MIN-MAX THEOREMS RELATED TO GEOMETRIC REPRESENTATIONS OF GRAPHS AND THEIR SDPS

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Abstract. Lovász proved a nonlinear identity relating the theta number of a graph to its smallest radius hypersphere embedding where each edge has unit length. We use this identity and its generalizations to establish min-max theorems and to translate results related to one of the graph invariants above to the other.

Classical concepts in tensegrity theory allow good interpretations of the dual SDP for the problem of finding an optimal hypersphere embedding as above. We generate a spectrum of structured SDPs on which extensions of such interpretations are possible.

1. Introduction

Geometric representations of graphs is a beautiful area where combinatorial optimization, graph theory, and semidefinite optimization meet and connect with many other research areas. In this paper, we start by studying geometric representations of graphs where each node is mapped to a point on a hypersphere so that each edge has unit length and the radius of the hypersphere is minimum. Lovász proved that this graph invariant, denoted by t_h, is related to the Lovász ϑ number of the complement of the graph via a simple nonlinear equation. This tight relationship leads to min-max theorems and to a "dictionary" to translate existing results about the ϑ-function and its variants to the hypersphere representation setting and vice versa. Also, some properties of the hypersphere number t_h are more naturally observed in its own context. These properties can also be "translated back" to the ϑ-function setting.

We further illustrate that geometric representation setting allows the underlying dual SDPs to be interpreted in a useful way. In combinatorial optimization, min-max theorems and primal-dual algorithms rely heavily on suitable interpretation of the dual problem (see, for instance, Euclidean matching and generalizations [8], primal-dual method in approximation algorithms [29]). Even though a general theory of dual interpretations exists (even in the context of nonconvex nonlinear optimization) based on Lagrangian duals and/or perturbation theory, such interpretations are not as sharp or clean as in the setting of linear optimization or more specifically in the special cases of LPs related to combinatorial optimization problems.

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Using SDPs related to the geometric representations of graphs and their connection to tensegrity theory, we generate a spectrum of structured SDPs on which dual interpretations, extending those for special graph representations problems, are possible. (This is related to [25].)

In Section 2, we remind the reader preliminary results from SDP theory that we will use. In Section 3 we review relevant facts about Lovász ϑ number. Section 4 covers the new results expressing the hypersphere graph invariant t_h in terms of ϑ, min-max theorems, and translations of some results from one of the settings t_h, ϑ to the other. In Section 5, we discuss the consequences of our observations in the context of graph homomorphisms and sandwich theorems. We conclude the paper with Section 6 by presenting a spectrum of structured SDPs whose duals allow interpretations (of varying degrees of sharpness) based on geometric representations of graphs and tensegrity theory.

2. Preliminaries on Semidefinite Programming

Denote the set of symmetric $n \times n$ matrices by $\mathbb{S}^n$, the set of symmetric $n \times n$ positive semidefinite matrices by $\mathbb{S}^n_+$, and the set of symmetric $n \times n$ positive definite matrices by $\mathbb{S}^n_{++}$. For a finite set V, the set of symmetric $V \times V$ matrices is denoted by $\mathbb{S}^V$, and the notations $\mathbb{S}^V_+$ and $\mathbb{S}^V_{++}$ are defined analogously. For $A, B \in \mathbb{S}^n$, we write $A \succeq B$ if $A - B \in \mathbb{S}^n_+$.

A semidefinite programming problem (SDP) is an optimization problem of the form

$$\begin{array}{ll} \inf & \langle C, X \rangle \\ & \mathcal{A}(X) = b, \\ & X \in \mathbb{S}^n_+, \end{array} \tag{2.1}$$

where the data is given by a matrix $C \in \mathbb{S}^n$, a vector $b \in \mathbb{R}^m$ and a linear transformation $\mathcal{A}\colon \mathbb{S}^n \to \mathbb{R}^m$. Here, $\langle A, B \rangle$ is the inner product of A and B defined by the trace as follows: $\langle A, B \rangle := \mathrm{Tr}(A^T B) := \sum_i \sum_j A_{ij} B_{ij}$.

The *dual* of (2.1) is the following SDP:

$$\begin{array}{ll} \sup & b^T y \\ & \mathcal{A}^*(y) + S = C, \\ & y \in \mathbb{R}^m,\ S \in \mathbb{S}^n_+, \end{array} \tag{2.2}$$

where $\mathcal{A}^*(\cdot)$ is the adjoint of $\mathcal{A}(\cdot)$ defined by $\langle \mathcal{A}^*(y), X \rangle = \langle y, \mathcal{A}(X) \rangle$ for any $y \in \mathbb{R}^m$ and $X \in \mathbb{S}^n$. Note that we use interchangeably the notation $a^T b$ and $\langle a, b \rangle$ for vectors a and b.

It is easy to see that weak duality holds, i.e., for every X feasible in (2.1) and every (y, S) feasible in (2.2), we have $\langle C, X \rangle \geq b^T y$. A *Slater point* for (2.1) is a matrix $\bar{X} \in \mathbb{S}^n$ such that $\mathcal{A}(\bar{X}) = b$ and $\bar{X} \in \mathbb{S}^n_{++}$, i.e., a feasible solution that is positive definite. Similarly, a *Slater point* for (2.2) is a pair $(\bar{y}, \bar{S}) \in \mathbb{R}^m \oplus \mathbb{S}^n$ such that $\mathcal{A}^*(\bar{y}) + \bar{S} = C$ and $\bar{S} \in \mathbb{S}^n_{++}$. Sometimes we refer only to $\bar{y}$ or to $\bar{S}$ as a Slater point.

The following well-known result can be proved using a hyperplane separation theorem (see, for instance, [27]):

Theorem 2.1 (SDP Strong Duality)**.** *Suppose* (2.2) *has a Slater point. If the objective value of* (2.2) *is bounded from above, then* (2.1) *attains its optimal value and the optimal values of* (2.1) *and* (2.2) *coincide.*

Under the definition of the dual of an SDP, the dual of (2.2) is equivalent to (2.1). This fact yields the next corollary.

Corollary 2.2. *If* (2.1) *and* (2.2) *both have Slater points, then they both attain their optimal values and their optimal values coincide.*

We shall use the following lemma:

Lemma 2.3. *Let $C \in \mathbb{S}^n_+ \setminus \{0\}$ and $C' \in \mathbb{S}^n_{++}$. Let $\mathcal{A}\colon \mathbb{S}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that there exists $\bar{X} \in \mathbb{S}^n_{++}$ such that $\mathcal{A}(\bar{X}) = 0$. Define*

$$\begin{array}{rll} \beta := \sup & \langle C, X\rangle & \\ & \mathcal{A}(X) = 0, & \\ & \langle C', X\rangle = 1, & \\ & X \in \mathbb{S}^n_+, & \end{array} \tag{2.3}$$

and

$$\begin{array}{rll} \beta' := \inf & \langle C', Y\rangle & \\ & \mathcal{A}(Y) = 0, & \\ & \langle C, Y\rangle = 1, & \\ & Y \in \mathbb{S}^n_+. & \end{array} \tag{2.4}$$

Then both β and β' are attained, and $\beta\beta' = 1$.

Proof. The dual of (2.3) is

$$\begin{array}{ll} \inf & \eta \\ & \eta C' \succeq C + \mathcal{A}^*(y), \\ & \eta \in \mathbb{R},\ y \in \mathbb{R}^m, \end{array} \tag{2.5}$$

which has $(\bar{\eta}, \bar{y}) := (M, 0)$ as a Slater point for sufficiently large M. Note that $\langle C', \bar{X}\rangle > 0$, so $(\langle C', \bar{X}\rangle)^{-1}\bar{X}$ is feasible in (2.3) with positive objective value, since $\langle C, \bar{X}\rangle > 0$. Thus, by Theorem 2.1,

$$\beta \text{ is attained and } \beta > 0. \tag{2.6}$$

The dual of (2.4) is

$$\begin{array}{ll} \sup & \eta \\ & C' \succeq \eta C + \mathcal{A}^*(y), \\ & \eta \in \mathbb{R},\ y \in \mathbb{R}^m, \end{array} \tag{2.7}$$

which has $(\bar{\eta}, \bar{y}) := (0, 0)$ as a Slater point. In fact, for small enough $\varepsilon > 0$, $(\eta, y) := (\varepsilon, 0)$ is feasible for (2.7), so $\beta' > 0$. Moreover, $(\langle C, \bar{X}\rangle)^{-1}\bar{X}$ is feasible in (2.4), so Theorem 2.1 implies that

$$\beta' \text{ is attained and } \beta' > 0. \tag{2.8}$$

Finally, let X^* be an optimal solution for (2.3). Then $(\langle C, X^*\rangle)^{-1}X^*$ is feasible for (2.4), so $\beta' \le 1/\beta$. Let Y^* be an optimal solution for (2.4). Then $(\langle C', Y^*\rangle)^{-1}Y^*$ is feasible for (2.3), so $\beta \ge 1/\beta'$. □

Note that, if we replace $\mathbb{S}^n_+$ by an arbitrary pointed closed convex cone K with nonempty interior, and modify the assumptions analogously ($c \in K^* \setminus \{0\}$, $c' \in \operatorname{int}(K^*)$, and there exists $\bar{x} \in \operatorname{int}(K)$ such that $\mathcal{A}(\bar{x}) = 0$), the lemma continues to hold.

3. A quick review of Lovász's theta number

Notation 3.1. *For any function f on graphs, we denote by $\overline{f}$ the function defined by $\overline{f}(G) := f(\overline{G})$ for every graph G, where $\overline{G}$ denotes the complement of G.*

Let $G = (V, E)$ be a graph. An *orthonormal representation of G* is a function $p\colon V \to \mathbb{R}^d$ for some $d \geq 1$ such that

(i) $\|p(i)\| = 1$ for every $i \in V$, and
(ii) $\langle p(i), p(j)\rangle = 0$ for every $\{i, j\} \in \overline{E}(G)$.

These representations were introduced by Lovász in his seminal paper [19]. However, we shall mostly follow the slightly different treatment presented in [12].

Define the *theta body of G,* denoted by $\mathrm{TH}(G)$, to be the set of all vectors $x \in \mathbb{R}^V_+$ such that

$$\sum_{i \in V} \left(c^T p(i)\right)^2 x_i \leq 1$$

for every orthonormal representation p of G and unit vector c of the appropriate dimension. For any weight function $w \in \mathbb{R}^V_+$, define

$$\vartheta(G, w) := \max\{\, w^T x : x \in \mathrm{TH}(G)\}.$$

The *theta number of G* is $\vartheta(G) := \vartheta(G, \bar{e})$, where $\bar{e}$ denotes the vector of all ones.

The most striking motivation to study the theta number of a graph is that it is a polynomial-time approximately computable graph invariant that lies sandwiched between two graph invariants which are NP-hard to compute:

Theorem 3.2 (The Sandwich Theorem [19])**.** *For any graph G, we have $\omega(G) \leq \overline{\vartheta}(G) \leq \chi(G)$.*

Here $\omega(G) := \overline{\alpha}(G)$ is the clique number of G, and $\chi(G)$ is the chromatic number of G.

We shall use the following SDP characterization from [12], which may be used to prove that $\vartheta(G, w)$ can be computed up to an error of $\varepsilon > 0$ in time polynomial in $|V|$ and $\log(1/\varepsilon)$:

Theorem 3.3. *Let $G = (V, E)$ be a graph, and let $w \in \mathbb{R}^V_+$. Then*

$$\begin{aligned} \vartheta(G, w) = \max \quad & \langle W, X\rangle \\ & \langle I, X\rangle = 1, \\ & \langle B^{\{i,j\}}, X\rangle = 0, \quad \forall \{i, j\} \in E, \\ & X \in \mathbb{S}^V_+, \end{aligned} \tag{3.1}$$

where $W_{ij} := \sqrt{w_i w_j}$ for all $i, j \in V$.

Here we use the following notation. For $i, j \in V$, with e_i denoting the ith unit vector, we set $B^{\{i,j\}} := e_i e_j^T + e_j e_i^T$.

We will also need the following facts (see, e.g., [12, Remark (9.3.20) and Theorem (9.3.12)]):

Theorem 3.4. *Let $G = (V, E)$ be a graph, and let $w \in \mathbb{R}_+^V$. Then there exists an orthonormal representation q of G and a unit vector c of appropriate dimension such that*

$$\vartheta(G, w)\big(c^T q(i)\big)^2 = w_i, \quad \forall i \in V.$$

Theorem 3.5. *Let $G = (V, E)$ be a graph, and let $w \in \mathbb{R}_+^V$. Then*

$$\vartheta(G, w) = \min_{p,c} \max_{i \in V} \frac{w_i}{(c^T p(i))^2},$$

where the minimum ranges over all orthonormal representations p of G and unit vectors c of appropriate dimension.

The following combinatorial interpretation of $\vartheta(G, w)$ will also come in handy (see [23, ch. 67]). Given a graph $G = (V, E)$ and $w \in \mathbb{Z}_+^V$, define the graph G_w as follows. For each $i \in V$, let $V_i := \{\, (i, k) : k \in [w_i]\}$, where $[n] := \{1, \ldots, n\}$. Note that $V_i \cap V_j = \varnothing$ for all distinct $i, j \in V$. Then the node set of G_w is $\bigcup_{i \in V} V_i$ and there is an edge in G_w joining (i, k) to (j, ℓ) if and only if $\{i, j\} \in E$.

Theorem 3.6. *Let $G = (V, E)$ be a graph and let $w \in \mathbb{Z}_+^V$. Then $\vartheta(G, w) = \vartheta(G_w)$.*

4. Hypersphere representations

4.1. **A minmax relation.** Let $G = (V, E)$ be a graph. A *unit-distance representation of G* is a function $p\colon V \to \mathbb{R}^d$ for some $d \geq 1$ such that $\|p(i) - p(j)\| = 1$ whenever $\{i, j\} \in E$. A *hypersphere representation of G* is a unit-distance representation p of G that is contained in a hypersphere centered at the origin, i.e., $\|p(i)\| = \|p(j)\|$ for every $i, j \in V$. The *radius* of the hypersphere representation p is this common norm of the $p(i)$'s.

These geometric representations seem natural, and it is also natural to consider problems of finding the "best" such representations. For instance, we can consider the problem of finding the smallest radius of a hypersphere representation of a graph G, that is, the smallest radius of a hypersphere that contains a unit-distance representation of G. The usual Gram matrix trick allows us to formulate this problem as an SDP, as follows (see for instance, [17]):

$$\begin{aligned} t_h(G) := \min \quad & t \\ & \operatorname{diag}(X) = t\bar{e}, \\ & X_{ii} - 2X_{ij} + X_{jj} = 1, \quad \forall \{i, j\} \in E, \\ & X \in \mathbb{S}_+^V,\ t \in \mathbb{R}. \end{aligned} \tag{4.1}$$

Here diag(·) maps a matrix to the vector formed by its diagonal entries. Note that $t_h(G)$ is actually the square of the smallest radius of a hypersphere representation of G.

Before we write the dual of (4.1), let us set some notation. The *Laplacian of* G is the linear transformation $\mathcal{L}_G\colon \mathbb{R}^E \to \mathbb{S}^V$ defined by

$$(\mathcal{L}_G(z))_{ij} := \begin{cases} z(\delta(i)), & \text{if } i = j, \\ -z_{\{i,j\}}, & \text{if } \{i,j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\delta_G(i)$ denotes the set of edges of G incident to the node i; sometimes we write just $\delta(i)$ if the graph G is clear from the context. Also, for a vector x indexed by a set S and a subset I of S, we denote $x(I) := \sum_{i\in I} x_i$. Laplacians arise naturally in spectral graph theory and spectral geometry (see [2]).

If we denote the adjoint of diag(·) by Diag(·), the dual of (4.1) can be written as

$$\begin{array}{ll} \max & z(E) \\ & \mathrm{Diag}(y) \succeq \mathcal{L}_G(z), \\ & y(V) = 1, \\ & y \in \mathbb{R}^V,\ z \in \mathbb{R}^E, \end{array} \tag{4.2}$$

which has $(\bar{y}, \bar{z}) := (\frac{1}{n}\bar{e}, 0)$ as a Slater point, where $n := |V|$. Similarly, $(\bar{X}, \bar{t}\,) := \frac{1}{2}(I, 1)$ is a Slater point of (4.1). Thus, by Corollary 2.2, SDP Strong Duality holds for this dual pair of SDPs, so their optimal values coincide and both optima are attained.

The Laplacian plays a key role in geometric representations of graphs, so it is worthwhile to note that (4.1) may be rewritten as:

$$\begin{array}{lll} t_h(G) = \min & t & \\ & \mathrm{diag}(X) = t\bar{e}, & \\ & \mathcal{L}_G^*(X) = \bar{e}, & \\ & X \in \mathbb{S}^V_+,\ t \in \mathbb{R}. & \end{array} \tag{4.3}$$

One nice feature of the SDP (4.1) is that it models the desired problem exactly. Namely, every feasible solution (X, t) of (4.1) corresponds to a hypersphere representation p of G, and its objective value t is precisely the square of the radius of p. This is in contrast with many popular SDPs related to graphs, which are usually just relaxations of integer programming problems that model the combinatorial problem exactly.

This exact correspondence highlights some interesting properties of hypersphere representations. Since $(\bar{X}, \bar{t}\,) = \frac{1}{2}(I, 1)$ is a Slater point for (4.1), every graph $G = (V, E)$ has a hypersphere representation $p\colon i \mapsto 2^{-1/2}e_i \in \mathbb{R}^V$, and $t_h(G) < 1/2$.

It is well-known that the hypersphere representations of G are related to orthonormal representations of $\overline{G}$. To see this, note that, if $p\colon V \to \mathbb{R}^d$ is a

hypersphere representation of G with squared radius $t \le 1/2$, then the map

$$q\colon i \mapsto \sqrt{2}\begin{bmatrix}\sqrt{1/2-t}\\ p(i)\end{bmatrix} \in \mathbb{R}\oplus\mathbb{R}^d \tag{4.4}$$

is an orthonormal representation of $\overline{G}$. Also, if $q\colon V\to\mathbb{R}^d$ is an orthonormal representation of $\overline{G}$, then $p\colon i\to 2^{-1/2}q(i)$ is a hypersphere representation of G with squared radius $1/2$.

However, the connection runs deeper, as shown via the following formula mentioned by Lovász [18, p. 23]:

Theorem 4.1. *For any graph $G=(V,E)$, we have*

$$2t_h(G)+\frac{1}{\overline{\vartheta}(G)}=1. \tag{4.5}$$

Proof. We have already observed above that SDP Strong Duality holds for the pair of SDPs (4.1) and (4.2). We can rewrite the dual (4.2) as follows:

$$\begin{aligned} t_h(G)=\max\quad & \langle \tfrac{1}{2}A_G, Y\rangle \\ & \langle I+A_G, Y\rangle = 1,\\ & \langle B^{\{i,j\}}, Y\rangle = 0, \quad \forall\{i,j\}\in\overline{E}(G),\\ & Y\in\mathbb{S}^V_+. \end{aligned} \tag{4.6}$$

Here A_G denotes the adjacency matrix of G.

For every feasible solution Y of (4.6), the first constraint allows us to rewrite the objective value of Y as $\langle \frac{1}{2}A_G, Y\rangle = \frac{1}{2}(\langle A_G, Y\rangle + \langle I, Y\rangle - \langle I, Y\rangle) = \frac{1}{2}(1-\langle I, Y\rangle)$. Moreover, by adding each constraint of the form $\langle B^{\{i,j\}}, Y\rangle = 0$ to the first constraint, we can replace it by the constraint $\langle \bar{e}\bar{e}^T, Y\rangle = 1$. Thus, we can write

$$t_h(G)=\frac{1}{2}\Big(1-t'_h(G)\Big), \tag{4.7}$$

where

$$\begin{aligned} t'_h(G):=\min\quad & \langle I, Y\rangle \\ & \langle \bar{e}\bar{e}^T, Y\rangle = 1,\\ & \langle B^{\{i,j\}}, Y\rangle = 0, \quad \forall\{i,j\}\in\overline{E}(G),\\ & Y\in\mathbb{S}^V_+. \end{aligned} \tag{4.8}$$

To prove the theorem, it suffices to show that

$$t'_h(G)\overline{\vartheta}(G)=1, \tag{4.9}$$

which follows from Lemma 2.3 and Theorem 3.3. □

We can view Theorem 4.1 as strong duality for a nonlinear minmax relation, in view of the following:

Proposition 4.2. *Let $G=(V,E)$ be a graph. Then, for every hypersphere representation p of G with squared radius t and every nonzero $x\in \mathrm{TH}(\overline{G})$, we have*

$$2t+\frac{1}{\bar{e}^T x}\ge 1,$$

with equality if and only if $t = t_h(G)$ *and* $\bar{e}^T x = \overline{\vartheta}(G)$.

Proof. Let $p\colon V \to \mathbb{R}^d$ be a hypersphere representation of G with squared radius t. We may assume that $t < 1/2$. We have to prove that $(1-2t)\bar{e}^T x \leq 1$ for every $x \in \mathrm{TH}(\overline{G})$. Define an orthonormal representation $q\colon V \to \mathbb{R} \oplus \mathbb{R}^d$ of $\overline{G}$ from p as in (4.4). Put $c := 1 \oplus 0 \in \mathbb{R} \oplus \mathbb{R}^d$. Then

$$(1-2t)\sum_{i \in V} x_i = \sum_{i \in V} \left(c^T q(i)\right)^2 x_i \leq 1$$

by the definition of $\mathrm{TH}(\overline{G})$.

The equality case now follows from Theorem 4.1. □

The equality case for Proposition 4.2 used Theorem 4.1, whose proof in turn shows explicitly that $\overline{\vartheta}(G)$ and orthonormal representations of $\overline{G}$ are natural dual objects for $t_h(G)$ and hypersphere representations of G. We can also use a well-known result about theta number to derive (4.5) more quickly, but in a way that hides duality. We shall use the following construction to get hypersphere representations of G from a special class of orthonormal representations of $\overline{G}$.

Suppose that $q\colon V \to \mathbb{R}^d$ is an orthonormal representation of $\overline{G}$ such that, for some positive $\mu \in \mathbb{R}$ and $p\colon V \to \mathbb{R}^{d-1}$, we have

$$q(i) = \sqrt{2}\begin{bmatrix} (2\mu)^{-1/2} \\ p(i) \end{bmatrix} \tag{4.10}$$

for every $i \in V$. Then p is a hypersphere representation of G with squared radius $\frac{1}{2}(1 - 1/\mu)$.

Now the proof of equality in Proposition 4.2 goes as follows. By Theorem 3.4, there exists an orthonormal representation q of $\overline{G}$ and a unit vector c of appropriate dimension such that $\left(c^T q(i)\right)^2 = 1/\overline{\vartheta}(G)$ for every $i \in V$. By rotation, we may assume that $c = e_1$, and by replacing some $q(i)$'s by their opposites if necessary, we may assume that q has the form (4.10) with $\mu = \overline{\vartheta}(G)$. Thus, p is a hypersphere representation of G with squared radius $\frac{1}{2}(1 - 1/\overline{\vartheta}(G))$.

It can thus be argued that the proof of (4.5) hinges essentially on the transformations (4.4) and (4.10) between hypersphere representations of G with squared radius smaller than $1/2$ and a special class of orthonormal representations of $\overline{G}$ together with unit vectors.

We shall now use these transformations to obtain a Gallai-type identity involving these objects.

Proposition 4.3. *Let* $G = (V, E)$ *be a graph. Then*

$$2t_h(G) + \max_{p,c} \min_{i \in V} \left(c^T p(i)\right)^2 = 1, \tag{4.11}$$

where p *ranges over all orthonormal representations of* $\overline{G}$ *and* c *over unit vectors of the appropriate dimension.*

Proof. We first prove "$\leq$" in (4.11). It suffices to prove that

$$t_h(G) \leq \frac{1}{2}\Big(1 - \min_{i \in V} \big(c^T u(i)\big)^2\Big) \tag{4.12}$$

for any orthonormal representation u of $\overline{G}$ and unit vector c of appropriate dimension. It is easy to see that there exists an orthonormal representation q of $\overline{G}$ and a unit vector d such that

$$\beta := \big(d^T q(j)\big)^2 = \min_{i \in V} \big(c^T u(i)\big)^2, \quad \forall j \in V.$$

If $\beta = 0$, then $i \mapsto 2^{-1/2} e_i \in \mathbb{R}^V$ shows that $t_h(G) \leq 1/2$ as desired, so assume that $\beta > 0$. By rotation, we may assume that $d = e_1$, and by replacing some $q(i)$'s by their opposites if necessary, we may assume that $d^T q(i) \geq 0$ for every $i \in V$. Now use (4.10) with $\mu = 1/\beta$ to get a hypersphere representation p of G from q with squared radius $\frac{1}{2}(1-\beta)$. This proves (4.12).

Next we prove "$\geq$" in (4.11). It suffices to find an orthonormal representation q of $\overline{G}$ and a unit vector c such that $\big(c^T q(i)\big)^2 \geq 1 - 2t_h(G)$ for every $i \in V$. Let $p\colon V \to \mathbb{R}^d$ be a hypersphere representation of G with squared radius $t_h(G)$. Build an orthonormal representation q of $\overline{G}$ as in (4.4) and pick $c := 1 \oplus 0 \in \mathbb{R} \oplus \mathbb{R}^d$. Then $\big(c^T q(i)\big)^2 = 1 - 2t_h(G)$ for every $i \in V$. □

Note that (4.11) does not provide a good characterization of either $t_h(G)$ or the maximization problem. In this sense, Proposition 4.3 is akin to Gallai's identities for graphs, which say that $\alpha(G) + \tau(G) = |V(G)|$ for every graph G, and $\nu(G) + \rho(G) = |V(G)|$ for every graph G without isolated nodes; see [20, Lemmas 1.0.1 and 1.0.2].

Thus, as expected, the proof of Proposition 4.3 makes no use of duality. Together with Theorem 3.5, which has SDP duality at its core, we get yet another proof of Theorem 4.1.

4.2. **Unit-distance representations in hyperspheres and balls.** For a graph $G = (V, E)$, let $t_b(G)$ be the square of the smallest radius of an Euclidean ball containing a unit-distance representation of G. It is easy to modify (4.1) to obtain an SDP formulation for $t_b(G)$:

$$\begin{aligned} t_b(G) := \min \quad & t \\ & \operatorname{diag}(X) \leq t\bar{e}, \\ & X_{ii} - 2X_{ij} + X_{jj} = 1, \quad \forall \{i,j\} \in E, \\ & X \in \mathbb{S}^V_+,\ t \in \mathbb{R}. \end{aligned} \tag{4.13}$$

Its dual is the same as (4.2), with the extra constraint that $y \geq 0$. Thus, $(\bar{X}, \bar{t}) := (\frac{1}{2}I, \frac{1}{2} + \varepsilon)$ is a Slater point for (4.13) for every $\varepsilon > 0$, and $(\bar{y}, \bar{z}) := \frac{1}{n}(\bar{e}, 0)$ is a Slater point for the dual of (4.13), where $n := |V|$. Hence, Strong Duality also holds for this dual pair of SDPs, and both optima are attained.

It is obvious that $t_b(G) \leq t_h(G)$ for every graph G. However, we now point out that

$$t_b(G) = t_h(G), \quad \text{for every graph } G. \tag{4.14}$$

To see this, first note that, if we mimic the proof of Theorem 4.1 for $t_b(G)$, we find that

$$2t_b(G) + \frac{1}{\overline{\vartheta_b}(G)} = 1, \tag{4.15}$$

where $\vartheta_b(G) := \vartheta_b(G, \bar{e})$, and, for every $w \in \mathbb{R}^V_+$,

$$\begin{array}{rll} \vartheta_b(G,w) := \max & \langle W, X\rangle & \\ & \langle I, X\rangle = 1, & \\ & \langle B^{\{i,j\}}, X\rangle = 0, & \forall \{i,j\} \in E, \\ & \langle C^i, X\rangle \geq 0, & \forall i \in V, \\ & X \in \mathbb{S}^V_+, & \end{array} \tag{4.16}$$

with $W_{ij} := \sqrt{w_i w_j}$ for all $i, j \in V$, as usual, and $C^i := \frac{1}{2}(e_i \bar{e}^T + \bar{e} e_i^T)$ for all $i \in V$.

Thus, by (4.5) and (4.15), it suffices to prove that

$$\vartheta_b(G) = \vartheta(G), \qquad \text{for every graph } G. \tag{4.17}$$

This follows immediately from the following result from [7, Proposition 9] (this was pointed out to the first author by Fernando Mário de Oliveira Filho):

Proposition 4.4. *Let $\mathbb{K} \subseteq \mathbb{S}^n$ be a set such that $\operatorname{Diag}(h) X \operatorname{Diag}(h) \in \mathbb{K}$ whenever $X \in \mathbb{K}$ and $h \in \mathbb{R}^n_+$. If $\hat{X}$ is an optimal solution to the optimization problem*

$$\max \left\{ \bar{e}^T X \bar{e} : \operatorname{Tr}(X) = 1,\ X \in \mathbb{K} \cap \mathbb{S}^n_+ \right\}, \tag{4.18}$$

then there exists a positive $\mu \in \mathbb{R}$ such that $\operatorname{diag}(\hat{X}) = \mu \hat{X} \bar{e}$.

4.3. **Hypersphere proofs of ϑ facts.** The formula (4.5) relating $t_h(G)$ to $\overline{\vartheta}(G)$ allows us to regard some basic facts about the theta number from a geometrically simpler viewpoint. Let us look, for instance, at Theorem 3.2, known as the Sandwich Theorem. By Theorem 4.1, it is equivalent to the inequalities $t_h(K_{\omega(G)}) \leq t_h(G) \leq t_h(K_{\chi(G)})$ for every graph G, where K_n denotes the complete graph on n nodes (note that $t_h(K_n) = \frac{1}{2}(1 - 1/n)$). The first inequality is obvious: whenever H is a subgraph of G, we have $t_h(H) \leq t_h(G)$. The second one is also obvious: if we have a hypersphere representation of K_ℓ with radius t, say $p\colon [\ell] \to \mathbb{R}^d$, and a colouring $c\colon V(G) \to [\ell]$ of G, then we can get a hypersphere representation of G with radius t by mapping each node i of G to $p(c(i))$. This already hints at a strong connection with homomorphisms, which we will look at more closely in Section 5.

The next result shows that ϑ may be used to characterize bipartite graphs, as pointed out by Lovász [18, p. 34]:

Proposition 4.5. *A graph G is bipartite if and only if $\vartheta(\overline{G}) \leq 2$.*

Proof. We will show that $t_h(G) \leq 1/4$ precisely when G is bipartite.

If G is bipartite, then $t_h(G) \leq 1/4$, with a hypersphere representation of G with radius $1/2$ even in $\mathbb{R}^1$.

Suppose $t_h(G) \leq 1/4$. If $t_h(G) = 0$, then G has no edges. If $t_h(G) > 0$, then $t_h(G) \geq 1/4$. The only pairs of points at distance 1 in a hypersphere of radius $1/2$ are the pairs of antipodal points. So G is bipartite. □

Another example where it seems much simpler to prove something about ϑ through Theorem 4.1 is the behaviour of ϑ under direct cosums of graphs. We briefly recall the definitions here. Given graphs $G = (V, E)$ and $H = (W, F)$, the *direct sum of G and H* is the graph $G + H := (V \cup W, E \cup F)$, where we assume, by relabeling if necessary, that $V \cap W = \varnothing$. The *direct cosum of G and H* is the graph $G \mathbin{\overline{+}} H$ defined by $\overline{G \mathbin{\overline{+}} H} := \overline{G} + \overline{H}$. It is easy to check that $G \mathbin{\overline{+}} H$ is obtained from $G + H$ by adding all possible edges joining a node of G to a node of H.

It is proved in [14] that $\vartheta(G \mathbin{\overline{+}} H) = \max\{\vartheta(G), \vartheta(H)\}$. The proof is short, but the construction is a bit convoluted and does not seem to offer much insight into the geometry of the problem. Now note that this equation is equivalent to the geometrically obvious equation $t_h(G+H) = \max\{t_h(G), t_h(H)\}$ by Theorem 4.1.

Let us mention a few more geometric constructions that look simple from the point of view of hypersphere representations.

For every graph G, we have

$$t_h(G) = \max\{\, t_h(C) : C \text{ a component of } G\}. \tag{4.19}$$

Moreover,

$$t_h(G) = \max\{\, t_h(B) : B \text{ a block of } G\}.$$

This follows from the following result:

Proposition 4.6. *Let $G = (V, E)$ be a graph, and suppose $G = G_1 \cup G_2$ for graphs G_1 and G_2, with $G_1 \cap G_2$ a complete graph. Then*

$$t_h(G) = \max\{t_h(G_1), t_h(G_2)\}.$$

Proof. We obviously have '$\geq$' in the desired equation. Assume $t_h(G_1) \geq t_h(G_2)$. Note that, by convexity of the feasible region of (4.1) and the fact that $(\bar{X}, \bar{t}) = \frac{1}{2}(I, 1)$ is feasible for (4.1), there is a hypersphere representation q of G_2 with squared radius $t_h(G_1)$. Let p be a hypersphere representation of G_1 with squared radius $t_h(G_1)$. We may assume that the images of p and q live in the same Euclidean space.

Since the nodes of $G_1 \cap G_2$ are mapped into points with squared norm $t_h(G_1)$ and they are pairwise one unit apart, there is an orthogonal matrix Q such that $Qq(i) = p(i)$ for every $i \in V(G_1 \cap G_2)$. Thus, if we take the hypersphere representation $q' \colon i \mapsto Qq(i)$ of G_2 and glue it with p, we get a hypersphere representation of G with squared radius $t_h(G_1)$. □

Let us briefly describe the ϑ counterparts of (4.19) and Proposition 4.6. Given a graph $G = (V, E)$, call a subset $S \subseteq V$ *friendful* if every node of S is adjacent to every node of $V \setminus S$. Note that V is friendful. It is easy to check that (4.19) is equivalent to

$$\vartheta(G) = \max\big\{\, \vartheta(G[S]) : S \text{ a minimal friendful subset of } V \big\}.$$

The ϑ counterpart of Proposition 4.6 is as follows:

Proposition 4.7. *Let $G = (V, E)$ be a graph. Suppose $V_1, V_2 \subseteq V$ have the following properties:*

(i) $V = V_1 \cup V_2$,
(ii) $G[V_1 \cap V_2]$ has no edges,
(iii) every node of $V_1 \setminus V_2$ is adjacent in G to every node of $V_2 \setminus V_1$.

Then

$$\vartheta(G) = \max\big\{\vartheta(G[V_1]), \vartheta(G[V_2])\big\}.$$

The next proposition concerns the behaviour of t_h with respect to edge contraction:

Proposition 4.8. *Let $G = (V, E)$ be a graph and let $e \in E$. Let $(\bar{y}, \bar{z})$ be an optimal solution for (4.2). Then*

$$\bar{z}_e \geq t_h(G) - t_h(G/e).$$

Proof. It suffices to show a feasible solution for (4.2) applied to G/e with objective value $t_h(G) - \bar{z}_e$. Assume $e = \{a, b\}$ and $V' := V(G/e) = V \setminus \{b\}$, so we are denoting the contracted node of G/e by a. Let P be the $V' \times V$ matrix defined by $P := e_a e_b^T + \sum_{i \in V'} e_i e_i^T$. It is easy to check that $P\mathcal{L}_G(\bar{z})P^T = \mathcal{L}_{G/e}(\hat{z})$, where $\hat{z}\colon E(G/e) \to \mathbb{R}$ is obtained from $\bar{z}$ as follows. In taking the contraction G/e from G, immediately after we identify the ends of e, but before we remove resulting parallel edges, there are at most two edges between each pair of nodes of G/e, as we assume that G is simple. If there is exactly one edge between nodes i and j, we just set $\hat{z}_{\{i,j\}} := \bar{z}_{\{i,j\}}$. If there are two edges joining nodes i and j, say f and f', we put $\hat{z}_{\{i,j\}} := \bar{z}_f + \bar{z}_{f'}$.

Similarly, if we define $\hat{y}\colon V' \to \mathbb{R}$ by putting $\hat{y}_i := \bar{y}_i$ for $i \in V' \setminus \{a\}$ and $\hat{y}_a := \bar{y}_a + \bar{y}_b$, then $P\,\mathrm{Diag}(\bar{y})P^T = \mathrm{Diag}(\hat{y})$. Since $P\mathbb{S}_+^V P^T \subseteq \mathbb{S}_+^{V'}$, we see that $(\hat{y}, \hat{z})$ is a feasible solution for (4.2) applied to G/e, and its objective value is $\hat{z}(E(G/e)) = \bar{z}(E) - \bar{z}_e$. □

The ϑ counterpart of this is as follows.

Proposition 4.9. *Let $G = (V, E)$ be a graph and let i, j be distinct non-adjacent nodes of G. Let $\bar{X}$ an optimal solution for (3.1) applied to $\vartheta(G, \bar{e})$. Let G' be the graph obtained from G by creating a new node adjacent to all common neighbours of i and j, and then deleting i and j. Then*

$$\frac{\vartheta(G)}{\vartheta(G')} \leq 2\bar{X}_{ij} + 1.$$

Next we show that one obvious property of the stability number $\alpha(G)$ and the clique covering number $\overline{\chi}(G)$ remains true for $\vartheta(G)$. Given a graph $G = (V, E)$ and $i \in V$, it is obvious that $\alpha(G) \geq \alpha(G - i - N(i)) + 1$ and $\overline{\chi}(G) \geq \overline{\chi}(G - i - N(i)) + 1$, where $N(\cdot) := N_G(\cdot)$ denotes the set of neighbours of the node $\cdot$ in G. It is also easy to show that

$$\overline{\chi^*}(G) \geq \overline{\chi^*}(G - i - N(i)) + 1,$$

where $\chi^*(G)$ denotes the fractional chromatic number of G. The following is somewhat based on [13, Lemma 4.3]:

Proposition 4.10. *Let $G = (V, E)$ be a graph and $i \in V$. Then*

$$\vartheta(G) \geq \vartheta(G - i - N(i)) + 1.$$

Proof. The inequality holds if G is complete, so assume G is not complete. Note that $G - i - N_G(i) = G[N_{\overline{G}}(i)]$. By Theorem 4.1, the equation $\vartheta(G) \geq \vartheta(G[N_{\overline{G}}(i)]) + 1$ is equivalent to

$$t_h\Big(\overline{G}\big[N_{\overline{G}}(i)\big]\Big) \leq 1 - \frac{1}{4t_h(\overline{G})}.$$

Thus, it suffices to prove that, for every graph $G = (V, E)$ with at least one edge, and every $i \in V$, we have

$$t_h(G[N_G(i)]) \leq 1 - \frac{1}{4t_h(G)}. \tag{4.20}$$

Let $p\colon V \to \mathbb{R}^d$ be a hypersphere representation of G with squared radius $t := t_h(G)$. We may assume that $p(i) = \sqrt{t}e_1$. For every $j \in N(i)$, we have $1 = \|p(i) - p(j)\|^2 = \|p(i)\|^2 + \|p(j)\|^2 - 2\langle p(i), p(j)\rangle = 2t - 2\sqrt{t}p_1^{(j)}$. Hence,

$$p_1^{(j)} = \frac{2t - 1}{2\sqrt{t}}, \quad \forall j \in N(i). \tag{4.21}$$

Define the following hypersphere representation of $G[N(i)]$: for each $j \in N(i)$, let $q(j)$ be obtained from $p(j)$ by dropping the first coordinate. The squared radius of the resulting hypersphere representation is

$$t - \left(\frac{2t - 1}{2\sqrt{t}}\right)^2 = 1 - \frac{1}{4t}.$$

This proves (4.20), and thus the theorem. □

We point out that Proposition 4.10 is easy to prove from well-known results about ϑ, but the proof above involves a nice geometric construction.

4.4. **A weighted version.** We can use the proof of Theorem 4.1 as a guide to find a definition of a weighted hypersphere number. Namely, we want to define $t_h(G, w)$ so that the following remains true for every graph $G = (V, E)$ and nonzero $w \in \mathbb{R}^V_+$:

$$2t_h(G, w) + \frac{1}{\vartheta(\overline{G}, w)} = 1. \tag{4.22}$$

Let us follow the proof of Theorem 4.1 backwards. We want to define

$$t_h(G,w) = \frac{1}{2}\Big(1 - t'_h(G,w)\Big),$$

for a parameter $t'_h(G,w)$ that satisfies

$$t'_h(G,w)\vartheta(\overline{G},w) = 1$$

for the same reason as (4.9) holds. Using Theorem 3.3 in its full generality, we can define

$$\begin{array}{rll} t'_h(G,w) := \min & \langle I, Y\rangle & \\ & \langle W, Y\rangle = 1, & \\ & \langle B^{\{i,j\}}, Y\rangle = 0, & \forall \{i,j\} \in \overline{E}(G), \\ & Y \in \mathbb{S}^V_+, & \end{array} \tag{4.23}$$

where $W_{ij} := \sqrt{w_i w_j}$ for every $i, j \in V$.

Thus, we want the dual SDP for $t_h(G,w)$ (the one corresponding to (4.2)) to be:

$$\begin{array}{rll} t_h(G,w) = \max & \frac{1}{2}\Big(\langle W, Y\rangle - \langle I, Y\rangle\Big) & \\ & \langle W, Y\rangle = 1, & \\ & Y = \mathrm{Diag}(y) - \mathcal{L}_G(z) \succeq 0, & \\ & y \in \mathbb{R}^V,\ z \in \mathbb{R}^E. & \end{array} \tag{4.24}$$

So it is reasonable to define $t_h(G,w)$ as the dual of (4.24):

$$\begin{array}{rll} t_h(G,w) = \min & t & \\ & \mathrm{diag}(X) = \frac{1}{2}\bar{e} + (t - \frac{1}{2})w, & \\ & \mathcal{L}^*_G(X) = \bar{e} + (t - \frac{1}{2})\mathcal{L}^*_G(W), & \\ & X \in \mathbb{S}^V_+,\ t \in \mathbb{R}. & \end{array} \tag{4.25}$$

Note that $(\bar{X}, \bar{t}\,) := \frac{1}{2}(I, 1)$ and $(\bar{y}, \bar{z}) := (\frac{1}{\bar{e}^T w}\bar{e}, 0)$ are Slater points for (4.25) and (4.24), respectively, so we still have SDP Strong Duality in place.

We could not offer a very nice direct interpretation for this definition of $t_h(G,w)$. However, by construction, we get the following weighted version of Proposition 4.2:

Theorem 4.11. *Let $G = (V,E)$ be a graph and $w \in \mathbb{R}^V_+$ be nonzero. Then, for every feasible solution (X,t) of (4.25) and every nonzero $x \in \mathrm{TH}(\overline{G})$, we have*

$$2t + \frac{1}{w^T x} \geq 1,$$

with equality if and only if (X,t) is an optimal solution for (4.25) and $w^T x = \vartheta(\overline{G},w)$.

Proof. We may assume that $t < 1/2$. We have to prove that $(1-2t)w^T x \leq 1$. First decompose $X \succeq 0$ as $X = P^T P$ for some $[d] \times V$ matrix P, and define

$p\colon V \to \mathbb{R}^d$ by $p(i) := Pe_i$ for every $i \in V$. Define $q\colon V \to \mathbb{R} \oplus \mathbb{R}^d$ by

$$q\colon i \mapsto \sqrt{2}\begin{bmatrix}\sqrt{w_i(1/2-t)}\\ p(i)\end{bmatrix}.$$

It is easy to check that q is an orthonormal representation of $\overline{G}$. Put $c := 1 \oplus 0 \in \mathbb{R} \oplus \mathbb{R}^d$. Then

$$(1-2t)w^T x = \sum_{i\in V} \big(c^T q(i)\big)^2 x_i \le 1$$

by the definition of $\mathrm{TH}(\overline{G})$.

The equality case now follows by construction. □

Next, we present a geometric construction that provides a reasonable interpretation for $t_h(G,w)$ based on Theorem 3.6. First, we need to define another graph operation. Let $G = (V,E)$ be a graph, and let $w \in \mathbb{Z}_+^V$. Following the notation from Section 3, define the graph G^w to be the graph obtained from G_w by adding all the edges of the form $\{(i,k),(i,\ell)\}$ for $i \in V$ and $k \neq \ell$. Note that G^w is the graph complement of $(\overline{G})_w$. Thus, by (4.22) and Theorem 3.6, we get

$$t_h(G,w) = \frac{1}{2}\left(1 - \frac{1}{\vartheta(\overline{G},w)}\right) = \frac{1}{2}\left(1 - \frac{1}{\vartheta\big((\overline{G})_w\big)}\right) = t_h(G^w).$$

Now we show how to obtain a hypersphere representation of G^w from a feasible solution $(\bar{X},\bar{t}\,)$ for (4.25) with squared radius $\bar{t}$. Write $\bar{X} = P^T P$ for some $[d] \times V$ matrix P, and define $p\colon V \to \mathbb{R}^d$ by $p(i) := Pe_i$ for $i \in V$. For each $i \in V$, let $q_i\colon V_i \to \mathbb{R}^{d_i}$ be an optimal hypersphere representation of the complete graph on node set V_i. Here we note that such a representation is easy to compute from $|V_i| = w_i$, and the square of the optimal radius is $\frac{1}{2}(1 - 1/w_i)$ by Theorem 4.1.

Let us build a hypersphere representation $u\colon V(G^w) \to \mathbb{R}^d \oplus \Big(\bigoplus_{i\in V} \mathbb{R}^{d_i}\Big)$. We may assume that $w_i > 0$ for every $i \in V$. For $(i,k) \in V(G^w)$, with $i \in V$ and $k \in [w_i]$, set $u(i,k)$ to be the following vector. Its block in $\mathbb{R}^d$ is $w_i^{-1/2} p(i)$. Its block in $\mathbb{R}^{d_i}$ is $q_i(i,k)$. All the other blocks of $u(i,k)$ are zero. It is easy to check that u is a hypersphere representation of G^w with squared radius $\bar{t}$.

So we can think of $t_h(G,w)$ as a more economical way of computing the hypersphere number of a graph that can be encoded as G^w for some smaller graph G.

5. Graph homomorphisms and sandwich theorems

Let G and H be graphs. Recall that a *homomorphism from G to H* is a function $f\colon V(G) \to V(H)$ such that $\{f(i),f(j)\} \in E(H)$ whenever $\{i,j\} \in E(G)$. If there is a homomorphism from G to H, we write $G \to H$.

We have seen that the sandwich theorem for $\overline{\vartheta}(G)$ is equivalent to the inequality $t_h(K_{\omega(G)}) \leq t_h(G) \leq t_h(K_{\chi(G)})$ by Theorem 4.1. Moreover, the only property that we required from $t_h(\cdot)$ to prove this easy geometric inequality was the fact that t_h is monotone under taking homomorphisms, namely, we have $t_h(G) \leq t_h(H)$ whenever $G \to H$. To see why this is enough, note that we obviously have $K_{\omega(G)} \to G \to K_{\chi(G)}$.

Let us now prove that $t_h(G) \leq t_h(H)$ if $G \to H$. Let $f\colon V(G) \to V(H)$ be a homomorphism from G to H, and let $v\colon V(H) \to \mathbb{R}^d$ be a hypersphere representation of H. It is now easy to check that the map $i \mapsto v(f(i))$ is a hypersphere representation of G with the same radius as v.

Motivated by this, we are led to study graph invariants that are monotone under taking homomorphisms. We say a real-valued graph invariant f is *hom-monotone* if

(i) $f(G) \leq f(H)$ whenever $G \to H$, and
(ii) there is a non-decreasing function $g\colon \operatorname{Im}(f) \to \mathbb{R}$ such that $g(f(K_n)) = n$ for every integer $n \geq 1$.

Using these properties for an arbitrary graph G and the fact that $K_{\omega(G)} \to G \to K_{\chi(G)}$, we get $f(K_{\omega(G)}) \leq f(G) \leq f(K_{\chi(G)})$, and thus

$$\omega(G) \leq g(f(G)) \leq \chi(G). \tag{5.1}$$

Note that the function $g(x) := 1/(1-2x)$ is non-decreasing on $[0, 1/2) \supseteq \operatorname{Im}(t_h)$, so $f := t_h$ is hom-monotone. In this case, we recover from (5.1) the sandwich theorem for Lovász's theta number.

The second condition for a graph invariant to be hom-monotone does not seem completely natural, but it is a reasonable property to expect from "nondegenerate" graph invariants f satisfying the first condition, namely, $f(G) \leq f(H)$ whenever $G \to H$.

If we track down the reason why t_h satisfies the first condition of hom-monotonicity, we see that it roughly comes from the fact that we are looking for a geometric representation minimizing a certain parameter, and the only constraint on the geometric representation is for pairs of adjacent nodes of the graph. Moreover, the constraints for distinct edges are uniform. We are thus led to define other SDPs of the same type.

One such example is the parameter $t_b(\cdot)$. However, as we have seen in (4.14), this parameter is equal to $t_h(\cdot)$. Let us now define a new parameter:

$$\begin{aligned} t'(G) := \min \quad & t \\ & \operatorname{diag}(X) = t\bar{e}, \\ & X_{ii} - 2X_{ij} + X_{jj} \geq 1, \quad \forall \{i,j\} \in E, \\ & X \in \mathbb{S}^V_+,\ t \in \mathbb{R}. \end{aligned} \tag{5.2}$$

The dual of (5.2) can be written as

$$\begin{array}{ll} \max & z(E) \\ & \mathrm{Diag}(y) \succeq \mathcal{L}_G(z), \\ & y(V) = 1, \\ & y \in \mathbb{R}^V,\ z \in \mathbb{R}^E_+. \end{array} \tag{5.3}$$

Thus, $(\bar{X}, \bar{t}\,) := (\frac{1}{2} + \varepsilon)(I, 1)$ is a Slater point for (5.2) for any $\varepsilon > 0$, and $(\bar{y}, \bar{z}) := \frac{1}{n}(\bar{e}, \varepsilon\bar{e})$ is a Slater point for (5.3) for $\varepsilon > 0$ small enough, where $n := |V|$. Thus, Strong Duality for this dual pair of SDPs, and both optima are attained.

Clearly, $t'(G) \le t(G)$ for every graph G. Moreover, by applying the usual symmetrization operator (sometimes called the Reynolds operator)

$$X \mapsto \frac{1}{|\operatorname{Aut}(G)|} \sum_{P \cdot P^T \in \operatorname{Aut}(G)} PXP^T,$$

we see that $t'(G) = t_h(G)$ if G is node-transitive. In particular, $t'(K_n) = t_h(K_n)$ for every n. Thus, the function $g(x) := 1/(1-2x)$ proves that t' is hom-monotone.

Using (5.1), we obtain

$$\omega(G) \le \frac{1}{1 - 2t'(G)} \le \frac{1}{1 - 2t_h(G)} \le \chi(G) \tag{5.4}$$

for every graph G.

If we now mimic the proof of Theorem 4.1 for $t'(G)$, we obtain

$$2t'(G) + \frac{1}{\overline{\vartheta'}(G)} = 1. \tag{5.5}$$

where $\vartheta'(G) := \vartheta'(G, \bar{e})$ and, for any $w \in \mathbb{R}^V_+$,

$$\begin{array}{lll} \vartheta'(G, w) := \max & \langle W, X\rangle & \\ & \langle I, X\rangle = 1, & \\ & \langle B^{\{i,j\}}, X\rangle = 0, & \forall \{i,j\} \in E, \\ & X \in \mathbb{S}^V_+, & \\ & X \ge 0, & \end{array} \tag{5.6}$$

with $W_{ij} := \sqrt{w_i w_j}$ for all $i, j \in V(G)$. Note that $\vartheta'(G)$ is the graph parameter introduced in [21] and [24].

Let us point out one drawback for this framework of hom-monotone graph invariants yielding sandwich inequalities. The following strengthening of Sandwich Theorem holds: for any graph G, we have $\omega(G) \le \overline{\vartheta}(G) \le \chi^*(G) \le \chi(G)$. This stronger inequality fails for the hom-monotone graph invariant χ.

5.1. **Hypersphere representations and vector colourings.** The following relaxation of graph colouring was introduced in [13]. Let $G = (V, E)$ be a graph. For a real number $k \ge 1$, a *vector k-colouring of G* is a function $p\colon V \to \mathbb{R}^d$ for some $d \ge 1$ such that

(i) $\|p(i)\| = 1$ for every $i \in V$, and
(ii) $\langle p(i), p(j)\rangle \leq -1/(k-1)$ for every $\{i,j\} \in E$,

where we consider the fraction to be $-\infty$ if $k = 1$. In other words, the only graphs that have a vector 1-colouring are the graphs with no edges. A vector k-colouring p of G is *strict* if $\langle p(i), p(j)\rangle = -1/(k-1)$ for every $\{i,j\} \in E$, and a strict vector k-colouring p of G is *strong* if $\langle p(i), p(j)\rangle \geq -1/(k-1)$ for every $\{i,j\} \in \overline{E}(G)$.

The *vector chromatic number of* G is the smallest $k \geq 1$ for which there exists a vector k-colouring of G, and the strict vector chromatic number and strong vector chromatic number are defined analogously.

It is easy to show (see, e.g., [15]) that the vector chromatic number of G is $\overline{\vartheta'}(G)$, the strict vector chromatic number of G is $\overline{\vartheta}(G)$, and the strong vector chromatic number of G is $\overline{\vartheta^+}(G)$, known as Szegedy's number [26], and defined as $\vartheta^+(G) := \vartheta^+(G, \bar{e})$ and, for any $w \in \mathbb{R}^V_+$,

$$\begin{array}{rll} \vartheta^+(G,w) := \max & \langle W, X\rangle & \\ & \langle I, X\rangle = 1, & \\ & \langle B^{\{i,j\}}, X\rangle \leq 0, & \forall \{i,j\} \in E, \\ & X \in \mathbb{S}^V_+. & \end{array} \tag{5.7}$$

Here we note that a scaling map yields a correspondence between these variations of vector colourings and unit-distance representations, provided that the graph G has at least one edge.

Let p be a strict vector k-colouring of G. Then the map $i \mapsto tp(i)$, where $t^2 = \frac{1}{2}(1 - 1/k)$, is a hypersphere representation of G with squared radius t. Conversely, if q is a hypersphere representation of G with squared radius $t < 1/2$, then the map $i \mapsto t^{-1/2}q(i)$, is a strict vector k-colouring of G, where $k = 1/(1-2t)$. This correspondence shows that $t_h(G) = \frac{1}{2}(1 - 1/\chi_v(G))$, where $\chi_v(G)$ denotes the strict vector chromatic number of G.

The same scaling maps as above yield correspondences between vector k-colourings and the geometric representations arising from the graph invariant t', and also between strong vector k-colourings and geometric representations arising from the graph invariant

$$\begin{array}{rll} t^+(G) := \min & t & \\ & \operatorname{diag}(X) = t\bar{e}, & \\ & X_{ii} - 2X_{ij} + X_{jj} = 1, & \forall \{i,j\} \in E(G), \\ & X_{ii} - 2X_{ij} + X_{jj} \leq 1, & \forall \{i,j\} \in \overline{E}(G), \\ & X \in \mathbb{S}^V_+,\ t \in \mathbb{R}. & \end{array} \tag{5.8}$$

Note however, that the parameter t^+ does not fit into the framework of hom-monotone graph invariants since the SDP (5.8) has non-edge constraints.

We point out here that, while these equivalences between variants of vector chromatic number and variants of theta number are easy to prove, they are not as widely known as they should be. For instance, in [1] it is shown

that the vector chromatic number $\chi'_v(G)$ of G satisfies

$$\chi'_v(G) \geq \max\left\{ 1 - \frac{\lambda_{\max}(B)}{\lambda_{\min}(B)} : B \in \mathcal{A}_G,\, B \geq 0 \right\}, \tag{5.9}$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the largest and smallest eigenvalue, respectively, and $\mathcal{A}_G$ denotes the set of all weighted adjacency matrices of G, i.e., all symmetric $V \times V$ matrices A such that $A_{ij} \neq 0 \implies \{i,j\} \in E$. However, since $\chi'_v(G) = \overline{\vartheta'}(G)$, it is possible to adapt the proof of the Hoffman bounds for $\vartheta(G)$ (see, e.g., [14, Corollary 33]) to show that (5.9) actually holds with equality.

Also, in [22, Remark 3.1] it is reported that a certain graph G has vector chromatic number strictly smaller than its strict vector chromatic number, and that it was unknown whether some such graph existed. However, this statement about the vector chromatic numbers is equivalent to $\overline{\vartheta'}(G) < \overline{\vartheta}(G)$, and the existence of graphs satisfying this strict inequality was already known as far back as 1979 (see [24]).

We also mention that one of the characterizations of $\vartheta'(G)$ in [6] is inaccurate. Define an *obtuse representation of a graph* $G = (V,E)$ to be a map $p\colon V \to \mathbb{R}^d$ for some $d \geq 1$ such that

(i) $\|p(i)\| = 1$ for every $i \in V$, and
(ii) $\langle p(i), p(j)\rangle \leq 0$ for every $\{i,j\} \in \overline{E}(G)$.

In [6, p. 133] it is claimed that

$$\vartheta'(G) = \min_{p,c} \max_{i \in V} \frac{1}{\left(c^T p(i)\right)^2}, \tag{5.10}$$

where p ranges over obtuse representations of G and c ranges over unit vectors of appropriate dimension. Let G be a $2n$-partite graph with color classes $C_1, \ldots, C_{2n}$ such that $\omega(G) = 2n$. Thus, by (5.4) and (5.5), we have $\vartheta'(\overline{G}) \geq 2n$. Let $p(j) := e_i \in \mathbb{R}^n$ for every $j \in C_i$ and $i \in [n]$, and $p(j) := -e_i \in \mathbb{R}^n$ for every $j \in C_{n+i}$ and $i \in [n]$. Set $c := n^{-1/2}\bar{e} \in \mathbb{R}^n$. By (5.10), we get $\vartheta'(\overline{G}) \leq n$, a contradiction.

Now we show how to fix the formula (5.10). Given an obtuse representation $p\colon V \to \mathbb{R}^d$ of a graph $G = (V,E)$, we say that a vector $c \in \mathbb{R}^d$ is *consistent with* p if $c^T p(i) \geq 0$ for every $i \in V$. The next result is a Gallai-type identity involving $t'(G)$, parallel to Proposition 4.3 for $t_h(G)$.

Proposition 5.1. *Let $G = (V,E)$ be a graph. Then*

$$2t'(G) + \max_{p,c} \min_{i \in V} \left(c^T p(i)\right)^2 = 1, \tag{5.11}$$

where p ranges over all obtuse representations of $\overline{G}$ and c over unit vectors consistent with p.

Proof. This proof is analogous to the proof of Proposition 4.3, with the following slight adjustments. In the notation of the proof of (4.12), the

vector d may be chosen to be consistent with the obtuse representation q, so we do not need to replace any of the $q(i)$'s by their opposites. □

Corollary 5.2. *Let $G = (V, E)$ be a graph. Then $\vartheta'(G)$ is given by* (5.10)*, where p ranges over obtuse representations of G and c ranges over unit vectors consistent with p.*

Proof. This follows from Proposition 5.1 together with (5.5). □

5.2. **Optimal hypersphere dimension.** For a graph $G = (V, E)$, let $\dim_h(G)$ denote the smallest d for which G has a hypersphere representation in $\mathbb{R}^d$. A related parameter was introduced by Erdős, Harary, and Tutte [4]: they defined $\dim(G)$ to be the smallest d for which there is an injective unit-distance representation of G in $\mathbb{R}^d$. In [5] a geometric construction is presented that shows that $\dim(G) \leq \Delta(G)+2$. In fact, this geometric construction embeds G into a hypersphere, so that the same proof implies $\dim_h(G) \leq \Delta(G) + 2$.

Here we point out that a better bound for $\dim_h(G)$ may be obtained from the fact that $G \to H$ implies $\dim_h(G) \leq \dim_h(H)$. Since $G \to K_{\chi(G)}$, we get $\dim_h(G) \leq \dim_h(K_{\chi(G)}) \leq \chi(G) - 1 \leq \Delta(G)$. In fact, we get a slight improvement using Brooks' Theorem: if G is connected, and G is not a clique nor an odd cycle, then $\dim_h(G) \leq \Delta(G) - 1$.

6. Duality and energy functions

In polyhedral combinatorics, one of the most fundamental tools for gaining insight into problems is duality. Often a good understanding of the dual problem allows one to improve a polyhedral description, or obtain better approximation algorithms.

For several natural combinatorial optimization problems, the dual of a natural formulation of the problem as an LP turns out to be a natural-looking combinatorial optimization problem, i.e., a problem that is interesting in its own right, regardless of it being the dual of another problem. An example of this is the dual pair of LPs associated with $\alpha^*(G)$ and $\overline{\chi^*}(G)$.

The situation seems to be different for SDP duality. For instance, even though problem (4.1) seems extremely natural, it is not obvious why anyone would be interested in solving (4.2) in the first place, were it not for the fact that it is the dual of (4.1). Namely, it is not obvious how to interpret (4.2) as a natural-looking problem on graphs, or as an SDP-free problem on graphs; recall that the original problem of computing $t_h(G)$ can be formulated in a natural way not as an SDP.

This is not surprising, as SDP duality theory is more complex than LP duality theory. Indeed, LP duality applied to combinatorial optimization problems often reduces to double counting, so the resulting dual problems are usually natural. Since SDP duality theory is richer, we may expect that obtaining nice interpretations from SDP duals to be harder, but to pay off perhaps even more significantly.

In this section, we investigate a possible interpretation of the dual (4.2) of (4.1) related to graph rigidity. We note that some attempts in this direction of interpreting duals of SDPs have already been made; see, e.g., [11, Remark 1] and [10, Remark 1].

6.1. **Optimal energy functions.** Let $G = (V, E)$ be a graph and let $p\colon V \to \mathbb{R}^d$ for some $d \geq 1$. A function $\sigma\colon E \to \mathbb{R}$ is a *stress function for* p (describing a stress coefficient for each edge) if

$$\sum_{j \in N(i)} \sigma_{\{i,j\}}\big(p(j) - p(i)\big) = 0, \quad \forall i \in V. \tag{6.1}$$

This condition, for a fixed $i \in V$, can be interpreted as follows. For an edge $\{i, j\} \in E$ with $\sigma_{\{i,j\}} > 0$, we regard the edge $\{i, j\}$ as a rubber band pulling node i towards node j. If $\sigma_{\{i,j\}} < 0$, then the edge $\{i, j\}$ can be thought of as a strut pushing nodes i and j apart. An edge $\{i, j\} \in E$ with $\sigma_{\{i,j\}} = 0$ is effectively non-existing. Then each of the terms of the sum in (6.1) can be seen as the force acting on node i arising from the physical structure associated with the corresponding edge. In this context, condition (6.1) means that the physical structure is in equilibrium.

The above interpretation shows why stress functions show up naturally in graph rigidity. A related concept is that of an "energy function" (see, e.g., [17, ch. 4] and [3]). Fix a function $\sigma\colon E \to \mathbb{R}$. We can associate to each map $p\colon V \to \mathbb{R}^d$ the energy of p as

$$\mathcal{E}_\sigma(p) := \sum_{\{i,j\} \in E} \sigma_{\{i,j\}} \|p(j) - p(i)\|^2.$$

A nice interpretation of this energy is given as follows. Suppose $\sigma_e > 0$ for every $e \in E$. Then, as above, we can interpret each edge as a rubber band pulling its ends closer together, and the term $\sigma_{\{i,j\}}\|p(j)-p(i)\|^2$ can be seen as the contribution of edge $\{i, j\}$ to the total potential energy $\mathcal{E}_\sigma(p)$ of the system.

In [16], the following problem is considered, in connection with Tutte's barycentric mappings [28]: for a certain subset $S \subseteq V$ of nodes, fix a position $p^0\colon S \to \mathbb{R}^d$, and find an extension $p\colon V \to \mathbb{R}^d$ of p^0 that minimizes the energy $\mathcal{E}_\sigma(p)$, where $\sigma\colon E \to \mathbb{R}_{++}$ is fixed. This corresponds to nailing down the nodes of S into their prescribed positions, then taking each edge as a rubber band with "constant of elasticity" given by σ, and letting the system vibrate until it reaches equilibrium. Thus, an optimal solution p of the above optimization problem corresponds to a configuration in static equilibrium. Optimality conditions then show that σ is "close to" a stress function for p, namely, (6.1) holds for all $i \in V \setminus S$.

The situation is a bit more complicated when we allow some entries of σ to be negative. Indeed, if $\sigma_e < 0$, then we should interpret e as a strut pushing its ends further apart, but then the contribution $\sigma_e\|p(j) - p(i)\|^2$ of edge e to the total potential energy $\mathcal{E}_\sigma(p)$ of the system is negative. This might

seem counterintuitive, but given the fact that edge e is constantly pushing its ends apart, it somewhat makes sense. The most important property that we must preserve for the above ideas to carry through is that the energy function $\mathcal{E}_\sigma$ must have a minimum.

Let us briefly investigate for which functions $\sigma\colon E \to \mathbb{R}$ the energy function $\mathcal{E}_\sigma$ has a minimum. Given $p\colon V \to \mathbb{R}^d$, define a $[d] \times V$ matrix P by setting $Pe_i := p(i)$ for every $i \in V$. Let D be an arbitrary orientation of G, and let B_D denote its node-arc incidence matrix. Then

$$\begin{aligned}\mathcal{E}_\sigma(p) &= \sum_{k=1}^{d} \sum_{\{i,j\}\in E} \big(p_k^{(j)} - p_k^{(i)}\big)\sigma_{\{i,j\}}\big(p_k^{(j)} - p_k^{(i)}\big) \\ &= \sum_{k=1}^{d} e_k^T P B_D \operatorname{Diag}(\sigma) B_D^T P^T e_k = \operatorname{Tr}\left(P\mathcal{L}_G(\sigma)P^T\right)\end{aligned}$$

where we used the fact that

$$\mathcal{L}_G(z) = B_D \operatorname{Diag}(z) B_D^T. \tag{6.2}$$

Now it is easy to see that

$$\mathcal{E}_\sigma \text{ has a minimum if and only if } \mathcal{L}_G(\sigma) \succeq 0. \tag{6.3}$$

Indeed, suppose $\mathcal{L}_G(\sigma) \succeq 0$. Then $\mathcal{E}_\sigma(p) \geq 0$ for every $p\colon V \to \mathbb{R}^d$, so $p = 0$ is a minimum of $\mathcal{E}_\sigma$. Now suppose $h^T\mathcal{L}_G(\sigma)h < 0$ for some $h \in \mathbb{R}^V$. Set $P := e_1 h^T$, and define $p\colon V \to \mathbb{R}^d$ accordingly. Then $\mathcal{E}_\sigma(p) = h^T\mathcal{L}_G(\sigma)h \operatorname{Tr}\left(e_1 e_1^T\right) = h^T\mathcal{L}_G(\sigma)h < 0$, so $\mathcal{E}_\sigma(\lambda p) \to -\infty$ as $\lambda \to \infty$.

Let us go back to our SDP (4.1). In fact, we will start by looking at the augmented SDP

$$\begin{aligned} t_h(G) = \min \quad & t \\ & X_{00} - 2X_{0i} + X_{ii} = t, \quad \forall i \in V, \\ & X_{ii} - 2X_{ij} + X_{jj} = 1, \quad \forall \{i,j\} \in E, \\ & X \in \mathbb{S}_+^{\{0\}\cup V},\ t \in \mathbb{R}, \end{aligned} \tag{6.4}$$

and its dual

$$\begin{aligned} \max \quad & -\sigma(E(G)) \\ & \mathcal{L}_H(\sigma) \succeq 0, \\ & \sigma(\delta_H(0)) = 1, \\ & \sigma \in \mathbb{R}^{E(H)}, \end{aligned} \tag{6.5}$$

where H denotes the cosum of G and the graph containing a single node, called 0.

Note that (6.4) really models $t_h(G)$, and the only difference between this formulation of $t_h(G)$ and the one given by (4.1) is that here we do not insist that the optimal hypersphere is centered at the origin. Thus, (6.4) has an optimal solution. Moreover, since $(\bar{X}, \bar{t}) := (\frac{1}{2}I, 1)$ is a Slater point of (6.4) and (6.5) is feasible, it follows from SDP Strong Duality that there is no duality gap, and the dual is attained.

It should be easy to interpret (6.5) from our previous discussion of energy functions. Among all vectors $\sigma\colon E(H) \to \mathbb{R}$ giving rise to an energy function $\mathcal{E}_\sigma$ that has a minimum, normalized so that $\sigma(\delta_H(0)) = 1$, choose one that minimizes $\sigma(E(G))$.

Let (X, t) be an optimal solution for (6.4) and let σ be an optimal solution for (6.5). Write $X = P^T P$ for some $[d] \times V(H)$ matrix P, and put $p(i) := Pe_i$ for every $i \in V(H)$. Now $t = -\sigma(E(G))$ is equivalent to $0 = \mathrm{Tr}(X\mathcal{L}_H(\sigma)) = \mathrm{Tr}(P\mathcal{L}_H(\sigma)P^T)$. Since $\mathcal{L}_H(\sigma) \succeq 0$, we have $P\mathcal{L}_H(\sigma)P^T \succeq 0$, and thus $P\mathcal{L}_H(\sigma)P^T = 0$. Hence, for each $h \in \mathbb{R}^d$, we have $\|\mathcal{L}_H(\sigma)^{1/2}P^T h\|^2 = 0$, so $\mathcal{L}_H(\sigma)^{1/2}P^T = 0$, and $P\mathcal{L}_H(\sigma) = 0$. It is easy to see that this is equivalent to the fact that σ is a stress function for p.

Note that, if we decompose $\sigma\colon E(H) \to \mathbb{R}$ by setting $\sigma = y \oplus -z$, with $y\colon V \to \mathbb{R}$ and $z\colon E(G) \to \mathbb{R}$ in an obvious way, then

$$\mathcal{L}_H(\sigma) = \begin{bmatrix} y(V) & -y^T \\ -y & \mathrm{Diag}(y) - \mathcal{L}_G(z), \end{bmatrix}$$

so the matrix $\mathrm{Diag}(y) - \mathcal{L}_G(z)$ that appears in the positive semidefiniteness constraint of (4.2) is a principal submatrix of $\mathcal{L}_H(\sigma)$.

Let us now translate these interpretations to the original dual problem (4.2). Let $y\colon V \to \mathbb{R}$ and $z\colon E(G) \to \mathbb{R}$ be such that $y(V) = 1$. Given any function $p\colon V \to \mathbb{R}^d$, define a $[d] \times V$ matrix P by setting $Pe_i := p(i)$ for every $i \in V$ and define the energy of p to be

$$\begin{aligned} \mathcal{E}_{y,z}(p) &:= \mathrm{Tr}\left(P(\mathrm{Diag}(y) - \mathcal{L}_G(z))P^T\right) \\ &= \sum_{i \in V} y_i \|p(i) - 0\|^2 + \sum_{\{i,j\} \in E} (-z_{\{i,j\}})\|p(i) - p(j)\|^2. \end{aligned}$$

This can be seen as the energy function of p on the augmented graph H arising from $\sigma := y \oplus -z$ that considers the node 0 as mapped into the origin.

As before, the parameters (y, z) that give rise to energy functions that have a minimum are precisely the ones for which $\mathrm{Diag}(y) \succeq \mathcal{L}_G(z)$. Thus, (4.2) can be seen as the search for the "best" such parameters, normalized so that $y(V) = 1$. The optimal solution (y, z) to (4.2) also yields a stress function for any optimal hypersphere representation of G, where we assume an extra node has been placed at the origin.

6.2. **Dual interpretation in more general frameworks.** We can extend the interpretation of duality presented above by regarding the constraint $y(V) = 1$ as an instance of a general constraint of the form $y \oplus z \in P$ for some polyhedron P.

Thus, in our extended formulation, the dual SDP is defined as:

$$\begin{array}{ll} \sup & \langle w^{(0)}, y\rangle + \langle w^{(1)}, z\rangle \\ & \mathrm{Diag}(y) \succeq \mathcal{L}_G(z) \\ & A_{00}y + A_{01}z \leq b^{(0)} \\ & A_{10}y + A_{11}z = b^{(1)}, \\ & y \in \mathbb{R}^V,\ z \in \mathbb{R}^E. \end{array} \tag{6.6}$$

Here

(i) $G = (V, E)$ is a graph,
(ii) k and ℓ are nonnegative integers,
(iii) $R_0 := [k]$ and $R_1 := [\ell]$,
(iv) $C_0 := V$ and $C_1 := E$,
(v) $w^{(j)} \in \mathbb{R}^{C_j}$,
(vi) A_{ij} is an $R_i \times C_j$ matrix,
(vii) $b^{(i)} \in \mathbb{R}^{R_i}$,

The primal SDP is thus:

$$\begin{array}{ll} \inf & \langle b^{(0)}, u\rangle + \langle b^{(1)}, v\rangle \\ & \mathrm{diag}(X) - A_{00}^T u - A_{10}^T v = -w^{(0)} \\ & \mathcal{L}_G^*(X) + A_{01}^T u + A_{11}^T v = w^{(1)} \\ & X \in \mathbb{S}_+^V,\ u \in \mathbb{R}_+^k,\ v \in \mathbb{R}^\ell. \end{array} \tag{6.7}$$

Note that, whenever SDP Strong Duality holds for this pair of SDPs, e.g., whenever both (6.7) and (6.6) have Slater points, our interpretation via energy functions still makes sense. That is, among all energy functions $\mathcal{E}_{y,z}(\cdot)$ that have a minimum, and such that $y \oplus z$ lies in a certain polyhedron, choose one that maximizes the objective function $\langle w^{(0)}, y\rangle + \langle w^{(1)}, z\rangle$.

It is easily seen that the parameters t_h, t_b, t', and t^+ can be modelled in this framework. Moreover, for each of these parameters, the weight vectors $w^{(0)}$ and $w^{(1)}$ and the polyhedron P where we require $y \oplus z$ to lie in are symmetric in the following sense: for any $V \times V$ permutation matrix $P^{(0)}$ and any $E \times E$ permutation matrix $P^{(1)}$, and for any $y \oplus z \in \mathbb{R}^V \oplus \mathbb{R}^E$ we have

(i) $\langle w^{(0)}, y\rangle + \langle w^{(1)}, z\rangle = \langle w^{(0)}, P^{(0)}y\rangle + \langle w^{(1)}, P^{(1)}z\rangle$, and
(ii) $y \oplus z \in P$ if and only if $P^{(0)}y \oplus P^{(1)}z \in P$.

In other words, except possibly for the constraint $\mathrm{Diag}(y) \succeq \mathcal{L}_G(z)$, the SDP (6.6) treats all the nodes of G in the same way, and similarly for all the edges.

This symmetry makes the dual interpretation look very natural. For instance, consider the following modification of (5.2):

$$\begin{array}{ll} \min & t \\ & \mathrm{diag}(X) \leq t\bar{e}, \\ & X_{ii} - 2X_{ij} + X_{jj} \geq 1, \quad \forall \{i, j\} \in E, \\ & X \in \mathbb{S}_+^V,\ t \in \mathbb{R}. \end{array} \tag{6.8}$$

Its dual is the same as (5.3), with the extra constraint that $y \geq 0$. (In fact, the constraint $y \geq 0$ is implied in (5.3).) Thus, besides the normalization constraint $y(V) = 1$, all we require is that $y \geq 0$ and that $z \geq 0$, that is, we require the edges joining the nodes of G to the new node at the origin to be rubber bands, whereas the edges in the original graph should be struts.

The parameter $t^+(G)$ can also be formulated in this model, where the graph in the SDP is taken to be the complete graph on node set $V(G)$. However, then the resulting polyhedron P would not have the above symmetry property, since the dual variables corresponding to edges are free, whereas the dual variables corresponding to non-edges are nonpositive. Nevertheless, the dual SDP of $t^+(G)$ still has a large amount of symmetry, namely, all edges are treated in the same way, and all non-edges are treated in the same way. Thus, this dual would fit the following format:

$$\begin{array}{ll} \sup & \langle w^{(0)}, y\rangle + \langle w^{(1)}, z\rangle + \langle w^{(2)}, \bar{z}\rangle \\ & \mathrm{Diag}(y) \succeq \mathcal{L}_G(z) + \mathcal{L}_{\overline{G}}(\bar{z}) \\ & A_{00}y + A_{01}z + A_{02}\bar{z} \leq b^{(0)} \\ & A_{10}y + A_{11}z + A_{12}\bar{z} = b^{(1)}, \\ & y \in \mathbb{R}^V,\ z \in \mathbb{R}^E,\ \bar{z} \in \mathbb{R}^{\overline{E}(G)}, \end{array} \tag{6.9}$$

where the following type of symmetry is "required." Let P denote the corresponding polyhedron. For every $V \times V$ permutation matrix $P^{(0)}$, every $E \times E$ permutation matrix $P^{(1)}$, and every $\overline{E} \times \overline{E}$ permutation matrix $P^{(2)}$, and for any $y \oplus z \oplus \bar{z} \in \mathbb{R}^V \oplus \mathbb{R}^E \oplus \mathbb{R}^{\overline{E}}$, we have

(i) $\langle w^{(0)}, y\rangle + \langle w^{(1)}, z\rangle + \langle w^{(2)}, \bar{z}\rangle = \langle w^{(0)}, P^{(0)}y\rangle + \langle w^{(1)}, P^{(1)}z\rangle + \langle w^{(2)}, P^{(2)}\bar{z}\rangle$, and

(ii) $y \oplus z \oplus \bar{z} \in P$ if and only if $P^{(0)}y \oplus P^{(1)}z \oplus P^{(2)}\bar{z} \in P$.

The interpretation for (6.9) extends easily. Namely, among all vectors $y \oplus z \oplus \bar{z}$ in a certain polyhedron that treats all the components of y, z and $\bar{z}$ in the same way, and such that the energy function

$$\begin{aligned} \mathcal{E}_{y,z,\bar{z}}(p) &:= \mathrm{Tr}\left(P(\mathrm{Diag}(y) - \mathcal{L}_G(z) - \mathcal{L}_{\overline{G}}(\bar{z}))P^T\right) \\ &= \sum_{i \in V} y_i \|p(i) - 0\|^2 + \sum_{\{i,j\} \in E} (-z_{\{i,j\}})\|p(i) - p(j)\|^2 \\ &\qquad + \sum_{\{i,j\} \in \overline{E}} (-\bar{z}_{\{i,j\}})\|p(i) - p(j)\|^2 \end{aligned}$$

has a minimum, choose one that maximizes a certain linear function of $y \oplus z \oplus \bar{z}$, where the linear function also treats each component of y, z and $\bar{z}$ in the same way.

To illustrate the symmetry in this case, consider the following variant of $t^+(G)$:

$$\begin{array}{ll} \min & t \\ & \operatorname{diag}(X) \leq t\bar{e}, \\ & X_{ii} - 2X_{ij} + X_{jj} \geq 1, \quad \forall \{i,j\} \in E(G), \\ & X_{ii} - 2X_{ij} + X_{jj} \leq 1, \quad \forall \{i,j\} \in \overline{E}(G), \\ & X \in \mathbb{S}^V_+,\ t \in \mathbb{R}. \end{array} \tag{6.10}$$

If we now formulate this SDP as the primal of (6.9), then the constraints of the dual, besides the normalization constraint $y(V) = 1$, are equivalent to requiring the edges joining the nodes of G to the new node at the origin and the non-edges of G to be rubber bands, whereas the edges of G are required to be struts.

Thus, in moving from the model (6.6) to (6.9), and assuming both have the corresponding symmetry properties in their polyhedra, we are allowing two types of edges, namely, the edges of G and the edges of $\overline{G}$, each of which has to be treated in the same way. Also, there is no need to assume that the two types of edges partition the edge set of the complete graph on the same node set. With this in mind, we propose the following model.

Let V be a finite set, and let K_V be the complete graph on node set V. For some nonnegative integer q, let $E_1, \ldots, E_q$ be disjoint subsets of $E(K_V)$. Let $G_j := (V, E_j)$ for each $j \in [q]$. Our dual SDP is defined as:

$$\begin{array}{ll} \sup & \langle w^{(0)}, y\rangle + \sum_{j=1}^{e} \langle w^{(j)}, z^{(j)}\rangle \\ & \operatorname{Diag}(y) \succeq \sum_{j=1}^{q} \mathcal{L}_{G_j}(z^{(j)}) \\ & A_{00}y + \sum_{j=1}^{q} A_{0j} z^{(j)} \leq b^{(0)} \\ & A_{10}y + \sum_{j=1}^{q} A_{1j} z^{(j)} = b^{(1)}, \\ & y \in \mathbb{R}^V, \\ & z^{(j)} \in \mathbb{R}^{E_j}, \qquad\qquad \forall j \in [q]. \end{array} \tag{6.11}$$

Here

(i) k and ℓ are nonnegative integers,
(ii) $R_0 := [k]$ and $R_1 := [\ell]$,
(iii) $C_0 := V$ and $C_j := E_j$,
(iv) $w^{(j)} \in \mathbb{R}^{C_j}$,
(v) A_{ij} is an $R_i \times C_j$ matrix,
(vi) $b^{(i)} \in \mathbb{R}^{R_i}$,

and we "require" the following symmetry property. Let P denote the corresponding polyhedron P. For each choice of $C_j \times C_j$ permutation matrix $P^{(j)}$ for $j = 0, \ldots, q$, and for every $y \oplus \bigoplus_{j=1}^{q} z^{(j)}$, we have

(i) $\langle w^{(0)}, y\rangle + \sum_{j=1}^{q} \langle w^{(j)}, z^{(j)}\rangle = \langle w^{(0)}, P^{(0)}y\rangle + \sum_{j=1}^{q} \langle w^{(j)}, P^{(j)}z^{(j)}\rangle$, and
(ii) $y \oplus \bigoplus_{j=1}^{q} z^{(j)} \in P$ if and only if $P^{(0)}y \oplus \bigoplus_{j=1}^{q} P^{(j)}z^{(j)} \in P$.

The primal SDP is:

$$\begin{array}{ll} \inf & \langle b^{(0)}, u\rangle + \langle b^{(1)}, v\rangle \\ & \operatorname{diag}(X) - A_{00}^T u - A_{10}^T v = -w^{(0)} \\ & \mathcal{L}_{G_j}^*(X) + A_{0j}^T u + A_{1j}^T v = w^{(j)}, \qquad \forall j \in [q], \\ & X \in \mathbb{S}_+^V,\ u \in \mathbb{R}_+^k,\ v \in \mathbb{R}^\ell. \end{array} \tag{6.12}$$

We could even relax the symmetry property by allowing for several types of nodes, i.e., we would fix a partition of V. The generalization would be done in a straightforward way.

Note that any SDP can be written in the form (6.11) with the relaxed symmetry property, even if the cost is to have the symmetry property to become meaningless. Indeed, consider a general SDP with variable $X \in \mathbb{S}_+^V$ for some finite set V, with constraints $\langle A_i, X\rangle = b_i$ for $i = 1, \dots, m$, and with objective function $\langle C, X\rangle$. For each $\{i,j\} \in E(K_V)$, let $E_{\{i,j\}}$ be the singleton set with element $\{i,j\}$, and write $G_{\{i,j\}} := (V, E_{\{i,j\}})$. We can use the change of variables $X = \operatorname{Diag}(y) - \sum_{f \in E(K_n)} \mathcal{L}_{G_f}(z^{(f)})$. Now each constraint $\langle A_i, X\rangle = b_i$ becomes an affine constraint on the variables y and z_f for each $f \in E(K_n)$, and similarly for the objective function. Moreover, the relaxed symmetry property holds trivially if we consider each element of V to have its own type.

This implies that any SDP can have its dual interpreted using energy functions. We have a spectrum of dual interpretations, with one end of the spectrum being the case where all nodes and all edges are treated in the same way, and on the other end, each node and each edge is treated differently. As we move along the spectrum, the quality of the interpretation can be seen as degrading, until we reach a point where all the symmetry is lost.

However, even for some highly structured SDPs, this type of symmetry could be nonexistent and we could still have a very nice dual interpretation. For instance, the SDP used in the approximation algorithm for MaxCut [9] is

$$\begin{array}{ll} \max & \langle \frac{1}{4}\mathcal{L}_G(w), X\rangle \\ & \operatorname{diag}(X) = \bar{e} \\ & X \in \mathbb{S}_+^V, \end{array} \tag{6.13}$$

and its dual is

$$\begin{array}{ll} \min & \langle \frac{1}{4}\bar{e}, y\rangle \\ & \operatorname{Diag}(y) \succeq \mathcal{L}_G(w), \\ & y \in \mathbb{R}^V. \end{array} \tag{6.14}$$

Here we restrict ourselves to energy functions $\mathcal{E}_{y,z}(\cdot)$ that have a minimum and such that $y \oplus z$ lies in the polyhedron $\mathbb{R}^V \oplus \{w\}$. If all edges have distinct weights, the symmetry for the edges is lost, but we still have a very natural dual interpretation.

Another nice feature of the general model is that it sometimes allows us to preserve the block-diagonal structure of some SDPs. For instance, if we have the constraints $\operatorname{Diag}(y^{(1)}) \succeq \mathcal{L}_G(z^{(1)})$ and $\operatorname{Diag}(y^{(2)}) \succeq \mathcal{L}_H(z^{(2)})$, where G

and H are graphs on disjoint node sets, we can replace these constraints by the equivalent constraint $\mathrm{Diag}(y^{(1)} \oplus y^{(2)}) \succeq \mathcal{L}_{G+H}(z^{(1)} \oplus z^{(2)})$. It is clear that a constraint of this type encodes several constraints in block-diagonal form.

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