A PSEUDO–POLYNOMIAL COMPLEXITY ANALYSIS FOR INTERIOR–POINT ALGORITHMS

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Abstract

We provide a pseudo–polynomial iteration complexity analysis for interior–point methods under the assumption that the smaller of the dimensions is fixed. We show that the bounds proven can be independent of the larger of the dimensions defining the problem instance. The complexity analysis is based on the distance between two sets defined by some underlying oblique projections.

Keywords: Linear programming, primal-dual, interior-point methods, time-complexity.

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1 Introduction

Our motivation is to find new ways of explaining the slow growth in the number of iterations required by interior–point methods as one of the dimensions ($m :=$ the number of equality constraints) is fixed and the other ($n :=$ the number of variables) becomes very large (see Bixby, Gregory, Lustig, Marsten, Shanno [2] and Lustig, Marsten and Shanno [6]). We will use some recent results involving scaled projections to derive complexity bounds on the number of iterations that can be independent of n. The main result is showing that when a vector is projected onto the null space of a matrix, infinity norm of the projection cannot be more than a constant times the infinity norm of the initial vector, where the constant depends only on the entries and the number of rows in the matrix. So, if the matrix has integral entries and m is $O(1)$ then the constant will be independent of n (so if the constant depends on n, it is implied that the matrix has some entries growing with n).

We consider linear programming problems in the following primal (P) and dual (D) forms:

$$
\begin{array}{lll}
(P) & \text{minimize} & c^T x \\
 & & \begin{array}{rcl} Ax & = & b, \\ x & \geq & 0, \end{array} \\
 & & \\
(D) & \text{maximize} & b^T y \\
 & & \begin{array}{rcl} A^T y + s & = & c, \\ s & \geq & 0, \end{array}
\end{array}
$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. We will assume A has full row rank and that there exist interior solutions for both problems, i.e.,

$$\mathcal{F}_0 := \{(x, s) > 0 : x \in F(P), s \in F(D)\} \neq \emptyset,$$

where $F(P)$ and $F(D)$ denote the sets of feasible solutions for the primal and dual problems respectively. Most of the time we will deal only with s as a dual feasible solution. So, whenever we say $s \in F(D)$, we mean that $s \geq 0$ and there exists a $y \in \mathbb{R}^m$ such that $A^T y + s = c$. Given a vector denoted by a lower-case roman letter (e.g. x), the corresponding upper-case letter (e.g. X) will denote the diagonal matrix whose entries are the components of that vector (i.e. $X = \text{diag}(x)$), e will denote the vector of ones, and t will denote the desired improvement in the precision of the solution. We will denote the components of a vector using subscripts and the iterate numbers using superscripts. Note that given $x \in F(P)$ and $s \in F(D)$ duality gap corresponding to the pair (x, s) is $x^T s$. Henceforth, we will use μ to denote $\frac{x^T s}{n}$.

We first describe a generic primal–dual algorithm (see Kojima, Mizuno and Yoshise [4], Monteiro and Adler [10], and Mizuno, Todd and Ye [9]). The search direction

is chosen as a linear combination of the primal–dual affine–scaling direction and the primal–dual centering direction. dx and ds will denote the components of the search direction in the primal feasible region and the dual feasible region respectively.

Suppose we have an interior-point solution $(x, s) \in \mathcal{F}_0$ and we are given a value for the centering parameter γ ($\gamma \in [0, 1]$). Then a search direction (dx, ds) at (x, s) is given by the following set of equalities (see Kojima et al. [3]):

$$\begin{aligned} dx(\gamma) &:= -X^{1/2}S^{-1/2}P_{\bar{A}}(X^{1/2}S^{1/2} - \gamma\mu X^{-1/2}S^{-1/2})e, \\ ds(\gamma) &:= -X^{-1/2}S^{1/2}(I - P_{\bar{A}})(X^{1/2}S^{1/2} - \gamma\mu X^{-1/2}S^{-1/2})e, \end{aligned}$$

where $\bar{A} := AX^{1/2}S^{-1/2}$, and $P_{\bar{A}} := I - \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A}$, the projection matrix onto the null space of $\bar{A}$. The next iterates in terms of α can be written as

$$\begin{aligned} x(\alpha) &= x + \alpha dx(\gamma), \\ s(\alpha) &= s + \alpha ds(\gamma). \end{aligned}$$

A primal-dual interior-point algorithm takes an iterate $(x, s) \in \mathcal{F}_0$ and defines the next iterate as $(x(\alpha), s(\alpha))$ derived as above for some $\gamma \in [0, 1]$ and some $\alpha > 0$ such that $(x(\alpha), s(\alpha)) \in \mathcal{F}_0$. The centering parameter γ and the step size α may vary from one iteration to the next. When γ is set to zero for all iterations (no centering component in the search direction) we have a primal–dual affine–scaling algorithm and the corresponding search direction is called primal–dual affine–scaling direction. When γ is set to 1, we have what is called primal–dual centering direction (along this direction the duality gap stays constant).

Indeed, the interiority of the iterates is one of the most important issues for these algorithms. Given a positive constant μ, 'the most central pair' (x, s) with duality gap $n\mu$ is usually defined by $(x, s) \in \mathcal{F}_0$ such that $x_j s_j = \mu \; \forall j$. Taking different values for μ we can define a path of central solutions or the central path (see Megiddo [7]). Using the ∞-norm (also the 2-norm), neighbourhoods of the central path have been defined and used by Mizuno et al. [9]. Using the ∞-norm, we get

$$\mathcal{N}(\pi) := \{(x, s) \in \mathcal{F}_0 : \|Xs - \mu e\|_\infty \leq (1 - \pi)\mu\}.$$

Here, $\pi \in [0, 1]$. Alternatively, we can write:

$$\mathcal{N}(\pi) := \{(x, s) \in \mathcal{F}_0 : \min_j\{x_j s_j\} \geq \pi\mu \text{ and } \max_j\{x_j s_j\} \leq (2 - \pi)\mu\}.$$

Let $\mathcal{N}$ be a neighbourhood of the central path and $\gamma' \in (0, 1)$ be a constant. Suppose $(x^0, s^0) \in \mathcal{N}$ with $(x^0)^T s^0 \leq 2^t$ is given. We can now state a generic primal–dual algorithm that uses the neighbourhood $\mathcal{N}$ as follows:

Algorithm:

$k := 0$

While $((x^k)^T s^k > 2^{-t})$ do

$(x, s) := (x^k, s^k)$,

compute $dx(\gamma)$, $ds(\gamma)$

choose the maximum step size $\alpha \in (0, 1)$ such that $(x + \alpha dx(\gamma), s + \alpha ds(\gamma)) \in \mathcal{N}$,

$(x^{k+1}, s^{k+1}) := (x + \alpha dx(\gamma), s + \alpha ds(\gamma))$,

$k := k + 1$

end

In the next section we describe some recent results related to scaled projections and show how one might use these results to get new complexity bounds for interior–point algorithms. Section three includes two analyses : one for the algorithm stated here and one for a primal–dual affine–scaling algorithm. Section four provides a way of relating the complexity bound more directly to the input (namely the entries of A). The fifth section is a brief conclusion.

2 Oblique Projections

Let $\mathcal{D}$ be the set of all real $n \times n$ diagonal positive definite matrices. Define

$$\begin{aligned} N &:= \{x : AD^2x = 0,\ D \in \mathcal{D}\}, \\ R_p &:= \{y : y = A^T w,\ \|y\|_p = 1\}. \end{aligned}$$

Let $cl(.)$ denote the closure of a set and $d(\ .\ ,\ .\ )$ denote the Euclidean distance between two sets in $\mathbb{R}^n$. Stewart [13] and Todd [14] independently proved the following:

Theorem 2.1.

$$cl(N) \cap R_2 = \emptyset \ \text{ and } \ d(cl(N), R_2) = \rho_2 > 0.$$

We will see that the distance between the closure of N and R_p yields interesting relations on the norms of projections and the projected vectors. First, we would like to note that we can take $\hat{x} := 0 \in \mathbb{R}^n$ ($\hat{x} \in N$) to show that the distance is at most one in any $p-$norm. Let columns of U form an orthonormal basis for the range of A^T. Then U_J will denote the submatrix formed by the rows of U whose index set is defined by J. Stewart [13] provided an estimate for ρ_2 by proving the following:

$$\rho_2 \leq \min_{\emptyset \neq J \subseteq \{1,2,\ldots,n\}} \sigma_{\min}(U_J),$$

where $\sigma_{\min}(U_J)$ denotes the smallest nonzero singular value of U_J. O'Leary [12] later showed that

$$\rho_2 \geq \min_{\emptyset \neq J \subseteq \{1,2,\ldots,n\}} \sigma_{\min}(U_J);$$

hence, establishing

Theorem 2.2.

$$\rho_2 = \min_{\emptyset \neq J \subseteq \{1,2,\ldots,n\}} \sigma_{\min}(U_J).$$

We will show that when we measure the distances with the ∞-norm and take $p = \infty$ in the definition of the set R_p, the resulting distance ρ_∞ can be easily related to the iteration complexity of interior–point methods.

Let $r \in \mathbb{R}^n$, $r \notin N$. Define

$$y := A^T(AD^2A^T)^{-1}AD^2r, \text{ note } y \neq 0. \tag{1}$$

We have

$$r = y + r - y$$

$$\frac{r}{\|y\|_\infty} = \frac{y}{\|y\|_\infty} + \frac{r-y}{\|y\|_\infty}.$$

Now, note $\frac{y}{\|y\|_\infty} \in R_\infty$ and $AD^2(r-y) = 0$; so, $\frac{r-y}{\|y\|_\infty} \in N$. Hence, we have

$$\frac{\|r\|_\infty}{\|y\|_\infty} = \left\| \frac{y}{\|y\|_\infty} + \frac{r-y}{\|y\|_\infty} \right\|_\infty \geq \rho_\infty.$$

From which we get

$$\|y\|_\infty \leq \frac{\|r\|_\infty}{\rho_\infty}. \tag{2}$$

Getting from (1) to (2) we used the analysis given by Vavasis [16]. For a properly chosen r and D, inequality (2) provides a new way of estimating the coefficients of the second order terms in the complexity analysis of interior–point methods. We will confine ourselves to the analysis of primal–dual algorithms.

Given $v \in \mathbb{R}^n$, and A and D as before, v_p denotes the orthogonal projection of v onto the null space of AD and $v_q := v - v_p$ denotes the orthogonal projection of v onto the range of DA^T. Let $\kappa(D)$ be the condition number of D. With these definitions we have:

Lemma 2.3.

$$\|v_p\|_\infty \|v_q\|_\infty \leq \frac{2\kappa^2(D)\|v\|_\infty^2}{\rho_\infty^2}.$$

Proof: Choosing y as in (1), we get

$$\begin{aligned} Dy &= DA^T(AD^2A^T)^{-1}AD^2r \\ &= (Dr)_q. \end{aligned}$$

If $\|Dr\|_\infty$ is zero then there is nothing to prove. Otherwise, we have

$$\begin{aligned} \frac{\|(Dr)_q\|_\infty}{\|Dr\|_\infty} &= \frac{\|Dy\|_\infty}{\|Dr\|_\infty} \\ &\leq \frac{\kappa(D)\|y\|_\infty}{\|r\|_\infty} \\ &\leq \frac{\kappa(D)}{\rho_\infty}. \end{aligned}$$

To get the last inequality we used (2). Now, we let $v = Dr$, to get

$$\|v_q\|_\infty \leq \frac{\kappa(D)\|v\|_\infty}{\rho_\infty}.$$

Since $v_p + v_q = v$, we get

$$\begin{aligned} \|v_p\|_\infty \|v_q\|_\infty &= \|v_q\|_\infty \|v - v_q\|_\infty \\ &\leq \|v_q\|_\infty \left(\|v\|_\infty + \|v_q\|_\infty\right) \\ &\leq \frac{\kappa(D)\|v\|_\infty^2}{\rho_\infty} + \frac{\kappa^2(D)\|v\|_\infty^2}{\rho_\infty^2} \\ &\leq \frac{2\kappa^2(D)\|v\|_\infty^2}{\rho_\infty^2}. \end{aligned}$$

The last inequality uses the fact that $\rho_\infty \leq 1$ and $\kappa(D) \geq 1$. □

3 Analysis of Iteration Complexity

The following results (Lemma 3.1 - 3.2) are quite standard in the primal-dual framework (see for instance Kojima et al. [4], Monteiro and Adler [10], Mizuno, Todd and Ye [9]).

Lemma 3.1.
(a) $x_j(\alpha)s_j(\alpha) = (1-\alpha)x_j s_j + \alpha\gamma\mu + \alpha^2 dx_j(\gamma)ds_j(\gamma)$.

(b) $\mu(\alpha) := x(\alpha)^T s(\alpha)/n = [1-\alpha(1-\gamma)]\mu$.

Lemma 3.2.
$(x,s) \in \mathcal{N}(\pi)$ *and* $\alpha \leq \min\left\{\frac{(1-\pi)\gamma\mu}{\|dX(\gamma)ds(\gamma)\|_\infty}, 1\right\}$ *implies* $(x(\alpha), s(\alpha)) \in \mathcal{N}(\pi)$.

We will show that the second-order term $\|dXds\|_\infty$ can be bounded by a multiple of $\frac{\kappa^2(D)\mu}{\rho_\infty^2}$. This is yet a new way of estimating the second-order terms which enables us to present a complexity analysis giving bounds that can be independent of n.

Theorem 3.1. *A generic primal–dual interior–point algorithm using wide neighbourhoods terminates in* $O(\frac{\kappa^2(D)t}{\rho_\infty^2})$ *iterations.*

Proof: Define $v := (X^{1/2}S^{1/2} - \gamma\mu X^{-1/2}S^{-1/2})e$. Then $dX(\gamma)ds(\gamma) = V_p v_q$. We have

$$\begin{aligned} \|dX(\gamma)ds(\gamma)\|_\infty &\leq \|v_p\|_\infty \|v_q\|_\infty \\ &\leq \frac{2\|v\|_\infty^2 \kappa^2(D)}{\rho_\infty^2}. \end{aligned}$$

To get the last inequality, we used Lemma 2.3. Now we can bound $\|v\|_\infty^2$:

$$\begin{aligned} \|v\|_\infty^2 &= \max_j \left\{ x_j s_j - 2\gamma\mu + \frac{\gamma^2\mu^2}{x_j s_j} \right\} \\ &\leq (2-\pi)\mu - 2\gamma\mu + \frac{\gamma^2\mu}{\pi} \\ &= \mu\left[2(1-\gamma) - \pi + \frac{\gamma^2}{\pi}\right]. \end{aligned}$$

Define $C := \left[2(1-\gamma) - \pi + \frac{\gamma^2}{\pi}\right]$. Since π and γ are in $(0,1)$ constants, C is a constant depending only on π and γ; for instance, if $\pi \geq \gamma/2$ then $C < 2$. So, $\|dX(\gamma)ds(\gamma)\|_\infty \leq$

$2C\kappa^2(D)\mu/\rho_\infty^2$. By Lemma 3.2, the step size α can be taken as at least $\frac{(1-\pi)\gamma\rho_\infty^2}{2C\kappa^2(D)}$. Hence, by Lemma 3.1.(b), the duality gap decreases at least by a fraction of $\frac{(1-\pi)\gamma(1-\gamma)\rho_\infty^2}{2C\kappa^2(D)}$ at each iteration. Therefore, in $O(\frac{\kappa^2(D)t}{\rho_\infty^2})$ iterations we must have $x^T s < 2^{-t}$. □

3.1 Constant–Potential Affine–Scaling Algorithm

In this subsection we illustrate how primal–dual affine–scaling algorithm (see Monteiro, Adler and Resende [11]) will terminate in at most $O(\kappa^2(D)t^2/\rho_\infty^2)$ iterations. There are many ways of choosing a step size (see for instance Kojima, Megiddo, Noma and Yoshise [5]). To simplify the analysis, we will focus on an algorithm that keeps a potential function value fixed (hence the name constant–potential) to determine the step size (see Mizuno and Nagasawa [8], and Tunçel [15]). Since our analysis is based on the $\infty-$norm neighbourhoods, we will use the following potential function to determine the step size:

$$\psi(x,s;q) \;:=\; q\log\left(\frac{x^T s}{n}\right) - \log\left(\min\left\{\frac{\min\{x_j s_j\}}{\mu}, 2 - \frac{\max\{x_j s_j\}}{\mu}\right\}\right).$$

Using Theorem 4.1 from [15] for $q := 1/2t$ and $M := 4\kappa^2(D)/\rho_\infty^2$, we have the following result:

Theorem 3.2. *A constant–potential primal–dual affine–scaling algorithm using the potential function ψ terminates in $O(\frac{\kappa^2(D)t^2}{\rho_\infty^2})$ iterations.*

Proof: Define $v := X^{1/2}S^{1/2}e$. Then $dX(0)ds(0) = V_p v_q$. We have

$$\begin{aligned}\|dX(0)ds(0)\|_\infty &\leq \|v_p\|_\infty \|v_q\|_\infty \\ &\leq \frac{2\kappa^2(D)\|v\|_\infty^2}{\rho_\infty^2}\end{aligned}$$

To get the last inequality, we used Lemma 2.3. Now we can bound $\|v\|_\infty^2$:

$$\begin{aligned}\|v\|_\infty &= \max_j\{x_j s_j\} \\ &\leq (2-\pi)\mu.\end{aligned}$$

So, $\|dX(\gamma)ds(\gamma)\|_\infty \leq 4\kappa^2(D)\mu/\rho_\infty^2$. By Theorem 4.1 of [15] (taking $q := 1/2t$ and $M := 4\kappa^2(D)/\rho_\infty^2$) we conclude that $O(\frac{\kappa^2(D)t^2}{\rho_\infty^2})$ iterations suffices. □

Note that $\kappa(D)$ may be scale dependent; but in the following section we give an analysis which provides a lower bound on ρ_∞ in terms of the number of equality constraints, m, and the entries of A. Henceforth we will assume that all the entries of A are integral which also makes the overall measure of complexity invariant under scaling.

4 Analysis for ρ_∞

Let columns of V form an orthonormal basis for the null space of A. Then ρ_∞ is the optimal objective function value of the following optimization problem:

$$\begin{aligned} (OPT_1) \quad \inf \quad & \|D^{-2}Vv - A^Tw\|_\infty \\ & D \in \mathcal{D}, \qquad v \in \mathbb{R}^{n-m}, \quad w \in \mathbb{R}^m \\ & \|A^Tw\|_\infty = 1. \end{aligned}$$

For any choice of $v \in \mathbb{R}^{n-m}$ and $w \in \mathbb{R}^m$ such that $\|A^Tw\|_\infty = 1$, the signs of the entries of the vectors Vv and A^Tw must disagree on a set of indices J such that $\emptyset \neq J \subset \{1, 2, \ldots, n\}$. Otherwise, we can pick $D \in \mathcal{D}$ to make $D^2Vv - A^Tw = 0$ (which contradicts Theorem 2.1). For a given pair of v and w the best D is found by setting $d_{ii}^2 := \frac{(Vv)_i}{(A^Tw)_i}$ for $i \notin J$ (note that if $i \notin J$ and $(A^Tw)_i = 0$ then any d_{ii} will do) and arbitrarily large if $i \in J$. Given $x \in \mathbb{R}^n$, let $sign(x)$ be the vector in $\{-1, 0, 1\}^n$ such that $(sign(x))_j = -1 \iff x_j < 0$; $(sign(x))_j = 0 \iff x_j = 0$; $(sign(x))_j = 1 \iff x_j > 0$. Let $\{v^k, w^k, D_k\}$ be a sequence of feasible solutions of (OPT_1) converging to (v^*, w^*, D^*) such that $\inf \|D_k^{-2}Vv^k - A^Tw^k\|_\infty$ converges to optimal objective function value of (OPT_1). So, if the signs of Vv^* and A^Tw^* disagree on a set of indices J then the optimal objective function value of (OPT_1) is $\|(A^Tw^*)_J\|_\infty$. Let σ^* be the sign vector of A^Tw^*, i.e., $\sigma^* := sign(A^Tw^*)$. Then ρ_∞ is the optimal objective function value of the following optimization problem:

$$\rho_\infty \;=\; \inf_{\substack{w\,:\,\|A^Tw\|_\infty = 1 \\ sign(A^Tw) = \sigma^*}} \|(A^Tw)_J\|_\infty.$$

Note that the infimum given above is bounded away from zero, by the previous arguments. So, far we followed O'Leary [12] (O'Leary uses similar arguments for the 2-norm to get Theorem 2.2). Equivalently, we can solve (OPT_2) to get $\frac{1}{\rho_\infty}$ as the optimal objective function value of the objective function:

$$\begin{aligned} (OPT_2) \quad \max \quad & \|A^Tw\|_\infty \\ & sign(A^Tw) \;=\; \sigma^*, \\ & \|(A^Tw)_J\|_\infty \;\leq\; 1. \end{aligned}$$

The sign constraint, $sign(A^T w) = \sigma^*$, can be written as a system of linear inequalities; for instance, there exists an $\epsilon > 0$ (ϵ can be chosen as $\min\{|(A^T w^*)_j| : (A^T w^*)_j \neq 0\}$) such that the constraint

$$sign(A^T w) = \sigma^*$$

can be replaced by the linear inequality constraints:

$$\begin{aligned}(A^T w)_j &\geq \epsilon \text{ if } \sigma_j^* = 1,\\ (A^T w)_j &= 0 \text{ if } \sigma_j^* = 0,\\ (A^T w)_j &\leq -\epsilon \text{ if } \sigma_j^* = -1.\end{aligned}$$

Now, it is clear that (OPT_2) can be solved by solving a sequence of linear programming problems over the same feasible solution set. Let $F(\sigma^*, J)$ denote the set of feasible solutions for the optimization problem (OPT_2) when the sign constraint is expressed as a system of linear inequalities as described above. Let a^T be a row of A^T for which $|a^T w^*|$ equals the optimal objective function value of (OPT_2) (note $w^* \in F(\sigma^*, J)$). So, w^* solves the following linear programming problem:

$$\begin{aligned}(LP_1) \quad \max \quad & \tilde{a}^T w\\ & w \in F(\sigma^*, J),\end{aligned}$$

where $\tilde{a}$ is either a or $(-a)$. (LP_1) is equivalent to

$$\begin{aligned}(LP_2) \quad \max \quad & \eta\\ & w \in F(\sigma^*, J),\\ & \tilde{a}^T w - \eta \geq 0.\end{aligned}$$

Clearly, $F(\sigma^*, J)$ is a polyhedron in $\mathbb{R}^m$. Our first claim is that $F(\sigma^*, J)$ is pointed and hence has extreme points. Suppose not. Then there must exist a line L contained in $F(\sigma^*, J)$. This implies, there exists $d \in \mathbb{R}^m$, $d \neq 0$ such that $A^T d = 0$ (which contradicts the assumption that A has full row rank). Since (LP_2) has an optimum solution and $F(\sigma^*, J)$ is a pointed polyhedron, (LP_2) must have a basic feasible solution that is optimal. Let η^* correspond to such a basic feasible solution. We immediately get (using Cramer's rule)

$$\begin{aligned}\eta^* &= \left|\frac{subdet[A^T, f(\epsilon)]}{subdet[A^T]}\right|\\ &\leq m \max \left|subdet[A^T]\right|,\end{aligned}$$

where $f(\epsilon)$ is the vector representing the right–hand–side values in (LP_2) (entries of $f(\epsilon)$ are 0, ± 1, and ϵ). So, defining $\Delta := \max \left| subdet[A^T] \right|$, we get

Theorem 4.3. *Let $A \in Z^{m \times n}$ such that rank$(A) = m$ and define $\Delta := \max \left| subdet[A^T] \right|$ and ρ_∞ as before. Then*

$$\frac{1}{\rho_\infty} \;\leq\; m\,\Delta.$$

H. Wolkowicz pointed out to us that the estimation of ρ_∞ can be directly obtained from a result of Ben-Tal and Teboulle [1]. Indeed, they prove that the weighted least squares solution of an overdetermined system of linear equations lies in the convex hull of solutions to square subsystems of the original system.

Now, using Theorem 3.1 and 3.2 we have

Corollary 4.1. *A generic primal–dual interior–point algorithm using an $\infty-$norm neighbourhood terminates in $O(m^2\kappa^2(D)\Delta^2\ t)$ iterations.*

Corollary 4.2. *A constant–potential primal–dual affine–scaling algorithm using the potential function ψ terminates in $O(m^2\kappa^2(D)\Delta^2\ t^2)$ iterations.*

5 Conclusion

We provided a different way of estimating the second order terms for primal–dual interior–point algorihtms. When the number of constraints, m, is constant $O(1)$ the iteration bound depends (pseudo-polynomially) on the numbers in the coefficient matrix A and the scaling D. Indeed, no matter what the initial solution or the problem is as the iterates approach the optimal set, $\kappa^2(D)$ will go to infinity. So, the analysis presented here does not aim at replacing the existing analysis, but rather propose a new view which might help in explaining the practical performance of primal–dual interior–point algorithms. In particular, the results presented here should be interpreted for each iteration together with the existing analysis (rather than being interpreted as a global analysis).

Existing complexity analyses for the first algorithm prove that the step size is at least C_1/n for all iterations, where C_1 is a constant independent of the problem instance. Here we proved that the step size is at least $C_2/(m^2\kappa^2(D)\Delta^2)$, where C_2 is again a constant independent of the problem instance. So, we have (for all iterations)

$$\alpha \;\geq\; \max\left\{\frac{C_1}{n},\ \frac{C_2}{m^2\kappa^2(D)\Delta^2}\right\}.$$

We would finally like to remark that as it is pointed here, this analysis can be useful in explaining the good behaviour only for problems with small m and small Δ and for the iterates that have small $\kappa(D)$. However, as $\kappa(D)$ grows very large we know that the iterates must be getting close the set of optimal solutions and at this phase the behaviour of the algorithm can better be explained by asymptotic superlinear and quadratic convergence properties.

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