

Invariance and Efficiency of Convex Representations

Chek Beng Chua* Levent Tunçel†

C & O Research Report: CORR 2004-18
June 27, 2004 (revised: December 31, 2005)

Abstract

We consider two notions for the representations of convex cones: G -representation and lifted- G -representation. The former represents a convex cone as a slice of another; the latter allows in addition, the usage of auxiliary variables in the representation. We first study the basic properties of these representations. We show that some basic properties of convex cones are invariant under one notion of representation but not the other. In particular, we prove that lifted- G -representation is closed under duality when the representing cone is self-dual. We also prove that strict complementarity of a convex optimization problem in conic form is preserved under G -representations. Then we move to study efficiency measures for representations. We evaluate the representations of homogeneous convex cones based on the “smoothness” of the transformations mapping the central path of the representation to the central path of the represented optimization problem.

Keywords: convex optimization, semidefinite programming, semidefinite representations, interior-point methods, central path, weighted central path, homogeneous cones, duality theory, strict complementarity

AMS Subject Classification: 90C22, 90C51, 90C25, 52A41

* (cbchua@math.uwaterloo.ca) Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada. Research of this author was supported in part by a grant from the Faculty of Mathematics, University of Waterloo and by a Discovery Grant from NSERC.

† (ltuncel@math.uwaterloo.ca) Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada. Research of this author was supported in part by a Discovery Grant from NSERC and a PREA from Ontario, Canada.

1 Introduction, Motivation and Definitions

Problem formulation is one of the most important stages in the theory and practice of mathematical programming. Here we focus on convex optimization problems formulated as the optimization of a real-valued linear function over the intersection of a convex cone and an affine subspace of a finite dimensional Euclidean space. We call these problems *convex optimization problems in conic form* or *convex conic programming problems*. This formulation encompasses essentially all convex optimization problems on finite dimensional spaces. While this may not always be the best way to attack a given convex optimization problem, much of the modern theory focused on this model partly for expository reasons. In this paper, we study formulations (representations) of a given subclass of convex optimization problems in conic form as instances of another subclass.

Some convex conic programming problems and underlying convex cones are better understood than others. Therefore, there is a way of measuring the relative *easiness* of convex optimization problems based on the current state-of-the-art. For instance, there are many efficient and robust software for semidefinite programming. Then the results about representing a convex set as the intersection of a cone of symmetric, positive semidefinite matrices and an affine subspace, raise the hope of solving convex optimization problems involving such convex constraints by using the standard software. More interestingly, this opens the ways of thinking about specialized solution methods or theories to be developed for the new problem along the lines of those existing for semidefinite programming. From a more theoretical point of view, the representations can help us understand the *hierarchies* and the structures of convex optimization problems from easy to difficult.

Let \mathbb{S}^n denote the space of n -by- n symmetric matrices equipped with the inner product $\langle \cdot, \cdot \rangle : (U, V) \mapsto \text{tr}U^T V = \text{tr}UV$, where U^T denotes the transpose of U and $\text{tr}(\cdot)$ denotes the trace. Denote the (i, j) th entry of each $X \in \mathbb{S}^n$ by X_{ij} .

Let $\mathbb{S}_+^n \subset \mathbb{S}^n$ denote the cone of n -by- n symmetric, positive semidefinite matrices and let $\mathbb{S}_{++}^n := \text{int}(\mathbb{S}_+^n)$, where $\text{int}(\cdot)$ denotes interior. We also use the partial order notation. For $A, B \in \mathbb{S}^n$, we write $A \succeq B$ ($A \succ B$) to mean $(A - B)$ is positive semidefinite (positive definite). The *second-order cone* in \mathbb{R}^{n+1} is the cone

$$Q^n := \{(x_0, x) \in \mathbb{R} \oplus \mathbb{R}^n : x_0 \geq \|x\|_2\}.$$

Throughout this paper, let $K \subset \mathbb{R}^d$ be a pointed, closed, convex cone with nonempty interior.

Definition 1.1. *$K \subset \mathbb{R}^d$ is said to admit a G -representation via \mathcal{L} if $G \subset \mathbb{R}^N$ is a pointed, closed, convex cone with nonempty interior and $\mathcal{L} : \mathbb{R}^d \rightarrow \mathbb{R}^N$ is a linear map such that*

$$x \in \text{int}(K) \iff \mathcal{L}(x) \in \text{int}(G).$$

The relation

$$\mathcal{L}(x) \in G$$

is called a G -representation.

If \mathcal{G} is a collection of pointed, closed, convex cones, and \mathcal{P} is the subclass of convex conic programming problems with underlying cones in \mathcal{G} , then we say that a pointed, closed, convex cone $K \subset \mathbb{R}^d$ is \mathcal{P} -representable if it admits a G -representation for some $G \in \mathcal{G}$.

Especially when we deal with the cones \mathbb{S}_+^n and Q^n , we abuse these definitions slightly and talk about the *SDP-representation* and *SOCP-representation* of convex cones and sets (instead of \mathcal{G} -representation with $\mathcal{G} := \{\mathbb{S}_+^n : n \in \{1, 2, \dots\}\}$ and $\mathcal{G} := \{Q^n : n \in \{1, 2, \dots\}\}$ respectively).

Example 1.1. When \mathcal{G} is the collection $\{\mathbb{S}_+^n : n \in \{1, 2, \dots\}\}$, we say that K is SDP-representable if it admits a \mathbb{S}_+^n -representation for some n .

When \mathcal{G} is the collection of second-order cones and their direct sums, we say that K is SOCP-representable if it admits a $G_1 \oplus \dots \oplus G_k$ -representation for some second-order cones G_1, \dots, G_k .

Definition 1.2. $K \subset \mathbb{R}^d$ is said to admit a lifted- G -representation via \mathcal{L} if $G \subset \mathbb{R}^N$ is a pointed, closed, convex cone with nonempty interior and $\mathcal{L} : \mathbb{R}^d \oplus \mathbb{R}^m \rightarrow \mathbb{R}^N$ is a linear map such that

$$x \in \text{int}(K) \iff \mathcal{L}(x, u) \in \text{int}(G) \text{ for some } u \in \mathbb{R}^m.$$

For every linear subspace $V \subseteq \mathbb{R}^N$, let $V^\perp \subseteq \mathbb{R}^N$ denote its orthogonal complement and let $\text{Pr}_V(\cdot)$ denote the orthogonal projection onto V . For every convex cone $K \subseteq \mathbb{R}^d$, let

$$K^* := \{s \in \mathbb{R}^d : \langle s, x \rangle \geq 0, \forall x \in K\}$$

denote its dual cone.

The following two propositions present alternative definitions for G -representability and lifted- G -representability respectively.

Proposition 1.1. Suppose that $G \subset \mathbb{R}^N$ is a pointed, closed, convex cone with nonempty interior. Then the following are equivalent:

- (i) K admits a G -representation;
- (ii) there exists a linear subspace $V \subseteq \mathbb{R}^N$ such that $V \cap \text{int}(G) \neq \emptyset$ and K is linearly isomorphic to $G \cap V$.

Proposition 1.2. Suppose that $G \subset \mathbb{R}^N$ is a pointed, closed, convex cone with nonempty interior. Then the following are equivalent:

- (i) K admits a lifted- G -representation;
- (ii) there exist linear subspaces $V, W \subseteq \mathbb{R}^N$ such that $V \cap \text{int}(G) \neq \emptyset$, $W^\perp \subseteq V$ and K is linearly isomorphic to $\text{Pr}_W(G \cap V)$;
- (iii) there exist linear subspaces $V, W \subseteq \mathbb{R}^N$ such that $(V + W^\perp) \cap \text{int}(G) \neq \emptyset$ and K is linearly isomorphic to $\text{Pr}_W(G) \cap V$ (without loss of generality, $V \subseteq W$);

(iv) there exist linear subspaces $V, W \subseteq \mathbb{R}^N$ such that $V \cap (\text{int}(G) + W) \neq \emptyset$ and K is linearly isomorphic to $(G + W) \cap V$.

Proof. We shall show (i) \implies (ii) \implies (iii) \implies (iv) \implies (i).

Suppose K admits a lifted- G -representation via \mathcal{L} . Then $x \in \mathbb{R}^d \mapsto \text{Pr}_W \mathcal{L}(x, 0)$ is a linear isomorphism from K to $\text{Pr}_W(G \cap V)$, where $V := \mathcal{L}(\mathbb{R}^d \oplus \mathbb{R}^m)$ and $W := (\mathcal{L}(\{0\} \oplus \mathbb{R}^m))^\perp$ are linear subspaces satisfying $V \cap \text{int}(G) \neq \emptyset$ and $W^\perp \subseteq V$.

Suppose (ii) holds. Then K is linearly isomorphic to $\text{Pr}_W(G \cap V) = \text{Pr}_W(G) \cap \tilde{V}$, where $\tilde{V} := V \cap W$ satisfies $(\tilde{V} + W^\perp) \cap \text{int}(G) \neq \emptyset$ as desired.

Suppose (iii) holds. Then K is linearly isomorphic to $\text{Pr}_W(G) \cap V = (G + \tilde{W}) \cap V$, where $\tilde{W} := W^\perp$ satisfies $V \cap (\text{int}(G) + \tilde{W}) \neq \emptyset$ as desired.

Suppose (iv) holds with $\mathcal{L} : \mathbb{R}^d \rightarrow \mathbb{R}^N$ as the linear isomorphism from K to $(G + W) \cap V$. Then K admits a lifted- G -representation via $\tilde{\mathcal{L}} : (x, u) \in \mathbb{R}^d \oplus W \mapsto \mathcal{L}(x) + u$. Since \mathcal{L} is an isomorphism and $V \cap (\text{int}(G) + W) \neq \emptyset$, we have that

$$x \in \text{int}(K) \iff \exists u \in \mathbb{R}^m \text{ s.t. } \tilde{\mathcal{L}}(x, u) \in \text{int}(G).$$

□

Remark 1.1. In Part (ii) of Proposition 1.2, by observing that the linear operator Pr_W is an isomorphism from $\text{Pr}_V W$ to $\text{Pr}_W V$, it only takes a little extra effort to remove the condition $W^\perp \subseteq V$. □

We similarly define the SDP-representation, the lifted-SDP-representation, the G -representation and the lifted- G -representation of convex sets (via affine maps and affine restrictions). For elementary operations which preserve lifted-SDP-representability, see [1].

Our interests in this paper are

- to investigate representations as theoretical tools in establishing facts about K by utilizing known facts about \mathbb{S}_+^n or G and the properties of the representations,
- to evaluate the *goodness* of representations from a theoretical viewpoint.

The first path of the investigation is concerned more with the basic properties and the existential issues. For instance, if the cone G has property χ which is preserved under G -representation and if cone K violates property χ , then clearly K is *not* G -representable. Sections 2 and 3 deal with this item. Our second interest is a matter of efficiency. Once a \mathcal{P} -representation exists there may be many others. A main question is “how good are these representations?” To answer this, we need to find *measures of goodness* for representations. One obvious measure of goodness is the dimension of the representing cone. However, the dimension alone is a very rough measure of efficiency. We would like

to find more precise measures. To find more precise measures with rigorous justifications for them, we need to define the context for representations. We choose to focus on the modern theory of interior-point methods. We immediately have the fundamental notions of barrier functions, optimal barrier parameters and central paths. These notions lead to more precise measures of efficiency in the context of interior-point methods. Sections 4, 5, and 6 are geared towards establishing the efficiency measures. We give below a preview of our approach for studying the efficiency of convex representations.

We are interested in representations, mostly because they allow us to deal with *difficult* sets via *easier* sets. In the context of modern interior-point methods, we usually think of a convex set as *easy* if we know of efficiently computable barriers for it (see [14, 15]). For example, suppose that the cone G is very well-studied and understood. Further suppose that “the best” barrier function for G is $F : \text{int}(G) \mapsto \mathbb{R}$. Consider K which admits a G -representation via \mathcal{L} . Then we can study the barrier function $F(\mathcal{L}(\cdot)) : \text{int}(K) \mapsto \mathbb{R}$ and determine how good a barrier $F(\mathcal{L}(\cdot))$ is for K . This would give a notion of efficiency for the G -representation of K via \mathcal{L} . So, in some cases, the representation can be assumed to imply the treatment of the represented set using a particular barrier function.

A fundamental concept in the theory of interior-point methods is that of *the central path* (which is defined by a barrier function) and its *neighborhoods*. The first thing to understand here is what happens to the barrier function under representations.

In the case of \mathbb{S}_{++}^n , the standard barrier is

$$-\ln(\det(x)).$$

An obvious measure of goodness in this context is the *barrier parameter*.

Suppose that K is SDP-representable. Let $\{x(\mu) : \mu \in \mathbb{R}_{++}\}$ denote the central path for the problem

$$\inf \{ \langle c, x \rangle : \mathcal{A}(x) = b, x \in K \}$$

which is defined based on the “best” self-concordant barrier available for K . Let $\{\tilde{x}(\mu) : \mu \in \mathbb{R}_{++}\}$ denote the central path for an SDP-representation of the problem. Clearly, there exists a map $\mathcal{R} : \mathbb{R}^d \mapsto \mathbb{S}^n$ such that

$$\mathcal{R}(x(\mu)) = \tilde{x}(\mu), \text{ for every } \mu > 0.$$

- What good properties can be expected of such \mathcal{R} (depending on \mathcal{L})?
- Are there algebraic expressions of good, computable maps \mathcal{R} ?

In Section 4, we study SDP-representations of homogeneous cones. In Section 6, we consider the mapping \mathcal{R} above for the primal-dual central path. We also work with \mathcal{R}^{-1} (mapping the primal-dual central path for the SDP-representation to the primal-dual central path based on an optimal barrier for the homogeneous cone K). We show that there is a very nice choice for such mapping \mathcal{R} (this is the mapping $\mathcal{T}^{-1} \circ \tau \circ \mathcal{T}$ defined in Section 6). We then address the question “how *smooth* is this mapping \mathcal{R} ?” In our context, the primal-dual interior-point method theory, to evaluate the “smoothness,” we

measure the scale-invariant size of the neighbourhood of the primal-dual central path over which \mathcal{R} (and hence \mathcal{R}^{-1}) is a diffeomorphism. (See Corollary 6.1 and the discussion following it.)

Example 1.2. Consider as K the following cone, which is useful in describing the epigraph of matrix 2-norm over $\mathbb{R}^{k \times r}$:

$$K := \text{cl} \left\{ \begin{pmatrix} x \\ u \\ t \end{pmatrix} \in \mathbb{S}^r \oplus \mathbb{R}^{r \times k} \oplus \mathbb{R} : t > 0, \left(x - \frac{uu^T}{t} \right) \succ 0 \right\},$$

where $\text{cl}(\cdot)$ denotes the closure. An optimal barrier for K is

$$f(x, u, t) := -\ln \det \left(x - \frac{uu^T}{t} \right) - \ln(t).$$

The SDP-representation

$$\begin{pmatrix} tI & u^T \\ u & x \end{pmatrix} \succeq 0$$

implies the usage of the barrier

$$f(x, u, t) - (k-1) \ln(t).$$

Clearly, if the constraints of the problem imply that t is constant, then these two barriers are essentially the same. Indeed, the situation is more complicated in general even for this simple example. Recall that we allow for taking direct sums. So, a convex optimization problem over the family of such cones (epigraph of matrix 2-norms) could lead to the barrier

$$\sum_{i=1}^m f_i(x^{(i)}, u^{(i)}, t_i) - \sum_{i=1}^m (k_i - 1) \ln(t_i),$$

where we assumed that we have the direct sum of m cones. Moreover, we could let \mathcal{G} denote the family of cones linearly isomorphic to a direct sum of such cones K with varying r and k . Then the barrier functions for representations arising from the family \mathcal{G} may not look as separable over the direct sum as the above barrier (with m main components and the difference involving only $\ln(t_i)$).

For instance, consider $K := \mathbb{S}_+^3 \oplus \mathbb{Q}^3$ as a pointed, closed convex cone in \mathbb{R}^{10} with nonempty interior. Take any nonsingular linear transformation $L : \mathbb{R}^{10} \rightarrow \mathbb{R}^{10}$. Then the cone $L(K)$ is isomorphic to K . However, for most choices of L , at a first glance, the corresponding barrier function for $L(K)$ may not look as separable as the one above. \square

The above convex cone is an example of a *homogeneous cone*. A cone $K \subset \mathbb{R}^d$ is *homogeneous* if its automorphism group acts transitively on the interior of K .

A convex cone K is *self-dual* if there exists an inner product under which $K^* = K$. A convex cone is *symmetric* if it is homogeneous and self-dual.

Both \mathbb{S}_+^n and Q^n are symmetric. The family of homogeneous cones is significantly richer than symmetric cones (however every homogeneous cone can be represented as a slice of a possibly higher dimensional semidefinite cone; see [4, 7]).

The next family in the hierarchy is *hyperbolic cones*. A homogeneous polynomial $p : \mathbb{R}^d \mapsto \mathbb{R}$ is *hyperbolic* in the direction $h \in \mathbb{R}^d$, if the univariate polynomial (in $t \in \mathbb{R}$)

$$p(x + th)$$

has only real roots for every $x \in \mathbb{R}^d$. (More general notions exist for nonhomogeneous polynomials over \mathbb{C} ; see Güler [8].) A convex cone K is a *hyperbolic cone* if it is

$$\left\{ x \in \mathbb{R}^d : p(x + th) \neq 0, \forall t \in \mathbb{R}_+ \right\}$$

for a polynomial p which is hyperbolic in the direction $h \in \mathbb{R}^d$. Homogeneous cones make up a proper subset of hyperbolic cones; see [4].

Recently, Lax conjecture was proved by Lewis, Parrilo and Ramana [10] which implies that all three dimensional hyperbolic cones are SDP-representable. Also see [9, 23] for some of the foundational work related to this result. Generalized Lax conjecture states that “every hyperbolic cone can be represented as a slice of a possibly higher dimensional semidefinite cone.” As further progress is made along these lines and similar avenues of research in convex representations, we hope that our results on invariance properties of representations and our suggestions for evaluating the goodness/efficiency of representations continue to be fruitful.

2 Invariance for Convex Cones

We start with a series of basic definitions. A subset P of K is a *face* of K if for every $x, z \in K$ such that $(x + z) \in P$ we have $x, z \in P$ (equivalently, for every open line segment $(x, z) \subset K$ which intersects P , we have $[x, z] \subset P$). A face P of K is called *proper* if P is neither empty nor equal to K . A face P of K is *exposed* if $P = K \cap H$, for some supporting hyperplane H of K . We say that K is *facially exposed* if every proper face of K is exposed.

We continue with an elementary fact.

Proposition 2.1. *Suppose that $G \subset \mathbb{R}^N$ is a pointed, closed, convex cone with nonempty interior, and that $V \subset \mathbb{R}^N$ is a linear subspace such that $\text{int}(G) \cap V \neq \emptyset$. Then a subset $P \subseteq G \cap V$ is a proper face of $G \cap V$ if and only if it is the intersection of some proper face P' of G with V .*

Proof. Suppose that P' is a face of G . For any line segment that lies in $G \cap V$, if its relative interior intersects $P' \cap V$, then its endpoints must lie in P' . Thus, its endpoints lie in $P' \cap V$. This proves the “if” part of the statement.

For the “only if” part, let P' be the smallest face of G containing P . Since P is proper, so is P' . Clearly, $P \subseteq P' \cap V$. Since P' is the smallest face containing P , there exists $z \in \text{relint}(P') \cap P$, where $\text{relint}(\cdot)$ denotes the relative interior. Suppose that $x \in P' \cap V$. Then for sufficiently small $\varepsilon > 0$, $y := z + \varepsilon(z - x) \in P' \cap V \subseteq G \cap V$. Since $(x, y) \cap P \supseteq \{z\} \neq \emptyset$, we conclude that $x \in P$. \square

Corollary 2.1. *Facial exposedness is preserved under G -representability.*

The next example proves that facial exposedness is not preserved under lifted- G -representations.

Example 2.1. *We consider the example from [21] (pp. 239–240, Figure 1). This is a convex cone with four unexposed extreme rays. The cone admits a lifted-SDP-representation as follows (the interior of the cone equals) :*

$$\left\{ \left(\begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right) : \left(\begin{array}{cccc} x_3 & x_2 & 0 & 0 \\ x_2 & -x_3 & 0 & 0 \\ 0 & 0 & x_1 + x_3 & 0 \\ 0 & 0 & 0 & -x_1 + x_3 \end{array} \right) \succ \left(\begin{array}{cccc} -x_0 & 0 & 0 & 0 \\ 0 & -x_0 & 0 & 0 \\ 0 & 0 & -x_0 & 0 \\ 0 & 0 & 0 & -x_0 \end{array} \right), \text{ for some } x_3 \in \mathbb{R} \right\}.$$

Since the largest block in the above representation is (2×2) , and \mathbb{S}_+^2 admits an SOCP-representation, the above cone admits a lifted-SOCP-representation.

Another interesting facial property of a subclass of cones is

$$\left(K^* + P^\perp \right) \text{ is closed for every proper face } P \text{ of } K, \quad (FP_1)$$

where $P^\perp = (P - P)^\perp$ denotes the orthogonal complement of the linear span of P . The above is a property of semidefinite cones, and it is relevant in the duality theory of conic optimization.

Proposition 2.2. *Property (FP_1) is preserved under G -representability.*

Proof. Suppose that G satisfies (FP_1) and K is G -representable via \mathcal{L} . So, $\mathcal{L}(K) = G \cap V$ for some linear subspace V with $V \cap \text{int}(G) \neq \emptyset$. Let P be an arbitrary proper face of K . By Proposition 2.1, there exists a proper face P' of G such that $\mathcal{L}(P) = P' \cap V$. Moreover, we know from its proof that $V \cap \text{relint}(P') \neq \emptyset$. Let H be the linear span of P' , so that $P' = G \cap H$ and $(P')^\perp = H^\perp$. Using Corollary 16.4.2 of Rockafellar [18], we deduce that $(P')^* = (G \cap H)^* = \text{cl}(G^* + H^\perp) = G^* + H^\perp$ by (FP_1) . Moreover, since each of $\text{relint}(P')$, $\text{relint}(H)$ and $\text{int}(G)$ has nonempty intersection with V , we invoke Corollary 16.4.2 of Rockafellar [18] again to deduce that

$$\mathcal{L}(P)^\perp = (P' \cap V)^\perp = (H \cap V)^\perp = H^\perp + V^\perp$$

and

$$\mathcal{L}(P)^* = (P' \cap V)^* = (P')^* + V^\perp = G^* + H^\perp + V^\perp = \mathcal{L}(K)^* + \mathcal{L}(P)^\perp.$$

Consequently, using Corollary 16.3.2 of Rockafellar [18], $K^* + P^\perp = \mathcal{L}^*(\mathcal{L}(K)^* + \mathcal{L}(P)^\perp) = \mathcal{L}^*(\mathcal{L}(P)^*)$ is closed. \square

Proposition 2.3. *K is lifted- G -representable if and only if K^* is lifted- G^* -representable.*

Proof. Suppose that K admits a lifted- G -representation. According to Proposition 1.2, there exists a linear isomorphism \mathcal{L} from K to $\text{Pr}_W(G \cap V)$, where $V, W \subseteq \mathbb{R}^N$ are linear subspaces satisfying $V \cap \text{int}(G) \neq \emptyset$. Using Corollary 16.3.2 of Rockafellar [18],

$$\begin{aligned} K^* &= \mathcal{L}^*((\text{Pr}_W(G \cap V))^*) = \mathcal{L}^*((W \cap ((G \cap V) + W^\perp))^*) = \mathcal{L}^*(W^\perp + (G \cap V)^* \cap W) \\ &= \mathcal{L}^*((G^* + V^\perp) \cap W). \end{aligned}$$

Since \mathcal{L}^* is a bijection from W to \mathbb{R}^d , we conclude from Proposition 1.2 that K^* admits a lifted- G^* -representation. Since all closed, convex cones C satisfy $(C^*)^* = C$, the converse follows. \square

Corollary 2.2. *If K is G -representable, then K^* is lifted G^* -representable.*

Corollary 2.3. *If G is self-dual and K is G -representable, then K^* is lifted G -representable.*

Corollary 2.4. *If G is a pointed, closed, convex cone that is self-dual, then lifted- G -representability is closed under duality.*

However, the following example shows that in general, G -representability is not closed under duality, even when G is self-dual.

Example 2.2. *Consider the \mathbb{S}_+^3 -representable cone*

$$\left\{ \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} x_0 + x_1 & x_2 & 0 \\ x_2 & x_0 - x_1 & 0 \\ 0 & 0 & x_1 \end{pmatrix} \succeq 0 \right\},$$

which is exactly the intersection of the second-order cone Q_2 and the half-space $H := \{(x_0, x_1, x_2)^T : x_1 \geq 0\}$. (Note that the resulting cone is the intersection of two hyperbolic cones Q^2 and H ; therefore, it is also a hyperbolic cone.) The dual cone is then

$$(Q^2)^* + H^\perp = Q^2 + \left\{ \begin{pmatrix} 0 \\ s_1 \\ 0 \end{pmatrix} : s_1 \geq 0 \right\} = \left\{ \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix} : s_0 \geq |s_2|, s_1 \geq -\sqrt{s_0^2 - s_2^2} \right\},$$

which has two unexposed extreme rays (to locate the unexposed rays, set $s_0 := 1$ and plot), and thus is not even SDP-representable (of course, it admits a lifted-SDP-representation as well as a lifted-SOCP-representation, by Corollary 2.3 or directly from the structure of Example 2.1). This type of example was also used by Braams [2] to show that the pre-images of the cones \mathbb{S}_+^n under arbitrary linear maps is not equal to the topological closures of images of \mathbb{S}_+^n under arbitrary linear maps.

It is well-known that \mathbb{S}_+^n is facially exposed. It was proven in [19] that homogeneous cones are also facially exposed. This last fact also easily follows from the result that all homogeneous cones admit SDP-representations [4] and Proposition 2.1. Recently, Renegar [17] noted that all hyperbolic cones are facially exposed. Noticing that there are convex cones with unexposed faces (and hence not hyperbolic) which admit lifted-SDP-representations, we immediately conclude that the set of convex cones that admit lifted-SDP-representations differs from the set of *hyperbolic cones*.

3 Invariance for Convex (Conic) Optimization Problems

Suppose K admits a G -representation via \mathcal{L} . Then for any linear map $\mathcal{A}_{\mathcal{L}}$ whose kernel coincides with the range of \mathcal{L} , the interior of K is linearly isomorphic to

$$\mathcal{A}_{\mathcal{L}}(x) = 0, x \in \text{int}(G).$$

Then any cone programming problem over K can be represented by

$$\begin{aligned} \inf \quad & \langle c, x \rangle \\ & \mathcal{A}(x) = b, \\ & \mathcal{A}_{\mathcal{L}}(x) = 0, \\ & x \in G, \end{aligned}$$

for appropriate choices of \mathcal{A} , b and c .

Definition 3.1. (Pataki [16]) *Suppose that (CP) is a convex conic programming problem with underlying cone K . A feasible primal-dual pair (x, s) is called strictly complementary for (CP) if there exists a face P of K such that*

$$x \in \text{relint}(P) \text{ and } s \in \text{relint}(P^{\Delta}).$$

where $P^{\Delta} := P^{\perp} \cap K^*$ denotes the complementary face of P .

The above definition of strict complementarity is not symmetric under duality. In general, if K or its dual cone K^* is not facially exposed, then we can always find some \mathcal{A} , b and c such that one of $\inf\{\langle c, x \rangle : \mathcal{A}(x) = b, x \in K\}$ and $\sup\{b^T y : c - \mathcal{A}^*(y) \in K^*\}$ has at least one strictly complementary, feasible, primal-dual pair while the other has none. This is due to the fact that all complementary faces are exposed, and hence $P \neq (P^{\Delta})^{\Delta}$ whenever P is not exposed. This observation, together with the results of the previous section indicate that currently we do not have a primal-dual symmetric notion of strict complementarity for *hyperbolic programming*.

When both K and its dual cone are facially exposed, we shall briefly show that strict complementarity is preserved under duality. We need the following two lemmas. The first one is well-known and the next one is elementary.

Lemma 3.1. *Let K be as above. Then the following hold.*

1. *A proper face P of K is exposed if and only if $P = (P^{\Delta})^{\Delta}$.*
2. *For any proper face P of K and any subset $V \subset P$, the face $(V^{\Delta})^{\Delta}$ is the smallest exposed face of K containing V .*

Lemma 3.2. *There exists a strictly complementary, feasible primal-dual pair if and only if the dual optimal face intersects the relative interior of the complementary face of the primal optimal face.*

Theorem 3.1. *If K^* is facially-exposed, then there exists a strictly complementary, feasible, primal-dual pair if and only if the respective complementary faces of the primal and dual optimal faces are orthogonal. Consequently, if both K and K^* are facially-exposed, then strict complementarity is invariant under duality.*

Proof. Suppose that there exists a strictly complementary, feasible, primal-dual pair. By the preceding lemma, there exists $s' \in \mathcal{O}_d \cap \text{relint}(\mathcal{O}_p^\Delta)$, where \mathcal{O}_p and \mathcal{O}_d are the primal and dual optimal faces respectively (note that these are the faces of the corresponding feasible regions—they are not necessarily faces of the corresponding cones). Let $x \in \mathcal{O}_d^\Delta$ and $s \in \mathcal{O}_p^\Delta$ be arbitrary. Since $x \in K$ and $s \in K^*$, we have $\langle x, s \rangle \geq 0$. Since $s' \in \text{relint}(\mathcal{O}_p^\Delta)$, it follows that $s_\varepsilon := s' + \varepsilon(s' - s) \in \mathcal{O}_p^\Delta \subset K^*$, for ε positive and sufficiently small. If $\langle x, s \rangle > 0$, then $\langle x, s_\varepsilon \rangle = (1 + \varepsilon)\langle x, s' \rangle - \varepsilon\langle x, s \rangle < 0$ for all $\varepsilon > 0$, contradicting the fact that $x \in K$ and $s_\varepsilon \in K^*$ for ε positive and sufficiently small. Thus, we necessarily have $\langle x, s \rangle = 0$.

Conversely, suppose that $\mathcal{O}_d^\Delta \perp \mathcal{O}_p^\Delta$. Then $\mathcal{O}_d^\Delta \subseteq (\mathcal{O}_p^\Delta)^\Delta$. Since $\mathcal{O}_d \subseteq \mathcal{O}_p \implies \mathcal{O}_d^\Delta \supseteq (\mathcal{O}_p^\Delta)^\Delta$, we further have $\mathcal{O}_d^\Delta = (\mathcal{O}_p^\Delta)^\Delta$. Since \mathcal{O}_p^Δ is exposed, it follows that $\mathcal{O}_p^\Delta = ((\mathcal{O}_p^\Delta)^\Delta)^\Delta = (\mathcal{O}_d^\Delta)^\Delta$ is the smallest exposed face of K^* containing \mathcal{O}_d . Since K^* is facially-exposed, \mathcal{O}_p^Δ is in fact the smallest face of K^* containing \mathcal{O}_d . It then follows that $\mathcal{O}_d \cap \text{relint}(\mathcal{O}_p^\Delta) \neq \emptyset$, and thus the convex conic programming problem is strictly complementary.

Consequently, the last statement of the theorem follows from duality. \square

Consider the convex conic programming problem over K ,

$$\begin{aligned} \inf \quad & \langle c, x \rangle \\ \mathcal{A}(x) \quad & = \quad b, \\ x \quad & \in \quad K, \end{aligned} \tag{\mathcal{P}_K}$$

and its formulation as a convex conic programming problem over G

$$\begin{aligned} \inf \quad & \langle c_G, x \rangle \\ \mathcal{A}_G(x) \quad & = \quad b, \\ \mathcal{A}_\mathcal{L}(x) \quad & = \quad 0, \\ x \quad & \in \quad G, \end{aligned} \tag{\mathcal{P}_G}$$

where $c_G := \mathcal{L}(\mathcal{L}^*\mathcal{L})^{-1}(c)$ and $\mathcal{A}_G := \mathcal{A}(\mathcal{L}^*\mathcal{L})^{-1}\mathcal{L}^*$.

Let $\mathcal{O}_p(K)$ and $\mathcal{O}_p(G)$ denote the primal optimal faces of these problems, and let $\mathcal{O}_d(K)$ and $\mathcal{O}_d(G)$ denote the dual optimal faces of their respective dual problems. Clearly, $\mathcal{L}(\mathcal{O}_p(K)) = \mathcal{O}_p(G)$. By definition, the complementary optimal faces of $\mathcal{O}_p(K)$ and $\mathcal{O}_p(G)$ are $\mathcal{C}_d(K) = \{s \in K^* : \langle s, \bar{x}_K \rangle = 0\}$ and $\mathcal{C}_d(G) = \{s \in G^* : \langle s, \bar{x}_G \rangle = 0\}$ respectively, where $\bar{x}_K \in \text{relint}(\mathcal{O}_p(K))$ is arbitrary and $\bar{x}_G = \mathcal{L}(\bar{x}_K) \in \text{relint}(\mathcal{O}_p(G))$.

Let V be the range of \mathcal{L} .

Lemma 3.3. *With the above definitions, we have*

$$\mathcal{C}_d(K) = \mathcal{L}^*(\mathcal{C}_d(G)).$$

Proof. Using Corollaries 16.3.2 and 16.4.2 of Rockafellar [18], we deduce

$$\mathcal{L}^*(\mathcal{C}_d(G)) = \mathcal{L}^*(G^* \cap \{\bar{x}_G\}^\perp + V^\perp) = \mathcal{L}^*(G^*) \cap \mathcal{L}^*(\{\bar{x}_G\}^\perp) = K^* \cap \{\bar{x}_K\}^\perp = \mathcal{C}_d(K),$$

where the second equality follows from $\{\bar{x}_G\}^\perp \supseteq V^\perp = (\mathcal{L}^*)^{-1}(\{0\})$. \square

Let W denote the range of \mathcal{A}_G^* ; i.e., the image of the range of \mathcal{A}^* under $\mathcal{L}(\mathcal{L}^*\mathcal{L})^{-1}$. The respective dual optimal faces are

$$\mathcal{O}_d(K) = \mathcal{C}_d(K) \cap (\mathcal{L}^*(W) + \{c\}) \text{ and } \mathcal{O}_d(G) = \mathcal{C}_d(G) \cap (V^\perp + W + \{c_G\}).$$

Thus, according to Lemma 3.2, (\mathcal{P}_K) (resp. (\mathcal{P}_G)) has strictly complementary solutions if and only if $\text{relint}(\mathcal{C}_d(K)) \cap (\mathcal{L}^*(W) + \{c\})$ (resp. $\text{relint}(\mathcal{C}_d(G)) \cap (V^\perp + W + \{c_G\})$) is nonempty. Therefore

$$\begin{aligned} \mathcal{L}^*(\text{relint}(\mathcal{C}_d(G)) \cap (V^\perp + W + \{c_G\})) &= \mathcal{L}^*(\text{relint}(\mathcal{C}_d(G))) \cap \mathcal{L}^*(W + \{c_G\}) \\ &= \text{relint}(\mathcal{L}^*(\mathcal{C}_d(G))) \cap (\mathcal{L}^*(W) + \{c\}) \\ &= \text{relint}(\mathcal{C}_d(K)) \cap (\mathcal{L}^*(W) + \{c\}) \end{aligned}$$

where the first and second equality follows from $(\mathcal{L}^*)^{-1}(\{0\}) = V^\perp$ and Theorem 6.6 of Rockafellar [18] respectively, proves

Theorem 3.2. *Strict complementarity is invariant under G -representability.*

4 Some SDP Representations

4.1 An SDP Representation of Symmetric Cones

It is well-known that every symmetric cone admits an SDP-representation. Let $K \subset \mathbb{R}^d$ be an irreducible symmetric cone such that $\text{rank}(K) = r \geq 2$. Let \mathcal{J} denote the *Jordan Product representation* (see [6]) so that for every $x \in K$, $\mathcal{J}(x) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is represented by $\mathcal{J}_M(x) \in \mathbb{S}^d$. We have

$$x \in \text{int}(K) \iff \mathcal{J}_M(x) \succ 0.$$

Let $\lambda_i(x)$ denote the generalized eigenvalues of x . Then

$$f(x) := - \sum_{i=1}^r \ln(\lambda_i(x))$$

is an optimal barrier for K .

Using Theorem IV.2.1. and Corollary IV.2.6. of [6], we have

Theorem 4.1. For every $x \in \text{int}(K)$,

$$-\ln \det(\mathcal{J}_M(x)) = f(x) - \frac{2(d-r)}{r(r-1)} \sum_{i < j} \ln \left(\frac{\lambda_i(x) + \lambda_j(x)}{2} \right).$$

The parameter of the above barrier is exactly the dimension d . The eigenvalues of $\mathcal{J}_M(x)$ are $\lambda_1(x), \lambda_2(x), \dots, \lambda_d(x)$ and

$$\frac{\lambda_i(x) + \lambda_j(x)}{2}, \quad i \in \{1, 2, \dots, r-1\}, j \in \{i+1, i+2, \dots, r\}.$$

The eigenspace of $\mathcal{J}_M(x)$ corresponding to the eigenvalue $\frac{\lambda_i(x) + \lambda_j(x)}{2}$, for $i < j$, has dimension $\frac{2(d-r)}{r(r-1)}$.

The usual central path equation $S = -\mu X^{-1}$ is replaced by

$$S = -\mu X^{-1} - \mu \frac{2(d-r)}{r(r-1)} Q D_\lambda Q^T,$$

where the orthogonal matrix Q diagonalizes X and D_λ is the diagonal matrix with the i th entries given by

$$\sum_{j \neq i} \frac{1}{\lambda_i(x) + \lambda_j(x)},$$

repeated with the corresponding multiplicity as described above for the eigenspaces of $\mathcal{J}_M(x)$.

4.2 SDP Representations of Homogeneous Cones

It was shown in [4] that all homogeneous cones admit SDP-representations in the following way. A T -algebra is a matrix algebra \mathcal{A} with involution $*$ (i.e., \mathcal{A} a bi-graded algebra $\bigoplus_{i,j=1}^r \mathcal{A}_{ij}$ satisfying $\mathcal{A}_{ij}\mathcal{A}_{jk} \subseteq \mathcal{A}_{ik}$ for all i, j, k and $\mathcal{A}_{ij}\mathcal{A}_{k\ell} \subseteq \{0\}$ when $j \neq k$, and $*$ is an involutory, anti-automorphism of \mathcal{A}) satisfying the following seven axioms (see [22]).

- (I) \mathcal{A}_{ii} is isomorphic to \mathbb{R} ;
- (II) $e_i a_{ij} = a_{ij} e_j = a_{ij}$ for all $a_{ij} \in \mathcal{A}_{ij}$, where e_i denotes the unit of \mathcal{A}_{ii} ;
- (III) $\text{tr}ab = \text{tr}ba$ for all $a, b \in \mathcal{A}$, where $\text{tr} : (a_{ij})_{i,j=1}^r \in \bigoplus_{i,j}^r \mathcal{A}_{ij} \mapsto \sum_{i=1}^r a_{ii}$;
- (IV) $\text{tr}(bc) = \text{tr}(ab)c$ for all $a, b, c \in \mathcal{A}$;
- (V) $\text{tr}a^*a > 0$ for all $a \in \mathcal{A}$ unless $a = 0$;
- (VI) $t(uw) = (tu)w$ for all $t, u, w \in \mathcal{T} := \bigoplus_{1 \leq j \leq i \leq r} \mathcal{A}_{ij}$, the space of lower triangular elements;
- (VII) $t(uu^*) = (tu)u^*$ for all $t, u \in \mathcal{T}$.

Given a homogeneous cone K , Vinberg [22] showed that there is a T -algebra \mathcal{A} such that K is precisely the cone containing elements of the form $\ell^*\ell$, where ℓ is a lower triangular element with $\ell_{ii} > 0$ for all i . Moreover the map $\ell \in \{t \in \mathcal{T} : t_{ii} > 0\} \mapsto \ell^*\ell$ is a bijection.

For each $(i, j) \in \{1, \dots, r\}^2$, let n_{ij} denote the dimension of \mathcal{A}_{ij} as a vector subspace of \mathcal{A} . It follows from the definition of matrix algebra and Axiom (I) respectively that $n_{ij} = n_{ji}$ and $n_{ii} = 1$.

Let J denote a subset of $\{1, \dots, r\}$ satisfying the following property.

$$\text{For each } i \in \{1, \dots, r\} \text{ there exists } j \in J \text{ such that } j \leq i \text{ and } n_{ij} > 0. \quad (\star)$$

For instance, $J = \{1, \dots, r\}$ satisfies (\star) . Let \mathcal{T}_J denote the subspace $\bigoplus_{j \in J} \bigoplus_{i=j}^r \mathcal{A}_{ij}$ of lower triangular elements whose columns not indexed by J are zero columns. With each $x \in \mathcal{A}$, we associate the linear operator $\mathcal{L}_x : \mathcal{T}_J \rightarrow \mathcal{T}_J$ defined by $\mathcal{L}_x : \ell \mapsto \text{Pr}_{\mathcal{T}_J} x \ell$, where $\text{Pr}_{\mathcal{T}_J}$ denotes the projection onto \mathcal{T}_J . It was proved in [4] that, whenever J satisfies (\star) , the map $\mathcal{L} : x \mapsto \mathcal{L}_x$ is a bijection from K to the cone of self-adjoint, positive definite linear operators on \mathcal{T}_J . This gives an SDP representation of K for each ordered orthogonal basis of \mathcal{T}_J .

Since each $x \in K$ uniquely determines a lower triangular element ℓ with positive diagonal entries satisfying $x = \ell^*\ell$, the functional $f : K \mapsto \mathbb{R}$ defined by $\ell^*\ell \mapsto -\sum_{i=1}^r \ln \rho_i^2(\ell^*\ell)$ is well-defined. Furthermore, it is a logarithmically homogeneous, self-concordant barrier for K . In fact, it is optimal for K . We shall compare this barrier with the standard logarithmic barrier F of the representing SDP cone. In computing the barrier F , it suffices to compute $\det(\mathcal{L}_\ell)$ since $\det(\mathcal{L}_{\ell^*\ell}) = \det(\mathcal{L}_\ell)^2$, which is a direct consequence of the following proposition.

Proposition 4.1 (Proposition 3.4 of [4]). *Let $\mathcal{L} : x \in \mathcal{A} \rightarrow \mathcal{L}_x$ be as defined above. For every $a \in \mathcal{A}$ and $t, u \in \mathcal{T}$,*

1. $\mathcal{L}_{a^*} = (\mathcal{L}_a)^*$, the adjoint of \mathcal{L}_a under the inner product $\langle \cdot, \cdot \rangle$,
2. $\mathcal{L}_t \mathcal{L}_u = \mathcal{L}_{tu}$, and
3. $\mathcal{L}_t \mathcal{L}_{t^*} = \mathcal{L}_{tt^*}$.

Consider the orthogonal decomposition of $\mathcal{T}_J = \bigoplus_{j \in J} \mathcal{C}_j$ into the columns $\mathcal{C}_j = J \bigoplus_{i=j}^r \mathcal{A}_{ij}$, $j \in J$. Since $\mathcal{L}_\ell(\mathcal{C}_j) \subseteq \mathcal{C}_j$ for each $j \in \{1, \dots, r\}$, the determinant of \mathcal{L}_ℓ is the product of the determinants of $\mathcal{L}_\ell|_{\mathcal{C}_j}$, $j \in J$.

We now fix an arbitrary $j \in J$ and consider the determinant of $\mathcal{L}_\ell|_{\mathcal{C}_j}$.

For each $i \in \{j, \dots, r\}$, let \mathcal{B}_i be an arbitrary ordered orthogonal basis of \mathcal{A}_{ij} . Let L denote the matrix representing $\mathcal{L}_\ell|_{\mathcal{C}_j}$ under the ordered orthogonal basis $(\mathcal{B}_j, \dots, \mathcal{B}_r)$ of \mathcal{C}_j . We consider L in $(r - j + 1) \times (r - j + 1)$ block matrix form, where the $(i - j + 1, k - j + 1)$ -st block represents $\text{Pr}_{\mathcal{A}_{ij}}(\mathcal{L}_\ell|_{\mathcal{A}_{kj}})$.

Since $\mathcal{L}_\ell(\mathcal{A}_{ij}) = \{\ell y_{ij} : y_{ij} \in \mathcal{A}_{ij}\} \subseteq \bigoplus_{k=i}^r \mathcal{A}_{kj}$, the matrix L is a lower block-triangular matrix. Furthermore, $\text{Pr}_{\mathcal{A}_{ij}} \mathcal{L}_\ell y_{ij} = \text{Pr}_{\mathcal{A}_{ij}} \ell y_{ij} = \ell_{ii} y_{ij}$ for each $y_{ij} \in \mathcal{A}_{ij}$ implies that the $(i - j + 1)$ -st diagonal

block in L is $\ell_{ii}I_{n_{ij}}$, where $I_{n_{ij}}$ is the identity matrix of order n_{ij} . Thus, L is in fact a lower triangular matrix with n_{ij} copies of ℓ_{ii} on the diagonal for each $i \in \{j, \dots, r\}$. Consequently, the linear operator $\mathcal{L}_\ell|_{\mathcal{C}_j}$ has determinant $\prod_{i=j}^r \ell_{ii}^{n_{ij}}$. Thus, the standard logarithmic barrier of the representing semidefinite cone, when restricted to $\mathcal{L}(K)$, is given by

$$F(\mathcal{L}_{\ell^* \ell}) = - \sum_{j \in J} \sum_{i=j}^r \ln \rho_i(\ell^* \ell)^{2n_{ij}} = - \sum_{i=1}^r \sum_{\substack{j \in J \\ j \leq i}} n_{ij} \ln \rho_i(\ell^* \ell)^2 = f(\mathcal{L}_x) - \sum_{i=1}^r \left(\sum_{\substack{j \in J \\ j \leq i}} n_{ij} - 1 \right) \ln \rho_i(\ell^* \ell)^2,$$

where $\rho_i(\ell^* \ell)$ denotes ℓ_{ii} , and with a slight abuse of notation, $f(\mathcal{L}_x) := f(x) \forall x \in K$.

At this point, it is worth remarking that the standard barrier F is a special case of a family of logarithmically homogeneous, self-concordant barriers for K . This family is given by

$$x \mapsto - \sum_{i=1}^r w_i \ln \rho_i(x)^2,$$

where the weights w_i 's are real constants no less than one. This family will be referred to in Section 6 where we discuss the weighted central paths of positive semidefinite cones.

Now, let us specialize back into the case of irreducible symmetric cones of dimension d and rank r . It seems that $J = \{1\}$ yields the best barrier in terms of its parameter value:

$$F(x) = f(x) - \left(\frac{2(d-r)}{r(r-1)} - 1 \right) \sum_{i=2}^r \ln \rho_i(x)^2.$$

Note that when $K = \mathbb{S}_+^n$, the above barrier is the optimal barrier for K unlike the Jordan Product representation \mathcal{J} which leads to a barrier with parameter $n(n+1)/2$ in this case.

Another interesting choice is $J = \{1, 2, \dots, r\}$ which yields

$$F(x) = f(x) - \frac{2(d-r)}{r(r-1)} \sum_{i=1}^r (i-1) \ln \rho_i(x)^2.$$

The representation in this case also arises from [7] if one recognizes that every homogeneous cone is in fact the ‘‘cone of squares’’

$$\{\ell^* \ell : \ell \in \mathcal{T}\},$$

While this barrier is different from that obtained from the Jordan Product representation, it has the same barrier parameter which equals the dimension of the represented cone.

5 Monotone Solutions of Linear Matrix Equations

In this section, we study certain linear matrix equations whose unique solvability allows maps with nice properties between the usual central path and some weighted central paths.

Lemma 5.1. (Lemma 2.2 of Monteiro and Zanjácomo [13]) Let L be an $n \times n$ lower triangular matrix with $L_{ii} > 0$, $\forall i \in \{1, 2, \dots, n\}$ and $H \in \mathbb{S}^n$ be given. Then there exists a unique lower triangular matrix V such that

$$LV^T + VL^T = H$$

and

$$\|L^{-1}V\|_F^2 = \frac{\|L^{-1}HL^{-T}\|_F^2}{2} - \sum_{i=1}^n [(L^{-1}V)_{ii}]^2.$$

Now, suppose $X \succ 0$, $S \succ 0$. Let L_S denote the lower triangular matrix in the Cholesky decomposition of S (L_S satisfies $L_S L_S^T = S$), and let U_X denote the upper triangular matrix in the inverse Cholesky decomposition of X (U_X satisfies $U_X U_X^T = X$). Then, the following two systems (in unknowns $\Delta X, \Delta S \in \mathbb{S}^n$ and auxiliary variables: U , an upper triangular matrix, and V , a lower triangular matrix) are equivalent (under the substitution(s) $U \leftrightarrow -L_S^{-T} V^T U_X$).

$$(I) \begin{cases} L_S V^T + V L_S^T = \Delta S \\ L_S^T X V + V^T X L_S + L_S^T (\Delta X) L_S = 0 \\ \langle \Delta X, \Delta S \rangle \geq 0; \end{cases}$$

$$(II) \begin{cases} U_X U^T + U U_X^T = \Delta X \\ U_X^T S U + U^T S U_X + U_X^T (\Delta S) U_X = 0 \\ \langle \Delta X, \Delta S \rangle \geq 0. \end{cases}$$

Since we will only be concerned with nontrivial solutions to the above equivalent systems, we can replace the last bilinear inequality by the bilinear equation $\langle \Delta X, \Delta S \rangle = 0$ in each system.

We will use the following well-known fact.

Lemma 5.2. Let $A \in \mathbb{S}^n$, $B \in \mathbb{R}^{n \times n}$. Then

$$\|BAB^T\|_F \geq \lambda_n(BB^T)\|A\|_F.$$

The next theorem uses the techniques of Monteiro and Tsuchiya [11] and of Wolkowicz and the second author [20], and slightly improves the constant in Monteiro and Zanjácomo [13]. We include a proof for the sake of completeness.

Note that the matrices XS , $U_X^T S U_X$ and $L_S^T X L_S$ are similar under the similarity transformations $U_X^{-1} \cdot U_X$ and $L_S^T \cdot L_S^{-T}$ (respectively, for the last two). Therefore, they all have the same (real) eigenvalues. This fact will be used repeatedly.

Theorem 5.1. Let $X \succ 0$, $S \succ 0$ such that

$$\|L_S^T X L_S - \nu I\|_2 < (\sqrt{3} - 1)\nu, \text{ for some } \nu > 0.$$

Then both systems (I) and (II) have the unique solution: $\Delta X = \Delta S = 0$.

Proof. Let $(\Delta X, \Delta S)$ be a solution of (I) (or equivalently of (II)). Then the solution must satisfy

$$\nu L_S^{-1}(\Delta S)L_S^{-T} + L_S^T(\Delta X)L_S = -(L_S^T X L_S - \nu I)L_S^{-1}V - V^T L_S^{-T}(L_S^T X L_S - \nu I) \quad (1)$$

$$\nu U_X^{-1}(\Delta X)U_X^{-T} + U_X^T(\Delta S)U_X = -(U_X^T S U_X - \nu I)U_X^{-1}U - U^T U_X^{-T}(U_X^T S U_X - \nu I). \quad (2)$$

Note that $\langle \Delta X, \Delta S \rangle \geq 0$ implies

$$\langle L_S^{-1}(\Delta S)L_S^{-T}, L_S^T(\Delta X)L_S \rangle = \langle U_X^{-1}(\Delta X)U_X^{-T}, U_X^T(\Delta S)U_X \rangle = \langle \Delta X, \Delta S \rangle \geq 0.$$

Therefore, using Lemma 5.1 and (1), we obtain

$$\left(\nu^2 \|L_S^{-1}(\Delta S)L_S^{-T}\|_F^2 + \|L_S^T(\Delta X)L_S\|_F^2 \right)^{1/2} \leq \sqrt{2} \|L_S^{-1}(\Delta S)L_S^{-T}\|_F \|L_S^T X L_S - \nu I\|_2$$

and (using Lemma 5.1 and (2)) we obtain

$$\left(\nu^2 \|U_X^{-1}(\Delta X)U_X^{-T}\|_F^2 + \|U_X^T(\Delta S)U_X\|_F^2 \right)^{1/2} \leq \sqrt{2} \|U_X^{-1}(\Delta X)U_X^{-T}\|_F \|L_S^T X L_S - \nu I\|_2.$$

Using the last two inequalities and the facts

$$\|(U_X^T L_S)(L_S^{-1}(\Delta S)L_S^{-T})(L_S^T U_X)\|_F \geq \lambda_n(U_X^T S U_X) \|L_S^{-1}(\Delta S)L_S^{-T}\|_F,$$

$$\|(L_S^T U_X)(U_X^{-1}(\Delta X)U_X^{-T})(U_X^T L_S)\|_F \geq \lambda_n(L_S^T X L_S) \|U_X^{-1}(\Delta X)U_X^{-T}\|_F,$$

(we used Lemma 5.2), we conclude that every solution $(\Delta X, \Delta S)$ must satisfy

$$\left[\nu^2 - 2\|L_S^T X L_S - \nu I\|_2^2 + (\lambda_n(L_S^T X L_S))^2 \right] \left(\|U_X^{-1}(\Delta X)U_X^{-T}\|_F^2 + \|L_S^{-1}(\Delta S)L_S^{-T}\|_F^2 \right) \leq 0.$$

Let $\alpha := \frac{1}{\nu} \|L_S^T X L_S - \nu I\|_2$, $\nu := \frac{\lambda_n(L_S^T X L_S) + \lambda_1(L_S^T X L_S)}{2}$. Then $\lambda_n(L_S^T X L_S) = (1 - \alpha)\nu$. Hence, if $\nu^2 - 2\alpha^2\nu^2 + (1 - \alpha)^2\nu^2 > 0$ then $\Delta X = \Delta S = 0$ is implied. Since $\nu > 0$, the condition is equivalent to

$$-\alpha^2 - 2\alpha + 2 > 0.$$

Therefore, if $\alpha < (\sqrt{3} - 1)$ then $\Delta X = \Delta S = 0$ is the unique solution. \square

Note that the related results of [11] can also be improved slightly (the constant $(\sqrt{3} - 1)$ replaces $1/\sqrt{2}$).

6 Neighbourhoods of the Weighted Central Paths

In this section, we consider the SDP-representations of homogeneous cones as given in Section 4. Recall that the standard self-concordant barrier F for a SDP-representation of a homogeneous cone K , when restricted to $\mathcal{L}(K)$, is given by $F(\mathcal{L}_x) = -\sum_{i=1}^r w_i \ln \rho_i^2(x)$, where the integral weights w_i 's depend on the choice of the index set J . This expression was derived by arguing that if $\mathcal{L}_x = U_{\mathcal{L}_x} U_{\mathcal{L}_x}^T$

is the inverse Cholesky decomposition of $\mathcal{L}_x \in \mathbb{S}_{++}^n$, then $U_{\mathcal{L}_x}$ has w_i copies of $\rho_i(x)$ among its diagonal entries. Define the map $\pi : \{1, \dots, n\} \mapsto \{1, \dots, r\}$ by $(U_{\mathcal{L}_x})_{jj} =: \rho_{\pi(j)}(x)$. Recall that an optimal barrier for $\mathcal{L}(K)$ is given by

$$f(x) = f(\mathcal{L}_x) = - \sum_{i=1}^r \ln \rho_i(x)^2.$$

Using the map π , we may write f as

$$f(\mathcal{L}_x) = - \sum_{j=1}^n \frac{1}{w_{\pi(j)}} \ln (U_{\mathcal{L}_x})_{jj}^2.$$

Thus, the optimal barrier f for K is the restriction of a weighted barrier for its representing cone \mathbb{S}_{++}^n to $\mathcal{L}(K)$. Hence, in this section, we consider the relation between the standard central paths for SDP and weighted central paths given by the weighted barriers. (See [3] for some convergence properties of these weighted central paths.)

We consider the semidefinite programming (SDP) problem

$$\inf \{ \langle \bar{S}, X \rangle : X \in L + \bar{X}, X \in \mathbb{S}_+^n \},$$

where $\bar{S}, \bar{X} \in \mathbb{S}^n$ and $L \subset \mathbb{S}^n$ is a subspace, and its dual problem

$$\inf \{ \langle \bar{X}, S \rangle : S \in L^\perp + \bar{S}, S \in \mathbb{S}_+^n \}.$$

Let \mathcal{F}_p and \mathcal{F}_d denote the primal and dual feasible regions respectively, and let $\text{relint}(\mathcal{F}_p)$ and $\text{relint}(\mathcal{F}_d)$ denote the respective relative interiors. We assume that $\mathcal{F}_p \cap \mathbb{S}_{++}^n$ and $\mathcal{F}_d \cap \mathbb{S}_{++}^n$ are both nonempty.

For each $X \in \mathbb{S}^n$, let $\langle\langle X \rangle\rangle$ denote its lower triangular part, i.e., the unique lower triangular matrix L satisfying $L + L^T = X$.

For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, consider the functional $f_\alpha : \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ defined by

$$f_\alpha(X) = - \sum_{i=1}^n \alpha_i \ln (U_X)_{ii}^2.$$

This is called the *weighted logarithmic barrier for \mathbb{S}_{++}^n with weights α* .

To compute the gradient g_α and Hessian H_α of f_α , observe that for any upper triangular matrix U with positive diagonal entries,

$$f_\alpha(\tau_U^{-1} X) = f_\alpha(X) + c_U,$$

where τ_U denotes the automorphism $X \in \mathbb{S}_{++}^n \mapsto U^{-1} X U^{-T}$, and c_U is a constant independent of X . Thus,

$$Df_\alpha(x)[V] = Df_\alpha(\tau_{U_X}^{-1} I)[V] = Df_\alpha(I)[\tau_{U_X} V]$$

(here $V \in \mathbb{S}^n$ is arbitrary) and

$$D^2 f_\alpha(x)[V, W] = D^2 f_\alpha(\tau_{U_X}^{-1} I)[V, W] = D^2 f_\alpha(I)[\tau_{U_X} V, \tau_{U_X} W].$$

It remains to compute $Df_\alpha(I)$ and $D^2f_\alpha(I)$. Since f_α is analytic,

$$\begin{aligned} f_\alpha([I + h\langle\langle V \rangle\rangle]^T[I + h\langle\langle V \rangle\rangle]) &= f_\alpha(I + hV + h^2\bar{V}) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} D^k f_\alpha(I)[hV + h^2\bar{V}, \dots, hV + h^2\bar{V}] \\ &= hDf_\alpha(I)[V] + h^2 \left(Df_\alpha(I)[\bar{V}] + \frac{1}{2} D^2 f_\alpha(I)[V, V] \right) + O(h^3), \end{aligned}$$

where $\bar{V} = \langle\langle V \rangle\rangle^T \langle\langle V \rangle\rangle$. On the other hand,

$$\begin{aligned} f_\alpha([I + h\langle\langle V \rangle\rangle]^T[I + h\langle\langle V \rangle\rangle]) &= - \sum_{i=1}^n \alpha_i \ln(1 + hV_{ii}/2)^2 \\ &= h \left(- \sum_{i=1}^n \frac{1}{2} \alpha_i V_{ii} \right) + h^2 \left(\sum_{i=1}^n \frac{1}{4} \alpha_i V_{ii}^2 \right) + O(h^3) \\ &= -h \operatorname{tr} D_\alpha \langle\langle V \rangle\rangle + h^2 \operatorname{tr} D_\alpha \langle\langle V \rangle\rangle^2, \end{aligned}$$

where D_α denotes the diagonal matrix with α on its diagonal. Hence,

$$Df_\alpha(I)[V] = -\operatorname{tr} D_\alpha \langle\langle V \rangle\rangle$$

and

$$\begin{aligned} D^2 f_\alpha(I)[V, V] &= -2Df_\alpha(I)[\bar{V}] + 2\operatorname{tr} D_\alpha \langle\langle V \rangle\rangle^2 \\ &= 2\operatorname{tr} D_\alpha \langle\langle V \rangle\rangle^T \langle\langle V \rangle\rangle + 2\operatorname{tr} D_\alpha \langle\langle V \rangle\rangle^2 \\ &= \operatorname{tr} (\langle\langle V \rangle\rangle D_\alpha^{1/2} + D_\alpha^{1/2} \langle\langle V \rangle\rangle^T)^2 \\ &= \operatorname{tr} (\mathcal{M}_\alpha V)^2, \end{aligned}$$

where $\mathcal{M}_\alpha : V \mapsto \langle\langle V \rangle\rangle D_\alpha^{1/2} + D_\alpha^{1/2} \langle\langle V \rangle\rangle^T$. Consequently,

$$g_\alpha(X) = U_X^{-T} D_\alpha U_X^{-1} \quad \text{and} \quad H_\alpha(X) : V \mapsto U_X^{-T} \mathcal{M}_\alpha^2 (U_X^{-1} V U_X^{-T}) U_X^{-1}.$$

It is a known fact that the functional f_α is a self-concordant barrier for the convex cone \mathbb{S}_{++} if and only if $\alpha_i \geq 1$ for all $i \in \{1, \dots, n\}$ (see [5, Corollary 2.2]). Clearly, f_α is logarithmically homogeneous with parameter $\sum_i \alpha_i$.

From this point onward, we fix an $\alpha \in \mathbb{R}_{++}^n$ and drop all subscripts α .

It is well-known that the (modified) Legendre-Fenchel conjugate functional $f_* : \mathbb{S}^n \rightarrow \mathbb{R}$ of f , defined by

$$f_*(S) := - \inf_{X \in \mathbb{S}^n} \{ \langle X, S \rangle + f(X) \},$$

is a logarithmically homogeneous self-concordant barrier for $(\mathbb{S}_{++}^n)^*$, the dual cone of \mathbb{S}_{++}^n , which coincides with the cone \mathbb{S}_{++}^n . (Note that the self-concordance property may fail if some $\alpha_i < 1$;

however, the other properties are maintained, as long as $\alpha \in \mathbb{R}_{++}^n$.) The parameter of f_* is $\sum_i \alpha_i$, the same as that of f . The gradient g_* and Hessian H_* of the functional f_* are given by

$$g_*(S) = -L_S^{-T} D L_S^{-1}$$

and

$$H_*(S) : V \mapsto L_S^{-T} \mathcal{M}_*^2(L_S^{-1} V L_S^{-T}) L_S^{-1},$$

where $\mathcal{M}_* : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is the linear map defined by

$$\mathcal{M}_*(V) = D^{1/2} \langle\langle V \rangle\rangle + \langle\langle V \rangle\rangle^T D^{1/2}.$$

For each $\mu > 0$, the minimization problems

$$\inf\{\langle \bar{S}, X \rangle + \mu f(X) : X \in L + \bar{X}\}$$

and

$$\inf\{\langle \bar{X}, S \rangle + \mu f_*(X) : S \in L^\perp + \bar{S}\}$$

have unique solutions, which we denote by $\hat{X}(\mu)$ and $\hat{S}(\mu)$ respectively. The *weighted primal-dual central path with weights α* is the set

$$\{(\hat{X}(\mu), \hat{S}(\mu)) : \mu > 0\} \subset \text{relint}(\mathcal{F}_p) \oplus \text{relint}(\mathcal{F}_d).$$

Alternatively, we can define $(\hat{X}(\mu), \hat{S}(\mu))$ as the unique pair of matrices that satisfies the following set of constraints

$$\begin{aligned} X \in \text{relint}(\mathcal{F}_p), \quad S \in \text{relint}(\mathcal{F}_d), \quad \text{and} \\ L_S^T X L_S = \mu D. \end{aligned}$$

Note that the last constraint is equivalent to $U_X^T S U_X = \mu D$.

We are interested in looking at “nice” maps that take the weighted primal-dual central path to the *standard primal-dual central path*, i.e., the unweighted primal-dual central path.

We focus our attention on the map that takes $(\hat{X}(\mu), \hat{S}(\mu))$ to the pair $(X(\mu), S(\mu))$ on the standard primal-dual central path associated with the same μ . This can be viewed as a composition of the three maps

$$\begin{aligned} \mathcal{T} : \mathbb{S}_{++}^n \oplus \mathbb{S}_{++}^n &\rightarrow \mathbb{S}_{++}^n : (X, S) \mapsto L_S^T X L_S, \\ \tau : \mathbb{S}_{++}^n &\rightarrow \mathbb{S}_{++}^n : V \mapsto D^{-1/2} V D^{-1/2}, \end{aligned}$$

and \mathcal{T}^{-1} . Note that $D\mathcal{T}(X, S)[\Delta X, \Delta S] = 0$ is equivalent to the first two equations in system (I) (and (II)). Of course, \mathcal{T} is not necessarily invertible over $\text{relint}(\mathcal{F}_p) \oplus \text{relint}(\mathcal{F}_d)$ as the next example proves.

Example 6.1. *Let*

$$X := \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, S := \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \xi := -\frac{\sqrt{46} + 1}{2}$$

and

$$\Delta S := \begin{pmatrix} 2 & \xi \\ \xi & 0 \end{pmatrix}, \Delta X := \begin{pmatrix} \frac{1}{3}(10 - 16\xi) & 4 - 6\xi \\ 4 - 6\xi & \frac{10}{3}(\xi - 1) \end{pmatrix}.$$

We observe that with

$$V := \begin{pmatrix} 1 & 0 \\ (\xi - 1) & \frac{1}{\sqrt{3}}(1 - \xi) \end{pmatrix},$$

$\Delta X, \Delta S$ solve system (I) (and (II)).

However, we know from Theorem 5.1 that for $\gamma \in [0, \sqrt{3} - 1)$, the map \mathcal{T} is invertible over the neighbourhood

$$N(\gamma) := \{(X, S) \in \text{relint}(\mathcal{F}_p) \oplus \text{relint}(\mathcal{F}_d) : \|\mathcal{T}(X, S) - \mu I\|_2 \leq \gamma\mu \text{ for some } \mu > 0\}$$

around the standard primal-dual central path, where $\|\cdot\|_2$ is the operator 2-norm. So, for $\gamma \in [0, \sqrt{3} - 1)$, the map $\mathcal{T}^{-1} \circ \tau \circ \mathcal{T}$ is well-defined over the neighbourhood

$$\hat{N}(\gamma) := \{(X, S) \in \text{relint}(\mathcal{F}_p) \oplus \text{relint}(\mathcal{F}_d) : \|\tau(\mathcal{T}(X, S)) - \mu I\|_2 \leq \gamma\mu \text{ for some } \mu > 0\}$$

around the weighted primal-dual central path.

We now consider the inverse map $\mathcal{T}^{-1} \circ \tau^{-1} \circ \mathcal{T}$. For this map to be well-defined in a neighbourhood around the standard primal-dual central path, we need to show that \mathcal{T} is invertible over $\hat{N}(\gamma)$ for some $\gamma > 0$.

For simplicity of notation, let α_{\min} and α_{\max} denote $\min_i \{\alpha_i\}$ and $\max_i \{\alpha_i\}$ respectively. The next theorem uses the techniques of Monteiro and Tsuchiya [11] and generalizes a result in Monteiro and Zanjácomo [13] (also see [12] for some related results) that corresponds to the unweighted case.

Theorem 6.1. *Suppose $\|\tau(\mathcal{T}(X, S)) - \mu I\|_2 \leq \gamma\mu$ for some $\gamma \in [0, \sqrt{\alpha_{\min}/(2\alpha_{\max})})$ and some $\mu > 0$, and $\Delta X, \Delta S \in \mathbb{S}^n$ satisfy $\langle \Delta X, \Delta S \rangle \geq 0$. Then*

$$\max \left\{ \mu \left\| \mathcal{M}(L_S^{-1}(\Delta S)L_S^{-T}) \right\|_F, \left\| \mathcal{M}^{-1}(L_S^T(\Delta X)L_S) \right\|_F \right\} \leq \frac{\|D^{-1/4}(DT(X, S)[\Delta X, \Delta S])D^{-1/4}\|_F}{\left(\frac{\alpha_{\min}}{\alpha_{\max}}\right)^{1/4} - \sqrt{2}\left(\frac{\alpha_{\max}}{\alpha_{\min}}\right)^{1/4}\gamma},$$

where $DT(X, S)[\Delta X, \Delta S] = T\langle\langle L_S^{-1}(\Delta S)L_S^{-T} \rangle\rangle + \langle\langle L_S^{-1}(\Delta S)L_S^{-T} \rangle\rangle^T T + L_S^T(\Delta X)L_S$.

Consequently, the map \mathcal{T} is invertible over $\hat{N}(\gamma)$ whenever $\gamma \in [0, \sqrt{\alpha_{\min}/(2\alpha_{\max})})$.

Proof. For simplicity of notation, let T , V and W denote $\mathcal{T}(X, S)$, $L_S^{-1}(\Delta S)L_S^{-T}$ and $L_S^T(\Delta X)L_S$ respectively. It then follows from $\langle \mathcal{M}^{-1}(W), \mathcal{M}(V) \rangle = \langle W, V \rangle = \langle \Delta X, \Delta S \rangle \geq 0$ that

$$\begin{aligned} \max\{\mu \|\mathcal{M}(V)\|_F, \|\mathcal{M}^{-1}(W)\|_F\} &\leq \left(\|\mu \mathcal{M}(V)\|_F^2 + \|\mathcal{M}^{-1}(W)\|_F^2 \right)^{\frac{1}{2}} \\ &\leq \|\mu \mathcal{M}(V) + \mathcal{M}^{-1}(W)\|_F \\ &\leq \left(\frac{\alpha_{\max}}{\alpha_{\min}} \right)^{\frac{1}{4}} \delta, \end{aligned}$$

where

$$\begin{aligned} \delta &:= \left\| D^{-1/4} \langle \mu \mathcal{M}(V) + \mathcal{M}^{-1}(W) \rangle D^{1/4} + D^{1/4} \langle \mu \mathcal{M}(V) + \mathcal{M}^{-1}(W) \rangle^T D^{-1/4} \right\|_F \\ &= \left\| \begin{array}{c} D^{-1/4} (DT(X, S)[\Delta X, \Delta S]) D^{-1/4} \\ -D^{-1/4} (\tau(T) - \mu I) (\langle V \rangle D^{1/2}) D^{1/4} - D^{1/4} (D^{1/2} \langle V \rangle^T) (\tau(T) - \mu I) D^{-1/4} \end{array} \right\|_F \\ &\leq \left\| D^{-1/4} (DT(X, S)[\Delta X, \Delta S]) D^{-1/4} \right\|_F + 2 \left\| D^{-1/4} (\tau(T) - \mu I) (\langle V \rangle D^{1/2}) D^{1/4} \right\|_F \\ &\leq \left\| D^{-1/4} (DT(X, S)[\Delta X, \Delta S]) D^{-1/4} \right\|_F + 2 \left(\frac{\alpha_{\max}}{\alpha_{\min}} \right)^{\frac{1}{4}} \|\tau(T) - \mu I\|_2 \left\| \langle V \rangle D^{1/2} \right\|_F \\ &\leq \left\| D^{-1/4} (DT(X, S)[\Delta X, \Delta S]) D^{-1/4} \right\|_F + \sqrt{2} \left(\frac{\alpha_{\max}}{\alpha_{\min}} \right)^{\frac{1}{4}} \gamma \mu \|\mathcal{M}(V)\|_F \\ &\leq \left\| D^{-1/4} (DT(X, S)[\Delta X, \Delta S]) D^{-1/4} \right\|_F + \sqrt{2} \left(\frac{\alpha_{\max}}{\alpha_{\min}} \right)^{\frac{1}{4}} \gamma \max\{\mu \|\mathcal{M}(V)\|_F, \|\mathcal{M}^{-1}(W)\|_F\}. \end{aligned}$$

To see the first equation above, expand the first norm and note that $DT(X, S)[\Delta X, \Delta S] = 0$ is a representation of the first two equations in the linear system (I) (and (II)). Thus

$$\begin{aligned} &\left(\frac{\alpha_{\min}}{\alpha_{\max}} \right)^{\frac{1}{4}} \max\{\mu \|\mathcal{M}(V)\|_F, \|\mathcal{M}^{-1}(W)\|_F\} \\ &\leq \left\| D^{-1/4} (DT(X, S)[\Delta X, \Delta S]) D^{-1/4} \right\|_F + \sqrt{2} \left(\frac{\alpha_{\max}}{\alpha_{\min}} \right)^{\frac{1}{4}} \gamma \max\{\mu \|\mathcal{M}(V)\|_F, \|\mathcal{M}^{-1}(W)\|_F\}. \end{aligned}$$

Consequently, for any $\gamma \in [0, \sqrt{\alpha_{\min}/(2\alpha_{\max})}]$, the map \mathcal{T} has nonsingular Jacobian over $\hat{N}(\gamma)$, and hence is invertible over the same neighbourhood. \square

The next example shows that the bound $\sqrt{\alpha_{\min}/(2\alpha_{\max})}$ is tight up to a constant multiplicative factor.

Example 6.2. Suppose $\alpha_1 (= \alpha_{\max}) \geq \alpha_2 \geq \dots \geq \alpha_n (= \alpha_{\min})$. In the following, all matrices are expressed as 3-by-3 block matrices where the four corner blocks are scalars. Let

$$X = \begin{pmatrix} \alpha_{\max} & \mathbf{0} & 3\alpha_{\min} \\ \mathbf{0} & \text{Diag}(\alpha_2, \dots, \alpha_{n-1}) & \mathbf{0} \\ 3\alpha_{\min} & \mathbf{0} & \alpha_{\min} \end{pmatrix}, \quad S = I,$$

$$\Delta X = \begin{pmatrix} 12\alpha_{\min}\alpha_{\max} & \mathbf{0} & \alpha_{\min}(54\alpha_{\min} + 2\alpha_{\max}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \alpha_{\min}(54\alpha_{\min} + 2\alpha_{\max}) & \mathbf{0} & 4\alpha_{\min}\alpha_{\max} \end{pmatrix} \quad \text{and} \quad \Delta S = \begin{pmatrix} 36\alpha_{\min} & \mathbf{0} & -4\alpha_{\max} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -4\alpha_{\max} & \mathbf{0} & 4\alpha_{\max} \end{pmatrix}.$$

We observe that with

$$V = \begin{pmatrix} 18\alpha_{\min} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -4\alpha_{\max} & \mathbf{0} & 2\alpha_{\max} \end{pmatrix},$$

$\Delta X, \Delta S$ solve system (I) (and (II)). Moreover, $(X, S) \in \hat{N}(3\sqrt{\alpha_{\min}/\alpha_{\max}})$. \square

Corollary 6.1. For any $\gamma \in [0, \sqrt{\alpha_{\min}/(2\alpha_{\max})})$, the map $\mathcal{T}^{-1} \circ \tau \circ \mathcal{T}$ is a diffeomorphism over $\hat{N}(\gamma)$.

Proof. By Theorem 6.1 the map \mathcal{T} has nonsingular Jacobian at any $(X, S) \in \hat{N}(\gamma)$ whenever $\gamma \in [0, \sqrt{\alpha_{\min}/(2\alpha_{\max})})$. Furthermore, \mathcal{T}^{-1} is well-defined over $\tau(\mathcal{T}(\hat{N}(\gamma)))$ for any $\gamma \in [0, \sqrt{\alpha_{\min}/(2\alpha_{\max})}) \subset [0, \sqrt{3}-1)$. Moreover, it has nonsingular Jacobian at any $V \in \tau(\mathcal{T}(\hat{N}(\gamma)))$. Finally, τ is clearly a diffeomorphism over \mathbb{S}_{++}^n . \square

Consider a homogeneous cone programming problem (P) and one of its SDP-representations given in Section 4. Corollary 6.1 indicates that the mapping $\mathcal{T}^{-1} \circ \tau \circ \mathcal{T}$ and its inverse are well-defined and differentiable in a neighbourhood of the central path of the original problem (formulated using the optimal barrier for the underlying homogenous cone) and its SDP representation (using the barrier $-\ln \det(\cdot)$ on the underlying \mathbb{S}_+^n). The scale-invariant *size* of this neighbourhood (measured by γ above) is at least $\sqrt{\frac{\alpha_{\min}}{2\alpha_{\max}}}$. Example 6.2 proves that the ratio $\sqrt{\frac{\alpha_{\min}}{\alpha_{\max}}}$ is the largest (up to a small constant factor) such neighbourhood over which the mapping $\mathcal{T}^{-1} \circ \tau \circ \mathcal{T}$ is a diffeomorphism.

We deduce that the efficiency of the representation depends on the ratio $\frac{\alpha_{\min}}{\alpha_{\max}} \in (0, 1]$. Larger values of this ratio correspond to better representations in the above context, because the mapping is well-defined and is a diffeomorphism over a larger neighbourhood of the central path.

We conclude with another class of simple examples indicating again that the ratio $\frac{\alpha_{\min}}{\alpha_{\max}}$ is not just a by-product of our approach; but, it seems inherent in the way the representation deforms the usual central path.

Example 6.3. Consider as K an arbitrary homogeneous cone admitting an SDP-representation via \mathcal{L} . For a given set of \mathcal{A} , b , c and μ , we can look at how bad of an automorphism $A_x \in \text{Aut}(\mathbb{S}_+^n)$ we have to use to get

$$\tilde{x}(\mu) = A_x(\mathcal{L}(x(\mu))).$$

Let us pick a simple class of specific examples. Consider the primal SDP problem

$$\sup \{ \langle I, X \rangle : \langle I, X \rangle \leq 1, X \succeq 0 \}.$$

Then the dual central path is given by $S_y := (y-1)I$ depending on the dual variable y . For a fixed central path parameter $\mu > 0$, we have the primal central point $X(\mu) = \frac{\mu}{y(\mu)-1}I$. For weighted central

paths (weights described by the positive definite diagonal matrix D), using the central path equation $\hat{L}_S^T \hat{X} \hat{L}_S = \mu D$, we have $\hat{X}(\mu) = \frac{\mu}{y(\mu)-1} D$.

Now, if we focus on automorphisms mapping $\hat{X}(\mu)$ to $X(\mu)$, we see that their condition number must involve the ratio $\frac{\alpha_{\max}}{\alpha_{\min}}$; i.e., $D^{-1/2} \cdot D^{-1/2}$ has the condition number $\frac{\alpha_{\max}}{\alpha_{\min}}$. \square

References

- [1] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization*, MPS-SIAM Series on Optimization, SIAM, Philadelphia, 2001.
- [2] B. J. Braams, Dual descriptions of semidefinite and other non-polyhedral cones, *manuscript*, September 2002.
- [3] C. B. Chua, A new notion of weighted centers for semidefinite programming, *SIAM J. Optim.*, to appear.
- [4] C. B. Chua, Relating homogeneous cones and positive definite cones via T -algebras, *SIAM J. Optim.* 14 (2004) 500-506.
- [5] C. B. Chua, An algebraic perspective on homogeneous cone programming, and the primal-dual second order cone approximations algorithm for symmetric cone programming, *Ph.D. thesis*, Cornell University, May 2003.
- [6] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford University Press, NY, USA, 1994.
- [7] L. Faybusovich, On Nesterov's approach to semi-infinite programming, *Acta Appl. Math.* 74 (2002) 195–215.
- [8] O. Güler, Hyperbolic polynomials and interior point methods for convex programming, *Math. Oper. Res.* 22 (1997) 350–377.
- [9] J. W. Helton and V. Vinnikov, Linear matrix inequality representation of sets, *Technical Report*, 2002.
- [10] A. S. Lewis, P. A. Parrilo and M. V. Ramana, The Lax conjecture is true, *Proc. Amer. Math. Soc.* 133 (2005) 2495–2499.
- [11] R. D. C. Monteiro and T. Tsuchiya, Polynomial convergence of a new family of primal-dual algorithms for semidefinite programming, *SIAM J. Optim.* 9 (1999) 551–577.
- [12] R. D. C. Monteiro and P. Zanjácomo, General interior-point maps and existence of weighted paths for nonlinear semidefinite complementarity problems, *Math. Oper. Res.* 25 (2000) 381–399.

- [13] R. D. C. Monteiro and P. Zanjácomo, Implementation of primal-dual methods for semidefinite programming based on Monteiro and Tsuchiya Newton directions and their variants, *Optim. Method. Soft.* 11/12 (1999) 91–140.
- [14] Yu. E. Nesterov and A. S. Nemirovskii, *Interior-Point Polynomial Algorithms in Convex Programming*, SIAM, Philadelphia, PA, USA, 1994.
- [15] Yu. E. Nesterov and M. J. Todd, Self-scaled barriers and interior-point methods for convex programming, *Math. Oper. Res.* 22 (1997) 1–46.
- [16] G. Pataki, Cone LP's and semidefinite programs: Geometry and a simplex type method, *Proc. Fifth IPCO Conference*, W. H. Cunningham, S. T. McCormick and M. Queyranne (eds.), Springer-Verlag, 1996, pp. 161–174.
- [17] J. Renegar, Hyperbolic Programs and Their Derivative Relaxations, *Technical Report*, March 2004.
- [18] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, NJ, USA, 1970.
- [19] V. A. Truong and L. Tunçel, Geometry of homogeneous convex cones, duality mapping, and optimal self-concordant barriers, *Math. Prog. A* 100 (2004) 295–316.
- [20] L. Tunçel and H. Wolkowicz, Strengthened existence and uniqueness conditions for search directions in semidefinite programming, *Linear Algebra Appl.* 400 (2005) 31–60.
- [21] L. Tunçel and S. Xu, On homogeneous convex cones, the Carathéodory number, and the duality mapping, *Math. Oper. Res.* 26 (2001) 234–247.
- [22] È. B. Vinberg, The theory of homogeneous cones, *Trans. Moscow Math. Soc.* 12 (1965) 340–403.
- [23] V. Vinnikov, Self-adjoint determinantal representations of real plane curves, *Mathematische Annalen* 296 (1993) 453–479.