# Some Applications of Symmetric Cone Programming in Financial Mathematics

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To appear in: *Transactions on Operational Research*
(revised: September 2006)

**Abstract**

We review a few of the relativitely recent developments in cone programming that seem to have important applications in financial planning. In particular, we go over the semidefinite programming representation of a polynomial inequality as given by Nesterov. We mention some relevant references which show the power of cone programming in portfolio optimization. Then we turn to the recent work of Lobo et al. which showed how to use Second Order Cone Programs to model portfolio optimization problems with transaction costs. We extend their model to a multi-period decision making situation and we allow cash infusions into the portfolio every period. We conclude with some computational experiments using real data.

**Keywords:** convex optimization, semidefinite programming, second order cone, financial mathematics, portfolio optimization

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# 1 Introduction

One of the most important ingredients of successful applications of optimization is the proper forecasting of uncertain parameters of the problem. In many cases we deal with uncertainties in data and parameters by estimating high-order expected behavior. Indeed, in most cases when there is significant uncertainty, the expected value of a parameter is a poor way to represent the real problem as a mathematical, deterministic optimization problem.

Research in optimization under uncertainty has been flourishing during the last two decades. The main approaches are covered under the terms: *Stochastic Programming* and *Robust Optimization*.

Applications in the area of optimization under uncertainty have also been increasing in number as well as in practical impact. One of the most popular and visible applications is in the *Financial Markets*.

This paper is geared towards financial applications. In such application many restrictions on the variables based on the variance in the data can be expressed as variable vectors lying in nice convex cones. Many other restrictions based on higher-order moments can be expressed as certain scalar polynomials being nonnegative for every choice of its argument. Such positivity requirements can be equivalently expressed as certain variable matrix being symmetric positive definite.

In this paper, we first review these fundamental representation techniques (see Section 1.1 and Section 2). All the convex cones used in our formulations are unified under a well-behaved set of convex cones called *symmetric cones* (see the next section for a definition). We then turn to the financial applications and introduce portfolio optimization (see Section 1.2 and 2.2).

In Section 3, we focus on the portfolio optimization model proposed by Lobo et al. [14]. We (slightly) extend their model in two ways:

1. We allow for cash infusions in each planning period.
2. We allow for multiple periods.

In Section 4, we compare the performance of Lobo et al. model and our modifications using real data and computational experiments.

## 1.1 Symmetric Cone Optimization

Convex optimization problems, the problems of minimizing a convex function over a convex set, make up a very large and relatively well-behaved class of optimization problems.

Currently, many of the most successful approaches in the theory and algorithms treat convex optimization problems in *conic form.* A popular name for such form is *cone programming problems.*

Cone programming problem is the problem of optimizing (minimizing or maximizing) a linear function of finitely many real variables subject to the vector of real variables lying in the intersection of a prescribed affine subspace and a convex cone. Below, we introduce some notation and describe the cone programming problems in terms of the notation.

Given a linear operator $\mathcal{A} : \mathbb{R}^n \mapsto \mathbb{R}^m$ that is surjective, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, consider the cone programming problem

$$\begin{array}{lrcl} (P) & \inf & \langle c, x\rangle & \\ & \mathcal{A}(x) & = & b, \\ & x & \in & K, \end{array}$$

where $K \subset \mathbb{R}^n$ is a closed convex cone.

Note that $K \subseteq \mathbb{R}^n$ is a *cone* if $\forall x \in K$ and $\forall \lambda > 0$, $\lambda x \in K$.

Under very mild assumptions, all convex optimization problems can be formulated as cone programming problems, see for instance [18, 23].

Any linear operator $\mathcal{A} : \mathbb{R}^n \mapsto \mathbb{R}^m$ can be represented by $m$ elements of $\mathbb{R}^n$. That is, there exist $A_1, A_2, \ldots, A_m \in \mathbb{R}^n$ such that:

$$[\mathcal{A}(x)]_i = \langle A_i, x\rangle, \ \forall i \in \{1, 2, \ldots, m\}.$$

Then $\mathcal{A}$ being surjective is equivalent to $\{A_1, A_2, \ldots, A_m\}$ being linearly independent. Indeed the latter condition can be easily checked.

We denote by $\mathbb{S}^n$ the space of $n \times n$ symmetric matrices over the reals. $\mathbb{S}^n_+$ denotes the cone of symmetric, positive semidefinite matrices in $\mathbb{S}^n$. In the above optimization problem, setting

$$K := \mathbb{S}^{n_1}_+ \oplus \mathbb{S}^{n_2}_+ \oplus \cdots \oplus \mathbb{S}^{n_r}_+$$

yields a *Semidefinite Programming* (SDP) problem. For $x, s \in \mathbb{S}^n$, we write $x \preceq s$ to mean $(s - x) \in \mathbb{S}^n_+$.

Many financial applications can be treated via second order cones. An $(n+1)$ dimensional *second order cone* is defined as

$$SOC^n := \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^n : \ x_0 \geq \|x\|_2 \right\}.$$

We also call the cone $\mathcal{L}(SOC^n)$ a second order cone, for every nonsingular linear transformation $\mathcal{L} : \mathbb{R}^{n+1} \mapsto \mathbb{R}^{n+1}$. For example the cone

$$\text{cl} \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^n : \ x_0 x_1 \geq x_2^2 + x_3^2 + \cdots + x_n^2, \ x_0 > 0 \right\}$$

is equal to the image of $SOC^n$ under such a nonsingular linear transformation (cl$(\cdot)$ denotes the closure).

If we use

$$K := SOC^{n_1} \oplus SOC^{n_2} \oplus \cdots \oplus SOC^{n_k},$$

then we have a *Second Order Cone Programming* (SOCP) problem.

Note that every cross-section of $SOC^n$ with $x_0 := \alpha > 0$ gives an Euclidean Ball in $\mathbb{R}^n$. This cone is also called *ice-cream cone*, *light cone* or *Lorentz cone*.

The cone $K \subseteq \mathbb{R}^n$ is defined to be *pointed* if $K \bigcap (-K) = \{0\}$, which is equivalent to say that $K$ contains no lines.

Given $K \subseteq \mathbb{R}^n$, the *dual* cone of $K$ is

$$K^* \ := \{s \in \mathbb{R} : \ \langle x, s \rangle \geq 0, \forall x \in K\}.$$

Suppose that cone $K \subseteq \mathbb{R}^n$ has nonempty interior. Then:

- $K$ is *homogeneous* if for every pair $x$, $y \in \text{int}(K)$ (denoting the interior of $K$), there exists a nonsingular and linear transformation $\mathcal{L}$ such that , $\mathcal{L}(K) = K$ and $\mathcal{L}(y) = x$.
- $K$ is *self-dual* if there exists an inner product under which $K = K^*$.
- $K$ is *symmetric* if it is homogeneous and self-dual.

Now, we list some fundamental results on some of the elementary properties of convex cones.

**Theorem 1.1.** *Let $K \subseteq \mathbb{R}^n$. If $K$ is a pointed, closed convex cone with nonempty interior, then so is $K^*$.*

**Theorem 1.2.** *Let $K \subseteq \mathbb{R}^n$. Then $K$ is a closed convex cone iff $K^{**} = K$.*

**Corollary 1.1.** *Let $K \subseteq \mathbb{R}^n$. Then $K$ is a pointed, closed convex cone with nonempty interior iff $K^*$ is.*

**Theorem 1.3.** *Let $K \subseteq \mathbb{R}^n$ be a closed convex cone. Then*

$$\text{int}(K) \ = \{x \in K : \ \langle x, s \rangle > 0, \ \forall s \in K^* \backslash \{0\}\}.$$

Both $\mathbb{S}^n_+$ and the $SOC^n$ are pointed, closed convex cones with nonempty interior. Moreover, they are homogeneous and self-dual; hence, they are symmetric.

For the rest of the paper, the main convex cones we deal with will be $\mathbb{S}^n_+$ and $SOC^n$. Most of the models will only use second order cones. However, if we want to include more complicated constraints in our model, such as simple polynomial inequalities stipulating that a scalar polynomial be nonnegative, then we would utilize the results of Section 2 and the cone of symmetric positive definite matrices.

## 1.2 Portfolio Optimization

In his seminal paper [16], Markowitz introduced mathematical modeling techniques to solve the portfolio selection problem for a large private investor or an institutional investor. Markowitz's work provided a starting point for most of the work in the area of portfolio optimization.

In the book [15], Markowitz analyzed the Mean-Variance(M-V) portfolio selection problem. The problem is modelled as a parametric quadratic programming problem with general linear inequality constraints:

$$\begin{array}{ll} \min & f(x) := -t\mu^T x + \frac{1}{2}x^T V x \\ \text{s.t.} & Ax \le b \\ & c_1 \le x \le c_2, \end{array}$$

where $\mu$ is an $n$-vector of expected returns, $V$ is an $n \times n$ covariance matrix, $x$ is an $n$-vector of amount of asset holdings, $A$ is an $m \times n$ matrix, $b$ is an $m$-vector, $t$ is a parameter (usually nonnegative) and $c_1, c_2 \in \mathbb{R}^n_+$ are bounds on the holdings. Note that the objective function is quadratic and the constraints are linear. The function $f(x)$ is called *utility function* in [15].

An obvious drawback of the basic Markowitz model is that it needs the mean $\mu$ and variance $V$ computed (estimated), and then uses $\mu$, $V$ in a deterministic quadratic programming setting. During the last decade, a new area called *robust optimization* provided a very intriguing approach to uncertainty in optimization problems [2].

Instead of using a single estimate of an uncertain part of the data (or parameters) as in the Mean-Variance(M-V) portfolio selection model, robust optimization approach describes a set of possible values for that uncertain data (or parameters), which is called the *uncertainty region.* Then the robust portfolio selection problems try to find the optimal strategy under the assumption that the worst possible scenario in the uncertainty region can happen [1, 9, 21].

Many financial investment companies use the notion of *market driving factors* in their forecasting techniques. They choose a small set of indicators (say, order of 10) that reflect the basic tendencies of the financial market. Below, vector $f$ is the vector representing such market driving factors.

Define the return

$$r := \mu + V^T f + \varepsilon,$$

where $\mu$ is the vector of expected returns, $f \sim \aleph(0, F)$ is the vector of returns of the factors that drive the market, $V$ is the matrix of factor loadings of the $n$ assets and $\varepsilon \sim \aleph(0, D)$ is the vector of residual returns, where $x \sim \aleph(a, A)$ denotes that $x$ is a multivariate normal random variable with mean vector $a$ and covariance matrix $A$. Then

$$r \sim \aleph(\mu, V^T F V + D).$$

Goldfarb and Iyengar [9] noted that the eigenvalues of the residual covariance matrix $D$ are typically much smaller than those of the covariance matrix $V^T F V$ implied by the factors. Thus, the covariance matrix of return $r$ is usually dominated by $V^T F V$. In some cases, the lower-rank property of $F$ and $V$ can reduce the complexity of calculation for the covariance matrix.

We do not employ the robust portfolio selection approach in this paper. However, as we point out in Section 3, our formulations can be extended to the robust optimization model.

Our portfolio optimization problems with fixed transactions costs are cast as convex optimization problems. As we review in Section 3, convex portfolio optimization problems include those with linear transactions costs, margin and diversification constraints, and limits on variance and on shortfall risk. Recent theoretical and computational developments in SDP and SOCP, especially interior-point methods, provide us with fast algorithms, good modeling techniques and robust software for many nonlinear convex optimization problems.

# 2 SDP representation of positive polynomials

Moment problems, involving first $k$ order moment of random variables, have proven to be applicable to different areas such as computational finance, operations research and stochastic optimization. Employing duality theory and other representation tools, SDP can be used to represent the moment-type optimization problems. In this section, we will review the key connections between moment problems and SDP representations. In Section 2.1, some fundamental theorem and results are outlined. In Section 2.2, some examples of the financial applications are introduced.

## 2.1 Mathematical Foundations

Nesterov [17] showed that the set of coefficients of a degree $n$ univariate polynomial, which generate polynomials with non-negative values for every choice of the argument can be represented as an intersection of the positive semidefinite cone with an affine space. Here, we outline this result and some related theory.

$\mathcal{P}^n$ denotes the $(n+1)-$dimensional vector space. $p \in \mathcal{P}^n$ is written as

$$p(t) \; = \; \sum_{k=0}^{n} p_k t^k.$$

Defining

$$\tau_n \; := \; (1, t, t^2, \ldots, t^n)^T \in \mathcal{P}^n,$$

we have $p(t) = \langle p, \tau_n \rangle$.

The cone of non-negative polynomials is the cone of all coefficient vectors $p$ for which the underlying polynomial is non-negative for all values of $t$. (It is easy to show that such a polynomial must be of even degree.) That is

$$K_{2n} \; := \; \left\{ p \in \mathcal{P}^{2n} : \; p(t) \geq 0, \; \text{for all } t \in \mathbb{R} \right\}.$$

The following elementary result is well-known.

**Theorem 2.1.** *(i) A polynomial of odd degree can not be nonnegative on $\mathbb{R}$.(That is, if $p(t) \geq 0$, $\forall t \in \mathbb{R}$, then the degree of $p(t)$ is even.)*

*(ii) A polynomial is nonnegative on the whole of $\mathbb{R}$ iff the polynomial can be expressed as a sum of squares of polynomials.*

*Proof.*

(i) Suppose $p(t)$ has odd degree, $\deg(p) = 2k+1$ for some $k \in \mathbb{Z}_+$. Then, the coefficient of $t^{2k+1}$ is nonzero by definition. Let us denote it by $p_{2k+1}$.

If $p_{2k+1} > 0$, then $p(t) \to -\infty$, as $t \to -\infty$.

If $p_{2k+1} < 0$, then $p(t) \to -\infty$, as $t \to +\infty$.

Therefore, $p(t)$ is not nonnegative on the whole $\mathbb{R}$ when $\deg(p)$ is odd.

(ii) If $p(t)$ can be expressed as a sum of squares of polynomials, then clearly $p(t) \geq 0$ for all $t \in \mathbb{R}$.

Suppose that $p(t) \geq 0$ for all $t \in \mathbb{R}$. Let $\lambda_i$ be its real roots with multiplicity $m_i$, for $i \in \{1, \ldots, r\}$, and $a_j + \iota b_j, a_j - \iota b_j$ be its complex roots for $j \in \{1, \ldots, h\}$, where $\iota := \sqrt{-1}$. Further let $2k$ denote the degree of $p$. Then

$$\begin{aligned} p(t) &= p_{2k} \prod_{i=1}^{r} (t-\lambda_i)^{m_i} \prod_{j=1}^{h} \left[(t-a_j)^2 + b_j^2\right] \\ &= p_{2k}[(t-\lambda_1)^{m_1/2}(t-\lambda_2)^{m_2/2} \cdots (t-a_1) \cdots (t-a_h)]^2 \\ &\quad + p_{2k}[(t-\lambda_1)^{m_1/2} \cdots (t-a_{h-1})b_h]^2 \\ &\quad + p_{2k}[(t-\lambda_1)^{m_1/2} \cdots (t-a_{h-2})b_{h-1}b_h]^2 \\ &\quad + \cdots \\ &= \sum_{i=0}^{k} \left( \sum_{j=0}^{k} c_{ij} t^j \right)^2, \end{aligned}$$

where $c_{ij}$ is the coefficient for $x^j$ in the $i$th polynomial of the sum. We expressed $p(t)$ as a sum of squares of polynomials.

■

As it will become clearer, it seems more natural to treat cones like $K_{2n}$ as the squares of some other object. There are many versions of this. For instance, we may be interested only in non-negative values of $t$ or only in $t \in [0, 1]$, etc. Generalizations of this subject include representations of sets described as the solution set of systems of polynomial inequalities on several variables. For such studies, the *sum of squares representation* approach has been extremely fruitful. (See for instance, [11, 12, 20]; for systems of quadratic inequalities, see [10].)

Let $E_k$ denote the $k$th $(n+1)\times(n+1)$ cross diagonal matrix:

$$E_0 := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad E_1 := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$E_2 := \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \dots, E_{2n} := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

**Theorem 2.2.** *[17]*

$$K_{2n} = \left\{ p \in \mathcal{P}^{2n} : \ p_k = \langle X, E_k \rangle, \ k \in \{0,1,\dots,2n\}, \ X \succeq 0 \right\}.$$

*Proof.*

Let $T := \left\{ p \in \mathcal{P}^{2n} : \ p_k = \langle X, E_k \rangle, \ k \in \{0,1,\dots,2n\}, \ X \succeq 0 \right\}$, we will prove that $T = K_{2n}$.

(i) For $\forall p \in T$, $X := (x_{ij})$, $p_k = \langle X, E_k \rangle = \sum_{i+j=k+2} x_{ij}$,

$$\begin{aligned} p(t) &= \sum_{k=0}^{2n} p_k t^k = \sum_{k=0}^{2n} \sum_{i+j=k+2} x_{ij} t^k \\ &= \tau_n^T X \tau_n. \end{aligned}$$

As $X \succeq 0$, we have $\tau_{2n}^T X \tau_{2n} \geq 0$ for $\forall t \in \mathbb{R}$.
Hence, $p(t) \geq 0$ for $\forall t \in \mathbb{R}$. Therefore, $T \subseteq K_{2n}$.

(ii) Using the same notation as in the proof of the Theorem 2.1, for $\forall p \in K_{2n}$, $p(t) = \sum_{i=0}^n (\sum_{j=0}^n c_{ij} t^j)^2$.
Let $C$ be the $(n+1)\times(n+1)$ matrix whose $(ij)$th entry is $c_{ij}$.
Define $X := C^T C$, note that $\sum_{j=0}^n c_{ij} t^j = (C\tau_n)_i$, we obtain

$$\begin{aligned} p(t) &= \tau_n^T C^T C \tau_n = \tau_n^T X \tau_n \\ &= \sum_{k=0}^{2n} \left( \sum_{i+j=k+2} x_{ij} \right) t^k. \end{aligned}$$

Thus, $p_k = \sum_{i+j=k+2} x_{ij} = \langle X, E_k \rangle$, and $X = C^T C \succeq 0$.
So $p \in T$. Therefore, $K_{2n} \subseteq T$.

Combining (i) and (ii), we conclude that $K_{2n} = T$.

■

The dual of the cone $K_{2n}$ also has a nice description as the next result shows (it can be proved using Theorem 2.2).

**Lemma 2.1.** *[17]* $(K_{2n})^* = \{s \in \mathcal{P}^{2n} : \sum_{k=0}^{2n} s_k E_k \succeq 0\}$.

Using Theorem 1.3 and Lemma 2.1, we can establish the following fact.

**Lemma 2.2.** *[17]* $\mathrm{int}(K_{2n}) = \{p \in \mathcal{P}^{2n} : p_{2n} > 0, p(t) > 0, \forall t \in \mathbb{R}\}$.

Using Lemma 2.2 and Corollary 1.1, we can establish the next fact.

**Theorem 2.3.** *[17] $K_{2n}$ and $K_{2n}^*$ are pointed, closed convex cones with nonempty interiors.*

Here, we give a direct argument proving that $K_{2n}^*$ has nonempty interior.

**Proposition 2.1.** *There exists $\bar{s} \in \mathcal{P}^{2n}$ such that $\sum_{k=0}^{2n} \bar{s}_k E_k \succ 0$.*

*Proof.* Let $t_0 < t_1 < \ldots < t_n \in \mathbb{R}$. Define

$$\bar{s} \ := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ t_0 & t_1 & \ldots & t_n \\ t_0^2 & t_1^2 & \ldots & t_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_0^{2n} & t_1^{2n} & \ldots & t_n^{2n} \end{pmatrix} e \in \mathcal{P}^{2n}.$$

Take $\forall p \in \mathcal{P}^{2n} \neq 0$,

$$p^T(\sum_{k=0}^{2n} s_k E_k)p = \sum_{k=0}^{2n} s_k p^T E_k p = \sum_{i=0}^{n} p(t_i)^2 > 0,$$

because such a polynomial $p$ can have at most $n$ real roots unless $p$ is identically zero. Hence, $\bar{s} \in \mathrm{int}(K_{2n}^*)$, $K_{2n}^*$ has nonempty interior and $\sum_{k=0}^{2n} s_k E_k \succ 0$.

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The next result is useful in applying interior-point methods.

**Theorem 2.4.** *[17] If $p \in \operatorname{int}(K_{2n})$ then the set*

$$\{X \succeq 0 : \langle X, E_k \rangle = p_k, \quad \textit{for all } k \in \{0, 1, \ldots, 2n\}\}$$

*is bounded and there exists $X \succ 0$ such that $\langle X, E_k \rangle = p_k$ for all $k \in \{0, 1, \ldots, 2n\}$.*

*Proof.*

Fix $p \in \operatorname{int}(K_{2n})$, and let $\bar{X} \in \{X \succeq 0 : \langle X, E_k \rangle = p_k, \quad \text{for all } k \in \{0, 1, \ldots, 2n\}\}$. Pick $\bar{s} \in \operatorname{int}(K_{2n}^*)$, and define $\bar{S} := \sum_{k=0}^{2n} \bar{s}_k E_k > 0$,

$$\langle \bar{X}, \bar{S} \rangle = \langle \bar{X}, \sum_{k=0}^{2n} \bar{s}_k E_k \rangle = \sum_{k=0}^{2n} \bar{s}_k p_k.$$

Since $p$ and $\bar{s}$ are fixed, $\sum_{k=0}^{2n} \bar{s}_k p_k = \textit{ constant} > 0$. Then for every $\bar{X}$ as above, it satisfies $\langle \bar{X}, \bar{S} \rangle = \textit{constant}$. Note that

$$\left\{ X \succeq 0 : \langle X, \bar{S} \rangle = \sum_{k=0}^{2n} \bar{s}_k p_k = \textit{ constant} \right\}$$

is compact. Therefore, $\{X \succeq 0 : \langle X, E_k \rangle = p_k, \quad \text{for all } k \in \{0, 1, \ldots, 2n\}\}$ is bounded for every $p \in \operatorname{int}(K_{2n})$.

By Lemma 2.1, $(K_{2n})^* = \{s \in \mathcal{P}^{2n} : \sum_{k=0}^{2n} s_k E_k \in \mathbb{S}_+^n\}$. By Lemma 2.1, $\exists \bar{s} \in K_{2n}^*$ such that $\sum_{k=0}^{2n} s_k E_k \in \mathbb{S}_{++}^n$. Besides, by Theorem 2.3, $K_{2n}$ and $K_{2n}^*$ are pointed, closed convex cones with nonempty interiors. Therefore by Theorem 2.2,

$$K_{2n} = K_{2n}^{**} = \{\mathbb{A}(X) : X \in \mathbb{S}_+^n\},$$

where $\mathbb{A} : \mathbb{S}^n \to \mathbb{R}^{2n+1}$ such that $[\mathbb{A}(X)]_i = \langle E_i, X \rangle$, $\forall i \in \{0, 1, \ldots, 2n\}$. Then,

$$K_{2n}^* = \{s : \mathbb{A}^*(s) \in \mathbb{S}_+^n\}.$$

($\mathbb{A}^*$ denotes the adjoint of $\mathbb{A}$, such that

$$\langle \mathbb{A}^*(s), X \rangle = \langle s, \mathbb{A}(X) \rangle, \, \forall X \in \mathbb{S}^n, s \in \mathbb{R}^{2n+1}).$$

By standard duality theory, there exists $\bar{X} \in \mathbb{S}_{++}^n$ such that $\mathbb{A}(\bar{X}) \in K_{2n}$. Then there exists $\hat{X} \in \mathbb{S}_+^n$ such that $\mathbb{A}(\hat{X}) \in \operatorname{int}(K_{2n})$.

■

Polynomials that are non-negative on the half-line, $\mathbb{R}_+$, or on an interval $[0, 1]$ can be treated similarly. Trigonometric polynomials

$$p(t) \;=\; \sum_{k=0}^{n} p_k \, (\cos t + \iota \sin t)$$

can also be dealt with in an analogous way.

A central fact related to the above theorem (and its proofs) is that a polynomial is nonnegative on the whole of $\mathbb{R}$ iff the polynomial can be expressed as a sum of squares of polynomials. These types of results go back at least a hundred years.

It has been well known that a polynomial $p$, with coefficients from $\mathbb{R}$, is nonnegative on the whole real line iff there exist polynomials $p_1$ and $p_2$ with real coefficients such that

$$p(t) = [p_1(t)]^2 + [p_2(t)]^2 .$$

If we only require that $p(t) \geq 0$, for all $t \in \mathbb{R}_+$, then there exist polynomials $p_1, p_2, p_3$, and $p_4$ such that

$$p(t) = [p_1(t)]^2 + [p_2(t)]^2 + t\left([p_3(t)]^2 + [p_4(t)]^2\right).$$

A related, interesting question goes back to Hermite (in 1894). He asked whether every polynomial $p$ of degree at most $n$, with the property

$$p(t) > 0, \qquad \forall t \in (-1, 1),$$

can be expressed as

$$p(t) = \sum_{i,j \geq 0:\, i+j \leq n} a_{ij}(1-t)^i(1+t)^j,$$

where $a_{ij} \geq 0$. It was quickly answered "no." However, Hausdorff (in 1921) proved that if the restriction $i+j \leq n$ on the maximum degree of the representing polynomial is relaxed then the answer is "yes." That is, there exist $a_{ij} \geq 0$, for all $i, j$ such that

$$p(t) = \sum_{i,j \geq 0} a_{ij}(1-t)^i(1+t)^j.$$

## 2.2 Financial Applications

Black-Scholes formula is successful in pricing derivatives under the assumption that the underlying asset follows a *Geometric Brownian Motion* and no arbitrage profit exists [5].

Let $S$ be the price of underlying asset, one typical kind of random walk that $S$ follows is *Geometric Brownian Motion*, which can be expressed as:

$$\frac{dS}{S} = \mu dt + \sigma\phi\sqrt{dt},$$

where $\mu$ is the drift rate, $\sigma$ is the volatility and $\phi \sim \aleph(0, 1)$. Moreover, there are no arbitrage opportunities when all risk-free portfolios earn the risk-free rate of return. By constructing an hedging portfolio consists of 1 share of asset and $\alpha$ share of option, the Black-Scholes differential equation can be derived as:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV,$$

where $V$ is the price of option and $r$ is risk-free interest rate.

The Black- Scholes formula was extremely influential in forming the succeeding research in the area as well as the practice of the companies' operations in the markets, for instance [4][8]. As a result, many questions arose. If we do not assume the motion of underlying asset, only some moments of the price of the asset were given, such as expected price, variance, can we get a reliable value or bound for the price of the derivatives using only no arbitrage theory?

In order to apply the no-arbitrage theory, Cox and Ross in [6] showed that it is sufficient to assume the existence of a probability distribution $\pi$ for the price of underlying asset. Under this probability measure, we can define and calculate the moments of the asset price $X$. Based on the definition and properties of moments, we can formulate an optimization model for the underlying financial problems. The model is solvable efficiently, utilizing together the developments in algorithms for SDP, and the theory of SDP representations in Section 2.1.

Bertsimas and Sethuraman [3] discussed the applications of such SDP representations in financial mathematics. One example is to maximize the expected payoff of a call option given $n$ moments $(q_1, q_2, \ldots, q_n)$ for the price of asset. Let $q_0 := 1$, then the model can be expressed as:

$$\begin{aligned} \max \quad & E_\pi[\max(0, X-k)] = \int_0^\infty \max(0, t-k)\pi(t)dt \\ \text{s.t.} \quad & E_\pi[X^i] = \int_0^\infty t^i \pi(t)dt = q_i, \ i \in \{0, 1, 2, \ldots, n\} \\ & \pi(t) \geq 0, \end{aligned}$$

where $k$ is the strike price for the call option, $X$ is the spot price for underlying asset on maturity, and $\max(0, X-k)$ is the payoff for the call option.

The dual of the problem is:

$$\begin{aligned} \min \quad & \sum_{i=0}^n y_i q_i \\ \text{s.t.} \quad & \sum_{i=0}^n y_i t^i \geq \max(0, X-k), \ \forall t \geq 0. \end{aligned}$$

Note that the constraints of dual problem can be expressed as cones of non-negative polynomials. Therefore, by Theorem 2.2 discussed before, the above optimization problem can be formulated as an instance of SDP.

# 3 Portfolio Selection Model

## 3.1 Lobo et al. Model for single-period

Considering an investment on different types of portfolio, we want to maximize the expected return, taking transaction costs into account, and subject to several kinds of constraints for feasibility.

The single-period portfolio selection model below was introduced by Lobo et al. [14]. The current holdings in $n$ assets are $w := (w_1, \dots, w_n)^T$. The amounts (in number of units, not dollars) transacted in these assets are given in the vector $x := (x_1, \dots, x_n)^T$. After the transactions, the new holdings in the portfolio is $(w + x)$. Let $\phi(x)$ denote the sum of all transaction costs. The problem can be expressed as:

$$\begin{array}{rl} \max & \bar{a}^T(w+x) \\ s.t. & p^T x + \phi(x) \leq \xi \\ & (w+x) \in \mathbf{S}, \end{array}$$

where $\bar{a}$ is the vector of expected returns on each asset, $p$ is the price for assets at the beginning of the period, $\xi$ is the cash amount invested in this period. Then $p^T x$ is the investment needed for $x$, plus the transaction costs $\phi(x)$, must be less than or equal to the budget $\xi$. $\mathbf{S}$ is the set of feasible portfolios. We will discuss a variety of transaction cost functions and portfolio constraints later.

We can also add one asset $w_{n+1}$ to express the holding of cash on hand and $x_{n+1}$ is the cash transacted during this period to involve the cash invested in this period. Then the above problem becomes:

$$\begin{array}{rl} \max & \bar{a}^T(w+x) \\ \text{s.t.} & p^T x + \phi(x) \leq \xi + w_{n+1} \\ & (w+x) \in \mathbf{S}, \end{array}$$

where $\bar{a}_{n+1} = 1$, $p_{n+1} = 1$ and $\phi_{n+1}(x_{n+1}) = 0$.

Assume that the transaction costs can be separated, $\phi(x)$ is the sum of the transaction cost associated with each asset, $\phi(x) = \sum_{i=1}^n \phi_i(x_i)$, where $\phi_i$ is the transaction cost function for asset $i$.

There are several types of functions $\phi_i(x_i)$ for real world applications, and we will focus on the linear transaction cost functions, such as

$$\phi_i(x_i) = \begin{cases} a_i^+ x_i & \text{if } x_i \geq 0 \\ -a_i^- x_i & \text{if } x_i \leq 0 \end{cases} \quad \text{or } \phi_i(x_i) = \begin{cases} 0 & \text{if } x_i = 0 \\ \beta_i^+ + a_i^+ x_i & \text{if } x_i > 0 \\ \beta_i^- - a_i^- x_i & \text{if } x_i < 0, \end{cases}$$

where $a_i^+, a_i^-$ are the different transaction costs associated with buying and selling asset $i$, and $x_i = x_i^+ - x_i^-$ with $x_i^+ \geq 0, x_i^- \geq 0$ are used to express the amount of buying and selling of the asset $i$.

In practice, the transaction costs can be nonconvex (especially when we are planning for small investors). However, we can use convex relaxation methods to approximate them.

The feasible set of portfolios **S** can be discussed in different ways, we will focus on the expression through convex constraints. With convexity, the underlying optimization problems can be solved efficiently by special software based on interior-point methods.

*Diversification constraints*, limit the amount invested on each asset $i$. Suppose, we require that no more than a fraction $\gamma$ of cash can be invested on fewer than $r$ assets so that we can avoid the concentration on any small subset of assets to hedge the risk for investment.

$$\sum_{i=1}^{r}(p \odot x)_{[i]} \leq \gamma p^T x,$$

where $p \odot x := \begin{pmatrix} p_1 x_1 \\ p_2 x_2 \\ \vdots \\ p_n x_n \end{pmatrix}$, $x_{[i]}$ denotes the $i$th largest component of $x$.

An alternative way to express the $i$th largest component is via introducing new variables $y \in \mathbb{R}^n, t \in \mathbb{R}$. In what follows, $e \in \mathbb{R}^n$ denotes the vector of all ones.

$$\begin{aligned} \gamma p^T x &\geq rt + e^T y \\ t + y_i &\geq p_i x_i \ , \qquad \forall\, i \in \{1, \ldots, n\} \\ y_i &\geq 0 \ , \qquad \forall\, i \in \{1, \ldots, n\}. \end{aligned} \tag{1}$$

If the constraints (1) are satisfied, $\gamma p^T x \geq rt + e^T y$, and $rt + e^T y$ is greater than the sum of any $r$ items of $t + y_i$. Combining with $t + y_i \geq p_i x_i$, it is also greater than or equal to any $r$ items of $p \odot x$, so that $\gamma p^T x$ is greater than or equal to the sum of $r$ largest components of $p \odot x$, which is what we need. (See Lobo et al. [14].)

*Short selling constraints*, limit the maximum amount of short selling allowed on asset $i$.

$$w_i + x_i \geq -s_i \ , \quad \forall\, i \in \{1, \ldots, n\}$$

*Variance constraints*, are based upon the variance matrix $\Sigma$. The variance matrix is calculated based upon the historical data.

Note that the value of holdings at the end of the period is $W = a^T(w + x)$, where $a$ is a random vector for price of assets. The value of holdings is also a random vector

$$W = a^T(w + x) \sim \aleph(\mu, \sigma^2),$$

where $\mu = \bar{a}^T(w+x)$ and $\sigma^2 = E(W - EW)^2 = (w+x)^T\Sigma(w+x)$.

Note that $\bar{a}$ and $\Sigma$ can be estimated by the mean vector and the variance matrix of historical data [14]. If covariance terms are estimated independently and the variance matrix $\tilde{\Sigma}$ is not positive semidefinite, we can compute $W \in \mathbb{S}^n_+$, such that $\|W - \tilde{\Sigma}\|$ is minimized, to estimate $\Sigma$.

One method is to compute the eigenvalue decomposition of $\tilde{\Sigma}$:

$$\tilde{\Sigma} = \sum_{i=1}^{n} \lambda_i q_i q_i^T,$$

where $\lambda_i$ is an eigenvalue of $\tilde{\Sigma}$ and $q_i \in \mathbb{R}^n$ is the corresponding eigenvector. Then

$$W := \sum_{i:\lambda_i \geq 0} \lambda_i q_i q_i^T \in \mathbb{S}^n_+,$$

is a semidefinite estimation of the variance matrix.

Denoting the maximum standard deviation by $\sigma_{\max}$, we express the variance constraints as:

$$(w+x)^T\Sigma(w+x) \leq \sigma^2_{\max} \Longleftrightarrow \| \Sigma^{1/2}(w+x) \| \leq \sigma_{\max}.$$

This constraint can be expressed as a second-order cone constraint

$$\begin{pmatrix} \sigma_{\max} \\ \Sigma^{1/2}(w+x) \end{pmatrix} \in SOC^n.$$

Instead of the matrix square root $\Sigma^{1/2}$ of $\Sigma$, we can also use the Choleski factor $G^T$ of $\Sigma$, where $G$ is the unique lower triangular matrix such that $GG^T = \Sigma$. The speed of calculation of the Choleski decomposition in practice is much faster than the calculation of square root, even though they are both $O(n^3)$ in theory. Moreover, the computation of the Choleski factor seems more numerically stable than the computation of the square root. Then the constraints can be expressed as:

$$\| G^T(w+x) \| \leq \sigma_{\max} \Longleftrightarrow \begin{pmatrix} \sigma_{\max} \\ G^T(w+x) \end{pmatrix} \in SOC^n.$$

*Short risk constraints:* Assume that the return vector $a$ has a Gaussian distribution, $a \sim \aleph(\bar{a}, \Sigma)$. We want to require that the wealth $W$ at the end of the period be larger than $W^{low}$ under a probability no-less than $\eta$:

$$Prob(W \geq W^{low}) \geq \eta.$$

We have $W = a^T(w+x) \sim \aleph(\mu, \sigma^2)$, and let $\Phi(z)$ denote the cumulative distribution function of a zero mean, unit variance Gaussian variable. Then,

$$\begin{aligned} Prob\left(\frac{W-\mu}{\sigma} \leq \frac{W^{low}-\mu}{\sigma}\right) \leq (1-\eta) \quad &\Longrightarrow \quad \frac{W^{low}-\mu}{\sigma} \leq \Phi^{-1}(1-\eta) = -\Phi^{-1}(\eta) \\ &\Longrightarrow \quad \mu - W^{low} \geq \Phi^{-1}(\eta)\sigma. \end{aligned}$$

Combining with $\mu = \bar{a}^T(w+x)$ and $\sigma^2 = (w+x)^T\Sigma(w+x)$, we obtain

$$\Phi^{-1}(\eta) \parallel \Sigma^{1/2}(w+x) \parallel \leq \bar{a}^T(w+x) - W^{low} \Longleftrightarrow \begin{pmatrix} (\bar{a}^T(w+x) - W^{low})/\Phi^{-1}(\eta) \\ \Sigma^{1/2}(w+x) \end{pmatrix} \in SOC^n.$$

Using Choleski decomposition, such constraints can be expressed as

$$\Phi^{-1}(\eta) \parallel G^T(w+x) \parallel \leq \bar{a}^T(w+x) - W^{low} \Longleftrightarrow \begin{pmatrix} (\bar{a}^T(w+x) - W^{low})/\Phi^{-1}(\eta) \\ G^T(w+x) \end{pmatrix} \in SOC^n.$$

## 3.2 Portfolio selection model for multi-period with cash infusions

There are many situations in practice when we need to discuss the selection of portfolios problem under a multi-period model, which means that we will deal with a long time in the future, and divide it into several periods, such as 12 months in one year. During each period, the investor might have some scheduled income to be invested on assets. Based on the partial information about the future periods, we can consider certain utility function for the whole planning horizon. In each period, the assets transacted must be subject to the constraints on the feasible portfolios.

Based upon the previous discussion about the single-period model, we can design similar constraints for the new model. We will deal with the multi-period model using $m$ separate single periods, with different mean vectors and variance matrices for each period. We require that the amount of assets at the end of each period also be feasible, satisfying the constraints of transaction costs, diversification, etc.

We define $\hat{x} \in \mathbb{R}^{n\times m} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m)$ to be our variable for the new model, where the vector $\hat{x}_i$ denotes the amount of assets transacted during the $i$th period. In order to express different amounts for buying and selling assets, we transform the variable space to $\mathbb{R}^{2mn}$, so that the vector $\hat{x}_i^+$ corresponds to the amounts of assets bought in the period $i$ and the vector $\hat{x}_i^-$ corresponds to the amounts of assets sold in the period $i$. The variable can be expressed as:

$$\hat{x} := \begin{pmatrix} \hat{x}_1^+ \\ \hat{x}_2^+ \\ \vdots \\ \hat{x}_m^+ \\ \hat{x}_1^- \\ \vdots \\ \hat{x}_m^- \end{pmatrix} \in \mathbb{R}^{2mn} \geq 0.$$

We then define $x_i := \hat{x}_i^+ - \hat{x}_i^-$ to be the transaction amounts during the $i$th period. $y_j := (\sum_{i=1}^{j} x_i) + w$ is the total asset holdings at the end of period $j$.

In order to express this relationship in matrix notation, we can define a transfer matrix $T_j$ such that $y_j = T_j \hat{x} + w$, where $T_j \in \mathbb{R}^{n\times(2mn)}$,

$$T_j := \left( \underbrace{\begin{matrix} 1 & 0 & \dots & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \cdots & 0 & 0 & \dots & 1 \end{matrix}}_{j} \begin{matrix} 0 & \dots \\ 0 & \dots \\ \vdots & \ddots \\ 0 & \dots \end{matrix} \underbrace{\begin{matrix} -1 & 0 & \dots & 0 & -1 & \dots & 0 \\ 0 & -1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -1 & 0 & \dots & -1 \end{matrix}}_{j} \begin{matrix} 0 & \dots \\ 0 & \dots \\ \vdots & \ddots \\ 0 & \dots \end{matrix} \right).$$

*Objective function:* Our objective is to maximize the expected return at the end of the whole planning horizon

$$\max \quad \bar{a}_m^T y_m \quad = \bar{a}_m^T (w + \textstyle\sum_{i=1}^m x_i),$$

where $\bar{a}_m$ is the expected return on each asset for period $m$, $w$ is the vector of holdings in each asset at the beginning of the planning horizon.

*Transaction cost constraints:* Using the same notation as above

$$p_j^T x_j + \phi(x_j) \leq \xi_j \ , \quad \forall j \in \{1, \dots, m\},$$

where $\phi(x_j)$ is under the same definition as in the single-period model, and we will still focus on the linear form of the transaction cost. $\xi_j$ is the fixed investment for the period $j$. Note that we can use real asset prices $p_1$ for the first period, and $p_2, \dots, p_m$ can be estimated by the mean historic prices for periods $2, \dots, m$, which are $\bar{a}_1, \dots, \bar{a}_{m-1}$.

We can also add one dummy asset to express the holding of cash on hand, and denote it by $\zeta_i$ for $i \in \{1, \dots, m\}$. Then the above system of inequalities becomes:

$$p_j^T x_j + \phi(x_j) + \zeta_j \leq \xi_j + \zeta_{j-1} \ , \quad \forall j \in \{1, \dots, m\}.$$

*Diversification constraints:* Using the constraint discussed in the single-period model:

$$\textstyle\sum_{i=1}^r (p_j \odot x_j)_{[i]} \leq \gamma p_j^T (x_j) \ , \quad \forall j \in \{1, \dots, m\}.$$

The alternative way to express the $i$th largest component can also be applied here, as in (1).

*Short selling constraints:* $s \in \mathbb{R}^n$ is the vector of lower bounds (which could also represent a credit line).

$$y_j \geq -s \ , \quad \forall j \in \{1, \dots, m\}.$$

*Variance constraints:*

$$\| \Sigma_j^{1/2} (y_j) \| \leq (\sigma_{\max})_j \ , \quad \forall j \in \{1, \dots, m\}.$$

Note that when we begin to forecast the future return for a long time horizon, the variance matrix $\Sigma_j$ and mean vector $\bar{a}_j$ are calculated from previous data for each period, and they can only reflect partial information. Throughout the course of the horizon, we can update them to include the new information. Also, different variance matrices and mean vectors for each period can include the monthly or seasonal changes on the values of the assets in the portfolio.

*Short risk constraints:* based on the same assumption as in the single-period model, can be represented as

$$\Phi^{-1}(\eta) \parallel \Sigma_j^{1/2}(y_j) \parallel \leq \bar{a}_j^T(y_j) - W^{low} , \quad \forall j \in \{1, \ldots, m\}.$$

The previous discussion on the Choleski decomposition of the variance matrix is also applicable here. Moreover, the corresponding constraints are also representable as second order cone constraints.

We note that our models presented in this section can be extended using the robust optimization approach. Under some suitable assumptions, the resulting optimization problems are still SOCPs.

# 4 Computational results

## 4.1 Data description

The historical data for our experiment are obtained from the database of CRSP (The Center for Research in Security Prices), which creates and maintains premier historical US databases for publicly traded stocks (NASDAQ, AMEX, NYSE), indices, bond, and mutual fund securities. The database used is maintained and supported by SOAR (School of Accountancy Research) at the University of Waterloo.

## 4.2 Software package

We use Sedumi 1.05, an add-on toolbox for MATLAB. It implements the *self-dual* embedding technique for optimization problems over symmetric cones [22]. The version of MATLAB is 6.5 under the workstation of $Windows^{®}$ XP and *UNIX*. The machine for our experiment is powered by processor Intel $Pentium^{®}$ M 1.3 G, with $256MB$ DDR SDRAM. The server for *UNIX* is a Sun UltraSPARC IIe with speed 648 MHz and memory 1.5 GB.

## 4.3 Experiments

We present the results of two groups of runs. In the first group, we pick 20 stocks and we compare the performance of the multi-period model against the performance of the single-period model where the cash infusion varies. In the second group, to experiment with a larger SOCP problem, we pick 60 stocks.

For stocks on 20 companies with Nasdaq company number between 60006000 and 60006200, such as *RAYMOND, GRIFFON, KEY TRON, LIFELINE, MAIR HOL, UNIFIRST, LAFARGE, BURLINGT*, we selected the monthly average price between Jan $1st, 1993$ and Dec $31th, 2003$. We used the data from Jan $1st, 1993$ and Dec $31th, 2002$ to forecast the optimal investment strategy for the whole year 2003 (under a pre-determined cash flow for each month in 2003). The objective is to maximize the total profit earned at the end of 2003.

The process for the experiment can be stated as:

1. Estimate the mean vectors and the variance matrices $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m, \Sigma_1, \Sigma_2, \ldots, \Sigma_m$ from the historical data.
2. 
   - Use $\bar{a}_1, \Sigma_1$, under single-period model to find optimal decision $x^1$ for the $1st$ period
   - Use $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m, \Sigma_1, \Sigma_2, \ldots, \Sigma_m$, under multi-period model to find optimal decision $\tilde{x}^1$ for the $1st$ period.
3. Apply $x^1$ and $\tilde{x}^1$ to the $1st$ month and update mean vector and variance matrix $\bar{a}_2, \ldots, \bar{a}_m, \Sigma_2, \ldots, \Sigma_m$, go back to step 2 to forecast the optimal strategy beginning with the next month.

Repeat the process above until the end of $mth$ month.

For transaction cost and constraint parameters, we choose:

$$a_i^+ = 3.5 \quad a_i^- = 2 \quad \beta_i^+ = \beta_i^- = 0 \ , \quad \forall i \in \{1, \ldots, 20\}.$$

For the Diversification constraints, we used the formulation (1). The parameters in (1) and short selling constraints are:

$$r = 3 \quad \gamma = 0.7 \quad s_i = 0 \ , \quad \forall i \in \{1, \ldots, 20\}.$$

For the variance constraints and short risk constraints, we select:

$$\sigma_{\max} = \sqrt{1500} \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 80\% \\ 95\% \end{pmatrix} \quad W^{low} = \begin{pmatrix} 50 \\ 25 \end{pmatrix}.$$

We carry out the process described above. The first experiment is with cash flow 200 for each month and initial amount $w = 0$, and the second experiment is with cash flow 100 for each month and initial amount $w = 0$.

Adding another asset to express the cash holdings for each period, the third experiment follows the same process discussed above. The cash flow is 50 for each month while parameters stay the same except $W^{low} = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$ and $\sigma_{\max} = \sqrt{1000}$.

We increase the size of the problem as discussed before, still following the same process for the experiment.

## 4.4 Numerical result and analysis

The summary of the optimal strategy and holdings for the first three experiments are listed below:

| **Result for experiment with cash flow 200** | | |
|---|---|---|
| Single-period model | value of the commodity holdings | 1044.9 |
| | total transaction costs | 414.8612 |
| | total cash and commodity holdings | 1044.9+12×200-414.8612 =3030.04 |
| Multi-period model | value of the commodity holdings | 1131.4 |
| | total transaction costs | 371.8835 |
| | total cash and commodity holdings | 1131.4+12×200-371.8835=3159.51 |

| **Result for experiment with cash flow 100** | | |
|---|---|---|
| Single-period model | value of the commodity holdings | 722.2259 |
| | total transaction costs | 289.8983 |
| | total cash and commodity holdings | 722.2259+12×100-289.8983 =1632.36 |
| Multi-period model | value of the commodity holdings | 852.7675 |
| | total transaction costs | 281.7293 |
| | total cash and commodity holdings | 852.7675+12×100-281.7293=1771.04 |

| **Result for experiment with cash flow 50 and new variable for cash on hand** | | |
|---|---|---|
| Single-period model | value of the commodity holdings | 377.7366 |
| | cash holdings | 208.2462 |
| | total cash and commodity holdings | 585.9828 |
| Multi-period model | value of the commodity holdings | 231.3497 |
| | cash holdings | 415.2706 |
| | total cash and commodity holdings | 646.6203 |

Based on the above results, we can see the advantage of our multi-period model. The single-period model is more 'greedy', and the multi-period model considers future decisions when adjusting the portfolio in the current period. The multi-period model provides us with better investment strategy at the end of the planning horizon. Note that 'better' means bigger value of holdings at the end and smaller total transaction costs.

We selected another set of stocks from 100 companies with Nasdaq company number between 60007000 AND 60008000. The number of assets is 60. The underlying optimization problem for multi-period model had 1573 variables and 5293 constraints.

For the last experiment, we used the same parameters as above except:

$$\sigma_{\max} := \sqrt{5000},\ W^{low} := \begin{pmatrix} -50 \\ -100 \end{pmatrix},$$

and the cash infusion for each month was set to 300. Solving this largest model took 35 iterations of the interior-point code with total of 535.8 seconds computing time.

| **Result for experiment with 60 assets** | | |
|---|---|---|
| Single-period model | value of the commodity holdings | 4597.8 |
| | cash holdings | 300 |
| | total cash and commodity holdings | 4897.8 |
| Multi-period model | value of the commodity holdings | 5309.1 |
| | cash holdings | 0 |
| | total cash and commodity holdings | 5309.1 |

Further details on the data and the solutions can be found on the web [13].

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