another perturbed sums of squares theorem.

Theorem 4.7 *((iii) of Theorem 4.1 in [18]) Let $f \in \mathbb{R}[x]$ be a nonnegative polynomial. Then for every $\epsilon > 0$, there exists $\hat{r} \in \mathbb{N}$ such that for all $r \geq \hat{r}$,*

$$f + \epsilon\theta_r \in \Sigma,$$

where $\theta_r(x) := \sum_{i=1}^n \sum_{k=0}^r (x_i^{2k}/k!)$.

By incorporating Proposition 4.6 with Theorem 4.7, we obtain yet another perturbation theorem.

Theorem 4.8 *We assume that for $\rho \in \mathbb{R}$, we have $f - \rho > 0$ over K. Then we have*

i there exists $\tilde{r} \in \mathbb{N}$ such that for all $r \geq \tilde{r}$, $(f - \rho + \psi_r + \phi_{r,b})$ is positive over $\mathbb{R}^n$;
ii moreover, for every $\epsilon > 0$, there exists $\hat{r} \in \mathbb{N}$ such that for all $r \geq \hat{r}$,

$$(f - \rho + \psi_{\tilde{r}} + \phi_{\tilde{r},b} + \epsilon\theta_r) \in \Sigma.$$

We give an SDP relaxation analogous to (5), based on Theorem 4.8, as follows:

$$\eta(\epsilon, \tilde{r}, r) := \sup \left\{ \eta : \begin{array}{l} f - \eta + \epsilon\theta_r - \sum_{j=1}^m f_j\sigma_j - \sum_{i=1}^n (b^2 - x_i^2)\mu_i = \sigma_0, \\ \sigma_0 \in \Sigma_r, \sigma_j \in \Sigma(\tilde{r}\tilde{\mathcal{F}}_j), \mu_i \in \Sigma(\{d(\tilde{r}+1)e_i\}) \end{array} \right\}, \tag{23}$$

for some $r \geq \tilde{r}$, where e_i is the ith standard unit vector in $\mathbb{R}^n$. One of the differences between (5) and (23) is that (23) has n SOS variables $\mu_1, \ldots, \mu_n$. These variables correspond to nonnegative variables in the SDP formulation, but not positive semidefinite matrices, since these consist of a single monomial. On the other hand, it is difficult to estimate $\tilde{r}$ in the SDP relaxations (5) and (23), and thus we could not compare the size and the quality of the optimal value of (5) with (23) so far.

Next, we mention a result similar to Theorem 2.2. We omit the proof because we obtain the inequalities by applying a proof technique similar to that of Theorem 2.2.

Theorem 4.9 *For every $\epsilon > 0$, there exists $r, \tilde{r} \in \mathbb{N}$ such that $f^* - \epsilon \leq \eta(\epsilon, \tilde{r}, r) \leq f^* + \epsilon n e^{b^2}$.*

5. Concluding Remarks

We mention some other research related to our work in the context of Theorem 2.1. A common element in all of these approaches is to use perturbations $\epsilon\theta_r(x)$ or $\epsilon\Theta_r(x)$ for finding an approximate solution of a given POP.

In [10, 12], the authors added $\epsilon\Theta_r(x)$ to the objective function of a given unconstrained POP and used algebraic techniques to find a solution. In [13], the following equality constraints were added in the perturbed unconstrained POP and Lasserre's SDP relaxation was applied to the new POP:

$$\frac{\partial f_0}{\partial x_i} + 2r\epsilon x_i^{2r-1} = 0 \ (i = 1, \dots, n).$$

Lasserre in [20] proposed an SDP relaxation via $\theta_r(x)$ defined in Theorem 4.7 and a perturbation theorem for semi-algebraic set defined by equality constraints $g_k(x) = 0$ $(k = 1, \dots, m)$. The SDP relaxation can be applied to the following equality constrained POP:

$$\inf_{x \in \mathbb{R}^n} \left\{ f_0(x) : g_k(x) = 0 \ (k = 1, \dots, m) \right\}; \tag{24}$$

To obtain the SDP relaxations, $\epsilon\theta_r(x)$ is added to the objective function in POP (24) and the equality constraints in POP (24) are replaced by $g_k^2(x) \leq 0$. In the resulting SDP relaxations, $\theta_r(x)$ is explicitly introduced and variables associated with constraints $g_k^2(x) \leq 0$ are not positive semidefinite matrices, but nonnegative variables.

In this paper, we present a perturbed SOS theorem (Theorem 2.1) and its extensions, and propose a new sparse relaxation called Adaptive SOS relaxation. During the course of the paper, we have shed some light on why Lasserre's SDP relaxation calculates the optimal value of POP even if its SDP relaxation has a different optimal value. The numerical experiments show that there exist classes of POPs for which Adaptive SOS relaxation consumes significantly less CPU time compared to the sparse or Lasserre's relaxations, giving reasonable bounds. Therefore, we conclude that Adaptive SOS relaxation is promising, justifying the need for future research in this direction.

Of course, if the original POP is dense, i.e., $\tilde{F}_j$ contains many elements for almost all j, then the proposed relaxation has little effect in reducing the SDP relaxation. However, in real applications, such cases seem rare.

In the numerical experiments, the behaviors of SeDuMi and SDPT3 are sometimes very different. See, for example, Tables 4 and 5. In the column of Adaptive SOS of Table 4, SDPT3 solved significantly more problems than SeDuMi. On the other hand, for the largest problems in Table 5, SeDuMi obviously outperforms SDPT3. This is why we present the results of both solvers in every table. In solving a real problem, one should be very careful when choosing the appropriate SDP solver for the problem at hand.

Acknowledgements

We thank two anonymous referees for thier valuable comments to improve the paper. The first author was supported in part by JSPS KAKENHI Grant Numbers 19560063 and 26330025. The second author was supported by JSPS KAKENHI Grant Numbers 22740056 and 26400203. The third author was supported in part by a Discovery Grant from NSERC, a research grant from University of Waterloo and by ONR research grant N00014-12-10049.

Appendix A. Proof of Lemma 4.3

As we have already mentioned in Remark 1, Lemma 4.3 is slightly different from the original one in [9, Lemma 4]. To show the correctness of Lemma 4.3, we use the following lemma:

LEMMA A.1 *([9, Lemma 3]) Let $B \subseteq \mathrm{R}^n$ be a compact set. Assume that nonempty sets $C_1, \ldots, C_p \subseteq \{1, \ldots, n\}$ satisfy (RIP) and we can decompose f into $f = \hat{f}_1 + \cdots + \hat{f}_p$ with $\hat{f}_h \in \mathrm{R}[x_{C_h}]$ $(h = 1, \ldots, p)$. In addition, suppose that $f > 0$ on B. Then there exists $g_h \in \mathrm{R}[x_{C_h}]$ with $g_h > 0$ on B such that*

$$f = g_1 + \cdots + g_p.$$

We can prove Lemma 4.3 in a manner similar to [9, Lemma 4]. We define $F_r : \mathrm{R}^n \to \mathrm{R}$ as follows:

$$F_r = f - \sum_{h=1}^{p} \psi_{r,h}.$$

We recall that $\psi_{r,h} = \sum_{j \in C_h} (1 - f_j/R_j)^{2r} f_j$ for all $h = 1, \ldots, p$ and $r \in \mathrm{N}$, and that R_j is the maximum value of $|f_j|$ on B for all $j = 1, \ldots, m$. It follows from the definitions of $\psi_{r,h}$ and R_j that we have $\psi_{r,h} \geq \psi_{r+1,h}$ on B for all $h = 1, \ldots, p$ and $r \in \mathrm{N}$, and thus we have $F_r \leq F_{r+1}$ on B. In addition, we can prove that (i) on $B \cap K$, $F_r \to f$ as $r \to \infty$, and (ii) on $B \setminus K$, $F_r \to \infty$ as $r \to \infty$. Since B is compact, it follows from (i), (ii) and the positiveness of f on B that there exists $\tilde{r} \in \mathrm{N}$ such that for every $r \geq \tilde{r}$, $F_r > 0$ on B. Applying Lemma A.1 to F_r, we obtain the desired result.

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