Complexity Analyses of Discretized Successive Convex Relaxation Methods

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Abstract

We investigate the computational complexity of discretized successive convex relaxation methods in the way of upper bounding the number of major iterations required, in the worst case. Kojima and Takeda [2] earlier analyzed the computational complexity of semi-infinite successive convex relaxation methods (these methods require the solution of infinitely many linear programming or semidefinite programming problems with infinitely many constraints to be solved during each major iteration). Our analyses extend Kojima-Takeda analysis to the discretized successive convex relaxation methods which require the solution of finitely many ordinary linear programming or semidefinite programming problems in each major iteration. Our complexity bounds are within a small constant (four) multiple of theirs.

Keywords: non-convex quadratic optimization, computational complexity, convex relaxation, semidefinite programming, linear programming, lift-and-project methods

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1 Introduction

A very general problem in optimization is to compute the convex hull of non-convex sets. By *compute*, we mean to express the convex hull as the intersection of halfspaces or the intersection of well-understood convex cones and arbitrary affine subspaces. In the first case, we can essentially reduce the potentially very difficult non-convex optimization problem (say we want to maximize a linear function over the non-convex set) to a potentially very very large scale linear programming (LP) problem. In the second case, we may take the convex cone as the cone of symmetric, positive semidefinite matrices and essentially reduce the potentially very hard, non-convex optimization problem to a potentially very very large scale semidefinite programming (SDP) problem.

Some such general methods were developed by Kojima and the first author [4]. These successive convex relaxation methods laid the foundations; however, they required the solution of infinitely many LP problems or SDP problems with infinitely many linear inequality constraints in every major iteration. Later, *discretizations* of these methods were proposed [5]. The discretized methods required the solution of finitely many (even though potentially a very large number of) ordinary LP problems or SDP problems (with very large number of linear inequality constraints) in every major iteration and these methods are guaranteed to converge to an arbitrarily prescribed approximation of the convex hull in finitely many major iterations (the number depending on the desired precision among other things in the input). These implementable versions led to the development of heuristics and computer codes by Takeda, Dai, Fukuda and Kojima [3].

Recently, Kojima and Takeda [2] studied the computational complexity of semi-infinite successive convex relaxation methods of [4] for the first time and provided upper bounds on the number of major iterations, in terms of certain measures of the input, the feasible region, and the desired precision.

We use the approach of Kojima and Takeda [2] with essentially the same measures. Our analyses can also use their original measures. In this case, our complexity bounds are within a small constant (four) multiple of theirs. Our analyses extend Kojima-Takeda analysis in that discretized successive convex relaxation methods require the solution of finitely many ordinary LP or SDP problems in each major iteration.

We follow much of the notation and the approaches in [5, 2].

$\mathcal{S}^n$	:=	the set of $n \times n$ symmetric matrices.
$\mathcal{S}^n_+$	:=	the cone of $n \times n$ symmetric positive semidefinite matrices.
$qf(\cdot;\gamma,q,Q)$	:=	the quadratic function having the constant term γ, the linear term $2q^Tx$, and the quadratic term x^TQx; $qf(x;\gamma,q,Q) := \gamma + 2q^Tx + x^TQx$ for $x \in \mathbb{R}^n$, where $\gamma \in \mathbb{R}, q \in \mathbb{R}^n$, and $Q \in \mathcal{S}^n$.
$\mathcal{Q}_+$		denotes the set of all $qf(\cdot;\gamma,q,Q)$ such that $Q \in \mathcal{S}^n_+$.

$\mathcal{L}$		denotes the set of all $qf(\cdot;\gamma,q,Q)$ such that $Q=0$.
$\|\|v\|\|$		for $v\in\mathbb{R}^n$, denotes the Euclidean 2-norm.
$\|\|M\|\|$		for $M\in\mathbb{R}^{n\times n}$, denotes the operator 2-norm.
$\|\|M\|\|_F$		for $M\in\mathbb{R}^{n\times n}$, denotes the Frobenius norm.
$\mathrm{vec}(M)$		for $M\in\mathbb{R}^{n\times n}$, denotes the n^2-vector with elements M_{ij}.
$e_j\in\mathbb{R}^n$		denotes the j^{th} unit vector.
$e\in\mathbb{R}^n$		denotes the vector of all ones.
$\bar{D}$	$:=$	$\{d\in\mathbb{R}^n:\ \|\|d\|\|=1\}$.
$B(x_c,r)$	$:=$	$\{x\in\mathbb{R}^n:\ \|\|x-x_c\|\|\le r\}$
		: the closed Euclidean ball with center x_c and radius r.
$D(c,r)$	$:=$	$\bar{D}\cap B(c,r)$.

Given $\delta\ge 0$ and $D\subseteq\bar{D}$, a subset D' of $\bar{D}$ is a δ-net of D if for every $d\in D$, there exists $d'\in D'$ such that $\|d-d'\|\le\delta$.

The input data are a nonempty compact convex subset C_0 of $\mathbb{R}^n$ (e.g., an ellipsoid) and a set $\mathcal{P}_F$ of finitely many quadratic functions. The complexity bounds will depend on the diameters of C_0 and F denoted $\mathrm{diam}(C_0)$ (and $\mathrm{diam}(F)$), a given positive constant ϵ, describing the desired approximation of the convex hull of F (c.hull(F)) and the complexity measures given below (also see [2]):

$$
\begin{aligned}
\nu_{\mathrm{lip}} &:= \nu_{\mathrm{lip}}(\mathcal{P}_F,C_0) := \sup\left\{\frac{|p(x)-p(y)|}{\|x-y\|} : x,y,\in C_0, x\neq y, p(\cdot)\in\mathcal{P}_F\right\},\\
\nu_{\mathrm{nc}} &:= \nu_{\mathrm{nc}}(\mathcal{P}_F) := \max\{-\inf\{\lambda_{\min}(Q) : qf(\cdot;\gamma,q,Q)\in\mathcal{P}_F\},0\},\\
\nu_{\mathrm{nl}} &:= \nu_{\mathrm{nl}}(\mathcal{P}_F) := \sup\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}|Q_{i,j}| : qf(\cdot;\gamma,q,Q)\in\mathcal{P}_F\right\}.
\end{aligned}
$$

Here, $\lambda_{\min}(Q)$ denotes the minimum eigenvalue of $Q\in\mathcal{S}^n$. Note that ν_{lip}, ν_{nc}, and ν_{nl} are finite nonnegative numbers. If $\nu_{\mathrm{nl}}=0$, then we have a description of F in terms of the intersection of halfspaces and $F=\mathrm{c.hull}(F)$. Therefore, we can assume $\nu_{\mathrm{nl}}>0$. Now, we can normalize the data to ensure that that $\nu_{\mathrm{nl}}=1$. Let's call the related measures of the scaled data $\bar{\nu}_{\mathrm{lip}}$, $\bar{\nu}_{\mathrm{nc}}$ and $\bar{\nu}_{\mathrm{nl}}$. Note that $\bar{\nu}_{\mathrm{nc}}\le 1$. Our analyses can also be used (with very minor modifications) for the unscaled data (with $\nu_{\mathrm{nl}}\neq 1$).

Let C be a compact subset of $\mathbb{R}^n$. For every $d,d_1,d_2\in\bar{D}$ and for every $x\in\mathbb{R}^n$, define

$$
\begin{aligned}
\alpha(C,d) &:= \max\{d^Tx : x\in C\},\\
lsf(x,C,d) &:= d^Tx-\alpha(C,d),\\
r2f(x,C,d_1,d_2) &:= -(d_1^Tx-\alpha(d_1,C_0))(d_2^Tx-\alpha(d_2,C)).
\end{aligned}
$$

Let $D_1, D_2 \subseteq \bar{D}$, we also define

$$\begin{aligned} \mathcal{P}^L(C, D) &:= \{lsf(\cdot; C, d) : d \in D\}, \\ \tilde{\mathcal{P}}^2(C, D_1, D_2) &:= \{r2f(\cdot; C, d_1, d_2) : d_1 \in D_1, d_2 \in D_2\} \end{aligned}$$

Now, we describe the algorithms based on successive SDP relaxations and successive semi-infinite LP relaxations.

Algorithm 1(SSDP Relaxation Method [4, 5])

Step 0 : Choose a pair of direction sets $D_1, D_2 \subseteq \bar{D}$. Let $k = 0$.

Step 1 : If $C_k = \emptyset$ then stop.

Step 2 : Compute $\alpha(C_k, d) = \max\{d^T x : x \in C_k\}$ $(\forall d \in D_1 \cup D_2)$.

Step 3 : Let $\mathcal{P}_k = \mathcal{P}^L(C_k, D_1) \cup \tilde{\mathcal{P}}^2(C_k, D_1, D_2)$.

Step 4 : Let

$$\begin{aligned} C_{k+1} &= \hat{F}(C_0, \mathcal{P}_F \cup \mathcal{P}_k) \\ &:= \left\{ x \in C_0 : \begin{array}{l} \exists X \in \mathcal{S}^n \text{ such that } \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{S}^{1+n}_+ \text{and} \\ \gamma + 2q^T x + Q \bullet X \leq 0 \ (\forall qf(\cdot; \gamma, q, Q) \in \mathcal{P}_F \cup \mathcal{P}_k) \end{array} \right\}. \end{aligned}$$

Step 5 : Let $k = k + 1$, and go to Step 1.

Algorithm 2 (SSILP Relaxation Method [4, 5])

Step 0, 1, 2, 3: The same as Steps 0, 1, 2, 3, of Algorithm 1, respectively.

Step 4 : Let

$$\begin{aligned} C_{k+1} &= \hat{F}^L(C_0, \mathcal{P}_F \cup \mathcal{P}_k) \\ &:= \left\{ x \in C_0 : \begin{array}{l} \exists X \in \mathcal{S}^n \text{ such that} \\ \gamma + 2q^T x + Q \bullet X \leq 0 \ (\forall qf(\cdot; \gamma, q, Q) \in \mathcal{P}_F \cup \mathcal{P}_k) \end{array} \right\}. \end{aligned}$$

Step 5 The same as step 5 of Algorithm 1.

In Section 2, we extract the skeleton of the analysis of Kojima and Takeda [2] by separating it from the more technical part (which requires the construction of certain valid inequalities for Euclidean Balls containing the current convex relaxation) and make it slightly more general and abstract to enhance the understanding of the analyses. In Section 3, we cite two technical results also used in [5] and [2]. In Section 4, we present our technical results for Algorithm 1 and complete the analysis using the results of Sections 2 and 3. Section 5 serves the same purpose for Algorithm 2 in the same way.

2 The frame of the complexity analyses

In this section, we go over the skeleton of the complexity analysis of Kojima and Takeda [2]. We slightly unify a few of the ideas of the analysis.

Given $\xi \in \mathbb{R}^n$ and $G \subseteq \mathbb{R}^n$,

$$\rho(\xi, G) \ := \ \sup \{ \|x - \xi\| : \ x \in G \},$$

the radius of the smallest Euclidean ball, centered at ξ, containing G. Given $\epsilon > 0$ and $\psi > 0$, two relaxations of F are used:

$$F(\epsilon) \ := \ F(\epsilon; C_0, \mathcal{P}_F) \ := \ \{x \in C_0 : \ p(x) \ \leq \ \epsilon, \quad \text{for all } p(\cdot) \in \mathcal{P}_F\}$$

and

$$\text{c.relax}\,(F(\psi), G) \ := \ \bigcap_{\xi \in G} B\,(\xi, \rho(\xi, F(\psi)).$$

Note that by definition, $\text{c.relax}\,(F(\psi), G) \supseteq \text{c.hull}(F(\psi)) \supseteq \text{c.hull}(F)$. The first relaxation, $F(\epsilon)$, is used in describing the desired output of the algorithm, which is a convex set C_k containing F and contained in the convex approximation $\text{c.hull}(F(\epsilon))$. The second relaxation, $\text{c.relax}\,(F(\psi), G)$, even though implicit, is currently much easier to deal with in the complexity analyses than the first relaxation (see [5] and [2]). Kojima and Takeda [2] connected these two relaxations with the following result.

Lemma 2.1 *(Corollary 2.5 of [2]) Assume that*

$$\xi_0 \in C_0 \ \textit{and} \ 0 < \epsilon \leq \frac{\nu_{lip} diam(C_0)}{2}. \tag{1}$$

Let

$$\Xi \ := \ B\left(\xi_0, \frac{2\nu_{lip}\,(diam(C_0))^2}{\epsilon}\right). \tag{2}$$

Then $c.relax\,(F(\epsilon/2), \Xi) \ \subseteq \ c.hull(F(\epsilon))$.

We see from the conclusion of the above lemma that to establish the convergence in k iterations, it suffices for us to prove that $C_k \subseteq \text{c.relax}\,(F(\epsilon/2), \Xi)$ for the Algorithms 1 and 2.

Given $\bar{\nu}$ (to be described later, in terms of $\bar{\nu}_{\text{lip}}$ and $\bar{\nu}_{\text{nc}}$), we define

$$\bar{\kappa} \ := \ \begin{cases} \frac{8}{\pi} & \text{if } \bar{\nu} \leq \frac{\pi\epsilon^2}{2^{10} n (\text{diam}(C_0))^3}, \\ \frac{\epsilon^2}{2^7 n \bar{\nu} (\text{diam}(C_0))^3} & \text{otherwise,} \end{cases}$$

$$\bar{\theta} \;:=\; \begin{cases} \frac{\pi}{8} & \text{if } \bar{\nu} \leq \frac{\epsilon^2}{2^4 \pi n \left(\mathrm{diam}(C_0)\right)^3}, \\ \frac{\epsilon^2}{2^7 n \bar{\nu} \left(\mathrm{diam}(C_0)\right)^3} & \text{otherwise}, \end{cases}$$

$$\bar{\delta} \;:=\; 1 - \bar{\kappa}\bar{\theta}.$$

Note that $\bar{\delta} \geq 0$,

$$\bar{\delta} = 0 \qquad \text{iff} \qquad \bar{\nu} \leq \frac{\pi \epsilon^2}{2^{10} n \left(\mathrm{diam}(C_0)\right)^3} \tag{3}$$

and

$$0 \;<\; \bar{\delta} \;<\; 1 \qquad \text{iff} \qquad \bar{\nu} > \frac{\pi \epsilon^2}{2^{10} n \left(\mathrm{diam}(C_0)\right)^3}. \tag{4}$$

In particular, if

$$\frac{\pi \epsilon^2}{2^{10} n \left(\mathrm{diam}(C_0)\right)^3} \;<\; \bar{\nu} \;<\; \frac{\epsilon^2}{2^4 \pi n \left(\mathrm{diam}(C_0)\right)^3},$$

then

$$0 \;<\; \bar{\delta} \;<\; \left(1 - \frac{\pi^2}{2^6}\right). \tag{5}$$

If

$$\bar{\nu} \;\geq\; \frac{\epsilon^2}{2^4 \pi n \left(\mathrm{diam}(C_0)\right)^3}$$

then

$$-\ln(\bar{\delta}) \;\;\geq\;\; \bar{\kappa}\bar{\theta} \;=\; \frac{\epsilon^4}{2^{14} n^2 \bar{\nu}^2 \left(\mathrm{diam}(C_0)\right)^6}. \tag{6}$$

It is also easy to compute

$$(\bar{\kappa} + \bar{\theta})\bar{\nu} \;\;\leq\;\; \frac{\epsilon^2}{2^6 n (\mathrm{diam}(C_0))^3}. \tag{7}$$

For every $\xi \in \Xi$ and $k \geq 0$, define

$$\rho'_{k+1}(\xi) \;:=\; \max\left\{\rho\left(\xi, F(\epsilon/2)\right), \bar{\delta}\rho(\xi, C_k)\right\}.$$

Definition 2.1 *A successive convex relaxation algorithm is said to have the* $\bar{\delta}$-shrinking property *if for every iteration* $k \geq 0$,

$$C_{k+1} \;\subseteq \bigcap_{\xi \in \Xi} B\left(\xi, \rho'_{k+1}(\xi)\right).$$

Suppose we prove that an algorithm has the $\bar{\delta}$-shrinking property for some value of $\bar{\delta}$, bounded away from 1. Then, we see that for each $\xi \in \Xi$ either the termination condition holds, or the corresponding ρ value decreases as

$$\rho(\xi, C_{k+1})) \quad \leq \quad \bar{\delta}\rho(\xi, C_k). \tag{8}$$

Considering the relations (6) and (8), we can establish

Lemma 2.2 *([2]) Suppose $diam(F) > 0$, the assumptions of Lemma 2.1 hold and that the successive convex relaxation algorithm at hand has the $\bar{\delta}$-shrinking property for the above defined value of $\bar{\delta}$. Define*

$$k^* = \begin{cases} 1 & \textit{if } \ \bar{\nu} \leq \frac{\pi\epsilon^2}{2^{10}n(diam(C_0))^3}, \\ \left\lceil \left(\frac{2^7 n\bar{\nu}(diam(C_0))^3}{\epsilon^2}\right)^2 \ln \frac{8\bar{\nu}_{lip} diam(C_0)^2}{\epsilon \ diam(F)} \right\rceil & \textit{otherwise.} \end{cases} \tag{9}$$

If $k \geq k^*$, then $C_k \ \subseteq \ c.hull(F(\epsilon))$.

Proof. For every $\xi \in \Xi$ and $k = 0, 1, \ldots$, define

$$\rho_k(\xi) := \max\{\rho(\xi, F(\psi)), \rho(\xi, C_k)\}.$$

It suffices to show that if $k \geq k^*$ then

$$\rho_k(\xi) \leq \rho(\xi, F(\psi)), \quad \forall \xi \in \Xi. \tag{10}$$

In fact, if (10) holds, then

$$C_k \subseteq \bigcap_{\xi \in \Xi} B(\xi, \rho_k(\xi)) \subseteq \bigcap_{\xi \in \Xi} B(\xi, \rho(\xi, F(\psi))) = \text{c.relax}(F(\psi), \Xi).$$

By the $\bar{\delta}$-shrinking property,

$$\rho_{k+1}(\xi) \leq \max\{\rho(\xi, F(\psi)), \bar{\delta}\rho_k(\xi)\}$$

This implies that

$$\rho_k(\xi) \leq \max\{\rho(\xi, F(\psi)), \bar{\delta}^k \rho_0(\xi)\} \ \forall \xi \in \Xi \text{ and } \forall k = 0, 1, 2, \ldots.$$

Hence, for each $\xi \in \Xi$, if k satisfies the inequality

$$\bar{\delta}^k \rho_0(\xi) \leq \rho(\xi, F(\psi)), \tag{11}$$

then $\rho_k(\xi) \leq \rho(\xi, F(\psi))$. When $\bar{\nu} \leq \frac{\pi\epsilon^2}{2^{10}n(\mathrm{diam}(C_0))^3}$, we see from (3) that $\bar{\delta} = 0$. Hence (10) holds for $k = 1$. Next, we consider the case $\frac{\pi\epsilon^2}{2^{10}n(\mathrm{diam}(C_0))^3} < \bar{\nu} \leq \frac{\epsilon^2}{2^4\pi n(\mathrm{diam}(C_0))^3}$. In this case, we have, by (5),

$$-\ln(\bar{\delta}) \geq \frac{\pi^2}{2^6}.$$

Using this, and the below arguments for the third remaining case, we can prove that k^* can be taken as

$$\left\lceil \frac{2^6}{\pi^2} \ln \frac{8\bar{\nu}_{\mathrm{lip}} \mathrm{diam}(C_0)^2}{\epsilon \ \mathrm{diam}(F)} \right\rceil,$$

if $\frac{\pi\epsilon^2}{2^{10}n(\mathrm{diam}(C_0))^3} < \bar{\nu} \leq \frac{\epsilon^2}{2^4\pi n(\mathrm{diam}(C_0))^3}$. This bound is clearly better than the claim of the lemma. Now finally, assume that $\bar{\nu} > \frac{\pi\epsilon^2}{2^{10}n(\mathrm{diam}(C_0))^3}$. Then $\bar{\delta} > 0$ by (4). Since

$$F \subseteq F(\psi) \subseteq C_0 \subseteq \Xi,$$

we see that

$$\mathrm{diam}(F)/2 \leq \rho(\xi, F(\psi)) \leq \rho_0(\xi) \leq \mathrm{diam}(\Xi), \text{ for } \forall \xi \in \Xi,$$

Hence, if $\bar{\delta}^k \left(\frac{4\bar{\nu}_{\mathrm{lip}}(\mathrm{diam}(C_0))^2}{\epsilon} \right) \leq \mathrm{diam}(F)/2$, or equivalently, if

$$k(-\ln \bar{\delta}) \geq \ln \frac{8\bar{\nu}_{\mathrm{lip}}(\mathrm{diam}(C_0))^2}{\epsilon \ \mathrm{diam}(F)},$$

then (11) holds. Using (6) we conclude that if $k \geq k^*$, then

$$C_k \subseteq \text{c.relax}(F(\epsilon/2), \Xi) \subseteq \text{c.hull}(F(\epsilon))$$

as desired. □

3 Some technical results from the previous work

First we cite a theorem establishing the equivalence of two types of relaxations for each of the algorithms.

Theorem 3.1 *(see Theorem 2.1 of [1], and Theorem 4.2 and Corollary 4.3 of [4]) Under our assumptions, for Algorithm 1, we have for every $k \geq 1$,*

$$\hat{F}(C_0, \mathcal{P}_F \cup \mathcal{P}_k) = \{x \in \mathbb{R}^n : \ p(x) \leq 0, \ \forall p(\cdot) \in (c.cone(\mathcal{P}_F \cup \mathcal{P}_k) \cap \mathcal{Q}_+)\}.$$

Under our assumptions, for Algorithm 2, we have for every $k \geq 1$,

$$\hat{F}(C_0, \mathcal{P}_F \cup \mathcal{P}_k) = \{x \in \mathbb{R}^n : \ p(x) \leq 0, \ \forall p(\cdot) \in (c.cone(\mathcal{P}_F \cup \mathcal{P}_k) \cap \mathcal{L})\}.$$

Next, we will cite a result about the rank two valid inequalities for Euclidean balls. For every $\theta \in (0, \pi/8]$, $w \in \bar{D}$, and $i = 1, 2, \ldots, n$, define

$$\begin{array}{ll} v_i(\theta, w) := \|w \cos\theta + e_i \sin\theta\|, & \bar{v}_i(\theta, w) := \|w \cos\theta - e_i \sin\theta\|, \\ b_i(\theta, w) := \frac{w\cos\theta + e_i \sin\theta}{v_i(\theta, w)} \in \bar{D}, & \bar{b}_i(\theta, w) := \frac{w\cos\theta - e_i \sin\theta}{\bar{v}_i(\theta, w)} \in \bar{D}, \\ \lambda_i(\theta, w) := \frac{v_i(\theta, w)}{2\sin\theta}, & \bar{\lambda}_i(\theta, w) = \frac{\bar{v}_i(\theta, w)}{2\sin\theta}. \end{array} \quad (12)$$

For every $x \in \mathbb{R}^n, \rho > 0, \theta \in (0, \pi/8], w \in \bar{D}, i = 1, 2 \ldots, n$, and $j = 1, 2, \ldots, n$, define

$$\begin{array}{rcl} f_{ij}^{+}(x, \rho, \theta, w) & := & \lambda_i(\theta, w) r2f(x, -e_j, b_i(\theta, w), B(0, \rho)) \\ & & + \bar{\lambda}_i(\theta, w) r2f(x, e_j, \bar{b}_i(\theta, w), B(0, \rho)) \\ & = & -\lambda_i(\theta, w)(b_i(\theta, w)^T x - \rho)\left(-e_j^T x - \alpha(-e_j, C_0)\right) \\ & & -\bar{\lambda}_i(\theta, w)(\bar{b}_i(\theta, w)^T x - \rho)\left(e_j^T x - \alpha(e_j, C_0)\right), \end{array}$$

$$\begin{array}{rcl} f_{ij}^{-}(x, \rho, \theta, w) & := & \lambda_i(\theta, w) r2f(x, e_j, b_i(\theta, w), B(0, \rho)) \\ & & + \bar{\lambda}_i(\theta, w) r2f(x, -e_j, \bar{b}_i(\theta, w), B(0, \rho)) \\ & = & -\lambda_i(\theta, w)(b_i(\theta, w)^T x - \rho)\left(e_j^T x - \alpha(e_j, C_0)\right) \\ & & -\bar{\lambda}_i(\theta, w)(\bar{b}_i(\theta, w)^T x - \rho)\left(-e_j^T x - \alpha(-e_j, C_0)\right). \end{array}$$

We use the following further extension of Lemma 4.3 of [5], which was also used by [2].

Lemma 3.1 *(Lemma 4.1 of [2]) Let $\rho > 0, \theta \in (0, \pi/8], w \in \bar{D}, i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$.*

(i) Let $D' := \{b_j(\theta, w), \bar{b}_j(\theta, w)\}$. Then

$$f_{ij}^{+}(., \rho, \theta, w), f_{ij}^{-}(., \rho, \theta, w) \in c.cone(\mathcal{P}^2(B(0, \rho), D_1, D')),$$

for every $\rho > 0, \theta \in (0, \pi/8]$ and $w \in \bar{D}$.

$(ii)^+$ The Hessian matrix of the quadratic function $f_{ij}^{+}(., \rho, \theta, w) : \mathbb{R}^n \to \mathbb{R}$ coincides with the $n \times n$ matrix $\frac{e_i e_j^T + e_j e_i^T}{2}$.

$(ii)^-$ The Hessian matrix of the quadratic function $f_{ij}^{-}(., \rho, \theta, w) : \mathbb{R}^n \to \mathbb{R}$ coincides with the $n \times n$ matrix $-\frac{e_i e_j^T + e_j e_i^T}{2}$.

(iii) Suppose that $\tilde{\kappa} \geq 0, 1 \geq \tilde{\delta} \geq 1 - \tilde{\kappa}\theta \geq 0$ and $\tilde{\delta}\rho w \in C_0$. Then

$$f_{ij}^{+}(\tilde{\delta}\rho w, \rho, \theta, w), f_{ij}^{-}(\tilde{\delta}\rho w, \rho, \theta, w) \in [-2\rho\, diam(C_0)(\tilde{\kappa} + \theta), 0].$$

4 Complexity analyses for SDP based algorithms

In this section, when we use the results from Section 2, we always let $\bar{\nu} := \bar{\nu}_{\rm lip}\bar{\nu}_{\rm nc}$. Also, in this section, we can assume $\bar{\nu}_{\rm nc} > 0$. (If $\bar{\nu}_{\rm nc} = 0$, then F is convex and the first SDP relaxation is exact.)

Let $\tau\rho_k\bar{w} \in C_0$ with $\tau \le 1$ and $\|\bar{w}\| = 1$. Then

$$b_i(\bar{\theta}, \bar{w}) \text{ and } \bar{b}_i(\bar{\theta}, \bar{w}) \in \bar{D}.$$

Let

$$\delta \le \min\left\{\frac{\sin\bar{\theta}}{4n}, \frac{\epsilon^2 \sin\bar{\theta}}{2^6 n \bar{\nu}_{\rm lip}\bar{\nu}_{\rm nc}({\rm diam}(C_0))^3}\right\} \tag{13}$$

and D_2 be a $\delta-$net of $\bar{D}$. Choose

$$a_i \in D(b_i(\bar{\theta}, \bar{w}), \delta) \cap D_2,$$

$$\bar{a}_i \in D(\bar{b}_i(\bar{\theta}, \bar{w}), \delta) \cap D_2.$$

Let $A := [a_1, a_2, \ldots, a_n] \in \mathbb{R}^{n\times n}$, $\bar{A} := [\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n] \in \mathbb{R}^{n\times n}$,

$$\begin{aligned} g_{ij}(x) &:= \lambda_i(\bar{\theta}, \bar{w}) r2f(x; -e_j, a_i, C_k) + \bar{\lambda}_i(\bar{\theta}, \bar{w}) r2f(x; e_j, \bar{a}_i, C_k) \\ &= -\lambda_i(\bar{\theta}, \bar{w})\left(a_i^T x - \alpha(a_i, C_k)\right)\left(-e_j^T x - \alpha(-e_j, C_0)\right) \\ &\quad -\bar{\lambda}_i(\bar{\theta}, \bar{w})\left(\bar{a}_i^T x - \alpha(\bar{a}_i, C_k)\right)\left(e_j^T x - \alpha(e_j, C_0)\right). \end{aligned}$$

Let

$$g(x) := \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{ij} g_{ij},$$

where $M := [\mu_{ij}] \in \mathcal{S}^n$.

Proposition 4.1 *Let* $\rho_k \in \left(0, \frac{2\bar{\nu}_{lip}(diam(C_0))^2}{\epsilon}\right)$ *and* $C_k \subseteq B(0, \rho_k)$. *Then there exists a*

$$\delta^* \in \left(0, \min\left\{\frac{\sin\bar{\theta}}{4n}, \frac{\epsilon^2 \sin\bar{\theta}}{2^6 n \bar{\nu}_{lip}\bar{\nu}_{nc}(diam(C_0))^3}\right\}\right)$$

such that when we take D_2 *to be a* δ^**-net of* $\bar{D}$, *we have the following properties: There exists* $[\mu_{ij}] \in \mathcal{S}^n$ *such that*

(i) $\mu_{ij} \ge 0$, *for all* i, j,

(ii) the Hessian of $g(x)$ is $\bar{\nu}_{nc}\left(I + \frac{1}{\sqrt{n}}ee^T\right)$,

(iii) $\|vec(M)\|_1 \leq 4n\bar{\nu}_{nc}$.

Proof.

The Hessian of $g(x)$ is

$$
\begin{aligned}
&= -\sum_{i=1}^n\sum_{j=1}^n \mu_{ij}\frac{\left(-\lambda_i(\bar{\theta},\bar{w})a_ie_j^T + \bar{\lambda}_i(\bar{\theta},\bar{w})\bar{a}_ie_j^T\right) + \left(-\lambda_i(\bar{\theta},\bar{w})a_ie_j^T + \bar{\lambda}_i(\bar{\theta},\bar{w})\bar{a}_ie_j^T\right)^T}{2} \\
&= \sum_{i=1}^n\sum_{j=1}^n \mu_{ij}\frac{\left(\lambda_i(\bar{\theta},\bar{w})a_i - \bar{\lambda}(\bar{\theta},\bar{w})\bar{a}_i\right)e_j^T + e_j\left(\lambda_i(\bar{\theta},\bar{w})a_i - \bar{\lambda}(\bar{\theta},\bar{w})\bar{a}_i\right)^T}{2} \\
&= \sum_{i=1}^n\sum_{j=1}^n \mu_{ij}\frac{h_i(\bar{\theta},\bar{w},A,\bar{A})e_j^T + e_jh_i(\bar{\theta},\bar{w},A,\bar{A})^T}{2} \\
&= \frac{H(\bar{\theta},\bar{w},A,\bar{A})M + MH(\bar{\theta},\bar{w},A,\bar{A})^T}{2},
\end{aligned}
$$

where

$$h_i(\bar{\theta},\bar{w},A,\bar{A}) := \lambda_i(\bar{\theta},\bar{w})a_i - \bar{\lambda}_i(\bar{\theta},\bar{w})\bar{a}_i, \quad i = 1,2,\ldots,n, \tag{14}$$

and

$$H(\bar{\theta},\bar{w},A,\bar{A}) := \left(h_1(\bar{\theta},\bar{w},A,\bar{A}), h_2(\bar{\theta},\bar{w},A,\bar{A}),\ldots,h_n(\bar{\theta},\bar{w},A,\bar{A})\right). \tag{15}$$

Since

$$
\begin{aligned}
h_i(\bar{\theta},\bar{w},A,\bar{A}) &= \lambda_i(\bar{\theta},\bar{w})a_i - \bar{\lambda}(\bar{\theta},\bar{w})\bar{a}_i \\
&= \lambda_i(\bar{\theta},\bar{w})b_i(\bar{\theta},\bar{w}) - \bar{\lambda}_i(\bar{\theta},\bar{w})\bar{b}_i(\bar{\theta},\bar{w}) \\
&\quad +\lambda_i(\bar{\theta},\bar{w})\left(a_i - b_i(\bar{\theta},\bar{w})\right) - \bar{\lambda}_i(\bar{\theta},\bar{w})\left(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w})\right),
\end{aligned}
$$

and

$$\lambda_i(\bar{\theta},\bar{w})b_i(\bar{\theta},\bar{w}) - \bar{\lambda}_i(\bar{\theta},\bar{w})\bar{b}_i(\bar{\theta},\bar{w}) = e_i \quad (i = 1,2,\ldots,n).$$

we have $H = I - P$ with $P := [p_1, p_2, \ldots, p_n]$ and

$$p_i := \lambda_i(\bar{\theta},\bar{w})\left(a_i - b_i(\bar{\theta},\bar{w})\right) - \bar{\lambda}_i(\bar{\theta},\bar{w})\left(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w})\right). \tag{16}$$

Using (16) and the definitions, we obtain

$$
\begin{aligned}
\|P\| &\leq \|P\|_F \\
&= \sqrt{\|p_1\|^2 + \|p_2\|^2 + \ldots + \|p_n\|^2} \\
&\leq \|p_1\| + \|p_2\| + \ldots + \|p_n\| \\
&\leq \sum_{i=1}^n \left(|\lambda_i(\bar{\theta},\bar{w})|\,\left\|a_i - b_i(\bar{\theta},\bar{w})\right\| + |\bar{\lambda}_i(\bar{\theta},\bar{w})|\,\left\|\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w})\right\|\right)
\end{aligned}
$$

$$\leq \sum_{i=1}^{n} \frac{2\delta^*}{\sin\bar{\theta}}$$
$$\leq \tfrac{1}{2},$$

where the second to last inequality follows from

$$\begin{aligned}
|\lambda_i(\bar{\theta},\bar{w})| &= \left|\frac{v_i(\bar{\theta},\bar{w})}{2\sin\bar{\theta}}\right| \\
&= \frac{\left\|\bar{w}\cos\bar{\theta} + e_i\sin\bar{\theta}\right\|}{2\sin\bar{\theta}} \\
&= \frac{(\left\|\bar{w}\cos\bar{\theta}\right\|^2 + \left\|e_i\sin\bar{\theta}\right\|^2 + 2\bar{w}^T e_i \sin\bar{\theta}\cos\bar{\theta})^{\frac{1}{2}}}{2\sin\bar{\theta}} \\
&= \frac{(1 + 2\bar{w}^T e_i \sin\bar{\theta}\cos\bar{\theta})^{\frac{1}{2}}}{2\sin\bar{\theta}} \\
&\leq \frac{1 + \bar{w}^T e_i \cos\bar{\theta}\sin\bar{\theta}}{2\sin\bar{\theta}} \\
&= \frac{1 + 0.5\bar{w}^T e_i \sin 2\bar{\theta}}{2\sin\bar{\theta}} \\
&\leq \frac{1}{\sin\bar{\theta}},
\end{aligned}$$

and

$$|\bar{\lambda}_i(\bar{\theta},\bar{w})| \leq \frac{1}{\sin\bar{\theta}}.$$

So, the real parts of the eigenvalues of H are at least $\frac{1}{2}$ and by Lyapunov Theorem, the equation

$$\frac{H(\bar{\theta},\bar{w},A,\bar{A})M + MH(\bar{\theta},\bar{w},A,\bar{A})^T}{2} = \bar{\nu}_{\mathrm{nc}}\left(I + \frac{1}{\sqrt{n}}ee^T\right) \tag{17}$$

has a unique solution $[\mu_{ij}] \in \mathcal{S}^n$. Note that when we set

$$a_i := b_i(\bar{\theta},\bar{w}), \quad \text{and} \quad \bar{a}_i := \bar{b}_i(\bar{\theta},\bar{w}),$$

we obtain $H = I$. In this case, the unique solution of (17) is $\bar{\nu}_{\mathrm{nc}}\left(I + \frac{1}{\sqrt{n}}ee^T\right)$. Clearly, every component of this matrix is positive. By continuity of the solution with respect to the perturbations in H (and hence $a_i, \bar{a}_i$), we see that there exists $\delta^* > 0$ depending on $\bar{\theta}, \bar{w}$ such that for every $a_i \in D(b_i(\bar{\theta},\bar{w}),\delta^*)$, and for every $\bar{a}_i \in D(\bar{b}_i(\bar{\theta},\bar{w}),\delta^*)$, the solution matrix M has all of its entries non-negative.

We now show the boundedness of $[\mu_{ij}]$. Using the Kronecker product, the Lyapunov equation (17) can be rewritten as follows:

$$\frac{I\otimes H(\bar{\theta},\bar{w},A,\bar{A}) + H(\bar{\theta},\bar{w},A,\bar{A})^T\otimes I}{2}\mathrm{vec}(M) = \mathrm{vec}\left(\bar{\nu}_{\mathrm{nc}}\left(I + \frac{1}{\sqrt{n}}ee^T\right)\right). \tag{18}$$

The coefficient matrix can be written as

$$\begin{aligned}\frac{I\otimes H(\bar\theta,\bar w,A,\bar A)+H(\bar\theta,\bar w,A,\bar A)^T\otimes I}{2} &= \frac{I\otimes(I-P)+(I-P)^T\otimes I}{2}\\ &= I\otimes I-\frac{I\otimes P+P^T\otimes I}{2}.\end{aligned}\tag{19}$$

Since

$$\|I\otimes P\|_F\le\sqrt{n\sum_{i=1}^n\|p_i\|^2}\le\frac{2n\delta^*}{\sin\bar\theta}$$

and

$$\left\|P^T\otimes I\right\|_F\le\sqrt{n\sum_{i=1}^n\|p_i\|^2}\le\frac{2n\delta^*}{\sin\bar\theta},$$

we have

$$\begin{aligned}\left\|\frac{I\otimes P+P^T\otimes I}{2}\right\| &\le \left\|\frac{I\otimes P+P^T\otimes I}{2}\right\|_F\\ &\le \frac{1}{2}\|I\otimes P\|_F+\frac{1}{2}\left\|P^T\otimes I\right\|_F\\ &\le \frac{2n\delta^*}{\sin\bar\theta}\le\frac{1}{2}.\end{aligned}$$

By (19),

$$\begin{aligned}\left\|\left(\frac{I\otimes H(\bar\theta,\bar w,A,\bar A)+H(\bar\theta,\bar w,A,\bar A)^T\otimes I}{2}\right)^{-1}\right\| &= \left\|\left(I\otimes I-\frac{I\otimes P+P^T\otimes I}{2}\right)^{-1}\right\|\\ &\le \frac{1}{1-\left\|\frac{I\otimes P+P^T\otimes I}{2}\right\|}\\ &\le \frac{1}{1-\frac{1}{2}}\\ &= 2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathrm{vec}(M)\| &\le \bar\nu_{\mathrm{nc}}\left\|\left(\frac{I\otimes H(\bar\theta,\bar w,A,\bar A)+H(\bar\theta,\bar w,A,\bar A)^T\otimes I}{2}\right)^{-1}\right\|\left\|\mathrm{vec}\left(I+\frac{1}{\sqrt n}ee^T\right)\right\|\\ &\le 4\sqrt{n}\bar\nu_{\mathrm{nc}}.\end{aligned}$$

So, $\|\mathrm{vec}(M)\|_1\le\sqrt{n}\,\|\mathrm{vec}(M)\|\le 4n\bar\nu_{\mathrm{nc}}$. □

Note that by the above construction and the above proposition (implying $\mu_{ij} \geq 0$) we have that $g(\cdot) \in \text{c.cone}\left(\tilde{\mathcal{P}}^2(C_k, D_1, D_2)\right)$.

Proposition 4.2 *Under the same conditions as in Proposition 4.1, we have*

$$g(\tau\rho_k\bar{w}) \geq -\epsilon/2. \tag{20}$$

Proof.

$$\begin{aligned}
g_{ij}(\tau\rho_k\bar{w}) &= -\lambda_i(\bar{\theta},\bar{w})\left(a_i^T\tau\rho_k\bar{w} - \alpha(a_i, C_k)\right)\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j, C_0)\right) \\
&\quad -\bar{\lambda}_i(\bar{\theta},\bar{w})\left(\bar{a}_i^T\tau\rho_k\bar{w} - \alpha(\bar{a}_i, C_k)\right)\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j, C_0)\right) \\
&= -\lambda_i(\bar{\theta},\bar{w})\left(b_i(\bar{\theta},\bar{w})^T\tau\rho_k\bar{w} - \alpha(a_i, C_k)\right)\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j, C_0)\right) \\
&\quad -\bar{\lambda}_i(\bar{\theta},\bar{w})\left(\bar{b}_i(\bar{\theta},\bar{w})^T\tau\rho_k\bar{w} - \alpha(\bar{a}_i, C_k)\right)\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j, C_0)\right) \\
&\quad -\lambda_i(\bar{\theta},\bar{w})(a_i - b_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j, C_0)\right) \\
&\quad -\bar{\lambda}_i(\bar{\theta},\bar{w})(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j, C_0)\right) \\
&\geq -\lambda_i(\bar{\theta},\bar{w})\left(b_i(\bar{\theta},\bar{w})^T\tau\rho_k\bar{w} - \rho_k\right)\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j, C_0)\right) \\
&\quad -\bar{\lambda}_i(\bar{\theta},\bar{w})\left(\bar{b}_i(\bar{\theta},\bar{w})^T\tau\rho_k\bar{w} - \rho_k\right)\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j, C_0)\right) \\
&\quad -\lambda_i(\bar{\theta},\bar{w})(a_i - b_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j, C_0)\right) \\
&\quad -\bar{\lambda}_i(\bar{\theta},\bar{w})(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j, C_0)\right) \\
&= f_{ij}^{+}(\tau\rho_k\bar{w};\rho_k,\bar{\theta},\bar{w}) \\
&\quad -\lambda_i(\bar{\theta},\bar{w})(a_i - b_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j, C_0)\right) \\
&\quad -\bar{\lambda}_i(\bar{\theta},\bar{w})(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j, C_0)\right)
\end{aligned}$$

where the inequality above follows from the assumption that $C_k \subseteq B(0,\rho_k)$ and the facts that $\|a_i\| = \|\bar{a}_i\| = 1$.

Therefore,

$$\begin{aligned}
g(\tau\rho_k\bar{w}) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mu_{ij}g_{ij}(\tau\rho_k\bar{w})\right) \\
&\geq \sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij}f_{ij}^{+}(\tau\rho_k\bar{w};\rho_k,\bar{\theta},\bar{w}) \\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij}\left[-\lambda_i(\bar{\theta},\bar{w})(a_i - b_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j, C_0)\right)\right. \\
&\quad \left.-\bar{\lambda}_i(\bar{\theta},\bar{w})(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j, C_0)\right)\right].
\end{aligned}$$

By Lemma 3.1 and Proposition 4.1

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij} f_{ij}^{+}(\tau\rho_k\bar{w};\rho_k,\bar{\theta},\bar{w}) \\
= \;& -2\rho_k \mathrm{diam}(C_0)(\bar{\kappa}+\bar{\theta})\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij} \\
\geq \;& -2\rho_k \mathrm{diam}(C_0)(\bar{\kappa}+\bar{\theta})\,\|\mathrm{vec} M\|_1 \\
\geq \;& \frac{-2^4\bar{\nu}_{\mathrm{lip}}\bar{\nu}_{\mathrm{nc}}(\mathrm{diam}(C_0))^3(\bar{\kappa}+\bar{\theta})n}{\epsilon} \\
\geq \;& -\frac{\epsilon}{4}.
\end{aligned}$$

where the last inequality follows from (7). It is easy to see that

$$\begin{aligned}
-\mathrm{diam}(C_0) &\leq -e_j^T(\tau\rho_k\bar{w}) - \alpha(-e^j, C_0) \leq 0, \\
-\mathrm{diam}(C_0) &\leq e_j^T(\tau\rho_k\bar{w}) - \alpha(e^j, C_0) \leq 0.
\end{aligned}$$

We also have

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij}\left[-\lambda_i(\bar{\theta},\bar{w})(a_i - b_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j,C_0)\right)\right. \\
&\left. -\bar{\lambda}_i(\bar{\theta},\bar{w})(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j,C_0)\right)\right]\Bigg| \\
\leq \;& \sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij}\left[\left|\lambda(\bar{\theta},\bar{w})\right|\left\|a_i - b_i(\bar{\theta},\bar{w})\right\|\tau\rho_k\|\bar{w}\|\,|-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j,C_0)|\right. \\
&\left. + \left|\bar{\lambda}(\bar{\theta},\bar{w})\right|\left\|\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w})\right\|\tau\rho_k\|\bar{w}\|\,|e_j^T\tau\rho_k\bar{w} - \alpha(e_j,C_0)|\right], \\
\leq \;& \sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij}\left[\frac{2}{\sin\bar{\theta}}\delta^*\tau\rho_k \mathrm{diam}(C_0)\right] \\
\leq \;& \frac{2}{\sin\bar{\theta}}\delta^*\rho_k\mathrm{diam}(C_0)\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij} \\
\leq \;& \frac{2}{\sin\bar{\theta}}\delta^*\rho_k\mathrm{diam}(C_0)\left(\|\mathrm{vec}M\|_1\right) \\
\leq \;& \frac{8n}{\sin\bar{\theta}}\delta^*\rho_k\mathrm{diam}(C_0)\bar{\nu}_{\mathrm{nc}} \\
\leq \;& \frac{\epsilon}{4}.
\end{aligned}$$

Therefore,

$$g(\tau\rho_k\bar{w}) \geq -\frac{\epsilon}{4} - \frac{\epsilon}{4} \geq -\epsilon/2.$$

□

Theorem 4.1 *Assume that* $\xi_0 \in C_0$ *and* $0 < \epsilon \leq \frac{\bar{\nu}_{lip} diam(C_0)}{2}$ *and* $diam(F) > 0$. *Then there exists* $\delta^* \in \left(0, \min\left\{\frac{\sin\theta}{4n}, \frac{\epsilon^2 \sin\theta}{2^6 n \bar{\nu}_{lip} \bar{\nu}_{nc} (diam(C_0))^3}\right\}\right)$ *such that using* D_2, *a* δ^**-net of* $\bar{D}$ *in Algorithm 1, yields the iterates* $\{C_k : k \geq 1\}$ *such that*

$$c.hull(F) \subseteq C_{k+1} \subseteq C_k \text{ for all } k \geq 0,$$

and

$$C_k \subseteq c.hull(F(\epsilon)) \text{ for all } k \geq k^*,$$

where

$$k^* = O\left(\frac{n^2 \bar{\nu}_{lip}^2 \bar{\nu}_{nc}^2 (diam(C_0))^6}{\epsilon^4} \ln\left(\frac{8\bar{\nu}_{lip}(diam(C_0))^2}{\epsilon\ diam(F)}\right)\right).$$

Proof. The monotonicity of the iterates $\{C_k : k \geq 1\}$ was proved in [5]. Let

$$\rho'(\xi) := \max\{\rho(\xi, F(\psi)), \bar{\delta}\rho(\xi, C_k)\}.$$

We will first prove

$$C_{k+1} \subseteq \bigcap_{\xi \in \Xi} B(\xi, \rho'(\xi)).$$

It suffices to show that $C_{k+1} \subseteq B(\xi, \rho'(\xi))$ for every $\xi \in \Xi$. For an arbitrarily fixed $\xi \in \Xi$, let

$$\rho_k = \rho(\xi, C_k) \text{ and } \rho' = \max\{\rho(\xi, F(\psi)), \bar{\delta}\rho_k\}.$$

We may assume without loss of generality that $\xi = 0$ because both of the algorithms (1 and 2) are invariant under parallel transformation. See [5] for more details. Since $\xi = 0 \in \Xi$ and $C_k \subset C_0 \subset \Xi$, we see that

$$\rho_k \leq \frac{2\bar{\nu}_{\text{lip}}(\text{diam}(C_0))^2}{\epsilon}.$$

If $\rho(0, F(\psi)) \geq \rho_k$, then $\rho' = \rho(0, F(\psi)) \geq \rho_k$. In this case, the desired result follows from $C_{k+1} \subseteq C_k$. Now, suppose that $\rho(0, F(\psi)) < \rho_k$. Assuming that $\bar{x} \notin B(0, \rho')$, we will derive that $\bar{x} \notin C_{k+1}$. If $\bar{x} \notin C_k$, we obviously see $\bar{x} \notin C_{k+1}$ because $C_{k+1} \subseteq C_k$. Hence we only need to deal with the case that

$$\bar{x} \in C_k \subseteq C_0, \quad \rho(0, F(\psi)) \leq \rho' < \|\bar{x} - 0\| \leq \rho_k. \tag{21}$$

The relations (21) imply that $\bar{x} \in C_0$ and $\bar{x} \notin F(\epsilon/2)$. Hence there exists a quadratic function $qf(.; \bar{\gamma}, \bar{q}, \bar{Q}) \in \mathcal{P}_F$ such that $qf(\bar{x}; \bar{\gamma}, \bar{q}, \bar{Q}) > \epsilon$. Let $\bar{w} = \bar{x}/\|\bar{x}\|$, and $\tau = \|\bar{x}\|/\rho_k$. Then we see that

$$\begin{aligned} \bar{x} &= \|\bar{x}\|\,\bar{w} = \tau\rho_k\bar{w}, \\ 1 &\geq \tau = \|\bar{x}\|/\rho_k > \rho'/\rho_k \geq \bar{\delta} = 1 - \bar{\kappa}\bar{\theta}. \end{aligned}$$

By Propositions 4.1, there exists a δ^* positive, bounded away from zero, such that when we take D_2 to be a δ^*-net of $\bar{D}$, we can find $g(\cdot) \in \text{c.cone}\left(\tilde{\mathcal{P}}^2(C_k, D_1, D_2)\right)$ with Hessian matrix $\bar{\nu}_{\text{nc}}\left(I + \frac{1}{\sqrt{n}}ee^T\right)$. By the definition of $\bar{\nu}_{nc}$, and since every eigenvalue of the Hessian matrix of $g(\cdot)$ is at least $\bar{\nu}_{nc}$, all eigenvalues of $qf(\cdot;\bar{\gamma},\bar{q},\bar{Q}) + g(\cdot)$ are non-negative. Moreover, by Proposition 4.2 such $g(\cdot)$ can be chosen to satisfy

$$qf(\bar{x};\bar{\gamma},\bar{q},\bar{Q}) + g(\bar{x}) > 0.$$

Therefore, $\bar{x} \notin C_{k+1}$ (we used Theorem 3.1). We proved $C_{k+1} \subseteq \bigcap_{\xi\in\Xi} B(\xi,\rho'(\xi))$. Thus, Algorithm 1 has the $\bar{\delta}$-shrinking property for certain $\delta^* > 0$ described above. Now, choosing $\bar{\nu} := \bar{\nu}_{\text{lip}}\bar{\nu}_{\text{nc}}$ in Lemma 2.2 implies the claimed bound on the number of iterations. □

5 Complexity analysis for LP based methods

In this section, when we use the results from Section 2, we always let $\bar{\nu} := 2\bar{\nu}_{\text{lip}}$. Let $\tau\rho_k\bar{w} \in C_0$ with $\tau \le 1$. Then

$$b_i(\bar{\theta},\bar{w}) \text{ and } \bar{b}_i(\bar{\theta},\bar{w}) \in \bar{D}.$$

Let

$$\delta \le \min\left\{\frac{\sin\bar{\theta}}{4n}, \frac{\epsilon^2\sin\bar{\theta}}{2^7 n\bar{\nu}_{\text{lip}}(\text{diam}(C_0))^3}\right\} \tag{22}$$

and D_2 be a δ-net of $\bar{D}$. Choose

$$a_i \in D(b_i(\bar{\theta},\bar{w}),\delta) \cap D_2,$$

$$\bar{a}_i \in D(\bar{b}_i(\bar{\theta},\bar{w}),\delta) \cap D_2.$$

Let $A := [a_1, a_2, \ldots, a_n] \in \mathbb{R}^{n\times n}$, $\bar{A} := [\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n] \in \mathbb{R}^{n\times n}$,

$$\begin{aligned}
g^+_{ij}(x) &:= \lambda_i(\bar{\theta},\bar{w})r2f(x;-e_j,a_i,C_k) + \bar{\lambda}_i(\bar{\theta},\bar{w})r2f(x;e_j,\bar{a}_i,C_k)\\
&= -\lambda_i(\bar{\theta},\bar{w})\left(a_i^Tx - \alpha(a_i,C_k)\right)\left(-e_j^Tx - \alpha(-e_j,C_0)\right)\\
&\quad -\bar{\lambda}_i(\bar{\theta},\bar{w})\left(\bar{a}_i^Tx - \alpha(\bar{a}_i,C_k)\right)\left(e_j^Tx - \alpha(e_j,C_0)\right),
\end{aligned}$$

$$\begin{aligned}
g^-_{ij}(x) &:= \lambda_i(\bar{\theta},\bar{w})r2f(x;e_j,a_i,C_k) + \bar{\lambda}_i(\bar{\theta},\bar{w})r2f(x;-e_j,\bar{a}_i,C_k)\\
&= -\lambda_i(\bar{\theta},\bar{w})\left(a_i^Tx - \alpha(a_i,C_k)\right)\left(e_j^Tx - \alpha(e_j,C_0)\right)\\
&\quad -\bar{\lambda}_i(\bar{\theta},\bar{w})\left(\bar{a}_i^Tx - \alpha(\bar{a}_i,C_k)\right)\left(-e_j^Tx - \alpha(-e_j,C_0)\right).
\end{aligned}$$

Proposition 5.1 *Let*

$$g^+(x) := \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{ij}^+ g_{ij}^+,$$
$$g^-(x) := \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{ij}^- g_{ij}^-,$$
$$g(x) := g^+(x) + g^-(x),$$

where $M^+ := \left[\mu_{ij}^+\right] \in \mathcal{S}^n$ *and* $M^- := \left[\mu_{ij}^-\right] \in \mathcal{S}^n$. *Let* $\rho_k > 0$ *and* $C_k \subseteq B(0, \rho_k)$. *Then there exists a*

$$\delta^* \in \left(0, \min\left\{\frac{\sin\bar{\theta}}{4n}, \frac{\epsilon^2 \sin\bar{\theta}}{2^6 n \bar{\nu}_{lip} \bar{\nu}_{nc} (diam(C_0))^3}\right\}\right)$$

such that when we take D_2 *to be a* δ^**-net of* $\bar{D}$, *we have the following properties:*

(i) *Given* $\bar{Q}^- \in \mathcal{S}^n$ *with all entries positive, satisfying* $\|\bar{Q}^-\|_F \le 2$, *there exists* $\left[\mu_{ij}^+\right] \in \mathcal{S}^n$ *such that* $\mu_{ij} \ge 0$ *for all* i, j, *the Hessian of* $g^+(x)$ *is* $\bar{Q}^-$ *and* $\left\|\left[\mu_{ij}^+\right]\right\|_F \le 4$.

(ii) *Given* $\bar{Q}^+ \in \mathcal{S}^n$ *with all entries positive, satisfying* $\|\bar{Q}^+\|_F \le 2$, *there exists* $\left[\mu_{ij}^-\right] \in \mathcal{S}^n$ *such that* $\mu_{ij} \ge 0$ *for all* i, j, *the Hessian of* $g^-(x)$ *is* $-\bar{Q}^+$ *and* $\left\|\left[\mu_{ij}^-\right]\right\|_F \le 4$.

Proof. We prove that part (i) holds and part (ii) can be proved similarly. As in the proof of Proposition 4.1, we obtain

$$\begin{aligned} & \text{The Hessian of } g^+(x) \text{ is} \\ = \; & \frac{H(\bar{\theta}, \bar{w}, A, \bar{A}) M^+ + M^+ H(\bar{\theta}, \bar{w}, A, \bar{A})^T}{2}, \end{aligned}$$

where

$$h_i(\bar{\theta}, \bar{w}, A, \bar{A}) := \lambda_i(\bar{\theta}, \bar{w}) a_i - \bar{\lambda}_i(\bar{\theta}, \bar{w}) \bar{a}_i, \quad i = 1, 2, \ldots, n, \tag{23}$$

and

$$H(\bar{\theta}, \bar{w}, A, \bar{A}) := \left(h_1(\bar{\theta}, \bar{w}, A, \bar{A}), h_2(\bar{\theta}, \bar{w}, A, \bar{A}), \ldots, h_n(\bar{\theta}, \bar{w}, A, \bar{A})\right). \tag{24}$$

Since

$$\begin{aligned} h_i(\bar{\theta}, \bar{w}, A, \bar{A}) &= \lambda_i(\bar{\theta}, \bar{w}) a_i - \bar{\lambda}(\bar{\theta}, \bar{w}) \bar{a}_i \\ &= \lambda_i(\bar{\theta}, \bar{w}) b_i(\bar{\theta}, \bar{w}) - \bar{\lambda}_i(\bar{\theta}, \bar{w}) \bar{b}_i(\bar{\theta}, \bar{w}) \\ &\quad + \lambda_i(\bar{\theta}, \bar{w}) \left(a_i - b_i(\bar{\theta}, \bar{w})\right) - \bar{\lambda}_i(\bar{\theta}, \bar{w}) \left(\bar{a}_i - \bar{b}_i(\bar{\theta}, \bar{w})\right), \end{aligned}$$

and

$$\lambda_i(\bar{\theta}, \bar{w}) b_i(\bar{\theta}, \bar{w}) - \bar{\lambda}_i(\bar{\theta}, \bar{w}) \bar{b}_i(\bar{\theta}, \bar{w}) = e_i \quad (i = 1, 2, \ldots, n).$$

we have $H = I - P$ with $P := [p_1, p_2, \dots, p_n]$ and

$$p_i := \lambda_i(\bar{\theta}, \bar{w}) \left(a_i - b_i(\bar{\theta}, \bar{w})\right) - \bar{\lambda}_i(\bar{\theta}, \bar{w}) \left(\bar{a}_i - \bar{b}_i(\bar{\theta}, \bar{w})\right). \tag{25}$$

As in the proof of Proposition 4.1, we obtain

$$\|P\| \leq \frac{1}{2}.$$

So the real parts of the eigenvalues of H are at least $\frac{1}{2}$ and by Lyapunov Theorem, the equation

$$\frac{H(\bar{\theta}, \bar{w}, A, \bar{A})M^+ + M^+ H(\bar{\theta}, \bar{w}, A, \bar{A})^T}{2} = \bar{Q}^- \tag{26}$$

has a unique solution $\left[\mu_{ij}^+\right] \in \mathcal{S}^n$. Note that when we set

$$a_i := b_i(\bar{\theta}, \bar{w}), \quad \text{and} \quad \bar{a}_i := \bar{b}_i(\bar{\theta}, \bar{w}),$$

we obtain $H = I$. In this case, the unique solution of (26) is $\bar{Q}^-$. Every component of this matrix is positive. By continuity of the solution with respect to the perturbations in H (and hence $a_i, \bar{a}_i$), we see that there exists $\delta^* > 0$ depending on $\bar{\theta}, \bar{w}$ such that for every $a_i \in D(b_i(\bar{\theta}, \bar{w}), \delta^*)$, and for every $\bar{a}_i \in D(\bar{b}_i(\bar{\theta}, \bar{w}), \delta^*)$, the solution matrix M^+ has all of its entries non-negative. We now show the boundedness of $\left[\mu_{ij}^+\right]$. Using the Kronecker product, the Lyapunov equation (26) can be rewritten as follows.

$$\frac{I \otimes H(\bar{\theta}, \bar{w}, A, \bar{A}) + H(\bar{\theta}, \bar{w}, A, \bar{A})^T \otimes I}{2} \operatorname{vec}(M^+) = \operatorname{vec}(\bar{Q}^-). \tag{27}$$

As in the proof of Proposition 4.1, we obtain

$$\left\| \left(\frac{I \otimes H(\bar{\theta}, \bar{w}, A, \bar{A}) + H(\bar{\theta}, \bar{w}, A, \bar{A})^T \otimes I}{2} \right)^{-1} \right\| \leq 2.$$

Therefore,

$$\begin{aligned} \left\|\operatorname{vec}(M^+)\right\| &\leq \left\| \left(\frac{I \otimes H(\bar{\theta}, \bar{w}, A, \bar{A}) + H(\bar{\theta}, \bar{w}, A, \bar{A})^T \otimes I}{2} \right)^{-1} \right\| \left\|\operatorname{vec}(\bar{Q}^-)\right\| \\ &\leq 2 \left\|\bar{Q}^-\right\|_F \\ &\leq 4. \end{aligned}$$

So $\|M^+\|_F \leq 4$. □

Proposition 5.2 *Under the same conditions as in Proposition 5.1, we have*

$$g(\tau \rho_k \bar{w}) \geq -\epsilon/2. \tag{28}$$

Proof. As in the proof of Proposition 4.2, we obtain

$$\begin{aligned} g_{ij}^+(\tau\rho_k\bar{w}) \quad \geq \quad & f_{ij}^+(\tau\rho_k\bar{w};\rho_k,\bar{\theta},\bar{w}) \\ & -\lambda_i(\bar{\theta},\bar{w})(a_i - b_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j,C_0)\right) \\ & -\bar{\lambda}_i(\bar{\theta},\bar{w})(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j,C_0)\right) \end{aligned}$$

where the second to last inequality follows from the assumption that $C_k \subseteq B(0,\rho_k)$.

Similarly,

$$\begin{aligned} g_{ij}^-(\tau\rho_k\bar{w}) \quad \geq \quad & f_{ij}^-(\tau\rho_k\bar{w};\rho_k,\bar{\theta},\bar{w}) \\ & -\lambda_i(\bar{\theta},\bar{w})(a_i - b_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j,C_0)\right) \\ & -\bar{\lambda}_i(\bar{\theta},\bar{w})(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j,C_0)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} g(\tau\rho_k\bar{w}) \quad \geq \quad & \sum_{i=1}^n\sum_{j=1}^n\left(\mu_{ij}^+ f_{ij}^+(\tau\rho_k\bar{w};\rho_k,\bar{\theta},\bar{w}) + \mu_{ij}^- f_{ij}^-(\tau\rho_k\bar{w};\rho_k,\bar{\theta},\bar{w})\right) \\ & \sum_{i=1}^n\sum_{j=1}^n \mu_{ij}^+\left[-\lambda_i(\bar{\theta},\bar{w})(a_i - b_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j,C_0)\right)\right. \\ & \left.-\bar{\lambda}_i(\bar{\theta},\bar{w})(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j,C_0)\right)\right] \\ & +\sum_{i=1}^n\sum_{j=1}^n \mu_{ij}^-\left[-\lambda_i(\bar{\theta},\bar{w})(a_i - b_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(e_j^T\tau\rho_k\bar{w} - \alpha(e_j,C_0)\right)\right. \\ & \left.-\bar{\lambda}_i(\bar{\theta},\bar{w})(\bar{a}_i - \bar{b}_i(\bar{\theta},\bar{w}))^T\tau\rho_k\bar{w}\left(-e_j^T\tau\rho_k\bar{w} - \alpha(-e_j,C_0)\right)\right]. \end{aligned}$$

By Lemma 3.1 and Proposition 5.1 we obtain

$$\begin{aligned} & \sum_{i=1}^n\sum_{j=1}^n\left(\mu_{ij}^+ f_{ij}^+(\tau\rho_k\bar{w};\rho_k,\bar{\theta},\bar{w}) + \mu_{ij}^- f_{ij}^-(\tau\rho_k\bar{w};\rho_k,\bar{\theta},\bar{w})\right) \\ \geq \quad & \sum_{i=1}^n\sum_{j=1}^n\left(\mu_{ij}^+\left(-2\rho_k\mathrm{diam}(C_0)(\bar{\kappa}+\bar{\theta})\right) + \mu_{ij}^-\left(-2\rho_k\mathrm{diam}(C_0)(\bar{\kappa}+\bar{\theta})\right)\right) \\ = \quad & -2\rho_k\mathrm{diam}(C_0)(\bar{\kappa}+\bar{\theta})\sum_{i=1}^n\sum_{j=1}^n\left(\mu_{ij}^+ + \mu_{ij}^-\right) \\ \geq \quad & -2\rho_k\mathrm{diam}(C_0)(\bar{\kappa}+\bar{\theta})n\left(\left\|M^+\right\|_F + \left\|M^-\right\|_F\right) \\ \geq \quad & -\frac{2^4 n(\mathrm{diam}(C_0))^3(\bar{\kappa}+\bar{\theta})(2\bar{\nu}_{\mathrm{lip}})}{\epsilon} \\ \geq \quad & -\frac{\epsilon}{4}. \end{aligned}$$

where the last inequality follows from (7). As in the proof of Proposition 4.2, we obtain

$$
\begin{aligned}
& \Bigg| \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{ij}^{+} \left[-\lambda_i(\bar\theta,\bar w)(a_i - b_i(\bar\theta,\bar w))^T \tau\rho_k\bar w \left(-e_j^T\tau\rho_k\bar w - \alpha(-e_j,C_0)\right)\right. \\
& \left. -\bar\lambda_i(\bar\theta,\bar w)(\bar a_i - \bar b_i(\bar\theta,\bar w))^T \tau\rho_k\bar w \left(e_j^T\tau\rho_k\bar w - \alpha(e_j,C_0)\right)\right] \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{ij}^{-} \left[-\lambda_i(\bar\theta,\bar w)(a_i - b_i(\bar\theta,\bar w))^T \tau\rho_k\bar w \left(e_j^T\tau\rho_k\bar w - \alpha(e_j,C_0)\right)\right. \\
& \left. -\bar\lambda_i(\bar\theta,\bar w)(\bar a_i - \bar b_i(\bar\theta,\bar w))^T \tau\rho_k\bar w \left(-e_j^T\tau\rho_k\bar w - \alpha(-e_j,C_0)\right)\right] \Bigg|, \\
\leq\ & \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{ij}^{+} \left[\frac{2}{\sin\bar\theta}\delta^*\tau\rho_k \mathrm{diam}(C_0)\right] \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{ij}^{-} \left[\frac{2}{\sin\bar\theta}\delta^*\tau\rho_k \mathrm{diam}(C_0)\right] \\
\leq\ & \frac{2}{\sin\bar\theta}\delta^*\rho_k \mathrm{diam}(C_0) \sum_{i=1}^{n}\sum_{j=1}^{n} (\mu_{ij}^{+} + \mu_{ij}^{-}) \\
\leq\ & \frac{2}{\sin\bar\theta}\delta^*\rho_k \mathrm{diam}(C_0) n \left(\left\|M^{+}\right\|_F + \left\|M^{-1}\right\|_F\right) \\
\leq\ & \frac{2^4 n}{\sin\bar\theta}\delta^*\rho_k \mathrm{diam}(C_0) \\
\leq\ & \frac{\epsilon}{4}.
\end{aligned}
$$

Therefore,

$$g(\tau\rho_k\bar w) \geq -\frac{\epsilon}{4} - \frac{\epsilon}{4} \geq -\epsilon/2.$$

□

Theorem 5.1 *Assume that $\xi_0 \in C_0$ and $0 < \epsilon \leq \frac{\bar\nu_{lip}\, diam(C_0)}{2}$ and $diam(F) > 0$. Then there exists $\delta^* \in \left(0, \min\left\{\frac{\sin\bar\theta}{4n}, \frac{\epsilon^2 \sin\bar\theta}{2^7 n\bar\nu_{lip}(diam(C_0))^3}\right\}\right)$ such that using D_2, a δ^*-net of $\bar D$ in Algorithm 2, yields the iterates $\{C_k :\ k \geq 1\}$ such that*

$$c.hull(F) \ \subseteq\ C_{k+1} \ \subseteq\ C_k \quad \text{for all } k \geq 0,$$

and

$$C_k \ \subseteq\ c.hull(F(\epsilon)) \quad \text{for all } k \geq k^*,$$

where

$$k^* \ =\ O\left(\frac{n^2\bar\nu_{lip}^2(diam(C_0))^6}{\epsilon^4} \ln\left(\frac{8\bar\nu_{lip}(diam(C_0))^2}{\epsilon\ diam(F)}\right)\right).$$

Proof. The monotonicity of the iterates $\{C_k : k \geq 1\}$ was proved in [5]. The first half of the proof is identical to the proof of Theorem 4.1. Let

$$\rho'(\xi) := \max\{\rho(\xi, F(\psi)), \bar{\delta}\rho(\xi, C_k)\}.$$

We will first prove

$$C_{k+1} \subseteq \bigcap_{\xi \in \Xi} B(\xi, \rho'(\xi)).$$

It suffices to show that $C_{k+1} \subseteq B(\xi, \rho'(\xi))$ for every $\xi \in \Xi$. For an arbitrarily fixed $\xi \in \Xi$, let

$$\rho_k = \rho(\xi, C_k) \text{ and } \rho' = \max\{\rho(\xi, F(\psi)), \bar{\delta}\rho_k\}.$$

We may assume without loss of generality that $\xi = 0$ because both of the algorithms (1 and 2) are invariant under parallel transformation. See [5] for more details. Since $\xi = 0 \in \Xi$ and $C_k \subset C_0 \subset \Xi$, we see that

$$\rho_k \leq \frac{2\bar{\nu}_{\text{lip}}(\text{diam}(C_0))^2}{\epsilon}.$$

If $\rho(0, F(\psi)) \geq \rho_k$, then $\rho' = \rho(0, F(\psi)) \geq \rho_k$. In this case the desired result follows from $C_{k+1} \subseteq C_k$. Now, suppose that $\rho(0, F(\psi)) < \rho_k$. Assuming that $\bar{x} \notin B(0, \rho')$, we will derive that $\bar{x} \notin C_{k+1}$. If $\bar{x} \notin C_k$, we obviously see $\bar{x} \notin C_{k+1}$ because $C_{k+1} \subseteq C_k$. Hence we only need to deal with the case that

$$\bar{x} \in C_k \subseteq C_0, \quad \rho(0, F(\psi)) \leq \rho' < \|\bar{x} - 0\| \leq \rho_k. \tag{29}$$

The relations (29) imply that $\bar{x} \in C_0$ and $\bar{x} \notin F(\epsilon/2)$. Hence there exists a quadratic function $qf(.; \bar{\gamma}, \bar{q}, \bar{Q}) \in \mathcal{P}_F$ such that $qf(\bar{x}; \bar{\gamma}, \bar{q}, \bar{Q}) > \epsilon$. Let $\bar{w} = \bar{x}/\|\bar{x}\|$, and $\tau = \|\bar{x}\|/\rho_k$. Then we see that

$$\begin{aligned} \bar{x} &= \|\bar{x}\|\bar{w} = \tau\rho_k\bar{w}, \\ 1 &\geq \tau = \|\bar{x}\|/\rho_k > \rho'/\rho_k \geq \bar{\delta} = 1 - \bar{\kappa}\bar{\theta}. \end{aligned}$$

Note that $\|\bar{Q}\|_1 \leq 1$. Therefore, we can find symmetric matrices $\bar{Q}^+, \bar{Q}^-$ with all entries positive such that $\|\bar{Q}^+\| \leq 2$, $\|\bar{Q}^-\| \leq 2$, and $\bar{Q} = \bar{Q}^+ - \bar{Q}^-$. Now, Proposition 5.1 implies that there exists a $\delta^* > 0$, bounded away from zero, such that when we take D_2 to be a δ^*-net of $\bar{D}$, we can find $g(\cdot) \in \text{c.cone}\left(\tilde{\mathcal{P}}^2(C_k, D_1, D_2)\right)$ with Hessian matrix $\bar{Q}^- - \bar{Q}^+ = -\bar{Q}$. Hence, $qf(\cdot; \bar{\gamma}, \bar{q}, \bar{Q}) + g(\cdot)$ is a linear function. Moreover, by Proposition 5.2, such $g(\cdot)$ can be chosen to satisfy

$$qf(\bar{x}; \bar{\gamma}, \bar{q}, \bar{Q}) + g(\bar{x}) > 0.$$

Therefore, $\bar{x} \notin C_{k+1}$ (we used Theorem 3.1). We proved $C_{k+1} \subseteq \bigcap_{\xi \in \Xi} B(\xi, \rho'(\xi))$. Thus, Algorithm 1 has the $\bar{\delta}$-shrinking property for certain $\delta^* > 0$ described above. Now, choosing $\bar{\nu} := \bar{\nu}_{\text{lip}}\bar{\nu}_{\text{nc}}$ in Lemma 2.2 implies the claimed bound on the number of iterations. □

6 Conclusion

For the given input $\mathcal{P}_F$ (finite), C_0, and $\epsilon > 0$, there exists a largest value of $\delta > 0$ for which the convergence analysis of [5] proves the (correctness and) finiteness of both algorithms. Let's call this critical value of δ, $\delta_{\rm cr}$. In our analysis in the current paper, we established that for the same input, there exists $\delta^* > 0$ such that our complexity bounds apply. The proof techniques we used here are very similar to those of [5], and as a result, the values of $\delta_{\rm cr}$ and δ^* implied by the respective proofs are essentially the same (even though, we clearly have $\delta_{\rm cr} \geq \delta^*$).

As in the previous analysis by Kojima and Takeda [2] of the semi-infinite successive convex relaxation methods, our analyses of the implementable, discretized versions, also indicate that the current complexity bounds can be much better for SDP based algorithms than for LP based algorithms if $\bar{\nu}_{\rm nc} << 1$.

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