A NEW TRIANGULATION FOR SIMPLICIAL ALGORITHMS *

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Abstract

Triangulations are used in simplicial algorithms to find the fixed points of continuous functions or upper semi-continuous mappings; applications arise from economics and optimization. The performance of simplicial algorithms is very sensitive to the triangulation used. Using a facetal description, we modify Dang's D_1 triangulation to obtain a more efficient triangulation of the unit hypercube in R^n and then by means of translations and reflections we derive a new triangulation, D_1', of R^n. We show that D_1' uses fewer simplices (asymptotically 30% fewer) than D_1 while achieving comparable scores for other performance measures such as the diameter and the surface density. We also compare the results of Haiman's recursive method for getting asymptotically better triangulations from D_1, D_1' and other triangulations.

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1 Introduction

Scarf[Sc67] was the first to provide a constructive proof of Brouwer's and Kakutani's fixed-point theorems, which have important applications in proving the existence of competitive price equilibria in certain economic models. Scarf used the notion of primitive sets, but most subsequent work used triangulations to discretize the continuous problem. The resulting methods to compute the approximate fixed-points , known as simplicial algorithms, are described, for instance in Allgower and Georg[AG80,AG90], Eaves[Ea84], and Todd[To76]. The performance of such methods depends critically on the triangulation used, and this led to much work on devising efficient triangulations of R^n. Among those used in simplicial algorithms are those of Freudenthal[Fr42], Tucker(Lefschetz[Lef49] p. 140), Todd[To84], and Dang[Da89], known as K_1, J_1, $J_1^{'}$, and D_1 respectively. These triangulations have relatively simple descriptions of their simplices and their pivoting rules, i.e., rules indicating the adjacent simplex found when a specified vertex of a given simplex of the triangulation is dropped. Other triangulations, with attractive properties but with much more complicated descriptions and pivoting rules, are one independently devised by Sallee[Sa82], and by Lee[Le85], and Sallee's middle cut triangulation[Sa84]. In this paper we will modify Dang's triangulation to get a more efficient triangulation which we denote $D_1^{'}$.

A triangulation of an $n-$dimensional convex subset of R^n is a locally finite collection of $n-$dimensional simplices which cover the subset and any two of which intersect in a common face (possibly empty). All of the triangulations above (except $J_1^{'}$) also triangulate the unit cube $I^n := [0,1]^n$, in that their simplices in I^n form a triangulation of R^n. The triangulations of R^n are then obtained by replicating this triangulation using reflections and/or translations. One basic measure of such a triangulation is the number of simplices used to triangulate I^n. This is $n!$ for K_1 and J_1, about $(\mathrm{e}-2)n!$ for D_1, and about $(\mathrm{e}-2)^2 n!$ for $D_1^{'}$. The triangulation of Lee[Le85] and Sallee[Sa82] is slightly better, and that of Sallee[Sa84] is considerably better, but at a price of increased complexity.

An $n-$simplex can be described as the convex hull of $n+1$ affinely independent vertices or, alternatively, as the solution set of $n+1$ linear inequalities, provided it is bounded with nonempty interior. The latter description, called a facetal description, often provides a simpler proof that a given collection of simplices forms a triangulation see, e.g., Todd[To76,To84]. We use this description to derive $D_1^{'}$. A typical simplex of K_1 or J_1 in I^n has the form

$$\{x \in R^n : 1 \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\};$$

all possible orderings of the components give the $n!$ simplices in I^n (the triangulations differ in how this triangulation of I^n is replicated to cover R^n). A typical simplex of D_1 in I^n can easily be shown to be of the form

$$\{x \in R^n : x_1, x_2, \ldots, x_p \geq \frac{\sum_{i=1}^{p} x_i - 1}{p-1} \geq x_{p+1} \geq \ldots \geq x_n \geq 0\};$$

where $1 < p \leq n$. As we shall see, the typical simplices of $D_1^{'}$ have a more symmetrical facetal description which also distinguishes the last $n-q+1$ components of x, where $1 < p < q < n$.

Section 2 defines $D_1^{'}$ and proves that it is indeed a triangulation. In section 3 we provide the pivot rules of $D_1^{'}$. Finally, section 4 compares all the triangulations mentioned above according to the number of simplices in the unit cube, their diameters, and their average directional or surface densities. We conclude by comparing the results of applying Haiman's recursive method[Ha91] for obtaining yet better triangulations from these.

2 The Triangulation D_1'

We first describe how we triangulate the unit cube $I^n := [0,1]^n$. Copies of the triangulation are then constructed by standard methods (using reflections and translations) to give a triangulation of R^n.

Let $e^1, e^2, \ldots, e^n$ denote the standard basis of R^n and let $e := \sum_j e^j$. We divide the unit cube into a shell S and a core C, which is a neighborhood of the diagonal from 0 to e. We triangulate S and C separately; the collection of all the resulting simplices triangulates the unit cube.

C is the convex hull of 0, e, e^i for $i \in N := \{1, 2, \ldots, n\}$, and $e - e^k$ for $k \in N$. We triangulate it into $2^n + 2$ simplices as follows: First the hyperplane $\{x : e^T x = 1\}$ cuts off the simplex

$$\sigma_- := conv\{0, e^1, e^2, ..., e^n\} \tag{1}$$

and the hyperplane $\{x : e^T x = n - 1\}$ cuts off the simplex

$$\sigma_+ := conv\{e, e - e^1, e - e^2, ..., e - e^n\}. \tag{2}$$

What remains is $conv\{e^1, e^2, ..., e^n, e - e^1, e - e^2, ..., e - e^n\}$ which is an affine transformation of the standard octahedron $conv\{\pm e^1, \pm e^2, ..., \pm e^n\}$. We triangulate this into 2^n simplices, corresponding to the 2^n partitions of N into $I \cup K$; a typical simplex is

$$\sigma_{I,K} = conv\{\frac{1}{2}e, e^i, i \in I, e - e^k, k \in K\} \tag{3}$$

(which corresponds to the simplex $conv\{0, -e^i, i \in I, e^k, k \in K\}$ of the canonical triangulation of the standard octahedron). It is clear that this provides a triangulation of C (note that it is possible to use just $2^{n-1} + 2$ simplices by joining the center simplices in pairs; if $1 \in I$, replace $\frac{1}{2}e$ by $e - e^1$, while if $1 \in K$, replace $\frac{1}{2}e$ by e^1. Then all simplices include e^1 and $e - e^1$. For symmetry, we have retained the central vertex $\frac{1}{2}e$).

For future reference we need a facetal (by linear inequalities) description of the octahedron $conv\{e^i, i \in N, e - e^k, k \in N\}$ as well as the simplices described above. We write $x(I)$ for $\sum_{i \in I} x_i$, etc.

Lemma 2.1. *With the notation above,* $conv\{e^i, i \in N, e - e^k, k \in N\} = \{x | (|K| - 1)x(I) - (|I| - 1)x(K) \leq |K| - 1$ *for all partitions* $I \cup K$ *of* $N\}$.

Proof : It suffices to check that the inequality given is satisfied at equality by e^i, $i \in I$, and $e - e^k, k \in K$, and strictly by the other vertices, so that it describes the facet $conv\{e^i, i \in I, e - e^k, k \in K\}$ of the octahedron. □

For the next result, we call 0, $\frac{1}{2}e$, or e the 0^{th} vertex of any simplex it appears in, while e^j or $e - e^j$ is the j^{th} vertex of a simplex it appears in.

Lemma 2.2.
a) $\sigma_- = \{x \in R^n | x(N) \leq 1, x_j \geq 0, j \in N\}$.
Moreover, if $x \in \sigma_-$*, the* j^{th} *barycentric coordinate of* x *is positive iff the inequality indexed by* j *is satisfied strictly. (Here* $x(N) \leq 1$ *is indexed* 0*.)*
b) $\sigma_+ = \{x \in R^n | x(N) \geq n-1, x_j \leq 1, j \in N\}$*. Moreover, if* $x \in \sigma_+$*, the* j^{th} *barycentric coordinate of* x *is positive iff the inequality indexed by* j *is satisfied strictly. (Again, the first-listed inequality is indexed by* 0*.)*
c) $\sigma_{I,K} = \{x \in R^n | (|K|-1)x(I) - (|I|-1)x(K) \leq |K|-1, x_i \geq \frac{x(N)-1}{n-2}, i \in I, x_k \leq \frac{x(N)-1}{n-2}, k \in K\}$.
Moreover, if $x \in \sigma_{I,K}$*, the* j^{th} *barycentric coordinate of* x *is positive iff the inequality indexed by* j *is satisfied strictly. (Again, the first-listed inequality is indexed by* 0*.)*

Proof : In each case, we merely check that all the vertices satisfy all the inequalities, strictly iff the indices correspond. □

Now we define the shell S and its subdivision. Let $1 < p < q < n$ and let π be a permutation of N. Then let

$\sigma_{p,q,\pi} := \{x | x_{\pi(1)}, \ldots, x_{\pi(p)} \geq \frac{x(\{\pi(1),\ldots,\pi(p)\})-1}{p-1} \geq x_{\pi(p+1)} \geq \ldots \geq x_{\pi(q-1)} \geq \frac{x(\{\pi(q),\ldots,\pi(n)\})}{n-q} \geq x_{\pi(q)}, \ldots, x_{\pi(n)}\}$

and let S be the union of all such $\sigma_{p,q,\pi}$.(Note that the order of $\{\pi(1), \ldots, \pi(p)\}$ and of $\{\pi(q), \ldots, \pi(n)\}$ is immaterial.) By summing the first p inequalities above, except that indexed by $\pi(i)$,we can deduce that $x_{\pi(i)} \leq 1$ for each i less than or equal to p. Proceeding similarly with the last $n-q+1$ inequalities yields $x_{\pi(k)} \geq 0$ for each k greater than or equal to q. Hence $\sigma_{p,q,\pi}$ is in the unit cube. It is easy to find an x satisfying all inequalities strictly, whence one can see that $\sigma_{p,q,\pi}$ contains an open ball. Since it is defined by $n+1$ inequalities, it is an $n-$simplex. We label the inequalities $\pi(1), \ldots, \pi(p); \pi(p+\frac{1}{2}), \ldots, \pi(q-\frac{1}{2}); \pi(q), \ldots, \pi(n)$, as they appear above. Of course, $\pi(p+\frac{1}{2})$ is a purely formal notation, connoting that it is "between $\pi(p)$ and $\pi(p+1)$" in some sense.

Lemma 2.3. *The vertices of* $\sigma_{p,q,\pi}$ *are*

$$e^{\pi(i)}, i = 1, 2, \ldots, p; \tag{4.1}$$

$$\sum_{i=1}^{j} e^{\pi(i)}, j = p, p+1, \ldots, q-1; \tag{4.2}$$

and

$$e - e^{\pi(k)}, k = q, q+1, \ldots, n. \tag{4.3}$$

If indexed by $\pi(1), \ldots, \pi(p) : \pi(p+\frac{1}{2}), \ldots, \pi(q-\frac{1}{2})$*;and* $\pi(q), \ldots, \pi(n)$*, then they correspond to the facets with the same index. That is, each vertex is off just the facet with the same index.*

Proof : Again, merely check the inequalities. □

Given $x \in [0,1]^n$, let us suppose the components of x are ordered:

$$1 \geq x_1 \geq \ldots \geq x_n \geq 0.$$

For $p > 1$ and $q < n$ let us write

$$x_{1p} := x(\{1, 2, \ldots, p\}),$$

$$x_{qn} := x(\{q, \ldots, n\}),$$

$$f(p) := \frac{x_{1p} - 1}{p - 1}, \text{ and}$$

$$g(q) := \frac{x_{qn}}{n - q}.$$

(We suppress the dependence of f and g on x.) We can think of $f(p)$ as approximately the average of p largest components of x and $g(q)$ as approximately that of the $n - q + 1$ smallest. In fact, if $1 = x_1$ and $x_n = 0$, $f(p)$ is the average of the p largest components of x without the largest, and similarly for $g(q)$.

Clearly $f(p)$ and $g(q)$ are important in the description of $\sigma_{p,q,\iota}$, where ι is the identity permutation. The following result is very useful.

Lemma 2.4. *Let $p > 1$ and $q < n$. Then*
a) $f(p+1) = \frac{p-1}{p} f(p) + \frac{1}{p} x_{p+1}$;
b) $g(q-1) = \frac{n-q}{n-q+1} g(q) + \frac{1}{n-q+1} x_{q-1}$;
c) $x_{p+1}\{<, =, >\} f(p)$ according as $x_{p+1}\{<, =, >\} f(p+1)$;
d) $x_{q-1}\{<, =, >\} g(q)$ according as $x_{q-1}\{<, =, >\} g(q-1)$.
e) If $2 < p < n-1$, then $f(p-1) \leq g(p)$ and $g(p) \geq x_p$ imply $f(p) \leq g(p+1)$, and the third inequality is strict if either of the first two is.
f) If $2 < p < n-1$, then $f(p) \leq g(p+1)$ and $x_p \geq f(p)$ imply $f(p-1) \leq g(p)$, and the third inequality is strict if either of the first two is.

Proof : Parts (a) and (b) follow directly from the definition. Since $f(p+1)$ is a strict convex combination of $f(p)$ and x_{p+1}, part (c) follows; similarly, part (d) follows from (b). For part (e), the hypotheses imply that $f(p)$, as a strict convex combination of $f(p-1)$ and x_p, is at most $g(p)$. But $g(p)$ is a convex combination of $g(p+1)$ and x_p, so $g(p) \geq x_p$ implies $g(p+1) \geq g(p)$. This gives the weak inequality, and the claim on when it is strict follows also. Part (f) is similar. □

We can now show that our simplices cover the unit cube:

Proposition 2.1. *The simplices σ_-, σ_+, $\sigma_{I,K}$ and $\sigma_{p,q,\pi}$, where I, K, p, q, and π range over all appropriate values, cover the unit cube.*

Proof : Choose $x \in [0,1]^n$, and, without loss of generality, assume

$$1 \geq x_1 \geq x_2 \geq \ldots \geq x_{n-1} \geq x_n \geq 0.$$

Since $x_1, x_2 \leq 1$, we find that $x_1, x_2 \geq f(2) = \frac{x_1 + x_2 - 1}{2-1}$, and since $x_{n-1}, x_n \geq 0$, we see that $x_{n-1}, x_n \leq g(n-1) = \frac{x_{n-1} + x_n}{n-(n-1)}$.

Now we proceed as follows. We have $x_1, x_2, \ldots, x_p \geq f(p)$ for $p = 2$, and $g(p) \geq x_q, \ldots, x_{n-1}, x_n$ for $q = n-1$. If $x_{p+1} > f(p)$ and $p < q - 1$ we replace p by $p + 1$. Then if $x_{q-1} < g(q)$ and

$p < q-1$ we replace q by $q-1$. By (c) and (d) in lemma 2.4, we see that $x_1, \ldots, x_p \geq f(p)$ and $g(q) \geq x_q, \ldots, x_n$ are preserved.

Suppose the procedure ends with $p < q$ and

$$x_1 \geq \ldots \geq x_p \geq \frac{x_{1p}-1}{p-1} \geq x_{p+1} \geq \ldots \geq x_{q-1} \geq \frac{x_{qn}}{n-q} \geq x_q \geq \ldots \geq x_n.$$

Then $x \in \sigma_{p,q,\iota}$, where ι is again the identity permutation.

Otherwise, we want to increase p or decrease q, but we cannot since $p = q-1$. Hence

$$x_1 \geq \ldots \geq x_p \geq f(p);$$

$$g(p+1) \geq x_{p+1} \geq \ldots \geq x_n;$$

$x_p < g(p+1)$ or $x_{p+1} > f(p)$.

In either case, $g(p+1) > f(p)$. Now (e) in lemma 2.4 implies that $g(j+1) > f(j)$ for $p \leq j < n-1$, and (f) implies that $g(j+1) > f(j)$ for $2 \leq j \leq p$. We can now show that $x \in C$.

If $x(N) \leq 1$ or $x(N) \geq n-1$ then $x \in \sigma_-$ or $x \in \sigma_+$ respectively. If $1 \leq x(N) \leq n-1$ then the inequality

$$(|K|-1)x(I) - (|I|-1)x(K) \leq |K|-1$$

is satisfied for $I = \emptyset$ and $I = N$. Since $x \geq 0$ and $x \leq e$, this inequality is satisfied for I or K a singleton. So assume I has j elements, $1 < j < n-1$, then the inequality is certainly satisfied if

$$(|K|-1)x_{1j} - (|I|-1)x_{j+1,n} \leq |K|-1,$$

since the left hand side only increases by taking the indices of the j largest components of x as I and those of the $n-j+1$ smallest as K. But this inequality is exactly equivalent to $f(j) - g(j+1) \leq 0$, which holds as shown above. Hence if x lies in no $\sigma_{p,q,\pi}$, nor in σ_- or σ_+, it lies in the octahedron $conv\{e^i, i \in N, e-e^k, k \in N\}$ and hence in some $\sigma_{I,K}$. □

Since there are clearly only a finite number of simplices in our description, to show that we have a triangulation it only remains to show that any point in the unit cube lies in the relative interior of just one face of a simplex of our collection. First, we need the following lemma.

Lemma 2.5. *Suppose $x \in \sigma := \sigma_{p,q,\pi}$ and $x \in \sigma' := \sigma_{p',q',\pi'}$. Then*

$$\frac{x(\{\pi(1),\ldots,\pi(p)\})-1}{p-1} \frac{x(\{\pi'(1),\ldots,\pi'(p')\})-1}{p'-1}$$

and

$$\frac{x(\{\pi(q),\ldots,\pi(n)\})}{n-q} = \frac{x(\{\pi'(q'),\ldots,\pi'(n)\})-1}{n-q'}.$$

Proof : Without loss of generality we assume that π is the identity, so that

$$x_1 \geq x_2 \geq \ldots \geq x_n.$$

Since also

$$x_{\pi'(1)} \geq \ldots \geq x_{\pi'(n)},$$

it follows that $x(\{\pi'(1),\ldots,\pi'(p')\})$ is the sum of the p' largest components of x. We therefore need to show that $f(p)=f(p')$, and similarly that $g(q)=g(q')$. We prove just the first. Assume that $p'>p$. By lemma 2.4(a), $f(j)\geq x_{j+1}$ implies $f(j+1)\leq f(j)$ and $f(j+1)\geq x_{j+1}\geq x_{j+2}$. Hence $f(p)\geq f(p+1)\geq\ldots\geq f(p')$. Now either $x_{p+1}=f(p)$ or $x_{p+1}<f(p)$. In the first case, $f(p+1)=f(p)$, while in the second, lemma 2.4(c) shows that $x_{p+2}\leq x_{p+1}<f(p+1)$. Thus, as we proceed from p to p', either $f(p)=f(p+1)=\ldots=f(p')$, as desired, or at some stage $x_{j+1}<f(j)$, in which case $x_{j+2}<f(j+1),\ldots,x_{p'}<f(p'-1)$, which implies $x_{p'}<f(p')$. But $x\in\sigma'$ shows that the p' largest components of x are at least $f(p')$, a contradiction. Hence $f(p)=f(p')$.

A similar argument yields $g(q)=g(q')$. □

Proposition 2.2. *Each $x\in[0,1]^n$ lies in the relative interior of just one face of a simplex of our collection.*

Proof : If $x\in\sigma$, then the face of σ containing x in its relative interior is called the carrier of x in σ; its vertices are just those corresponding to the positive barycentric coordinates of x in σ.

If x lies in no $\sigma_{p,q,\pi}$, then by proposition 2.1 x lies in the core C, and the result is clear. Suppose therefore that $x\in\sigma:=\sigma_{p,q,\pi}$, and assume without loss of generality that π is the identity.

We show first that any vertex of the carrier of x in σ is a vertex of the carrier of x in any other simplex of our collection in which x lies, and then the converse follows easily. We distinguish several cases.

First, let e^i be a vertex of the carrier of x in σ. Then the i^{th} barycentric coordinate of x in σ is positive, so by lemma 2.3

$$x_i>\frac{x_{1p}-1}{p-1}.$$

If $x\in\sigma':=\sigma_{p',q',\pi'}$, then lemma 2.5 shows that the i^{th} barycentric coordinate of x in σ' is also positive, so e^i is also a vertex of the carrier of x in σ'. If $x\in\sigma'':=\sigma_{I,K}$, then the argument in the proof of lemma 2.5 shows that $f(p)\geq f(n)$ so that $x_i>\frac{x_{1n}-1}{n-1}$; hence e^i is also a vertex of the carrier of x in σ'' by lemma 2.2. If $x\in\sigma_-$, then $x_i>x_n\geq 0$ shows that e^i is a vertex of the carrier of x in σ_-. Finally, we show that x cannot belong to σ_+ as follows: For $j=1$ to $q-1$, $x_j\geq\frac{x_{qn}}{(n-q)}$, with at least one strict inequality. Hence $(n-q)x_{1,q-1}>(q-1)x_{qn}$. Adding $(q-1)x_{1,q-1}$ to both sides gives

$$(q-1)x_{1n}<(n-1)x_{1,q-1}\leq(n-1)(q-1),$$

so $x_{1n}<n-1$ and $x\notin\sigma_+$.

Next, let $v:=e^1+\ldots+e^p+\ldots+e^j$ be a vertex of the carrier of x in σ, so that the inequality indexed $j+\frac{1}{2}$ of σ is strict:

$$x_j>x_{j+1},$$

where x_j replaced by $f(p)$ if $j=p$, and x_{j+1} is replaced by $g(q)$ if $j=q-1$. Suppose $x\in\sigma':=\sigma_{p',q',\pi'}$. There is a gap between the j^{th} largest component of x (or $f(p)$) and the

$(j+1)^{st}$ (or $g(q)$), and since $f(p) = f(p')$ and $g(q) = g(q')$, this also holds true when x is regarded as a member of σ'. The vertex v is just the sum of the coordinate vectors corresponding to the j largest components of x, this is also a vertex of the carrier of x in σ'. Also, $f(p) > g(q)$ and $x_{p+1} \geq g(q)$ if $p < q-1$, so in this case $f(p+1) > g(q)$. Continuing, $f(q-1) > g(q)$, which implies that x violates one of the inequalities defining C, so lies in none of its simplices.

Now let $e - e^k$ be a vertex of the carrier of x in σ. Then we have

$$x_k < \frac{x_{qn}}{n-q}.$$

The argument follows exactly the lines of that for the first case. (Alternatively, we may replace x by $e-x$, the permutation $\pi = \iota$ by its reverse, p by $n+1-q$ and q by $n+1-p$; the argument is then identical.)

Hence every vertex of the carrier of x in σ is also a vertex of the carrier of x in every other simplex containing it. To show the reverse, we simply observe that if x lies in a simplex then the barycentric coordinates of x in that simplex is unique. This completes the proof. □

We have proved the following result.

Theorem 2.1. *The simplices σ_-, σ_+, $\{\sigma_{I,K}\}$, and $\{\sigma_{p,q,\pi}\}$ triangulate the unit cube $[0,1]^n$.*

To triangulate R^n, we first reflect our triangulation in each of the coordinate hyperplanes $x_j = 0$, to get a triangulation of $[-1,1]^n$. Then we translate this triangulation by each vector in $(2Z)^n$ (with even integer components) to triangulate R^n. Each unit cube corresponds to a vector $v \in (2Z)^n$ and a sign vector $s \in \{-1,+1\}^n$, and is the set $\{x | x_j$ between v_j and $v_j + s_j$, $j \in N\}$. This is the image of the unit cube $[0,1]^n$ under the nonsingular affine transformation $x \to (v + \Sigma x)$, where Σ is the nonsingular diagonal matrix whose diagonal entries are the components of s. Then an explicit description of the vertices of the resulting simplex is obtained by applying the same transformation to the vertices of σ_-, σ_+, $\{\sigma_{I,K}\}$, and $\{\sigma_{p,q,\pi}\}$ given in equations (1), (2), (3), (4.1), (4.2), (4.3). We call the resulting triangulation of R^n D_1'; it is a modification of Dang's D_1 triangulation [Da89].

3 Pivot Rules

Here we describe the rules for obtaining the adjacent simplex $\sigma' \in D_1'$, which contains all vertices of $\sigma \in D_1'$ except a specified one v. We confine ourselves to the case where $\sigma \subseteq [0,1]^n$.

Case 1: $\sigma = \sigma_-$. If $v = 0$, it is replaced by $v' = \frac{1}{2}e$, and $\sigma' = \sigma_{I,K}$ where $I = N$, $K = \emptyset$. If $v = e^i$, then it is replaced by $v' = -e^i$, and σ' is the reflection of σ in $x_i = 0$.

Case 2: $\sigma = \sigma_+$. If $v = e$, it is replaced by $v' = \frac{1}{2}e$, and $\sigma' = \sigma_{I,K}$ where $I = \emptyset$, $K = N$. If $v = e - e^k$, then it is replaced by $v' = e + e^k$, and σ' is the reflection of σ in $x_k = 1$.

Case 3: $\sigma = \sigma_{I,K}$. If $v = e^i$, then it is replaced by $v' = e - e^i$, and $\sigma' = \sigma_{I',K'}$ where $I' = I\backslash\{i\}$, $K' = K \cup \{i\}$. If $v = e - e^k$, then it is replaced by $v' = e^k$, and $\sigma' = \sigma_{I',K'}$ with $I' = I \cup \{k\}$, $K' = K\backslash\{k\}$. Finally, if $v = \frac{1}{2}e$, then if $I = N$ $v' = 0$ and $\sigma' = \sigma_-$; if $I = \emptyset$ $v' = e$ and $\sigma' = \sigma_+$; else $v' = \sum_{i\in I} e^i$ and $\sigma' = \sigma_{p,q,\pi}$, where $p = |I| = q - 1$ and π is any permutation placing all $i \in I$ before all $k \in K$.

Case 4: $\sigma = \sigma_{p,q,\pi}$. Suppose $v = e^j$. Then $v' = \sum_{i=1}^{p} e^{\pi(i)} - e^j$ and $\sigma' = \sigma_{p-1,q,\pi'}$, where π' moves j to position p, i.e.,$\pi'(p) = j$, (if it was not already there), as long as $p > 2$. If $p = 2$, then $\{\pi(1), \pi(2)\} = \{j, j'\}$, $v' = e^j + 2e^{j'}$, and σ' is the reflection of σ in $x_{j'} = 1$.

Suppose $v = \sum_{i=1}^{p} e^{\pi(i)}$. Then $v' = e^{\pi(p+1)}$, and $\sigma' = \sigma_{p+1,q,\pi}$, as long as $p < q - 1$. If $p = q - 1$, then $v' = \frac{1}{2}e$ and $\sigma' = \sigma_{I,K}$, where $I = \{\pi(1), \ldots, \pi(p)\}$, $K = \{\pi(q), \ldots, \pi(n)\}$.

Suppose $v = \sum_{i=1}^{j} e^{\pi(i)}$, $p < j < q - 1$. Then $v' = \sum_{i=1}^{j-1} e^{\pi(i)} + e^{\pi(j+1)}$ and $\sigma' = \sigma_{p,q,\pi'}$, where $\pi' = (\pi(1), \ldots, \pi(j-1), \pi(j+1), \pi(j), \pi(j+2), \ldots, \pi(n))$.

Suppose $v = \sum_{i=1}^{q-1} e^{\pi(i)}$. Then $v' = e - e^{\pi(q-1)}$ and $\sigma' = \sigma_{p,q-1,\pi}$, as long as $p < q - 1$. (The case $p = q - 1$ was considered earlier.)

Finally, suppose $v = e - e^j$. Then $v' = \sum_{i=1}^{q-1} e^{\pi(i)} + e^j$ and $\sigma' = \sigma_{p,q+1,\pi'}$, where π' moves j to position q (if it was not already there), as long as $q < n - 1$. If $q = n - 1$, then $\{\pi(n-1), \pi(n)\} = \{j, j'\}$, $v' = e - e^j - 2e^{j'}$, and σ' is the reflection of σ in $x_{j'} = 0$.

4 Efficiency Measures

The performance of simplicial algorithms is very sensitive to the triangulation used. To evaluate the triangulations several measures of efficiency have been proposed in the literature, see Todd[To76]. In this section we calculate the values of the efficiency measures for the new triangulation D_1' and compare them with those of D_1 and other previously developed triangulations. Here, we consider D_1' with "paired" simplices in the core, i.e., without the interior vertex $\frac{1}{2}e$.

4.1 The Number of Simplices in the Unit Cube

Let $P_n(D_1')$ be the number of simplices used by D_1' to triangulate I^n. The number of simplices in the core is $2 + 2^{n-1}$, and we count the number of simplices in the shell as follows: We know $2 \leq p < q \leq n-1$ and the order of the indices $\pi(j)$ for $j \in \{1, \ldots, p\}$ (and similarly for $j \in \{q, \ldots, n\}$) is irrelevant. Therefore, given p and q we choose p indices out of n indices, then we choose $(n-q+1)$ indices out of $(n-p)$ indices, and finally we have $(q-p-1)!$ different ways of ordering indices $\pi(j)$ for $j \in \{p+1, \ldots, q-1\}$.

So, for any given p and q we have

$$\binom{n}{p}\binom{n-p}{n-q+1}(q-p-1)! = \frac{n!}{(n-p)!p!}\frac{(n-p)!}{(q-p-1)!(n-q+1)!}(q-p-1)! = \frac{n!}{p!(n-q+1)!}$$

simplices. Hence

$$\begin{aligned}
P_n(D_1') &= 2 + 2^{n-1} + \sum_{q=3}^{n-1}\sum_{p=2}^{q-1}\frac{n!}{p!(n-q+1)!} \\
&\leq 2 + 2^{n-1} + \sum_{q=3}^{n-1}\frac{n!}{(n-q+1)!}\sum_{p=2}^{\infty}\frac{1}{p!} \\
&= 2 + 2^{n-1} + (\mathrm{e}-2)n!\sum_{q=3}^{n-1}\frac{1}{(n-q+1)!} \\
&\leq 2 + 2^{n-1} + (\mathrm{e}-2)n!\sum_{k=2}^{\infty}\frac{1}{k!} \\
&= 2 + 2^{n-1} + (\mathrm{e}-2)^2 n!.
\end{aligned}$$

(We use e for the base of the natural logarithm since e is reserved for the vector of ones.) Moreover, it is easy to see that the ratio of the left hand side and the right hand side approaches 1 as $n \to \infty$. Hence we have

Theorem 4.1. *$P_n(D_1') \leq (\mathrm{e}-2)^2 n! + 2^{n-1} + 2$, and*

$$\lim_{n\to\infty}\frac{P_n(D_1')}{n!} = (\mathrm{e}-2)^2.$$

4.2 The Diameter of $D_1^{'}$

Let τ and $\tau^{'}$ be the two facets of a triangulation. The distance between τ and $\tau^{'}$ is defined as the minimum number of adjacent simplices that must be visited to get from τ to $\tau^{'}$, i.e. if $\sigma_0, \sigma_1, \ldots, \sigma_m$ is a sequence of simplices in the triangulation such that $\tau \subset \sigma_0$, $\tau^{'} \subset \sigma_m$ and σ_i and σ_{i+1} are adjacent for all $i \in \{0, 1, \ldots, m-1\}$, then this sequence of simplices define a path of length $(m+1)$. So,the distance between τ and $\tau^{'}$ is defined as the minimum length of such a path. The diameter of a triangulation is the distance between the farthest two facets, or in other words, the maximum of all such distances.

For our analysis, it is easier to work with full-dimensional simplices. We will find the maximum distance between two simplices in $D_1^{'}$; the diameter is then one more.

If $I \cup K$ is a partition of $N^{'} := \{2, \ldots, n\}$, we will denote by $\sigma_{I,K}^{'}$ the simplex $conv\{e^1, e - e^1, e^i, i \in I, e - e^k, k \in K\}$. Let $\sigma_-^{'} := \sigma_{N^{'},\emptyset}^{'}$ and $\sigma_+^{'} := \sigma_{\emptyset,N^{'}}^{'}$. Then σ_- is adjacent to $\sigma_-^{'}$, σ_+ is adjacent to $\sigma_+^{'}$, and clearly any $\sigma_{I,K}^{'}$ is a distance of at most $n/2$ from either $\sigma_-^{'}$ or $\sigma_+^{'}$.

Now let $\sigma := \sigma_{p,q,\pi}$ be in the shell, and assume that π is the identity permutation. Let $I = \{1, 2, \ldots, p\}$, $J = \{p+1, \ldots, q-1\}$, and $K = \{q, \ldots, n\}$. From σ we can reach $\sigma_-^{'}$ in at most $n-1$ steps as follows. First cross the facet defined by $f(p) = x_{p+1}$, so that index $p+1$ moves from J to I. Then successively move $p+2, \ldots, q-1$ from J to I; $|J|$ steps are necessary. Now p has become $q-1$; move across the facet defined by $f(p) = g(q)$. The vertex $e^1 + e^2 + \ldots + e^p$ is replaced by $e - e^1$, and we have entered the core. Finally, move the elements of K one by one into I, in $|K|$ steps. The total is $|J| + 1 + |K| = n - |I| + 1$. Since $|I| \geq 2$, at most $n-1$ steps are necessary. Similarly, at most $n-1$ steps are necessary to move from σ to $\sigma_+^{'}$ (actually only $n-2$, since $1 \in I$ does not have to be moved).

Since $n - 1 \geq n/2$, it follows that we can move from any simplex to any other simplex in at most $2n-2$ steps, via either $\sigma_-^{'}$ or $\sigma_+^{'}$.

We now show that $2n-2$ steps are necessary to go from $\sigma^{'} := \sigma_{p,q,\pi^{'}}$, where $p = 2$, $q = n-1$, and $\pi^{'} = (2, 3, 1, 4, 5, \ldots, n)$, (here $n \geq 5$) to $\sigma^{''} := \sigma_{p,q,\pi^{''}}$, where $\pi^{''} = (n, n-1, \ldots, 5, 4, 1, 3, 2)$. Let $I^{'} = \{2, 3\}$, $J^{'} = \{1, 4, 5, \ldots, n-2\}$, $K^{'} = \{n-1, n\}$, and $I^{''} = K^{'}$, $J^{''} = J^{'}$, $K^{''} = I^{'}$. We let I, J, K denote the index sets during a typical simplex on the path from $\sigma^{'}$ to $\sigma^{''}$. First consider an index $j \in J^{'}$. If it leaves J at some step, it has to return at a later step, so we charge this index two steps. If it remains in J at all steps, then each index in $I^{'}$ and $K^{'}$ must cross this index, so we charge this index four steps. This accounts four at least $2|J^{'}| = 2n - 8$ steps.

Next, if we never reach the core, then each index in $I^{'} \cup K^{'}$ must enter J then leave at the other end, for two steps each or eight in total. This gives $2n$ steps in all. Hence we must reach the core and leave it again; this costs two steps.

Finally, each index in $I^{'} \cup K^{'}$ must cross from one end to the other. (Notice that none of the indices is the special index 1, which is "at both ends" in the core.) This takes at least one step for each such index, for a total of 4. Hence $2n-2$ steps in all are necessary.

When we add the extra one to account for the diameter for the facets, we have

Theorem 4.2. *$diam(D_1^{'}) = 2n - 1$.*

Note that even though the diameter of $D_1^{'}$ is $2n-1$, when we take a line that goes through the unit cube it might intersect as many as $\frac{1}{2}(n-4)(n-5)$ simplices. In diameter calculations, we free ourselves in taking the shortest distance between two facets, as a result the shortest path does not necessarily follow a line.

4.3 The Surface Density of $D_1^{'}$

The average directional density of a triangulation, a measure introduced by Todd[To76], was shown to be equivalent to the surface density of the same triangulation by Eaves and Yorke[EY84], as long as it satisfies certain regularity conditions, which hold for $D_1^{'}$. In fact they showed the equivalence for a larger class. The equivalence holds for tilings which do not have to have convex cells. They concluded that given a subdivision of R^n, the average directional density does not depend on how the cells are assembled, but it does depend on the cells used, and they give the following relationship:

$$\text{Average directional density} = (\text{Surface density}).g_n, \text{ where}$$

$$g_n = \frac{\Gamma(n/2)}{(n-1)\Gamma(1/2)\Gamma((n-1)/2)}.$$

Here, we calculate the volumes and the surface areas of the simplices in $D_1^{'}$. Then we can compute the surface density of $D_1^{'}$, $SD(D_1^{'})$, by two means:

$$SD(D_1^{'}) = \frac{\sum_{\sigma\in D_1^{'},\sigma\subset I^n} SA(\sigma)}{\sum_{\sigma\in D_1^{'},\sigma\subset I^n} Vol(\sigma)} = \sum_{\sigma\in D_1^{'},\sigma\subset I^n} SA(\sigma)$$

or

$$SD(D_1^{'}) = \frac{\sum_{\sigma\in D_1^{'},\sigma\subset I^n} SD(\sigma)Vol(\sigma)}{\sum_{\sigma\in D_1^{'},\sigma\subset I^n} Vol(\sigma)} = \sum_{\sigma\in D_1^{'},\sigma\subset I^n} SD(\sigma)Vol(\sigma).$$

Here $SA(\sigma)$, $SD(\sigma)$, and $Vol(\sigma)$ denote the surface area, the surface density, and the volume of simplex σ. Note that the second equation implies that the worst surface density over all individual simplices cannot be better than the surface density of the triangulation.

In order to calculate the volume of a simplex, we construct an $(n+1)$ by $(n+1)$ matrix M_σ whose columns are the vertices of that particular simplex σ augmented with a $+1$ in the $(n+1)^{st}$ position. Then the absolute value of the determinant of the constructed matrix divided by $n!$ is the volume of the simplex.

To calculate the area of a particular facet, we take the vertices of the facet, find the normal of the hyperplane defined by the facet, and create a new point by taking a unit step (in Euclidean norm) from a vertex of the facet in the direction of the normal. Then the convex hull of the vertices of the facet and the new point define an $n-$simplex, and n times the volume of this simplex is the same as the surface area of the facet.

4.3.1 The Simplices in the Core

We have two different types of simplices in the core. $\sigma_- = conv\{0, e^1, e^2, \dots, e^n\}$ and $\sigma_+ = conv\{e - e^1, e - e^2, \dots, e - e^n, e\}$ are of type 1 and the rest of type 2.

For type 1 simplices we have

$$Vol(\sigma_-) = \frac{1}{n!}.$$

One of the facets of σ_- is $conv\{e^1, \dots, e^n\}$, and all other n facets are congruent to $conv\{0, e^1, \dots, e^{n-1}\}$. Hence

$$SA(\sigma_-) = SA(conv\{e^1, \dots, e^n\}) + nSA(conv\{0, e^1, \dots, e^{n-1}\}) = \frac{n + \sqrt{n}}{(n-1)!}.$$

So we get the surface density of type 1 simplices:

$$SD(\sigma_-) = \frac{SA(\sigma_-)}{Vol(\sigma_-)} = (n + \sqrt{n})n.$$

Let $\sigma^{'}$ be a type 2 simplex. Then we have

$$Vol(\sigma^{'}) = \frac{(n-2)}{n!}.$$

Note that any type 2 simplex has e^1 and $e - e^1$ as its vertices. Let τ_1 and τ_2 be the facets that we get from $\sigma^{'}$ by throwing away e^1 and $e - e^1$ respectively. All other facets of $\sigma^{'}$ have the same surface area; let τ_3 denote such a facet. Let p be the number of e^i's that are vertices of $\sigma^{'}$; then the surface areas of the facets of $\sigma^{'}$ are as follows:

$$SA(\tau_1) = \frac{\sqrt{(n-p+1)(p-2)^2 + (p-1)(n-p)^2}}{(n-1)!},$$

$$SA(\tau_2) = \frac{\sqrt{p(n-p-1)^2 + (n-p)(p-1)^2}}{(n-1)!},$$

$$SA(\tau_3) = \frac{\sqrt{(n-2)(n-3)+2}}{(n-1)!}.$$

So if $\sigma_p^{'}$ is a type 2 simplex with parameter p we get

$$SA(\sigma_p^{'}) = SA(\tau_1) + SA(\tau_2) + (n-1)SA(\tau_3).$$

From this formula we can easily get an upper bound on the surface densities of the type 2 simplices independent of p:

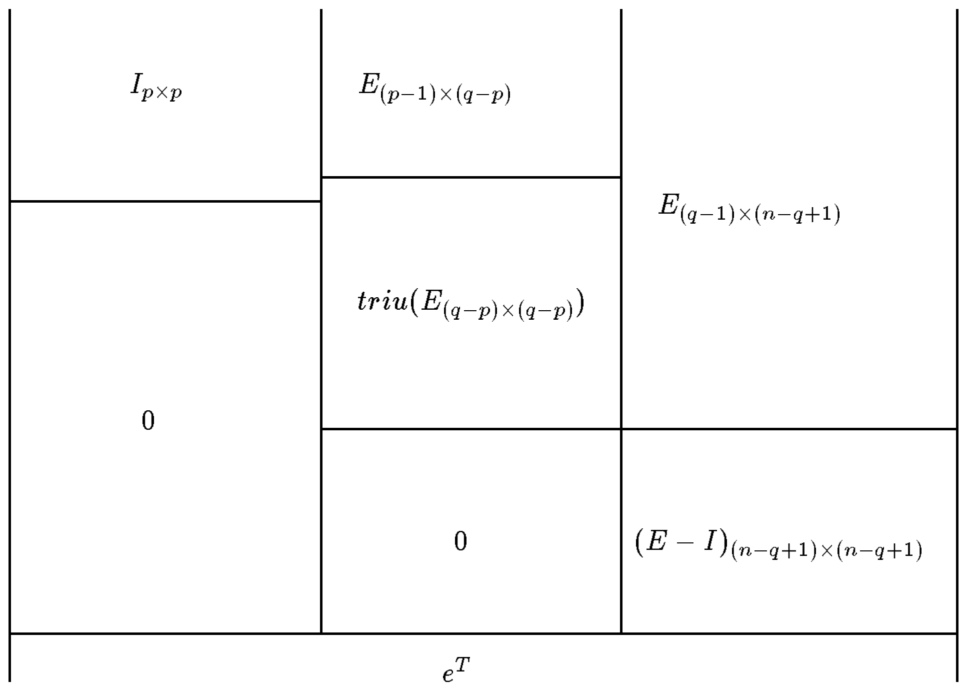
$$SA(\sigma_p^{'}) \leq \frac{n(n-2) + n\sqrt{n}}{(n-1)!},$$

$$SD(\sigma_p^{'}) \leq n^2 + \frac{n^2\sqrt{n}}{n-2}.$$

4.3.2 The Simplices in the Shell

For a generic simplex $\sigma_{p,q,\iota}$ in the core, we construct the corresponding matrix $M_{p,q,n}$ as described at the beginning of this section.

$M_{p,q,n} :=$



where $E_{r\times t}$ is the $r \times t$ matrix of ones, $I_{r\times r}$ is the $r \times r$ identity matrix, and $triu(A)$ is an upper triangular matrix which is the upper triangular portion of A. Hence

$$Vol(\sigma_{p,q,\iota}) = \frac{1}{n!}|det(M_{p,q,n})| = \frac{(p-1)(n-q)}{n!}.$$

Let $\tau_{p-1,q-1,n-1}$ be a facet of $\sigma_{p,q,\iota}$ which does not have one of the first p vertices of $\sigma_{p,q,\iota}$ (all such facets are congruent). We find

$$SA(\tau_{p-1,q-1,n-1}) = \frac{(n-q)\sqrt{p^2-3p+3}}{(n-1)!}.$$

Similarly, we get $\tau_{p,q,n-1}$ as a facet of $\sigma_{p,q,\iota}$ when we throw away one of the last $(n-q+1)$ vertices of $\sigma_{p,q,\iota}$ (again all such facets are congruent). We find

$$SA(\tau_{p,q,n-1}) = \frac{(p-1)\sqrt{(n-q+1)^2 - 3(n-q+1)+3}}{(n-1)!}.$$

Finally, we define $\tau^j_{p,q-1,n-1}$ as the facet obtained when the j^{th} vertex, $j \in \{p+1, \ldots, q\}$ of $\sigma_{p,q,\iota}$ is thrown away. We find

$$SA(\tau^j_{p,q-1,n-1}) = \begin{cases} \frac{\sqrt{2}(n-q)(p-1)}{(n-1)!} & j \neq q, j \neq p+1; \\ \frac{(n-q)\sqrt{p^2-p+1}}{(n-1)!} & j = q \neq p+1; \\ \frac{(p-1)\sqrt{(n-q+1)^2-(n-q+1)+1}}{(n-1)!} & j = p+1 \neq q; \\ \frac{\sqrt{(n-q+1)(p-1)^2+p(n-q)^2}}{(n-1)!} & j = q = p+1. \end{cases}$$

So, we have p facets like $\tau_{p-1,q-1,n-1}$, $(n-q+1)$ facets like $\tau_{p,q,n-1}$, and $(q-p)$ facets like $\tau^j_{p,q-1,n-1}$. Thus the total surface area for the simplex $\sigma_{p,q,\iota}$ is

$$SA(\sigma_{p,q,\iota}) = pSA(\tau_{p-1,q-1,n-1}) + (n-q+1)SA(\tau_{p,q,n-1}) + \sum_{p+1}^{q} SA(\tau^j_{p,q-1,n-1})$$

As $n \to \infty$ the worst surface density is given by the simplices which have *small* p and *large* q as parameters. In particular, the worst simplices are those with $p = 2$ and $q = n-1$ giving

$$SD(\sigma_{2,n-1,\iota}) = \sqrt{2}n^2 + o(n^2).$$

Note that the surface density of the triangulation cannot be worse than the worst simplex in the triangulation, therefore

$$SD(D_1^{'}) \leq \sqrt{2}n^2 + o(n^2).$$

(In fact, there are $\frac{n!}{4}$ simplices with $p = 2$ and $q = n-1$, with total volume $\frac{1}{4}$. If we next consider the simplices with $p = 3$ and $q = n-1$ or $p = 2$ and $q = n-2$, which have almost as bad a surface density, the volume increases to $\frac{7}{12}$. Continuing, we find that $SD(D_1^{'}) = \sqrt{2}n^2 + o(n^2)$.)

4.4 Comparison of the Triangulations in Terms of the Efficiency Measures

We define P_∞ of a triangulation as $\lim_{n\to\infty} \frac{P_n}{n!}$, where P_n is the number of simplices of the triangulation in I^n. Then we have the following table:

Triangulation	P_∞	Diameter
Freudenthal(1942), K_1	1	$O(n^2)$
Tucker(1949), J_1	1	$O(n^2)$
Sallee(1982) and Lee(1985)	0.4762	not known
Sallee(1984)	0	$O(n^2)$
Dang(1989), D_1	0.7183	$2n-3$
$D_1^{'}$	0.5159	$2n-1$

In terms of P_∞, $D_1^{'}$ is superior to J_1, K_1, and D_1. In terms of their diameters D_1 and $D_1^{'}$ are the only ones which are known to have $O(n)$ bounds. In terms of the surface densities, $D_1^{'}$ is slightly better than J_1, K_1, and D_1, yet asymptotically they all have the same surface density $\sqrt{2}n^2 + o(n^2)$.(We note that Dang[Da89] made an error in computing the surface density of D_1.)

4.5 Asymptotically Better Triangulations

We first mention an elegant result by Haiman[Ha91]:

Theorem 5.1. *If I^n can be triangulated into P_n simplices then I^{kn} can be triangulated into $[(kn)!/(n!)^k]P_n^k = \rho^{kn}(kn)!$ simplices, where $\rho = (P_n/n!)^{1/n}$.*

Note that according to the measure $R_n := (P_n/n!)^{1/n}$, $R_\infty = \lim_{n\to\infty} R_n$ we have $R_\infty = 1$ for all triangulations in the previous table. Haiman's result implies that if a triangulation achieves some $R_n = \rho$ for some n then the same number ρ is asymptotically achievable, i.e. $R_\infty = \rho$. In other words, this result enables us to get triangulations with $P_\infty = 0$ from those which have $P_\infty < 1$.(Note that this is weaker than saying that $R_\infty = \rho < 1$ which is also true.)

Using this result we can define new triangulations recursively using those in the previous table and choose the best possible ρ for each triangulation:

n	Sallee[82] and Lee[85]		Sallee [84]		D_1		D_1'	
	P_n	R_n	P_n	R_n	P_n	R_n	P_n	R_n
3	5	.9410	5	.9410	5	.9410	6	1
4	16	.9036	16	.9036	18	.9306	16	.9036
5	67	.8900	67	.8900	87	.9377	68	.8926
6	364	.8925	324	.8754	518	.9466	384	.9005
7	2445	.9018	1962	.8739	3621	.9539	2628	.9112
8	19296	.9120	13248	.8701	28962	.9595	20864	.9201
9	173015	.9210	106181	.8724	260651	.9639	187356	.9292
10	1720924	.9281	931300	.8728	2606502	.9675	1872496	.9360

We observe that for each triangulation R_n converges to 1 very fast. As a result the best value for ρ is achieved for $n < 10$ for all these triangulations (as expected, smaller ρ values are achieved by those triangulations which have smaller P_∞ values).

Finally, we note that all triangulations in table 2 except D_1' achieve the minimum value of P_3, all except D_1 achieve the minimum for P_4, and all except D_1 achieve(or are within 1 of) the minimum for P_5. See Mara[Ma76], Cottle[Co82], Böhm[Bö88], and Hughes[Hu90]. Hughes also shows that any triangulation that slices alternate corners off the unit cube in R^6 cannot achieve fewer than 324 simplices which is achieved by Sallee's middle-cut triangulation; however, Hughes[Hu92] recently showed that 6-cube can be triangulated into 312 simplices.

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