Clarke Generalized Jacobian of the Projection onto Symmetric Cones and Its Applications *

Lingchen Kong[1], Levent Tunçel[2], and Naihua Xiu[3]

May 27, 2008 (revised: April 14, 2009)
C & O Research Report CORR 2008–04

Abstract

In this paper, we give an exact expression for Clarke generalized Jacobian of the projection onto symmetric cones, which is linked to rank-1 matrices. As an application, we employ the projection operator to design a semismooth Newton algorithm for solving nonlinear symmetric cone programs. The algorithm is proved to be locally quadratically convergent without assuming strict complementarity of the solution.

Keywords: Clarke generalized Jacobian, Projection, Symmetric cones, Euclidean Jordan algebra, Semismooth Newton algorithm.

MSC2000 Subject Classification: Primary: 65K05, 90C30; Secondary: 26B05.

Abbreviated Title: Clarke Generalized Jacobian of the Projection

1 Introduction

This paper focuses on Clarke generalized Jacobian of the projection onto symmetric cones. First, we recall some basic concepts. Let $F : \Omega \subseteq \mathcal{X} \to \mathcal{Y}$ be a locally Lipschitz function on an open set Ω, where $\mathcal{X}$ and $\mathcal{Y}$ are two finite dimensional inner product spaces over the field $\mathbb{R}$. Let $\nabla F(x)$ denote the *derivative* of F at x if F is differentiable at x. The *Clarke generalized Jacobian* of F at x is defined by $\partial F(x) := \mathrm{conv}\{\partial_B F(x)\}$, where $\partial_B F(x) := \{\lim_{\bar{x}\to x, \bar{x}\in D_F} \nabla F(\bar{x})\}$ is the *B-subdifferential* of F at x, and D_F is the set of points of Ω where F is differentiable. We assume that the reader is familiar with the concepts of (strong) semismoothness, and refer to [5, 6, 20, 21, 23] for details.

It is well-known that the scalar-valued function $g : \mathbb{R} \to \mathbb{R}_+$ with $g(t) = t_+ := \max\{0, t\}$ is strongly semismooth and its Clarke generalized Jacobian is specified by t_+/t if $t \neq 0$ and the interval $[0, 1]$ if $t = 0$. Note that the projection onto an arbitrary closed convex set is

*The work was partly supported by a Discovery Grant from NSERC, and the National Natural Science Foundation of China (10671010, 70640420143).

[1] Lingchen Kong, Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada; Department of Applied Mathematics, Beijing Jiaotong University, Beijing 100044, P. R. China (e-mail: konglchen@126.com)

[2] Levent Tunçel, Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada (e-mail: ltuncel@math.uwaterloo.ca)

[3] Naihua Xiu, Department of Applied Mathematics, Beijing Jiaotong University, Beijing 100044, P. R. China (e-mail: nhxiu@bjtu.edu.cn)

Lipschitz continuous. In the case of second-order cone, we know that the projection operator is strongly semismooth, see, e.g., [3, 4, 12]. In particular, Hayashi, Yamashita and Fukushima [12] gave an explicit representation for Clarke generalized Jacobian of the projection onto second-order cones. Malick and Sendov [18] studied the differentiability properties of the projection onto cone of positive semi-definite matrices and worked out its Clarke generalized Jacobian. Recently, Sun and Sun [26] showed that the projection onto symmetric cones, which include the nonnegative orthant, the cone of positive semi-definite matrices and the second-order cone as special cases, is strongly semismooth everywhere. Kong, Sun and Xiu [16] studied the Clarke generalized Jacobian of the projection onto symmetric cones and gave its upper bound and lower bound. Meng, Sun and Zhao [19] studied the Clarke generalized Jacobian of the projection onto nonempty closed convex sets and showed that any element V in its Clarke generalized Jacobian is self-adjoint and $V \succeq V^2$ (Proposition 1 of [19] for details).

One of the areas where projection operator is widely applied is the area of complementarity problems, see, e.g., [6]. Also, projection operator is a fundamental ingredient of many algorithms for solving convex optimization problems (see for instance the survey by Bauschke and Borwein [1] and the references therein). Our work is in the setting of symmetric cones which unifies and generalizes the special cases of second-order cones, cones of Hermitian positive semi-definite matrices over reals, complex numbers, quaternions as well as the direct sums of any subset of these cones.

On another front, consider the *nonlinear symmetric cone program* (NSCP), which is defined by

$$\begin{array}{cr} \min & f(x) \\ \text{s.t.} & \mathcal{A}x = q, \\ & x \in K, \end{array} \tag{1.1}$$

where $f : \mathcal{J} \to \mathbb{R}$ is a twice continuously differentiable function, $\mathcal{A} : \mathcal{J} \to \mathbb{R}^m$ is a linear operator, $q \in \mathbb{R}^m$, $\mathcal{J}$ is a n-dimensional inner product space over real field $\mathbb{R}$ and $\mathcal{V} := (\mathcal{J}, \langle \cdot, \cdot \rangle, \circ)$ is a Euclidean Jordan algebra (see Section 2), K is the symmetric cone in $\mathcal{V}$. When $f(x)$ is linear, NSCP becomes a *linear symmetric cone program* (SCP). NSCP provides a simple, natural, and unified framework for various optimization programs, such as nonlinear second-order cone programs (NSOCP) and nonlinear semidefinite programs (NSDP). The research activity in NSCP has been increasing recently, see, e.g., [9, 10, 11, 17, 26, 27, 28]. These papers deal with either certain theoretical properties of some functions on Euclidean Jordan algebras, or interior-point methods and smoothing methods. However, to the best of our knowledge, nonsmooth algorithms for NSCP have not enjoyed the same level of activity except for some special cases. For instance, in the case of NSOCP, Kanzow, Ferenczi and Fukushima [13] reformulated its optimality conditions as a nonsmooth system of equations by applying a projection operator, and established nonsmooth Newton methods which are locally quadratically convergent under some suitable conditions.

Motivated by all of the work cited above, we aim to give the exact expression of the Clarke generalized Jacobian of the projection onto symmetric cones x_+ (or $\Pi_K(x)$, see Section 2), and to establish nonsmooth Newton methods for NSCP. Our result generalizes corresponding results of [12] and [18] (from second-order cones and positive semi-definite cones respectively) to symmetric cones. Interesting enough, the expression of the Clarke generalized Jacobian of x_+ is linked to rank one matrices. This allows us to obtain the formulae of operators x_- and $|x|$ in a similar manner. As an application to NSCP, we design semismooth Newton methods by employing the projection operator.

This paper is organized as follows. In the next section, we establish the preliminaries and study the relationship among all the Jordan frames of an element. We also introduce the matrix representation of the Jacobian operator of x_+. Moreover, based on the partition of the index set, we give a decomposition of Euclidean Jordan algebra (see Theorem 2.8). In Section 3, we present the exact expression of the Clarke generalized Jacobian of the projection operator x_+ by studying its B-subdifferential. In Section 4, we investigate the relationships among B-subdifferentials of x_+, x_- and $|x|$. In Section 5, we design semismooth Newton methods for NSCP, and show that the proposed algorithm is locally quadratically convergent under some conditions which do not require strict complementarity of the solution.

2 Preliminaries

2.1 Euclidean Jordan algebras

We give a brief introduction to Euclidean Jordan algebras. More comprehensive introduction to the area can be found in Koecher's lecture notes [15] and in the monograph by Faraut and Korányi [7].

A *Euclidean Jordan algebra* is a triple $(\mathcal{J}, \langle\cdot,\cdot\rangle, \circ)$ ($\mathcal{V}$ for short), where $(\mathcal{J}, \langle\cdot,\cdot\rangle)$ is a n-dimensional inner product space over $\mathbb{R}$ and $(x, y) \mapsto x \circ y : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is a bilinear mapping which satisfies the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathcal{J}$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathcal{J}$ where $x^2 := x \circ x$,
(iii) $\langle x \circ y, w\rangle = \langle x, y \circ w\rangle$ for all $x, y, w \in \mathcal{J}$.

We call $x \circ y$ the *Jordan product* of x and y. In general, $(x \circ y) \circ w \neq x \circ (y \circ w)$ for all $x, y, w \in \mathcal{J}$. We assume that there exists an element e (called the *identity* element) such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{J}$. Given a Euclidean Jordan algebra $\mathcal{V}$, define *the set of squares* as $K := \{x^2 : x \in \mathcal{J}\}$. By Theorem III 2.1 in [7], K is the *symmetric cone*, i.e., K is a closed, convex, homogeneous and self-dual cone. For $x \in \mathcal{J}$, the *degree* of x denoted by $\deg(x)$ is the smallest positive integer k such that the set $\{e, x, x^2, \cdots, x^k\}$ is linearly dependent. The *rank* of $\mathcal{V}$ is defined as $\max\{\deg(x) : x \in \mathcal{J}\}$. In this paper, r will denote the rank of the underlying Euclidean Jordan algebra. An element $c \in \mathcal{J}$ is an *idempotent* if $c^2 = c \neq 0$, which is also *primitive* if it cannot be written as a sum of two idempotents. A *complete system of orthogonal idempotents* is a finite set $\{c_1, c_2, \cdots, c_k\}$ of idempotents where $c_i \circ c_j = 0$ for all $i \neq j$, and $c_1 + c_2 + \cdots + c_k = e$. A *Jordan frame* is a complete system of orthogonal primitive idempotents in $\mathcal{V}$.

We now review two spectral decomposition theorems for the elements in a Euclidean Jordan algebra.

Theorem 2.1 *(Spectral Decomposition Type I (Theorem III.1.1, [7])) Let $\mathcal{V}$ be a Euclidean Jordan algebra. Then for $x \in \mathcal{J}$ there exist unique real numbers $\mu_1(x), \mu_2(x), \cdots, \mu_{\bar{r}}(x)$, all distinct, and a unique complete system of orthogonal idempotents $\{b_1, b_2, \cdots, b_{\bar{r}}\}$ such that*

$$x = \mu_1(x)b_1 + \cdots + \mu_{\bar{r}}(x)b_{\bar{r}}.$$

Theorem 2.2 *(Spectral Decomposition Type II (Theorem III.1.2, [7])) Let $\mathcal{V}$ be a Euclidean Jordan algebra with rank r. Then for $x \in \mathcal{J}$ there exist a Jordan frame $\{c_1, c_2, \cdots, c_r\}$ and real numbers $\lambda_1(x), \lambda_2(x), \cdots, \lambda_r(x)$ such that*

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r. \tag{2.1}$$

The numbers $\lambda_i(x)$ $(i = 1, 2, \cdots, r)$ are the eigenvalues of x. We call (2.1) the spectral decomposition (or the spectral expansion) of x.

Note that the above $\{b_1, b_2, \cdots, b_{\bar{r}}\}$ and $\{c_1, c_2, \cdots, c_r\}$ depend on x. We do not write this dependence explicitly for the sake of simplicity in notation. Let $\mathcal{C}(x)$ be the set consisting of all Jordan frames in the spectral decomposition of x. Let *the spectrum* $\sigma(x)$ be the set of all eigenvalues of x. Then $\sigma(x) = \{\mu_1(x), \mu_2(x), \cdots, \mu_{\bar{r}}(x)\}$ and for each $\mu_i(x) \in \sigma(x)$, denoting $N_i(x) := \{j : \lambda_j(x) = \mu_i(x)\}$ we have $b_i = \sum_{j \in N_i(x)} c_j$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Define the vector-valued function $G : \mathcal{J} \to \mathcal{J}$ as

$$G(x) := \sum_{j=1}^{r} g(\lambda_j(x))c_j(x) = g(\lambda_1(x))c_1(x) + g(\lambda_2(x))c_2(x) + \cdots + g(\lambda_r(x))c_r(x), \tag{2.2}$$

which is a *Löwner operator*. In particular, letting $t_+ := \max\{0, t\}$, $t_- := \min\{0, t\}$ and noting $|t| = t_+ - t_-$ $(t \in \mathbb{R})$, respectively, we define

$$\Pi_K(x) := x_+ := \sum_{i=1}^{r} (\lambda_i(x))_+ c_i(x), \quad x_- := \sum_{i=1}^{r} (\lambda_i(x))_- c_i(x) \text{ and } |x| := \sum_{i=1}^{r} |\lambda_i(x)| c_i(x).$$

Note that $z \in K$ $(z \in \text{int}(K))$ if and only if $\lambda_i(z) \geq 0$ $(\lambda_i(z) > 0)$ $(\forall i \in \{1, 2, \cdots, r\})$, where $\text{int}(K)$ denotes the interior of K. It is easy to verify that

$$x_+ \in K, \quad x = x_+ + x_-, \text{ and } |x| = x_+ - x_-.$$

In other words, x_+ is the *projection* of x onto K.

For each $x \in \mathcal{J}$, we define the *Lyapunov transformation* $\mathcal{L}(x) : \mathcal{J} \to \mathcal{J}$ by $\mathcal{L}(x)y = x \circ y$ for all $y \in \mathcal{J}$, which is a symmetric self-adjoint operator in the sense that $\langle \mathcal{L}(x)y, w \rangle = \langle y, \mathcal{L}(x)w \rangle$ for all $y, w \in \mathcal{J}$. The operator $\mathcal{Q}(x) := 2\mathcal{L}^2(x) - \mathcal{L}(x^2)$ is called the *quadratic representation* of x. We say two elements $x, y \in \mathcal{J}$ *operator commute* if $\mathcal{L}(x)\mathcal{L}(y) = \mathcal{L}(y)\mathcal{L}(x)$. By Lemma X.2.2 in [7], two elements x, y operator commute if and only if they share a common Jordan frame.

Next, we recall the Peirce decomposition on the space $\mathcal{J}$. Let $c \in \mathcal{J}$ be a nonzero idempotent. Then $\mathcal{J}$ is the orthogonal direct sum of $J(c, 0), J(c, \frac{1}{2})$ and $J(c, 1)$, where

$$J(c, \varepsilon) := \{z \in \mathcal{J} : c \circ z = \varepsilon z\}, \quad \varepsilon \in \left\{0, \frac{1}{2}, 1\right\}.$$

This is called the *Peirce decomposition* of $\mathcal{J}$ with respect to the nonzero idempotent c. Fix a Jordan frame $\{c_1, c_2, \cdots, c_r\}$. Defining the following subspaces for $i, j \in \{1, 2, \cdots, r\}$,

$$J_{ii} := \{x \in \mathcal{J} : x \circ c_i = x\} \text{ and } J_{ij} := \left\{x \in \mathcal{J} : x \circ c_i = \frac{1}{2}x = x \circ c_j\right\}, \ i \neq j,$$

we have the Peirce decomposition theorem as follows. For more detail, see [7].

Theorem 2.3 *(Theorem IV.2.1, [7]) Let $\{c_1, c_2, \cdots, c_r\}$ be a given Jordan frame in a Euclidean Jordan algebra $\mathcal{V}$ of rank r. Then $\mathcal{J}$ is the orthogonal direct sum of spaces J_{ij} $(i \leq j)$. Furthermore,*

(i) $J_{ij} \circ J_{ij} \subseteq J_{ii} + J_{jj}$;

(ii) $J_{ij} \circ J_{jk} \subseteq J_{ik}$, *if* $i \neq k$;

(iii) $J_{ij} \circ J_{kl} = \{0\}$, *if* $\{i, j\} \cap \{k, l\} = \emptyset$.

For any $i \neq j \in \{1,2,\cdots,r\}$ and $k \neq l \in \{1,2,\cdots,r\}$, by Corollary IV.2.6 of [7], we have $\dim(J_{ij}) = \dim(J_{kl}) =: \bar{n}$. Then

$$n = r + \frac{\bar{n}}{2} r(r-1). \tag{2.3}$$

For a given Jordan frame $\{c_1, c_2, \cdots, c_r\}$ and $i,j \in \{1,2,\cdots,r\}$, let $\mathcal{C}_{ij}(x)$ be the orthogonal projection operator onto subspace J_{ij}. Then, by Theorem 2.3, we have

$$\mathcal{C}_{jj}(x) = \mathcal{Q}(c_j) \text{ and } \mathcal{C}_{ij}(x) = 4\mathcal{L}(c_i)\mathcal{L}(c_j) = 4\mathcal{L}(c_j)\mathcal{L}(c_i) = \mathcal{C}_{ji}(x),\ i,j \in \{1,2,\cdots,r\}, \tag{2.4}$$

and the orthogonal projection operators $\{\mathcal{C}_{ij}(x) : i,j \in \{1,2,\cdots,r\}\}$ satisfy

$$\mathcal{C}_{ij}(x) = \mathcal{C}_{ij}^*(x),\ \mathcal{C}_{ij}(x)^2 = \mathcal{C}_{ij}(x),\ \mathcal{C}_{ij}(x)\mathcal{C}_{kl}(x) = 0 \text{ if } \{i,j\} \neq \{k,l\}, i,j,k,l \in \{1,2,\cdots,r\}$$

and

$$\sum_{1 \le i \le j \le r} \mathcal{C}_{ij}(x) = \mathcal{I},$$

where $\mathcal{C}_{ij}^*(x)$ is the adjoint operator of $\mathcal{C}_{ij}(x)$ and $\mathcal{I}$ is the identity operator. Furthermore, we have the following spectral decomposition theorem for $\mathcal{L}(x)$, $\mathcal{L}(x^2)$ and $\mathcal{Q}(x)$. For a more detailed exposition, see [15, 26].

Theorem 2.4 *(Theorem 3.1, [26]) Let $x \in \mathcal{J}$ and $\sum_{j=1}^r \lambda_j(x) c_j(x)$ denote its spectral decomposition with $\lambda_1(x) \ge \lambda_2(x) \ge \cdots \ge \lambda_r(x)$. Then, the symmetric operators $\mathcal{L}(x)$, $\mathcal{L}(x^2)$ and $\mathcal{Q}(x)$ have the spectral decompositions:*

$$\begin{aligned}
\mathcal{L}(x) &= \sum_{j=1}^r \lambda_j(x)\mathcal{C}_{jj}(x) + \sum_{1 \le j < l \le r} \frac{1}{2}(\lambda_j(x) + \lambda_l(x))\mathcal{C}_{jl}(x), \\
\mathcal{L}(x^2) &= \sum_{j=1}^r \lambda_j^2(x)\mathcal{C}_{jj}(x) + \sum_{1 \le j < l \le r} \frac{1}{2}(\lambda_j^2(x) + \lambda_l^2(x))\mathcal{C}_{jl}(x), \\
\mathcal{Q}(x) &= \sum_{j=1}^r \lambda_j^2(x)\mathcal{C}_{jj}(x) + \sum_{1 \le j < l \le r} \lambda_j(x)\lambda_l(x)\mathcal{C}_{jl}(x).
\end{aligned}$$

Moreover, the spectra $\sigma(\mathcal{L}(x))$, $\sigma(\mathcal{L}(x^2))$ and $\sigma(\mathcal{Q}(x))$, respectively, consist of all distinct numbers $\frac{1}{2}(\lambda_j(x) + \lambda_l(x))$, $\frac{1}{2}(\lambda_j^2(x) + \lambda_l^2(x))$, and $\lambda_j(x)\lambda_l(x)$ for all $j,l \in \{1,2,\cdots,r\}$.

In the end of this subsection, we characterize a cone $K_{\Im}$ (see (2.5) below) with respect to x, which plays a key role in establishing the connection between $\partial_B \Pi_K(x)$ and $\partial_B \Pi_{K_{\Im}}(0)$ (see Theorem 3.3). Let $x = \sum_{j=1}^r \lambda_j(x) c_j = \sum_{i=1}^{\bar{r}} \mu_i(x) b_i$ with $\lambda_1(x) \ge \lambda_2(x) \ge \cdots \ge \lambda_r(x)$ and $\mu_1(x) > \mu_2(x) > \cdots > \mu_{\bar{r}}(x)$. In what follows, let $\aleph(x) := \{i : \lambda_i(x) < 0\}$, $\Im(x) := \{i : \lambda_i(x) = 0\}$ and $\wp(x) := \{i : \lambda_i(x) > 0\}$. For the sake of simplicity, let $\aleph := \aleph(x)$, $\Im := \Im(x)$ and $\wp := \wp(x)$. Thus $\aleph \cup \Im \cup \wp = \{1,2,\cdots,r\}$. Set

$$b_{\Im} := \sum_{j \in \Im} c_j.$$

By Theorem 2.1, it follows that $b_{\Im}$ is uniquely defined by x. In other words, for any Jordan frame $\{\bar{c}_1, \cdots, \bar{c}_r\} \in \mathcal{C}(x)$, $\sum_{j \in \Im} \bar{c}_j = b_{\Im}$. Therefore, we can define a subspace

$$J_{\Im} := J(b_{\Im}, 1) := \{w \in \mathcal{J} : w \circ b_{\Im} = w\}.$$

By Lemma 20 in [9], it follows that

$$J_{\Im} = \text{span}\left\{c_{|\wp|+1}, c_{|\wp|+2}, \cdots, c_{|\wp|+|\Im|}\right\} + \sum_{|\wp|+1 \le j < k \le |\wp|+|\Im|} J_{jk}.$$

Then we can verify by direct calculation that $J_{\Im}$ is also a Euclidean Jordan algebra with its identity element $b_{\Im}$. This leads us to define the cone of its squares as

$$K_{\Im} := \{w^2 : w \in J_{\Im}\}. \tag{2.5}$$

Therefore, it is well-defined that $\Pi_{K_{\Im}} : J_{\Im} \to J_{\Im}$ is the projection onto the symmetric cone $K_{\Im}$.

We can also define a cone by the finite set $\{c_j : j \in \Im\}$. Thus, we arrive at another cone associated with x, denoted by

$$K(x,0) := \bigcup_{\{c_1,\cdots,c_r\} \in \mathcal{C}(x)} \text{Cone}\{c_j : j \in \Im\}. \tag{2.6}$$

An interesting question occurs: what is the difference between the cones $K_{\Im}$ and $K(x,0)$? Clearly, when $x = 0$, $K_{\Im} = K(0,0)(= K)$. The following proposition states that they always equal to each other.

Proposition 2.5 *Let $x = \sum_{j=1}^{r} \lambda_j(x) c_j$ with $\lambda_1(x) \ge \lambda_2(x) \ge \cdots \ge \lambda_r(x)$ and $b_{\Im} = \sum_{j \in \Im} c_j$. Let $K_{\Im}$ and $K(x,0)$ be defined by (2.5) and (2.6), respectively. Then we have*

$$K_{\Im} = K(x,0) = \bigcup_{\{c_1,\cdots,c_r\} \in \mathcal{C}(x)} \text{Cone}\{c_j : j \in \Im\} \tag{2.7}$$

and $b_{\Im}$ is the identity element in the above symmetric cone.

Proof. It is clear that $K(x,0) \subseteq K_{\Im}$ and $b_{\Im}$ is the identity element in $K_{\Im}$.

We now prove $K_{\Im} \subseteq K(x,0)$. Let $y \in K_{\Im}$. Then there exists a Jordan frame in $J_{\Im(x)}$, say, $\{e_1, e_2, \cdots, e_{|\Im|}\}$, such that $y = y_1 e_1 + y_2 e_2 + \cdots + y_{|\Im|} e_{|\Im|}$ with $y_i \ge 0$. Clearly, elements $e_1, e_2, \cdots, e_{|\Im|}$ belong to K and the sum of them is $b_{\Im}$. We next show that, for any Jordan frame $\{c_i : i \in \{1, \cdots, r\}\} \in \mathcal{C}(x)$, we can replace $\{c_{|\wp|+1}, \cdots, c_{|\wp|+|\Im|}\}$ by $\{e_1, \cdots, e_{|\Im|}\}$ to get an element in $\mathcal{C}(x)$; I.e.,

$$\{c_1, \cdots, c_{|\wp|}, e_1, \cdots, e_{|\Im|}, c_{|\wp|+|\Im|+1}, \cdots, c_r\} \in \mathcal{C}(x). \tag{2.8}$$

In fact, by the above definitions, we have

$$c_k \circ b_{\Im} = 0, \text{ for all } k \in \{1, \cdots, |\wp|, |\wp| + |\Im| + 1, \cdots, r\}.$$

Then $\langle c_k, b_{\Im} \rangle = 0$. By $b_{\Im} = e_1 + \cdots + e_{|\Im|}$, it follows that $\langle c_k, b_{\Im} \rangle = \sum_{i=1}^{|\Im|} \langle c_k, e_i \rangle = 0$. Since $c_k, e_i \in K$, $\langle c_k, e_i \rangle \ge 0$. It then follows $\langle c_k, e_i \rangle = 0$. By Proposition 6 in [9], we have

$$c_k \circ e_i = 0, \quad \text{for all} \quad k \in \{1, \cdots, |\wp|, |\wp| + |\Im| + 1, \cdots, r\}, \; i \in \{1, \cdots, |\Im|\}.$$

This proves (2.8). Then $\text{Cone}\{e_j : j \in \{1, \cdots, |\Im|\}\} \subseteq K(x,0)$, and therefore $y \in K(x,0)$. $\square$

Remark From the above analysis, for $x = \sum_{j=1}^{r} \lambda_j(x) c_j = \sum_{i=1}^{\bar{r}} \mu_i(x) b_i$, we can also obtain that for any Jordan frame $\{c_1, \cdots, c_r\} \in \mathcal{C}(x)$, the set $\{c_j : \lambda_j(x) = \mu_i(x)\}$ is a Jordan frame in $J(b_i, 1)$, and a Jordan frame in each $J(b_i, 1)$ $(i \in \{1, \cdots, \bar{r}\})$, say, C_{b_i}, is a subset of some Jordan frame in $\mathcal{C}(x)$ and the union of $C_{b_1}, \cdots, C_{b_{\bar{r}}}$ is a Jordan frame in $\mathcal{C}(x)$.

2.2 Matrix Representation of $\nabla G(x)$

Let $G(x)$ be given by (2.2). Suppose that g is differentiable at $\tau_i, i \in \{1, 2, \cdots, r\}$. Define *the first divided difference* $g^{[1]}$ of g at $\tau := (\tau_1, \tau_2, \cdots, \tau_r)^T \in \mathbb{R}^r$ as the $r \times r$ symmetric matrix with the ij-th entry $(g^{[1]}(\tau))_{ij}$ $(i, j \in \{1, 2, \cdots, r\})$ given by

$$[\tau_i, \tau_j]_g := \begin{cases} \frac{g(\tau_i)-g(\tau_j)}{\tau_i-\tau_j} & \text{if } \tau_i \neq \tau_j, \\ g'(\tau_i) & \text{if } \tau_i = \tau_j. \end{cases}$$

Based on Theorem 3.2 of [26], we can directly derive the following Jacobian properties of the Löwner operator $G(\cdot)$.

Theorem 2.6 *Let $x = \sum_{j=1}^{r} \lambda_j(x)c_j = \sum_{i=1}^{\bar{r}} \mu_i(x)b_i(x)$. Then, $G(\cdot)$ is (continuously) differentiable at x if and only if for each $j \in \{1, 2, \cdots, r\}$, g is (continuously) differentiable at $\lambda_j(x)$. In this case, the Jacobian $\nabla G(x)$ is given by*

$$\nabla G(x) = 2 \sum_{i,j=1}^{r} [\lambda_i(x), \lambda_j(x)]_g \mathcal{L}(c_i)\mathcal{L}(c_j) - \sum_{i=1}^{r} g'(\lambda_i(x))\mathcal{L}(c_i), \tag{2.9}$$

or equivalently

$$\nabla G(x) = 2 \sum_{i \neq j, i,j=1}^{\bar{r}} [\mu_i(x), \mu_j(x)]_g \mathcal{L}(b_i(x))\mathcal{L}(b_j(x)) + \sum_{i=1}^{\bar{r}} g'(\mu_i(x))\mathcal{Q}(b_i(x)). \tag{2.10}$$

Furthermore, $\nabla G(x)$ is a symmetric linear operator from $\mathcal{J}$ into itself.

Suppose that Löwner operator $G(\cdot)$ is (continuously) differentiable at $x = \sum_{j=1}^{r} \lambda_j(x)c_j$ with $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$. For a given Jordan frame $\{c_1, c_2, \cdots, c_r\}$, let

$$\mathcal{L} := \begin{pmatrix} \mathcal{L}(c_1) \\ \vdots \\ \mathcal{L}(c_r) \end{pmatrix}$$

be an *operator vector*. Denote $\mathcal{L}^* := (\mathcal{L}^*(c_1), \cdots, \mathcal{L}^*(c_r))$, where $\mathcal{L}^*(c_i)$ is the adjoint operator of $\mathcal{L}(c_i)$. Since $\mathcal{L}(c_i)$ is self-adjoint, we have

$$\mathcal{L}^* = (\mathcal{L}^*(c_1), \cdots, \mathcal{L}^*(c_r)) = (\mathcal{L}(c_1), \cdots, \mathcal{L}(c_r)).$$

For the real $r \times r$ matrix $\Lambda(x) := ([\lambda_i(x), \lambda_j(x)]_g)_{r\times r}$, we define

$$\mathcal{L}^*\Lambda(x)\mathcal{L} := \sum_{i,j=1}^{r} [\lambda_i(x), \lambda_j(x)]_g \mathcal{L}(c_i)\mathcal{L}(c_j).$$

Similarly, for the r-vector $d(x) = (g'(\lambda_1(x)), \cdots, g'(\lambda_r(x)))^T$, define

$$d^*(x)\mathcal{L} := \sum_{i=1}^{r} g'(\lambda_i(x))\mathcal{L}(c_i).$$

Thus, by (2.9), we can give a *matrix representation* of the Jacobian $\nabla G(x)$ as follows:

$$\nabla G(x) = 2\mathcal{L}^*\Lambda(x)\mathcal{L} - d^*(x)\mathcal{L}. \tag{2.11}$$

Using the matrix expression above and Theorem 2.4, we can obtain the eigenvalues of $\nabla G(x)$ explicitly.

Lemma 2.7 *Let $x = \sum_{j=1}^{r} \lambda_j(x)c_j = \sum_{i=1}^{\bar{r}} \mu_i(x)b_i(x)$. Suppose that $G(\cdot)$ is differentiable at x. Let $\Lambda(x)$ be defined by (2.11) using $\nabla G(x)$. Then the spectrum of $\nabla G(x)$ consists of all the elements of matrix $\Lambda(x)$. Moreover, the operator $\nabla G(x)$ has r single eigenvalues $[\lambda_i(x), \lambda_i(x)]_g$ $(i \in \{1, 2, \cdots, r\})$ as well as the eigenvalue $[\lambda_i(x), \lambda_j(x)]_g$ $(i < j \in \{1, 2, \cdots, r\})$ with multiplicity d (unless some elements of $\Lambda(x)$ are equal, in which case the multiplicities change in an obvious way).*

Proof. By direct calculation, we obtain

$$\begin{aligned} \nabla G(x) &= 2\sum_{i,j=1}^{r} [\lambda_i(x), \lambda_j(x)]_g \mathcal{L}(c_i)\mathcal{L}(c_j) - \sum_{i=1}^{r} g'(\lambda_i(x))\mathcal{L}(c_i) \\ &= \sum_{i=1}^{r} g'(\lambda_i(x))\mathcal{Q}(c_i) + 4\sum_{1\le i<j\le r} [\lambda_i(x), \lambda_j(x)]_g \mathcal{L}(c_i)\mathcal{L}(c_j) \\ &= \sum_{i=1}^{r} g'(\lambda_i(x))\mathcal{C}_{ii}(x) + \sum_{1\le i<j\le r} [\lambda_i(x), \lambda_j(x)]_g \mathcal{C}_{ij}(x) \end{aligned}$$

where the second equality holds by the definition of $\mathcal{Q}(x)$ and the third by (2.4). By Theorem 2.4, the spectrum of $\nabla G(x)$ consists of all scalars $[\lambda_i(x), \lambda_j(x)]_g$ $(i, j \in \{1, 2, \cdots, r\})$ or the elements of matrix $\Lambda(x)$. Then the property about the multiplicities follows from (2.3). $\square$

In particular, consider the metric projection $\Pi_K(\cdot)$ at $x = \sum_{j=1}^{r} \lambda_j(x)c_j$. Note that $t = 0$ is the unique point where $g = t_+$ is not smooth but strongly semismooth. Thus if $\Im = \emptyset$, i.e., x is nonsingular, $\Pi_K(\cdot)$ is continuously differentiable at x. In this case, from (2.11), we write $\nabla \Pi_K(x)$ as

$$\nabla \Pi_K(x) = 2\mathcal{L}^* \Lambda(x)\mathcal{L} - d^*(x)\mathcal{L}$$

where $\Lambda(x)$ is a symmetric matrix of the form

$$\Lambda(x) = \begin{pmatrix} E_{|\wp|\times|\wp|} & \Gamma \\ \Gamma^T & 0_{|\aleph|\times|\aleph|} \end{pmatrix} \tag{2.12}$$

with $E_{|\wp|\times|\wp|}$ being the all ones $|\wp| \times |\wp|$ matrix, the $|\wp| \times |\aleph|$ matrix $\Gamma = \left(\frac{\lambda_i(x)}{\lambda_i(x)-\lambda_j(x)}\right)_{|\wp|\times|\aleph|}$, and the vector $d(x) = \left(1_{|\wp|}^T, 0_{|\aleph|}^T\right)^T$. Throughout this paper, $E_{n_1\times n_2}$ denotes the $n_1 \times n_2$ all ones matrix.

2.3 A Decomposition Theorem of $\mathcal{J}$

Note that, for the operator $\nabla \Pi_K(x)$, any element of $\Lambda(x)$ as in (2.12) belongs to the interval $[0, 1]$. By (2.3) and Lemma 2.7, we can reset the elements of $\Lambda(x)$ in nonincreasing order, accounting for multiplicities as $a_1(x) \ge a_2(x) \ge \cdots \ge a_n(x)$. Thus, we can define a partition of $\{1, 2, \cdots, n\}$ as $\{1, 2, \cdots, n\} = \alpha \cup \beta \cup \gamma$ where

$$\alpha := \{i : a_i(x) = 1\},\ \beta := \{i : 0 < a_i(x) < 1\},\ \gamma := \{i : a_i(x) = 0\}. \tag{2.13}$$

Observing that $a_i(x)$ is the eigenvalue of some corresponding orthogonal projection operator in $\{\mathcal{C}_{jl}(x) : j < l, j, l \in \{1, 2, \cdots, r\}\}$, by (2.3) and Lemma 2.7, we rewrite $\nabla \Pi_K(x)$ as

$$\nabla \Pi_K(x) = \sum_{i=1}^{n} a_i(x)P_i(x),$$

where $P_i(x)$ is the corresponding orthogonal element in $\left\{\mathcal{C}_{jj},\ \frac{1}{d}\mathcal{C}_{jl}(x) : j < l, j, l \in \{1, 2, \cdots, r\}\right\}$. In view of the definitions of the index sets α, β, γ, we define three subspaces J_α, J_β and J_γ by

$$J_\alpha := \sum_{i\in\alpha} P_i(x)(\mathcal{J}), \quad J_\beta := \sum_{i\in\beta} P_i(x)(\mathcal{J}), \quad J_\gamma := \sum_{i\in\gamma} P_i(x)(\mathcal{J}). \tag{2.14}$$

Observe that

$$J_\alpha = \sum_{i\leq j\leq|\wp|} J_{ij}, \quad J_\beta = \sum_{i\leq|\wp|<j} J_{ij}, \quad J_\gamma = \sum_{j>i>|\wp|} J_{ij}.$$

Clearly, we have the following decomposition of the space $\mathcal{J}$.

Theorem 2.8 *Let the index sets α, β, γ be a partition of $\{1, 2, \cdots, n\}$ given by (2.13), and let J_α, J_β and J_γ be given by (2.14). Then $\mathcal{J}$ is the orthogonal direct sum of J_α, J_β and J_γ. In other words, every $y \in \mathcal{J}$ is of the form*

$$y = y_\alpha + y_\beta + y_\gamma, \tag{2.15}$$

where $y_\alpha \in J_\alpha$, $y_\beta \in J_\beta$ and $y_\gamma \in J_\gamma$.

Likewise, we define the corresponding operators from $\nabla\Pi_K(x)$ as follows:

$$[\nabla\Pi_K(x)]_\alpha := \sum_{i\in\alpha} a_i(x)P_i(x), \ \ [\nabla\Pi_K(x)]_\beta := \sum_{i\in\beta} a_i(x)P_i(x), \ \ [\nabla\Pi_K(x)]_\gamma := \sum_{i\in\gamma} a_i(x)P_i(x) = 0. \tag{2.16}$$

Note that

$$[\nabla\Pi_K(x)]_\beta \left[\sum_{i\in\beta} \frac{1}{a_i(x)} P_i(x)\right] y_\beta = \left[\sum_{i\in\beta} \frac{1}{a_i(x)} P_i(x)\right] [\nabla\Pi_K(x)]_\beta\, y_\beta = y_\beta, \forall y_\beta \in J_\beta.$$

Using the above definitions (2.16), we directly obtain the following fact.

Lemma 2.9 *Let the index sets α, β, γ be a partition of $\{1, 2, \cdots, n\}$ given by (2.13). Let the orthogonal operators $[\nabla\Pi_K(x)]_\alpha$, $[\nabla\Pi_K(x)]_\beta$ and $[\nabla\Pi_K(x)]_\gamma$ be defined by (2.16). For any $y \in J$ with $y = y_\alpha + y_\beta + y_\gamma$ as in (2.15),*

$$[\nabla\Pi_K(x)]_\alpha\, y = y_\alpha, \ \ [\nabla\Pi_K(x)]_\beta\, y = \bar{y}_\beta, \ \text{and} \ \ [\nabla\Pi_K(x)]_\gamma\, y = 0,$$

where $\bar{y}_\beta$ is a vector in J_β. Furthermore, $[\nabla\Pi_K(x)]_\beta$ is a one-to-one mapping from J_β to J_β and therefore it has an inverse $[\nabla\Pi_K(x)]_\beta^{-1}$ on J_β as follows:

$$[\nabla\Pi_K(x)]_\beta^{-1} = \left[\sum_{i\in\beta} \frac{1}{a_i(x)} P_i(x)\right].$$

3 Clarke Generalized Jacobian of Π_K

This section deals with the Clarke generalized Jacobian of Π_K. This is influenced by the recent work [13, 12, 18] in the special settings of second-order cones and positive semi-definite cones, where Kanzow, Ferenczi and Fukushima [13] gave an expression for the B-subdifferential of the projection onto second-order cones, Hayashi, Yamashita and Fukushima [12] gave an explicit representation for Clarke generalized Jacobian of the projection onto second-order cones; Malick and Sendov [18] worked out the Clarke generalized Jacobian of the projection onto the cone of symmetric positive semi-definite matrices. We generalize the above results to symmetric cones. For this purpose, we mainly look at the B-subdifferential of Π_K. First, we establish our notation.

For a given Jordan frame $\{c_1, c_2, \cdots, c_r\}$, we define

$$\partial_B^{\{c_1,\cdots,c_r\}}\Pi_K(x) := \left\{ \lim_{h\to 0, x+h\in D_{\Pi_K}} \nabla\Pi_K(x+h) : h \in \text{span}\{c_1, c_2, \cdots, c_r\}, \lim_{h\to 0} \nabla\Pi_K(x+h) \text{ exists} \right\}.$$

For a given integer $t \in \{0, 1, \cdots, |\Im|\}$ we define a r-vector $d_t(x)$ by

$$d_t(x) := \begin{pmatrix} 1_{|\wp|} \\ 1_t \\ 0_{|\Im|-t} \\ 0_{|\aleph|} \end{pmatrix} = \begin{pmatrix} 1_{|\wp|+t} \\ 0_{r-|\wp|-t} \end{pmatrix}, \tag{3.1}$$

and a set of $r \times r$ matrices $\Lambda_t(x)$ by

$$\Lambda_t(x) := \left\{ \begin{pmatrix} E_{|\wp|\times|\wp|} & E_{|\wp|\times|\Im|} & \Gamma_{|\wp|\times|\aleph|} \\ E_{|\Im|\times|\wp|} & \Lambda_0 & 0_{|\Im|\times|\aleph|} \\ \Gamma^T_{|\wp|\times|\aleph|} & 0_{|\aleph|\times|\Im|} & 0_{|\aleph|\times|\aleph|} \end{pmatrix} : \Lambda_0 = \begin{pmatrix} E & \Lambda_{00} \\ \Lambda_{00}^T & 0 \end{pmatrix}, \Lambda_{00} \in \Lambda(t, |\Im|) \right\}, \tag{3.2}$$

where $\Gamma_{|\wp|\times|\aleph|}$ is a given matrix by the ij-entry $\Gamma_{ij} = \frac{\lambda_i(x)}{\lambda_i(x)-\lambda_j(x)}$ $(i \in \wp, j \in \aleph)$, and $\Lambda(t, |\Im|)$ is a set of $t \times (|\Im| - t)$ matrices $(\theta_{ij})_{t\times(|\Im|-t)}$ (the rows are indexed by $|\wp| + 1, |\wp| + 2, \cdots, |\wp| + t$, and the columns by $|\wp| + t + 1, |\wp| + t + 2, \cdots, |\wp| + |\Im|$) specified by

$$\Lambda(t, |\Im|) := \left\{ (\theta_{ij})_{t\times(|\Im|-t)} \in [0, 1]^{t\times(|\Im|-t)} : \theta_{ij} \text{ satisfy } (a) \text{ and } (b) \text{ below} \right\}$$

- (a) $\theta_{i,|\wp|+t+1} \geq \theta_{i,|\wp|+t+2} \geq \cdots \geq \theta_{i,|\wp|+|\Im|}$ $(i \in \{|\wp| + 1, |\wp| + 2, \cdots, |\wp| + t\})$,
  $\theta_{|\wp|+1,j} \geq \theta_{|\wp|+2,j} \geq \cdots \geq \theta_{|\wp|+t,j}$ $(j \in \{|\wp| + t + 1, |\wp| + t + 2, \cdots, |\wp| + |\Im|\})$;
- (b) $\left(\frac{1}{\theta_{ij}} - 1\right)_{t\times(|\Im|-t)}$ is a matrix of rank at most one.

Clearly, when $x = 0$, we have a r-vector

$$d_t(0) := d_t := \begin{pmatrix} 1_t \\ 0_{r-t} \end{pmatrix}, \tag{3.3}$$

and a set of $r \times r$ matrices

$$\Lambda_t(0) := \Lambda_t := \left\{ \begin{pmatrix} E_{t\times t} & \Lambda_{t\times(r-t)} \\ \Lambda^T_{t\times(r-t)} & 0_{(r-t)\times(r-t)} \end{pmatrix} : \Lambda_{t\times(r-t)} \in \Lambda(t, r) \right\}, \tag{3.4}$$

where $\Lambda(t, r)$ is a set of $t \times (r - t)$ matrices $(\theta_{ij})_{t\times(r-t)}$ (the rows are indexed by $1, 2, \cdots, t$, and the columns by $t + 1, t + 2, \cdots, r$) specified by

$$\Lambda(t, r) := \left\{ (\theta_{ij})_{t\times(r-t)} \in [0, 1]^{t\times(r-t)} : \theta_{ij} \text{ satisfy } (a') \text{ and } (b') \text{ below} \right\}$$

- (a') $\theta_{i,t+1} \geq \theta_{i,t+2} \geq \cdots \geq \theta_{i,r}$ $(i \in \{1, 2, \cdots, t\})$,
  $\theta_{1j} \geq \theta_{2j} \geq \cdots \geq \theta_{tj}$ $(j \in \{t + 1, t + 2, \cdots, r\})$;
- (b') $\left(\frac{1}{\theta_{ij}} - 1\right)_{t\times(r-t)}$ is a matrix of rank at most one.

We are now ready to give the formula for the Clarke generalized Jacobian of $\Pi_K(x)$ by its B-subdifferential. Our next result generalizes Lemma 2.6 of [13] (for B-subdifferential), Theorem 3.7 of [18] and Proposition 4.8 of [12] (for Clarke generalized Jacobian).

Theorem 3.1 *The B-subdifferential of $\Pi_K(\cdot)$ at x is given by*

$$\partial_B \Pi_K(x) = \bigcup_{\{c_1,\cdots,c_r\}\in\mathcal{C}(x)} \bigcup_{t=0}^{|\Im|} \{2\mathcal{L}^*\Lambda_t(x)\mathcal{L} - d_t^*(x)\mathcal{L}\}$$

where $d_t(x)$ and $\Lambda_t(x)$ are specified by (3.1) and (3.2), respectively. Furthermore, the Clarke generalized Jacobian of $\Pi_K(\cdot)$ at x is

$$\partial\Pi_K(x) = \operatorname{conv}\{\partial_B\Pi_K(x)\} = \operatorname{conv}\left\{ \bigcup_{\{c_1,\cdots,c_r\}\in\mathcal{C}(x)} \bigcup_{t=0}^{|\Im|} \{2\mathcal{L}^*\Lambda_t(x)\mathcal{L} - d_t^*(x)\mathcal{L}\} \right\}.$$

Proof. First we prove $\partial_B\Pi_K(x) \supseteq \bigcup_{\{c_1,\cdots,c_r\}\in\mathcal{C}(x)} \bigcup_{t=0}^{|\Im|} \{2\mathcal{L}^*\Lambda_t(x)\mathcal{L} - d_t^*(x)\mathcal{L}\}$. Without loss of generality, fix a Jordan frame $\{c_1,\cdots,c_r\} \in \mathcal{C}(x)$. For any arbitrary but given integer $t \in \{0,1,\cdots,|\Im|\}$, set $V := 2\mathcal{L}^*A_t(x)\mathcal{L} - d_t^*(x)\mathcal{L}$ where $A_t(x) \in \Lambda_t(x)$ and $d_t(x)$ is given by (3.1). We need only to show that $V \in \partial_B\Pi_K(x)$. For this purpose, it suffices to demonstrate that there exists a vector $h = \sum_{i=1}^r \lambda_i(h)c_i$ such that

$$V = \lim_{h\to 0, x+h\in D_{\Pi_K}} \nabla\Pi_K(x+h).$$

Let $A_t(x) = (\theta_{ij})_{r\times r}$. We define $\varsigma_{ij} := \frac{1}{\theta_{ij}} - 1$ for $i \in \{|\wp|+1,\cdots,|\wp|+t\}$ and $j \in \{|\wp|+t+1,\cdots,|\wp|+|\Im|\}$, then $\mathcal{B}_{t\times(|\Im|-t)} := (\varsigma_{ij})_{t\times(|\Im|-t)}$ can be written as

$$\mathcal{B}_{t\times(|\Im|-t)} = \begin{pmatrix} \pi_{|\wp|+1} \\ \vdots \\ \pi_{|\wp|+t} \end{pmatrix} \left(\pi_{|\wp|+t+1},\cdots,\pi_{|\wp|+|\Im|}\right),$$

where $\pi_i \in [0,+\infty]$, $\pi_{|\wp|+1} \le \pi_{|\wp|+2} \le \cdots \le \pi_{|\wp|+t}$ and $\pi_{|\wp|+t+1} \le \pi_{|\wp|+t+2} \le \cdots \le \pi_{|\wp|+|\Im|}$. Taking

$$\lambda_i(h) = \begin{cases} 0 & \text{if } 1 \le i \le |\wp|, \\ \rho & \text{if } \pi_i = 0,\ |\wp| < i \le |\wp|+t, \\ \frac{1}{\pi_i}\rho^2 & \text{if } \pi_i \ne 0,+\infty,\ |\wp| < i \le |\wp|+t, \\ \rho^3 & \text{if } \pi_i = +\infty,\ |\wp| < i \le |\wp|+t, \\ -\rho^3 & \text{if } \pi_i = 0,\ |\wp|+t < i \le |\wp|+|\Im|, \\ -\pi_i\rho^2 & \text{if } \pi_i \ne 0,+\infty,\ |\wp|+t < i \le |\wp|+|\Im|, \\ -\rho & \text{if } \pi_i = +\infty,\ |\wp|+t < i \le |\wp|+|\Im|, \\ 0 & \text{if } |\wp|+|\Im| < i \le r, \end{cases}$$

we have $x+h = \sum_{i=1}^r \lambda_i(x+h)c_i = \sum_{i=1}^r [\lambda_i(x)+\lambda_i(h)]c_i$ and for sufficiently small $\rho \in (0,1)$,

$$\lambda_1(x+h) \ge \cdots \ge \lambda_{|\wp|+t}(x+h) > 0 > \lambda_{|\wp|+t+1}(x+h) \ge \cdots \ge \lambda_r(x+h).$$

It is easy to verify that $x+h \in D_{\Pi_K}$ and

$$\lim_{h\to 0} \nabla\Pi_K(x+h) = 2\mathcal{L}^*A_t(x)\mathcal{L} - d_t^*(x)\mathcal{L}.$$

Hence $V \in \partial_B\Pi_K(x)$.

We next prove that $\partial_B \Pi_K(x) \subseteq \bigcup_{\{c_1,\cdots,c_r\}\in\mathcal{C}(x)} \bigcup_{t=0}^{r} \{2\mathcal{L}^*\Lambda_t(x)\mathcal{L} - d_t^*(x)\mathcal{L}\}$. Let $W \in \partial_B \Pi_K(x)$. Then there is a vector $h := h(W) \in \mathcal{J}$ such that $W = \lim_{h\to 0, x+h\in D_{\Pi_K}} \nabla\Pi_K(x+h)$. For the above $h \in \mathcal{J}$, let $x+h =: \sum_{j=1}^{r} \lambda_j(x+h)c_j(x+h)$ with $\lambda_1(x+h) \geq \lambda_2(x+h) \geq \cdots \geq \lambda_r(x+h)$. In the sense of set convergence (see [24]), we have

$$\limsup_{h\to 0, x+h\in D_{\Pi_k}} \{(c_1(x+h),\cdots,c_r(x+h))\} \subseteq \mathcal{C}(x).$$

Let us pick $\{c_1,\cdots,c_r\} \in \limsup\{c_1(x+h),\cdots,c_r(x+h)\}$. Clearly,

$$\lim_{h\to 0, x+h\in D_{\Pi_k}} \lambda(x+h) = \lambda(x),$$

where $\lambda(x+h) := (\lambda_1(x+h),\cdots,\lambda_r(x+h))^T$ and none of $\lambda_i(x+h)$ is zero. Suppose that $\aleph(x+h)$, $\wp(x+h)$ are given and $t := |\wp(x+h)| - |\wp|$. Then $d_t(x) = \left(1_{|\wp|+t}^T, 0_{r-|\wp|-t}^T\right)^T$. Set

$$\Lambda_t(x+h) := \begin{pmatrix} E_{|\wp|\times|\wp|} & E_{|\wp|\times|\Im|} & \Gamma_{|\wp|\times|\aleph|}(h) \\ E_{|\Im|\times|\wp|} & \tilde{\Lambda} & 0_{|\Im|\times|\aleph|} \\ \Gamma^T_{|\wp|\times|\aleph|}(h) & 0_{|\aleph|\times|\Im|} & 0_{|\aleph|\times|\aleph|} \end{pmatrix} \text{ with } \tilde{\Lambda} := \begin{pmatrix} E_{t\times t} & \tilde{\Lambda}_{12} \\ \tilde{\Lambda}_{12}^T & 0_{(|\aleph|-t)\times(|\aleph|-t)} \end{pmatrix},$$

where $\Gamma_{|\wp|\times|\aleph|}(h) = \left(\frac{\lambda_i(x+h)}{\lambda_i(x+h)-\lambda_j(x+h)}\right)_{|\wp|\times|\aleph|}$ $(i\in\wp, j\in\aleph)$ and $\tilde{\Lambda}_{12} = \left(\tilde{\Lambda}_{ij}\right)_{t\times(|\Im|-t)}$ with $\tilde{\Lambda}_{ij} = \frac{\lambda_i(x+h)}{\lambda_i(x+h)-\lambda_j(x+h)}$ $(i\in\{|\wp|+1,\cdots,|\wp|+t\},\ j\in\{|\wp|+t+1,\cdots,|\wp|+|\Im|\})$. If $\lim_{h\to 0}\tilde{\Lambda}_{12}$ exists, then direct calculation yields

$$\Theta_t(x) := \lim_{h\to 0}\Lambda_t(x+h) = \begin{pmatrix} E_{|\wp|\times|\wp|} & E_{|\wp|\times|\Im|} & \Gamma_{|\wp|\times|\aleph|} \\ E_{|\Im|\times|\wp|} & \Lambda & 0_{|\Im|\times|\aleph|} \\ \Gamma^T_{|\wp|\times|\aleph|} & 0_{|\aleph|\times|\Im|} & 0_{|\aleph|\times|\aleph|} \end{pmatrix} \text{ with } \Lambda = \begin{pmatrix} E_{t\times t} & \Lambda_{12} \\ \Lambda_{12}^T & 0_{(|\aleph|-t)\times(|\aleph|-t)} \end{pmatrix},$$

where $\Lambda := \lim_{h\to 0}\tilde{\Lambda}$ and $\Lambda_{12} := \lim_{h\to 0}\tilde{\Lambda}_{12} = (\theta_{ij})_{t\times(|\Im|-t)}$ with $\theta_{ij} = \lim_{h\to 0}\frac{\lambda_i(x+h)}{\lambda_i(x+h)-\lambda_j(x+h)}$ $(i\in\{|\wp|+1,\cdots,|\wp|+t\},\ j\in\{|\wp|+t+1,\cdots,|\wp|+|\Im|\})$. Observing that

$$\lambda_{|\wp|+1}(h) \geq \lambda_{|\wp|+2}(h) \geq \cdots \geq \lambda_{|\wp|+t}(h) > 0$$

and

$$0 < -\lambda_{|\wp|+t+1}(h) \leq -\lambda_{|\wp|+t+2}(h) \leq \cdots \leq -\lambda_{|\wp|+|\Im|}(h),$$

we easily show $\theta_{ij}\in[0,1]$ and (a). In order to prove (b), let

$$\varsigma_{ij} := \lim_{h\to 0}\frac{-\lambda_j(x+h)}{\lambda_i(x+h)}$$

for $i\in\{|\wp|+1,\cdots,|\wp|+t\},\ j\in\{|\wp|+t+1,\cdots,|\wp|+|\Im|\}$. Then,

$$\theta_{ij} = \frac{1}{1+\varsigma_{ij}} \quad \text{or equivalently} \quad \varsigma_{ij} = \frac{1}{\theta_{ij}} - 1. \tag{3.5}$$

Note that $\theta_{ij}\in[0,1]$ implies $\varsigma_{ij}\in[0,+\infty]$ and $\theta_{ij}=0 \Leftrightarrow \varsigma_{ij}=\infty$, $\theta_{ij}=1 \Leftrightarrow \varsigma_{ij}=0$. Define $\mathcal{B} := (\varsigma_{ij})_{t\times(|\Im|-t)}$. Then, it is easy to see that

$$\mathcal{B} = \lim_{h\to 0}\begin{pmatrix} \frac{1}{\lambda_{|\wp|+1}(h)} \\ \vdots \\ \frac{1}{\lambda_{|\wp|+t}(h)} \end{pmatrix}\left(-\lambda_{|\wp|+t+1}(h),\cdots,-\lambda_{|\wp|+|\Im|}(h)\right).$$

This establishes (b), and hence $\Theta_t(x)\in\Lambda_t(x)$. The existence of $\lim_{h\to 0, x+h\in D_{\Pi_K}}\nabla\Pi_K(x+h)$ means that $\lim_{h\to 0}\tilde{\Lambda}_{12}(h)$ exists. This proves $W = 2\mathcal{L}^*\Theta_t(x)\mathcal{L} - d_t^*(x)\mathcal{L} \in \{2\mathcal{L}^*\Lambda_t(x)\mathcal{L} - d_t^*(x)\mathcal{L}\}$, as desired. $\square$

From the above proof, we also obtain that for a given Jordan frame $\{c_1, c_2, \cdots, c_r\} \in \mathcal{C}(x)$

$$\partial_B^{\{c_1,\cdots,c_r\}} \Pi_K(x) = \bigcup_{t=0}^{|\Im|} \{2\mathcal{L}^* \Lambda_t(x) \mathcal{L} - d_t^*(x) \mathcal{L}\}.$$

Note that when $x = 0$ the above result becomes the following, which provides a formula for B-subdifferential of the projection onto symmetric cone $\Pi_K(\cdot)$ at the origin.

Corollary 3.2 *The B-subdifferential of $\Pi_K(\cdot)$ at 0 is*

$$\partial_B \Pi_K(0) = \bigcup_{\{c_1,\cdots,c_r\} \in \mathcal{C}(0)} \bigcup_{t=0}^{r} \{2\mathcal{L}^* \Lambda_t \mathcal{L} - d_t^* \mathcal{L}\}$$

where d_t and Λ_t are given by (3.3) and (3.4), respectively. Furthermore, the Clarke generalized Jacobian of $\Pi_K(\cdot)$ at 0 is

$$\partial \Pi_K(0) = \text{conv}\,(\partial_B \Pi_K(0)) = \text{conv} \left\{ \bigcup_{\{c_1,\cdots,c_r\} \in \mathcal{C}(0)} \bigcup_{t=0}^{r} \{2\mathcal{L}^* \Lambda_t \mathcal{L} - d_t^* \mathcal{L}\} \right\}.$$

In general, $\partial_B \Pi_K(0) \neq \partial \Pi_K(0)$.

Proof. We only need show that $\partial_B \Pi_K(0) \neq \partial \Pi_K(0)$ in general. For this purpose, we look at an example in the case of the *second-order cone* $\Lambda_+^n := \{(x_1, x_2^T)^T : x_1 \geq \|x_2\|, x_1 \in \mathbb{R}, x_2 \in \mathbb{R}^{n-1}\}$ $(n \geq 2)$. Let $\{c_1, c_2\}$ be a Jordan frame given by

$$c_i = \frac{1}{2} \begin{pmatrix} 1 \\ (-1)^{i-1}\ \omega \end{pmatrix}, \text{ for } i \in \{1, 2\},$$

with ω being any vector in $\mathbb{R}^{n-1}$ satisfying $\|\omega\| = 1$. By direct calculation, we can derive that $\partial_B \Pi_{\Lambda_+^n}(0) = \{0, \mathcal{I}, \mathcal{T}\}$ and $\partial \Pi_{\Lambda_+^n}(0) = \text{conv}\{0, \mathcal{I}, \mathcal{T}\}$ where $\mathcal{T}$ satisfies

$$\mathcal{T} = 4[0, 1] \mathcal{L}(c_1) \mathcal{L}(c_2) + \mathcal{Q}(c_1) + 0 \times \mathcal{Q}(c_2).$$

This means that $\partial_B \Pi_K(0) \neq \partial \Pi_K(0)$ in general. □

Applying Proposition 2.5, Theorem 3.1 and Corollary 3.2, we immediately obtain the following result, which states the interesting connection between $\partial_B \Pi_K(x)$ and $\partial_B \Pi_{K_\Im}(0)$. In the case of $\mathbb{S}^n$, it reduces to Proposition 2.2 in [25].

Theorem 3.3 *Let $x = \sum_{i=1}^r \lambda_i(x) c_i$ with $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$. Let*

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{|\wp|} \\ \mathcal{L}_{|\Im|} \\ \mathcal{L}_{|\aleph|} \end{pmatrix}$$

where

$$\mathcal{L}_{|\wp|} := \begin{pmatrix} \mathcal{L}(c_1) \\ \vdots \\ \mathcal{L}(c_{|\wp|}) \end{pmatrix}, \ \mathcal{L}_{|\Im|} := \begin{pmatrix} \mathcal{L}(c_{|\wp|+1}) \\ \vdots \\ \mathcal{L}(c_{|\wp|+|\Im|}) \end{pmatrix}, \ \mathcal{L}_{|\aleph|} := \begin{pmatrix} \mathcal{L}(c_{|\wp|+|\Im|+1}) \\ \vdots \\ \mathcal{L}(c_r) \end{pmatrix}.$$

Then $V \in \partial_B \Pi_K(x)$ if and only if there exists a $V_{|\Im|} \in \partial_B \Pi_{K_\Im}(0)$ such that

$$V = 2\mathcal{L}^* A_t(x)\mathcal{L} - d_t^*(x)\mathcal{L}, \quad V_{|\Im|} = 2\mathcal{L}_{|\Im|}^* A(t,|\Im|)\mathcal{L}_{|\Im|} - d_t^* \mathcal{L}_{|\Im|}$$

where $r \times r$ matrix $A_t(x) \in \Lambda_t(x)$ and $|\Im| \times |\Im|$ matrix $A(t,|\Im|) \in \Lambda(t,|\Im|)$ satisfy

$$A_t(x) = \begin{pmatrix} E_{|\wp|\times|\wp|} & E_{|\wp|\times|\Im|} & \Gamma_{|\wp|\times|\aleph|} \\ E_{|\Im|\times|\wp|} & A(t,|\Im|) & 0_{|\Im|\times|\aleph|} \\ \Gamma^T_{|\wp|\times|\aleph|} & 0_{|\aleph|\times|\Im|} & 0_{|\aleph|\times|\aleph|} \end{pmatrix};$$

and $d_t(x)$ and d_t are, respectively, r-vector and $|\Im|$-vector such that

$$d_t(x) = \left(1^T_{|\wp|}, d_t^T, 0_{|\aleph|}\right)^T.$$

Proof. Let $\{c_1, \cdots, c_r\} =: \{c_\wp, c_\Im, c_\aleph\}$ with $c_\wp := \{c_1, \cdots, c_{|\wp|}\}$, $c_\Im := \{c_{|\wp|+1}, \cdots, c_{|\wp|+|\Im|}\}$, and $c_\aleph := \{c_{|\wp|+|\Im|+1}, \cdots, c_r\}$. By Proposition 2.5, any Jordan frame in $J_\Im$ is a subset of a Jordan frame in $\mathcal{C}(x)$, and the part $c_\Im$ of a Jordan frame $\{c_, \cdots, c_r\}$ in $\mathcal{C}(x)$ is a Jordan frame in $J_\Im$. The conclusion follows immediately from Theorem 3.1 and Corollary 3.2. □

4 B-subdifferentials of x_- and $|x|$

Employing the same technique as in the proof of Theorem 3.1, we give below the formulae for the B-subdifferentials of x_- and $|x|$.

Theorem 4.1 *Let $\bar{e}$ be the all ones vector of appropriate size. The B-subdifferential of x_- is given by*

$$\partial_B x_- = \bigcup_{\{c_1,\cdots,c_r\}\in\mathcal{C}(x)} \bigcup_{t=0}^{|\Im|} \{2\mathcal{L}^*(E_{r\times r} - \Lambda_t(x))\mathcal{L} - (\bar{e} - d_t(x))^*\mathcal{L}\}$$

where $d_t(x)$ and $\Lambda_t(x)$ are specified by (3.1) and (3.2), respectively. Furthermore,

$$\partial_B x_- = \mathcal{I} - \partial_B \Pi_K(x).$$

Proof. Note that for any pair of scalars τ_i, τ_j,

$$\frac{(\tau_i)_- - (\tau_j)_-}{\tau_i - \tau_j} + \frac{(\tau_i)_+ - (\tau_j)_+}{\tau_i - \tau_j} = 1 \quad \text{or} \quad \frac{(\tau_i)_- - (\tau_j)_-}{\tau_i - \tau_j} = 1 - \frac{(\tau_i)_+ - (\tau_j)_+}{\tau_i - \tau_j}.$$

In a similar way as in Theorem 3.1, we obtain the formula for the B-subdifferential of x_-. □

Before studying the B-subdifferential of $|\cdot|$ at x, we need the following notation. For a given integer $t \in \{0, 1, \cdots, |\Im|\}$ we define a r-vector $\hbar_t(x)$ by

$$\hbar_t(x) := \begin{pmatrix} 1_{|\wp|} \\ 1_t \\ -1_{|\Im|-t} \\ -1_{|\aleph|} \end{pmatrix} = \begin{pmatrix} 1_{|\wp|+t} \\ -1_{r-|\wp|-t} \end{pmatrix}, \tag{4.1}$$

and a set of $r \times r$ matrices $\mathring{A}_t(x)$ by

$$\mathring{A}_t(x) := \left\{ \begin{pmatrix} E_{|\wp|\times|\wp|} & E_{|\wp|\times|\Im|} & \Upsilon_{|\wp|\times|\aleph|} \\ E_{|\Im|\times|\wp|} & \Upsilon_0 & -E_{|\Im|\times|\aleph|} \\ \Upsilon^T_{|\wp|\times|\aleph|} & -E_{|\aleph|\times|\Im|} & -E_{|\aleph|\times|\aleph|} \end{pmatrix} : \Upsilon_0 = \begin{pmatrix} E_{t\times t} & \Upsilon_{00} \\ \Upsilon_{00}^T & -E_{(|\Im|-t)\times(|\Im|-t)} \end{pmatrix}, \Upsilon_{00} \in \Upsilon(t, |\Im|) \right\}, \tag{4.2}$$

where $\Upsilon_{|\wp|\times|\aleph|}$ is a given matrix with the ij-entry $\Upsilon_{ij} = \frac{\lambda_i(x)+\lambda_j(x)}{\lambda_i(x)-\lambda_j(x)}$ $(i \in \wp, j \in \aleph)$, and $\Upsilon(t, |\Im|)$ is a set of $t \times (|\Im| - t)$ matrices $(\xi_{ij})_{t\times(|\Im|-t)}$ (the rows are indexed by $|\wp|+1, |\wp|+2, \cdots, |\wp|+t$, and the columns by $|\wp|+t+1, |\wp|+t+2, \cdots, |\wp|+|\Im|$) specified by

$$\Upsilon(t, |\Im|) := \left\{ (\xi_{ij})_{t\times(|\Im|-t)} \in [-1, 1]^{t\times(|\Im|-t)} :\ \xi_{ij} \text{ satisfy (c) and (d) below} \right\},$$

(*c*) $\xi_{i,|\wp|+t+1} \geq \xi_{i,|\wp|+t+2} \geq \cdots \geq \xi_{i,|\wp|+|\Im|}$ $(i = |\wp|+1, |\wp|+2, \cdots, |\wp|+t)$,
$\xi_{|\wp|+1,j} \geq \xi_{|\wp|+2,j} \geq \cdots \geq \xi_{|\wp|+t,j}$ $(j = |\wp|+t+1, |\wp|+t+2, \cdots, |\wp|+|\Im|)$;
(*d*) $\left(\frac{1-\xi_{ij}}{1+\xi_{ij}}\right)_{t\times(|\Im|-t)}$ is a matrix of rank at most one.

We are in a position to give a formula for the B-subdifferential of $|x|$.

Theorem 4.2 *The B-subdifferential of $|x|$ is given by*

$$\partial_B|x| = \bigcup_{\{c_1,\cdots,c_r\}\in\mathcal{C}(x)} \bigcup_{t=0}^{|\Im|} \{2\mathcal{L}^* \mathring{A}_t(x)\mathcal{L} - \hbar_t^*(x)\mathcal{L}\},$$

where $\hbar_t(x)$ and $\mathring{A}_t(x)$ are specified by (4.1) and (4.2), respectively.

Proof. Note that for any pair of scalars $\tau_i > \tau_j$,

$$\frac{|\tau_i| - |\tau_j|}{\tau_i - \tau_j} = \begin{cases} 1 & \text{if } \tau_i > \tau_j \geq 0, \\ \frac{\tau_i+\tau_j}{\tau_i-\tau_j} & \text{if } \tau_i > 0 > \tau_j, \\ -1 & \text{if } \tau_i \geq 0 > \tau_j. \end{cases}$$

In the case of $\tau_i > 0 > \tau_j$, set $\xi_{ij} := \frac{\tau_i+\tau_j}{\tau_i-\tau_j}$. Then

$$\xi_{ij} = \frac{1 - \frac{-\tau_j}{\tau_i}}{1 + \frac{-\tau_j}{\tau_i}}, \quad \text{or equivalently} \quad \frac{-\tau_j}{\tau_i} = \frac{1-\xi_{ij}}{1+\xi_{ij}}.$$

This implies that

$$\frac{-\tau_j}{\tau_i} \in [0, +\infty] \iff \xi_{ij} \in [-1, 1].$$

Based on the above fact and a similar argument as in the proof of Theorem 3.1, we deduce the claimed identity. □

Note that $\partial_B x \neq \partial_B x_+ + \partial_B x_-$ and $\partial_B |x| \neq \partial_B x_+ - \partial_B x_-$, although $x = x_+ + x_-$ and $|x| = x_+ - x_-$, respectively. For instance, in the case of $\mathcal{J} = \mathbb{R}$, it is easy to derive

$$\partial_B x_+ = \partial_B x_- = \{0, 1\}, \quad \partial_B x = 1 \ \text{ and } \ \partial_B|x| = \{-1, 1\}.$$

However, $\partial_B x_+ + \partial_B x_- = \{0, 1, 2\}$ and $\partial_B x_+ - \partial_B x_- = \{-1, 0, 1\}$.

5 Semismooth Newton Methods

As an application, in this section we deal with the semismooth Newton methods for NSCP(1.1) by using the projection operator. We establish sufficient conditions for our methods to be locally quadratically convergent. In the setting of second-order cones, Kanzow, Ferenczi and Fukushima [13] reformulated NSOCP as a nonsmooth system of equations and investigated conditions for the local quadratic convergence of nonsmooth Newton methods. They also gave the related conditions for the special case of a linear second-order cone program. In the setting of positive semi-definite cones, we only know of some similar results for the *linear* semi-definite programs, see the recent work [2, 8, 14] and the references therein.

As it is well-known, under certain favorable conditions like convexity of f and a Slater-type constraint qualification (see, e.g., [6, 24]), we can reformulate NSCP as a nonsmooth system of equations by the corresponding KKT conditions, which can be written as follows:

$$\begin{aligned} \nabla f(x) - \mathcal{A}^*\varrho - y &= 0, \\ \mathcal{A}x - q &= 0, \\ x \in K,\ y \in K,\ \langle x, y\rangle &= 0. \end{aligned}$$

By Proposition 6 in [9], the complementarity condition $x \in K,\ y \in K,\ \langle x, y\rangle = 0$ is the same as $x-(x-y)_+ = 0$. Therefore, the above system is equivalent to the system of equations $H(z) = 0$, where the mapping $H : \mathcal{V} \times \mathbb{R}^m \times \mathcal{V} \to \mathcal{V} \times \mathbb{R}^m \times \mathcal{V}$ is specified by

$$H(z) := H(x, \varrho, y) := \begin{pmatrix} \nabla f(x) - \mathcal{A}^*\varrho - y \\ \mathcal{A}x - q \\ x - (x-y)_+ \end{pmatrix}. \tag{5.1}$$

Then we can apply the nonsmooth Newton method (see [21, 22, 23])

$$z^{k+1} := z^k - W_k^{-1} H(z_k),\ W_k \in \partial_B H(z_k),\ k \in \{0, 1, 2, \cdots\}, \tag{5.2}$$

to the system of equations $H(z) = 0$, and therefore solve the NSCP under some suitable conditions. In order to establish (5.2), we need to show the semismoothness of H and the nonsingularity of ∇H.

Theorem 5.1 *The mapping H defined by (5.1) is semismooth. Moreover, if the Hessian $\nabla^2 f$ is locally Lipschitz continuous, then the mapping H is strongly semismooth.*

Proof. Note that $\Pi_K(x)$ is strongly semismooth by Proposition 3.3 in [26]. The conclusion follows from the operation rules of semismooth functions. □

Lemma 5.2 *Let $z^* = (x^*, \varrho^*, y^*)$ be a strictly complementary KKT point of NSCP (1.1). Then Π_K is continuously differentiable at $(x^* - y^*)$.*

Proof. By assumption, $x^* \in K,\ y^* \in K$ and $\langle x^*, y^*\rangle = 0$. Then, by Proposition 6 in [9], x^* and y^* operator commute and $x^* \circ y^* = 0$. It follows that there is a Jordan frame $\{c_1, c_2, \cdots, c_r\}$ such that

$$x^* = \sum_{i=1}^r \lambda_i(x^*)c_i,\ y^* = \sum_{i=1}^r \lambda_i(y^*)c_i,\ \text{and } \lambda_i(x^*)\lambda_i(y^*) = 0,\ \forall i \in \{1, 2, \cdots, r\}. \tag{5.3}$$

Since $x^*,\ y^* \in K$, $\lambda_i(x^*) \geq 0,\ \lambda_i(y^*) \geq 0$ for all $i \in \{1, 2, \cdots, r\}$. Again, since $x^* + y^* \in \text{int}(K)$ by the strict complementarity assumption,

$$\lambda_i(x^*) + \lambda_i(y^*) > 0,\ \forall i \in \{1, 2, \cdots, r\}.$$

This together with (5.3) yields that $\lambda_i(x^*) - \lambda_i(y^*) \neq 0,\ \forall i \in \{1, 2, \cdots, r\}$. Thus

$$x^* - y^* = \sum_{i=1}^{r} [\lambda_i(x^*) - \lambda_i(y^*)] c_i, \text{ with } \lambda_i(x^*) - \lambda_i(y^*) \neq 0,\ \forall i \in \{1, 2, \cdots, r\}.$$

The desired result follows immediately from Theorem 2.6. □

We are ready to give our main result on the nonsingularity of ∇H at a strictly complementary KKT point of NSCP (1.1). In the setting of NSOCP, it reduces to Theorem 3.5 in [13] by considering the concrete forms of $\mathcal{C}_{jl}$ which are related to the decomposition of a symmetric matrix, for details, see [13] and the references therein.

Theorem 5.3 *Let $z^* = (x^*, \varrho^*, y^*)$ be a strictly complementary KKT point of NSCP (1.1). Let the index sets α, β, γ be defined by (2.13) with $a_i(x) = a_i(x^* - y^*)$, and let the corresponding subspaces $J_\alpha, J_\beta, J_\gamma$ be given by (2.14). Let $V := \nabla \Pi_K(x^* - y^*)$, and $V_\beta := [\nabla \Pi_K(x^* - y^*)]_\beta$. Assume the following conditions hold:*

(i) The operator $\nabla^2 f(x^*) + V_\beta^{-1}(\mathcal{I} - V)_\beta$ is positive definite on the subspace

$$S := \{s = s_\alpha + s_\beta + s_\gamma \in \mathcal{V} : \mathcal{A}s = 0, s_\gamma = 0\};$$

(ii) The following implication holds: $\mathcal{A}^* \varrho \in J_\gamma \Rightarrow \varrho = 0$.

Then the Jacobian ∇H exists and is nonsingular. In particular, for the linear SCP, the statement holds with condition (i) replaced by the following one:

(i′) $\{s_\alpha : \mathcal{A}s_\alpha = 0\} \cap J_\alpha = \{0\}$.

Proof. The first part is clear by Lemma 5.2. It remains to show that ∇H is nonsingular. It is easy to see that

$$\nabla H(z^*) = \nabla H(x^*, \varrho^*, y^*) = \begin{pmatrix} \nabla^2 f(x^*) & -\mathcal{A}^* & -\mathcal{I} \\ \mathcal{A} & 0 & 0 \\ \mathcal{I} - V & 0 & V \end{pmatrix}.$$

Observe that $V_\beta^{-1}(\mathcal{I} - V)_\beta$ is positive definite on the subspace J_β, i.e.,

$$\langle w, V_\beta^{-1}(\mathcal{I} - V)_\beta w \rangle > 0, \quad \forall w \in J_\beta,\ w \neq 0. \tag{5.4}$$

In fact, since $V_\beta = [\nabla \Pi_K(x^* - y^*)]_\beta$, by (2.16) and Lemma 2.9, we obtain that $0 < a_i(x^* - y^*) < 1$ for $i \in \beta$, $(\mathcal{I} - V)_\beta = \sum_{i \in \beta} [1 - a_i(x^* - y^*)] P_i(x^* - y^*)$ and

$$V_\beta^{-1} = [\nabla \Pi_K(x^* - y^*)]_\beta^{-1} = \sum_{i \in \beta} \frac{1}{a_i(x^* - y^*)} P_i(x^* - y^*).$$

Thus, by (2.4)

$$V_\beta^{-1}(\mathcal{I} - V)_\beta = \sum_{i \in \beta} \frac{1 - a_i(x^* - y^*)}{a_i(x^* - y^*)} P_i(x^* - y^*).$$

This proves (5.4).

Let $z = (x, \varrho, y) \in \mathcal{V} \times \mathbb{R}^m \times \mathcal{V}$ be any vector such that $\nabla H(z^*)z = 0$. Then

$$\nabla^2 f(x^*)x - \mathcal{A}^*\varrho - y = 0, \tag{5.5}$$

$$\mathcal{A}x = 0, \tag{5.6}$$

$$(\mathcal{I} - V)x + Vy = 0. \tag{5.7}$$

By Theorem 2.8, set $x := x_\alpha + x_\beta + x_\gamma$ and $y := y_\alpha + y_\beta + y_\gamma$ where $x_\alpha, y_\alpha \in J_\alpha$, $x_\beta, y_\beta \in J_\beta$ and $x_\gamma, y_\gamma \in J_\gamma$. In view of the definitions of α, β and γ, by (2.16) and (5.7), we have

$$[(\mathcal{I} - V)_\alpha x + V_\alpha y] + [(\mathcal{I} - V)_\beta x + V_\beta y] + [(\mathcal{I} - V)_\gamma x + V_\gamma y] = 0.$$

By Lemma 2.9, we obtain

$$y_\alpha = 0, \; x_\gamma = 0 \; \text{ and } \; (\mathcal{I} - V)_\beta x_\beta + V_\beta y_\beta = 0. \tag{5.8}$$

Therefore, we deduce

$$\begin{aligned}
\langle x, y\rangle &= \langle x_\alpha + x_\beta + x_\gamma, y_\alpha + y_\beta + y_\gamma\rangle \\
&= \langle x_\alpha, y_\alpha\rangle + \langle x_\beta, y_\beta\rangle + \langle x_\gamma, y_\gamma\rangle \\
&= \langle x_\beta, y_\beta\rangle \\
&= \langle x_\beta, -V_\beta^{-1}(\mathcal{I} - V)_\beta x_\beta\rangle \\
&= -\langle x_\alpha + x_\beta, V_\beta^{-1}(\mathcal{I} - V)_\beta(x_\alpha + x_\beta)\rangle,
\end{aligned}$$

where the second equality holds by the orthogonality property of the spaces J_α, J_β and J_γ, the third by (5.8) and the fourth by Lemma 2.9, the last one follows from $\langle x_\alpha, V_\beta^{-1}(\mathcal{I} - V)_\beta x_\beta\rangle = 0$ and $V_\beta^{-1}(\mathcal{I} - V)_\beta x_\alpha = 0$. In the meanwhile, by (5.5), (5.6) and (5.8), we have

$$\begin{aligned}
0 &= \langle x, \nabla^2 f(x^*)x - \mathcal{A}^*\varrho - y\rangle \\
&= \langle x, \nabla^2 f(x^*)x\rangle - \langle \mathcal{A}x, \varrho\rangle - \langle x, y\rangle \\
&= \langle x_\alpha + x_\beta, \nabla^2 f(x^*)(x_\alpha + x_\beta)\rangle + \langle x_\alpha + x_\beta, V_\beta^{-1}(\mathcal{I} - V)_\beta(x_\alpha + x_\beta)\rangle \\
&= \langle x_\alpha + x_\beta, [\nabla^2 f(x^*) + V_\beta^{-1}(\mathcal{I} - V)_\beta](x_\alpha + x_\beta)\rangle.
\end{aligned}$$

These together with assumption (i) yield that $x_\alpha + x_\beta = 0$, and hence $x = 0$. By (5.8), this implies $y_\beta = 0$. Thus, from (5.5), we have $\mathcal{A}^*\varrho = -y_\gamma \in J_\gamma$. By assumption (ii), $\varrho = 0$ and hence $y_\gamma = 0$. So, we finally obtain that $z = 0$. The desired conclusion follows.

In particular, for linear SCP, since $\nabla^2 f(x^*) = 0$, from (5.4) and $\langle x, y\rangle = \langle x_\beta, -V_\beta^{-1}(\mathcal{I} - V)_\beta x_\beta\rangle$, we deduce that $x_\beta = 0$. Then, by (5.6) and assumption (i'), $x_\alpha = 0$. Following the same steps as above, we complete the proof. □

Next, we extend the above theorem to the general case where strict complementarity may not hold. Let $z^* = (x^*, \varrho^*, y^*)$ be a (not necessarily strictly complementary) KKT point of NSCP (1.1). Clearly, in that case, Π_K is strongly semismooth at $(x^* - y^*)$. In order to establish (5.2), we need to provide suitable conditions which guarantee the nonsingularity of all elements of the B-subdifferential of H at z^*. For this purpose, we give the following notations.

For any element $V \in \partial_B \Pi_K(x)$ with $V = 2\mathcal{L}^*\Lambda_t(x)\mathcal{L} - d_t^*(x)\mathcal{L}$ as in Theorem 3.1, from the argument before (2.13) (see Subsection 2.3), we obtain the corresponding index sets $\alpha(V), \beta(V)$ and $\gamma(V)$ as

$$\alpha(V) := \left\{i : a_i^V(x) = 1\right\}, \; \beta(V) := \left\{i : 0 < a_i^V(x) < 1\right\}, \; \gamma(V) := \left\{i : a_i^V(x) = 0\right\}, \tag{5.9}$$

where the scalars $a_i^V(x)$ are the elements of $\Lambda_t(x)$, accounting for multiplicities as $a_1^V(x) \geq a_2^V(x) \geq \cdots \geq a_n^V(x)$.

From the proof of Theorem 5.3, we can obtain the following result on the nonsingularity of any element of the B-subdifferential of H where the strict complementarity assumption is replaced by some conditions on this element, which is a generalization of Theorem 3.8 in [13] from NSOCP to NSCP. For details, see [13] and the references therein.

Theorem 5.4 *Let $z^* = (x^*, \varrho^*, y^*)$ be a (not necessarily strictly complementary) KKT point of NSCP (1.1). For any $V \in \partial_B \Pi_K(x^* - y^*)$, let the index sets $\alpha(V), \beta(V)$ and $\gamma(V)$ be defined by (5.9) with $a_i^V(x) = a_i^V(x^* - y^*)$, and the corresponding subspaces by $J_{\alpha(V)}, J_{\beta(V)}, J_{\gamma(V)}$. Let $V_{\beta(V)} = \sum_{i \in \beta(V)} a_i(x^* - y^*) P_i(x^* - y^*)$. Assume the following conditions hold:*

(i) The operator $\nabla^2 f(x^*) + V_{\beta(V)}^{-1}(\mathcal{I} - V)_{\beta(V)}$ is positive definite on the subspace

$$S := \{s = s_{\alpha(V)} + s_{\beta(V)} + s_{\gamma(V)} \in \mathcal{V} : \mathcal{A}s = 0, s_{\gamma(V)} = 0\};$$

(ii) The following implication holds: $\mathcal{A}^* \varrho \in J_{\gamma(V)} \Rightarrow \varrho = 0$.
Then all the elements of $\partial_B H(z^*)$ are nonsingular. In particular, for the linear SCP, the statement holds with condition (i) replaced by the following one:
(i′) $\{s_{\alpha(V)} : \mathcal{A}s_{\alpha(V)} = 0\} \cap J_{\alpha(V)} = \{0\}$.

When the Hessian $\nabla^2 f(x^*)$ is positive definite on some subspace (see below Theorem 5.5), we will establish the following result which also generalizes Theorem 5.3 to the case where the condition of strict complementarity may not hold. For this purpose, using the above notation (5.9), we define

$$\underline{\alpha}(x) := \bigcap_{V \in \partial_B \Pi_K(x)} \alpha(V), \ \underline{\gamma}(x) := \bigcap_{V \in \partial_B \Pi_K(x)} \gamma(V), \ \overline{\beta}(x) := \{1, \cdots, n\} \setminus (\underline{\alpha}(x) \cup \underline{\gamma}(x)). \quad (5.10)$$

It is easy to see that the sets $\underline{\alpha}(x)$, $\overline{\beta}(x)$ and $\underline{\gamma}(x)$ form a partition of $\{1, 2, \cdots, n\}$. In view of Theorem 2.8, we obtain that $\mathcal{J}$ is the orthogonal direct sum of $J_{\underline{\alpha}(x)}$, $J_{\overline{\beta}(x)}$ and $J_{\underline{\gamma}(x)}$ where

$$J_{\underline{\alpha}(x)} := \sum_{i \in \underline{\alpha}(x)} P_i(x)(\mathcal{J}), \quad J_{\overline{\beta}(x)} := \sum_{i \in \overline{\beta}(x)} P_i(x)(\mathcal{J}), \quad J_{\underline{\gamma}(x)} := \sum_{i \in \underline{\gamma}(x)} P_i(x)(\mathcal{J}). \quad (5.11)$$

Theorem 5.5 *Let $z^* = (x^*, \varrho^*, y^*)$ be a (not necessarily strictly complementary) KKT point of NSCP (1.1). Let the index sets $\underline{\alpha} := \underline{\alpha}(x^* - y^*), \overline{\beta} := \overline{\beta}(x^* - y^*), \underline{\gamma} := \underline{\gamma}(x^* - y^*)$ be specified by (5.10), and let the corresponding subspaces $J_{\underline{\alpha}}, J_{\overline{\beta}}, J_{\underline{\gamma}}$ be given by (5.11). Assume the following conditions hold:*

(i) The Hessian $\nabla^2 f(x^*)$ is positive definite on the subspace

$$S := \{s = s_{\underline{\alpha}} + s_{\overline{\beta}} + s_{\underline{\gamma}} \in \mathcal{V} : \mathcal{A}s = 0, s_{\underline{\gamma}} = 0\};$$

(ii) Under the decomposition $\mathcal{A}^* \varrho = (\mathcal{A}^* \varrho)_{\underline{\alpha}} + (\mathcal{A}^* \varrho)_{\overline{\beta}} + (\mathcal{A}^* \varrho)_{\underline{\gamma}}$, $(\mathcal{A}^* \varrho)_{\underline{\alpha}} = 0 \Rightarrow \varrho = 0$.
Then all elements of $\partial_B H(z^*)$ are nonsingular.

Proof. Choose any $W \in \partial_B H(z^*)$, we need to show that W is nonsingular. It is easy to see that W has the form as

$$W = \begin{pmatrix} \nabla^2 f(x^*) & -\mathcal{A}^* & -\mathcal{I} \\ \mathcal{A} & 0 & 0 \\ \mathcal{I} - V & 0 & V \end{pmatrix},$$

where $V \in \partial_B \Pi_K(x^* - y^*)$ is the given element related to W. Without loss of generality, let $V = 2\mathcal{L}^*\Lambda_t(x^* - y^*)\mathcal{L} - d_t^*(x^* - y^*)\mathcal{L}$ as in Theorem 3.1 and let $\alpha(V), \beta(V), \gamma(V)$ be the corresponding index sets specified by (5.9) with $a_i^V(x^* - y^*)$ being the elements of $\Lambda_t(x^* - y^*)$.

Suppose that $z = (x, \varrho, y) \in \mathcal{V} \times \mathbb{R}^m \times \mathcal{V}$ is any vector such that $Wz = 0$. As in the proof of Theorem 5.3, letting $x := x_{\alpha(V)} + x_{\beta(V)} + x_{\gamma(V)}$ and $y := y_{\alpha(V)} + y_{\beta(V)} + y_{\gamma(V)}$ where $x_{\alpha(V)}, y_{\alpha(V)} \in J_{\alpha(V)}$, $x_{\beta(V)}, y_{\beta(V)} \in J_{\beta(V)}$ and $x_{\gamma(V)}, y_{\gamma(V)} \in J_{\gamma(V)}$, we obtain

$$y_{\alpha(V)} = 0,\ x_{\gamma(V)} = 0 \ \text{ and } \ (\mathcal{I} - V)_{\beta(V)} x_{\beta(V)} + V_{\beta(V)} y_{\beta(V)} = 0,$$

and $\langle x, y\rangle = \left\langle x_{\beta(V)}, y_{\beta(V)}\right\rangle$. Thus, we have

$$x_{\beta(V)} = V_{\beta(V)} x_{\beta(V)} - V_{\beta(V)} y_{\beta(V)}, \tag{5.12}$$

and

$$0 = \left\langle x_{\beta(V)}, (\mathcal{I} - V)_{\beta(V)} x_{\beta(V)} + V_{\beta(V)} y_{\beta(V)}\right\rangle = \left\langle x_{\beta(V)}, (\mathcal{I} - V)_{\beta(V)} x_{\beta(V)}\right\rangle + \left\langle x_{\beta(V)}, V_{\beta(V)} y_{\beta(V)}\right\rangle.$$

Noting that $(\mathcal{I} - V)_{\beta(V)}$ is positive semidefinite on the subspace $J_{\beta(V)}$, we obtain

$$\left\langle x_{\beta(V)}, V_{\beta(V)} y_{\beta(V)}\right\rangle = -\left\langle x_{\beta(V)}, (\mathcal{I} - V)_{\beta(V)} x_{\beta(V)}\right\rangle \le 0. \tag{5.13}$$

Similarly,

$$\left\langle V_{\beta(V)} y_{\beta(V)}, y_{\beta(V)}\right\rangle \ge 0. \tag{5.14}$$

By (5.12),

$$\left\langle x_{\beta(V)}, y_{\beta(V)}\right\rangle = \left\langle V_{\beta(V)} x_{\beta(V)}, y_{\beta(V)}\right\rangle - \left\langle V_{\beta(V)} y_{\beta(V)}, y_{\beta(V)}\right\rangle.$$

This together with (5.13) and (5.14) shows that

$$\langle x, y\rangle = \left\langle x_{\beta(V)}, y_{\beta(V)}\right\rangle \le 0. \tag{5.15}$$

On the other hand, set

$$S(V) := \{s = s_{\alpha(V)} + s_{\beta(V)} + s_{\gamma(V)} \in \mathcal{V} : \mathcal{A}s = 0, s_{\gamma(V)} = 0\}.$$

By (5.10), $\gamma(V) \supseteq \bar{\gamma}$, hence, $S(V) \subseteq S$. Thus, assumption (i) implies that the Hessian $\nabla^2 f(x^*)$ is positive definite on the subspace $S(V)$. Observing that

$$\begin{aligned}
\langle x, y\rangle &= \left\langle x, \nabla^2 f(x^*)x - \mathcal{A}^*\varrho\right\rangle \\
&= \left\langle x, \nabla^2 f(x^*)x\right\rangle - \langle \mathcal{A}x, \varrho\rangle \\
&= \left\langle x_{\alpha(V)} + x_{\beta(V)}, \nabla^2 f(x^*)(x_{\alpha(V)} + x_{\beta(V)})\right\rangle,
\end{aligned}$$

from (5.15), we obtain that $x_{\alpha(V)} + x_{\beta(V)} = 0$. Hence $x = 0$, and $\mathcal{A}^*\varrho = -y_{\gamma(V)} \in J_{\gamma(V)}$. This implies $(\mathcal{A}^*\varrho)_{\alpha(V)} = 0$. Since $\underline{\alpha} \subseteq \alpha(V)$ by (5.10), $(\mathcal{A}^*\varrho)_{\underline{\alpha}} = 0$. By assumption (ii), it follows that $\varrho = 0$. These, along with $\nabla^2 f(x^*)x - \mathcal{A}^*\varrho - y = 0$ by $Wz = 0$, yield $y = 0$. So, we finally obtain that $z = 0$. The conclusion follows. $\square$

We end this paper by stating the following result about the convergence of the semismooth method (5.2), which is based on Theorems 5.1, 5.4 and 5.5, and similar techniques as in [22, 23]. Note that the sufficient conditions in below Theorem 5.6 below do not require strict complementarity of the solution.

Theorem 5.6 *Let $z^* = (x^*, \varrho^*, y^*)$ be a (not necessarily strictly complementary) KKT point of NSCP (1.1), and suppose that the assumptions of Theorems 5.4 or 5.5 hold at this KKT point. Then the semismooth Newton method (5.2) applied to the system of equations $H(z) = 0$ is locally superlinearly convergent. If, in addition, f has a locally Lipschitz continuous Hessian, then it is locally quadratically convergent.*

Proof. From the assumptions and Theorems 5.4 (or 5.5), we know that, at a KKT point $z^* = (x^*, \varrho^*, y^*)$ of NSCP (1.1), any element of $\partial_B H(z^*)$ is nonsingular. Thus, as in the proof of Lemma 2.6 in [22] (or Proposition 3.1 in [23]), we obtain that there are a neighborhood $\delta(z^*)$ of z^* and a constant η such that, for any $z \in \delta(z^*)$ and $W \in \partial_B H(z)$, W is nonsingular and

$$\left\|W^{-1}\right\| \leq \eta.$$

Then, (5.2) is well-defined in a neighborhood of z^* for the first step $k = 0$. Next, in a similar way as in the proof of Theorem 3.1 in [22] (or Theorem 3.2 in [23]), we obtain that

$$\begin{aligned} & \left\|z^{k+1} - z^*\right\| \\ = & \left\|z^k - z^* - W_k^{-1} H(z^k)\right\| \\ \leq & \left\|W_k^{-1}[H(z^k) - H(z^*) - H'(z^*; z^k - z^*)]\right\| + \left\|W_k^{-1}[W_k(z^k - z^*) - H'(z^*; z^k - z^*)]\right\| \\ = & \ o(\|z^k - z^*\|). \end{aligned}$$

This establishes the locally superlinear convergence of $\{z^k\}$ to z^*.

If, in addition, f has a locally Lipschitz continuous Hessian, then from (5.1) and the strong semismoothness of $\Pi_K(x)$ we obtain that H is strongly semismooth at z^*. Hence, similarly, the semismooth Newton method (5.2) is proved to be locally quadratically convergent. □

Based on an exact expression for Clarke generalized Jacobian of the projection onto symmetric cones, we designed a semismooth Newton algorithm for solving NSCP, which is regarded as a generalization of the corresponding results of [13] from the setting of NSOCP. However, in the case of NSDP, Sun [25] shown that the strong second order sufficient condition, together with constraint nondegeneracy, is equivalent to many conditions, notably the strong regularity of the KKT point and the nonsingularity of Clarke generalized Jacobian of a nonsmooth system at a KKT point. Recently, for linear SDP, Chan and Sun [2] established several equivalent links among the primal and dual constraint nondegeneracy conditions, the strong regularity, and the nonsingularity of both the B-subdifferential and Clarke's generalized Jacobian of a nonsmooth system at a KKT point. What is the relationship between the conditions in our results and those in the setting of SDP? (This is tied to an open question proposed by Chan and Sun [2], where they asked whether the corresponding results can be extended to linear SCP.) We leave it as a future research topic.

References

[1] Bauschke, H.H., Borwein, J.M.: On projection algorithms for solving convex feasibility problems. SIAM Rev. 38, 367-426 (1996)

[2] Chan, Z., Sun, D.: Constraint nondegeneracy, strong regularity and nonsingularity in semidefinite programming. SIAM J. Optim. 19, 370-396 (2008)

[3] Chen, J.-S., Chen, X., Tseng, P.: Analysis of nonsmooth vector-valued functions associated with second-order cones. Math. Program. 101, 95-117 (2004)

[4] Chen, X.D., Sun, D., Sun, J.: Complementarity functions and numerical experiments on some smoothing Newton methods for second-order-cone complementarity problems. Comput. Optim. Appl. 25, 39-56 (2003)

[5] Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York, 1983

[6] Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, Volume I and II. Springer-Verlag, New York, 2003

[7] Faraut, J., Korányi, A.: Analysis on Symmetric Cones. Oxford University Press, New York, 1994

[8] Flegel, M.L., Kanzow, C.: Equivalence of two nondegeneracy conditions for semidefinite programs. J. Optim. Theory Appl. 135, 381-397 (2007)

[9] Gowda, M.S., Sznajder, R., Tao, J.: Some P-properties for linear transformations on Euclidean Jordan algebras. Linear Algebra Appl. 393, 203-232 (2004)

[10] Gowda, M.S., Sznajder, R.: Automorphism invariance of P and GUS properties of linear transformations on Euclidean Jordan algebras. Math. Oper. Res. 31, 109-123 (2006)

[11] Gowda, M.S., Tao, J.: Z-transformation on proper and symmetric cone. Math. Program. To appear, (2006)

[12] Hayashi, S., Yamashita, N., Fukushima, M.: A combined smoothing and regularization method for monotone second-order cone complementarity problems. SIAM J. Optim. 15, 593-615 (2005)

[13] Kanzow, C., Ferenczi, I., Fukushima, M.: Semismooth Methods for Linear and Nonlinear Second-Order Cone Programs. Preprint 264, Institute of Mathematics, University of Würzburg, Würzburg, April 2006 (revised January 10, 2007)

[14] Kanzow, C., Nagel, C.: Quadratic convergence of a Newton-type method for semidefinite programs without strict complementarity. SIAM J. Optim. 15, 654-672 (2005)

[15] Koecher, M.: The Minnesota Notes on Jordan Algebras and Their Applications, edited and annotated by A. Brieg and S. Walcher, Springer, Berlin, 1999

[16] Kong, L.C., Sun, J., Xiu, N.H.: A regularized smoothing Newton method for symmetric cone complementarity problems. SIAM J. Optim. To appear, (2006)

[17] Liu, Y., Zhang, L., Wang, Y.: Some properties of a class of merit functions for symmetric cone complementarity problems. Asia-Pacific Journal of Operational Research 23, 473-496 (2006)

[18] Malick, J., Sendov, H.S.: Clarke generalized Jacobian of the projection onto the cone of positive semidefinite matrices. Set-Valued Analysis 14, 273-293 (2006)

[19] Meng,F., Sun, D.,Zhao, G.: Semismoothness of solutions to generalized equations and the Moreau-Yosida regularization. Math. Program. 104, 561-581 (2005)

[20] Mifflin, R.: Semismooth and semiconvex functions in constrained optimization. SIAM J. Cont. Optim. 15, 957-972 (1977)

[21] Pang, J.-S., Qi, L.: Nonsmooth equations: motivation and algorithms. SIAM J. Optim. 3, 443-465 (1993)

[22] Qi, L.: Convergence analysis of some algorithms for solving nonsmooth equations. Math. Oper. Res. 18, 227-244 (1993)

[23] Qi, L., Sun, J.: A nonsmooth version of Newton's method. Math. Program. 58, 353-367 (1993)

[24] Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer, Second Version 2004

[25] Sun, D.: The strong second order sufficient condition and constraint nondegeneracy in nonlinear semidefinite programming and their implications. Math. Oper. Res. 31, 761-776 (2006)

[26] Sun, D., Sun, J.: Löwner's operator and spectral functions on Euclidean Jordan algebras. Math. Oper. Res. 33 (2008)

[27] Tao, J., Gowda, M.S.: Some P-properties for nonlinear transformations on Euclidean Jordan algebras. Math. Oper. Res. 30, 985-1004 (2005)

[28] Yoshise, A.: Interior point trajectories and a homogeneous model for nonlinear complementarity problems over symmetric cones. SIAM J. Optim. 17, 1129-1153 (2006)