An introduction to Banach algebras and operator algebras

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Preface

These notes were designed as lecture notes for a first course in Banach Algebras and Operator Algebras. The student is assumed to have already taken a first course in Linear Analysis. In particular, they are assumed to already know the Hahn-Banach Theorem, the Open Mapping Theorem, etc., all of which can be found in my notes on Functional Analysis [**35**]. A list of those results which will be used in the sequel is included in the second section of the first chapter.

I am indebted to a number of people who have found typos in earlier versions of the notes, including Dan Pollock, Paul Skoufranis, and Austin Shiner. Any remaining errors are clearly the fault of my friend and colleague Heydar Radjavi. (Ok, I might be exaggerating there – they *might* be my fault.)

April 30, 2021

The reviews are in!

He is a writer for the ages, the ages of four to eight.

Dorothy Parker

This paperback is very interesting, but I find it will never replace a hardcover book - it makes a very poor doorstop.

Alfred Hitchcock

It was a book to kill time for those who like it better dead.

Rose Macaulay

That's not writing, that's typing.

Truman Capote

Only the mediocre are always at their best.

Jean Giraudoux

Contents

CHAPTER 1

A Brief Review of Banach Space Theory

Somewhere on this globe, every ten seconds, there is a woman giving birth to a child. She must be found and stopped.

Sam Levenson

Definitions and examples

1.1. In this manuscript, we shall be studying the basic properties of Banach algebras and of operator algebras acting on a Hilbert space. The reader is assumed to have already taken a first course in Functional Analysis. In particular, we shall assume that the reader is familiar with the material contained in *An introduction to functional analysis* [**35**].

In an effort to keep these notes relatively self-contained, we shall list the main results contained therein. But first, let us also review the definition of a Banach space, as well as a number of examples thereof.

Those who study Banach spaces are as interested (at times even more interested) in real Banach spaces as they are in complex Banach spaces. From the point of view of Banach *algebras*, complex Banach algebras are considered the more important example. One reason for this is that complex Banach spaces have the property that each of their elements has non-empty spectrum, and the spectrum of an element is one of the most important tools used to study it and the algebra it generates. We shall much more to say about this in subsequent Chapters. For now, we are only trying to justify why we are only considering complex Banach spaces in the current Chapter.

1.2. Definition. *Let $\mathfrak{X}$ be a vector space over $\mathbb{C}$. A* ***seminorm*** *on $\mathfrak{X}$ is a map*

$$\nu : \mathfrak{X} \to \mathbb{R}$$

satisfying

(i) $\nu(x) \ge 0$ *for all* $x \in \mathfrak{X}$*;*

(ii) $\nu(\lambda x) = |\lambda|\ \nu(x)$ *for all* $x \in \mathfrak{X}$*,* $\lambda \in \mathbb{C}$*; and*

(iii) $\nu(x + y) \le \nu(x) + \nu(y)$ *for all* $x, y \in \mathfrak{X}$*.*

If ν satisfies the extra condition:

(iv) $\nu(x) = 0$ *if and only if* $x = 0$*,* *then we say that ν is a* ***norm****, and we usually denote $\nu(\cdot)$ by $\|\cdot\|$. In this case, we say that $(\mathfrak{X}, \|\cdot\|)$ (or, with a mild abuse of nomenclature, $\mathfrak{X}$) is a* ***normed linear space****.*

A ***Banach space*** *normed linear space which is complete with respect to the metric*

$$d(x,y) := \|x-y\|, \quad x,y \in \mathfrak{X}$$

induced by the norm.

1.3. Example. Let $n \in \mathbb{N}$, and let $1 \le p < \infty$ be a real number. We define the p-norm $\|\cdot\|_p$ on $\mathbb{C}^n$ as follows. Given $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$, we set

$$\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

We also define the ∞-norm

$$\|x\|_\infty := \max_{1\le i\le n} |x_i|.$$

For each $1 \le p \le \infty$, $(\mathbb{C}^n, \|\cdot\|_p)$ is complete, hence a Banach space.

1.4. Remark. Recall that two norms $\|\cdot\|$ and $|||\cdot|||$ on a complex vector space $\mathfrak{X}$ are said to be **equivalent** if there exist constants $\alpha, \beta > 0$ such that

$$\alpha\ \|x\| \le |||x||| \le \beta\ \|x\|$$

for all $x \in \mathfrak{X}$. It is a theorem that all norms on a finite-dimensional normed linear space are equivalent. In other words, they generate the same topology.

A consequence of this is that given $n \in \mathbb{N}$ and *any* norm $\|\cdot\|$ on $\mathbb{C}^n$, $(\mathbb{C}^n, \|\cdot\|)$ is complete, and thus a Banach space.

1.5. Example. Let X be a compact, Hausdorff topological space and set

$$\mathcal{C}(X) = \{ f : X \to \mathbb{C} : f \text{ is continuous} \}.$$

With respect to the **sup-norm** $\|f\|_\infty := \max_{x\in X} |f(x)|$, the space $\mathcal{C}(X)$ is complete, and thus a Banach space.

1.6. Example. The above example may be generalised somewhat. Let X be a locally compact Hausdorff space. Then $(\mathcal{C}_0(X), \|\cdot\|_\infty)$ is a Banach space, where

$$\begin{aligned}\mathcal{C}_0(X) &= \{ f \in \mathcal{C}(X) : f \text{ vanishes at } \infty \} \\ &= \{ f \in \mathcal{C}(X) : \forall \varepsilon > 0, \{ x \in X : |f(x)| \ge \varepsilon \} \text{ is compact in } X \}.\end{aligned}$$

As before, the norm under consideration is the supremum norm: $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

1.7. Example. If (X, Ω, μ) is a measure space and $1 \le p < \infty$. Set

$$\begin{aligned}\mathcal{L}^p(X,\Omega,\mu) \;=\; & \{\, f : X \to \mathbb{C} : f \text{ is Lebesgue measurable} \\ & \qquad \text{and } \int_X |f(x)|^p \, d\mu(x) < \infty \,\}.\end{aligned}$$

The map $\nu_p : f \mapsto \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p}$ defines a seminorm on $\mathcal{L}^p(X,\Omega,\mu)$, with kernel which we denote by $\mathcal{N}^p(X,\Omega,\mu)$. The quotient space

$$L^p(X,\Omega,\mu) := \mathcal{L}^p(X,\Omega,\mu)/\mathcal{N}^p(X,\Omega,\mu)$$

is then a normed linear space, and a standard result from measure theory asserts that it is complete, and therefore a Banach space. The norm here is given by $\|[f]\|_p := \nu_p(f)$. With (X, Ω, μ) as above, and with $f : X \to \mathbb{C}$ measurable, we also define

$$\nu_\infty(f) = \operatorname{ess\,sup}(f) := \inf\{\gamma > 0 : \mu\{x \in X : |f(x)| > \gamma\} = 0\},$$

and set

$$\mathcal{L}_\infty(X,\Omega,\mu) = \{f : X \to \mathbb{C} : f \text{ is Lebesgue measurable and } \nu_\infty(f) < \infty\}.$$

Similar to the construction above, we let $\mathcal{N}^\infty(X,\Omega,\mu)$ denote the kernel of ν_∞, and set

$$L^\infty(X,\Omega,\mu) := \mathcal{L}^\infty(X,\Omega,\mu)/\mathcal{N}^\infty(X,\Omega,\mu).$$

In this case the norm is $\|[f]\|_\infty := \nu_\infty(f) = \operatorname{ess\,sup}(f)$.

For the sake of readability, we subscribe to the usual abuse of notation and denote an element of $L^p(X,\Omega,\mu)$ as "f", rather than "$[f]$", although it is crucial that the reader be aware that elements of $L^p(X,\Omega,\mu)$ are *equivalence classes* of functions, rather than functions. Thus, in general, it does not make sense to speak of $f(x)$ for $f \in L^p(X,\Omega,\mu)$ and $x \in X$.

1.8. Example. Let $I \ne \varnothing$ be a set and let $1 \le p < \infty$. Define $\ell^p(I)$ to be the set of all functions

$$\{\, f : I \to \mathbb{C} : \sum_{i\in I} |f(i)|^p < \infty \,\}$$

and for $f \in \ell^p(I)$, define $\|f\|_p = (\sum_{i\in I} |f(i)|^p)^{1/p}$. Then $\ell^p(I)$ is a Banach space. If $I = \mathbb{N}$, we also write ℓ^p. Of course,

$$\ell^\infty(I) = \{\, f : I \to \mathbb{C} : \sup\{\, |f(i)| : i \in I \,\} < \infty \,\}$$

and $\|f\|_\infty = \sup\{\, |f(i)| : i \in I \,\}$. A closed subspace of particular interest here is

$$c_0(I) = \{\, f \in \ell^\infty(I) : \text{ for all } \varepsilon > 0, \text{ cardinality}\{\, i \in I : |f(i)| \ge \varepsilon \,\} < \infty \,\}.$$

Again, if $I = \mathbb{N}$, we write only ℓ^∞ and c_0, respectively.

The reader may wish to review what it means for $\sum_{i\in I} x_i$ to converge (where $x_i \in \mathbb{C}$ for all $i \in I$) when I is uncountable.

1.9. Example. Let μ denote normalised Lebesgue measure on the set $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$. From Example 1.7, for each $1 \le p \le \infty$, the space $L^p(\mathbb{T}, \Omega, \mu)$ is a Banach space, where the corresponding measure space Ω on $\mathbb{T}$ is the set of Lebesgue measurable subsets of $\mathbb{T}$. We may also define the so-called **Hardy spaces**

$$H^p(\mathbb{T}, \mu) = \{ [f] \in L^p(\mathbb{T}, \mu) : \int_0^{2\pi} f(\theta) e^{in\theta} \, d\theta = 0 \text{ for all } n \ge 1 \}.$$

These are Banach spaces for each $p \ge 1$, including $p = \infty$.

1.10. Example. Let $n \in \mathbb{N}$ and let

$$\mathcal{C}^{(n)}[0,1] = \{ f : [0,1] \to \mathbb{C} : f \text{ has } n \text{ continuous derivatives} \}.$$

Define $\|f\| = \max_{0 \le k \le n} \{ \sup\{ |f^{(k)}(x)| : x \in [0,1] \} \}$. Then $(\mathcal{C}^{(n)}[0,1], \| \cdot \|)$ is a Banach space.

1.11. Example. If $\mathfrak{X}$ is a Banach space and $\mathfrak{Y}$ is a closed subspace of $\mathfrak{X}$, then

- $\mathfrak{Y}$ is a Banach space under the inherited norm, and
- $\mathfrak{X}/\mathfrak{Y}$ is a Banach space — where $\mathfrak{X}/\mathfrak{Y} = \{ x + \mathfrak{Y} : x \in \mathfrak{X} \}$. The norm is the usual quotient norm, namely: $\|x + \mathfrak{Y}\| = \inf_{y \in \mathfrak{Y}} \|x + y\|$.

1.12. Example. Examples of Banach spaces can of course be combined. For instance, if $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces over $\mathbb{C}$, then we can form the so-called *ℓ^p-direct sum* of $\mathfrak{X}$ and $\mathfrak{Y}$ as follows:

$$\mathfrak{X} \oplus_p \mathfrak{Y} = \{ (x, y) : x \in \mathfrak{X}, \, y \in \mathfrak{Y} \},$$

and $\|(x, y)\| = \left(\|x\|_{\mathfrak{X}}^p + \|y\|_{\mathfrak{Y}}^p \right)^{\frac{1}{p}}$.

More generally, given a family $(\mathfrak{X}_n, \| \cdot \|_{\mathfrak{X}_n})_n$ of Banach spaces over $\mathbb{C}$, the space

$$\oplus_n \mathfrak{X}_n := \{ (x_n)_n : x_n \in \mathfrak{X}_n, n \ge 1 \}$$

is a Banach space with norm defined by $\|(x_n)_n\| := \left(\sum_n \|x_n\|_{\mathfrak{X}_n}^p \right)^{\frac{1}{p}}$.

1.13. Example. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces over $\mathbb{C}$. Then the set of continuous linear transformations $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ from $\mathfrak{X}$ into $\mathfrak{Y}$ is a Banach space under the *operator norm* $\|T\| = \sup_{\|x\| \le 1} \|Tx\|$. When $\mathfrak{X} = \mathfrak{Y}$, we also write $\mathcal{B}(\mathfrak{X})$ for $\mathcal{B}(\mathfrak{X}, \mathfrak{X})$.

In particular, $\mathcal{B}(\mathbb{C}^n)$ is isomorphic to the $n \times n$ complex matrices $\mathbb{M}_n$ and forms a complex Banach space under a variety of norms, including the operator norm from above. On the other hand, as we observed in Remark 1.4, all such norms must be equivalent to the operator norm.

1.14. Example. Suppose that $\mathfrak{X}$ is a Banach space over $\mathbb{K}$. Then $\mathfrak{X}^* = \mathcal{B}(\mathfrak{X}, \mathbb{C})$ is a Banach space, called the *dual space* of $\mathfrak{X}$.

For example, if μ is σ-finite measure on the measure space (X, Ω), $1 \le p < \infty$, and if q, $1 < q \le \infty$ the Lebesgue conjugate of p so that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} [L^p(X, \Omega, \mu)]^* &= L^q(X, \Omega, \mu) \\ [\ell^p]^* &= \ell^q \\ [c_0]^* &= \ell^1. \end{aligned}$$

This is referred to as the **Riesz Representation Theorem**. Note that in general, the first two equalities fail if $p = \infty$.

As a second example, suppose X is a compact, Hausdorff space. Then

$$\begin{aligned} \mathcal{C}(X)^* &= \mathcal{M}(X) \\ &:= \{\, \mu : \mu \text{ is a bounded, regular Borel measure on } X \,\} \end{aligned}$$

Note that for $\mu \in \mathcal{M}(X)$, the action on $\mathcal{C}(X)$ is given by $\Phi_\mu(f) = \int_X f\, d\mu$. This is known as the **Riesz-Markov Theorem**.

In the case where $X = [0, 1]$, the space of bounded, regular Borel measures may also be identified with the set $BV([0, 1], \mathbb{K})$ of left-continuous complex-valued functions of bounded variation on $[0, 1]$.

For our purposes, one of the most important examples of a Banach space will be:

1.15. Definition. *A complex* ***Hilbert space*** $\mathcal{H}$ *is a Banach space over* $\mathbb{C}$ *whose norm is generated by an* ***inner product*** $\langle \cdot, \cdot \rangle$, *which is a map from* $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ *satisfying:*

(i) $\langle x, x\rangle \ge 0$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$*;*
(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$*; and*
(iii) $\langle \lambda x + \beta y, z\rangle = \lambda \langle x, z\rangle + \beta \langle y, z\rangle$

for all $x, y, z \in \mathcal{H}$*;* $\lambda, \beta \in \mathbb{C}$*. The norm on* $\mathcal{H}$ *is given by* $\|x\| = \langle x, x\rangle^{1/2}$, $x \in \mathcal{H}$.

Each Hilbert space $\mathcal{H}$ is equipped with a **Hilbert space basis**, also referred to as an **orthonormal basis**. This is an orthonormal set $\{e_\alpha\}_{\alpha \in \Lambda}$ in $\mathcal{H}$ with the property that any $x \in \mathcal{H}$ can be expressed as

$$x = \sum_{\alpha \in \Lambda} x_\alpha e_\alpha$$

in a unique way. It is a standard result that any two orthonormal bases for $\mathcal{H}$ are of the same cardinality, which allows one to define the **dimension** of the space $\mathcal{H}$ as the cardinality of any orthonormal basis of $\mathcal{H}$. From this it follows that any two Hilbert spaces of the same dimension are isomorphic. Examples of Hilbert spaces include:

- The spaces $(\mathbb{C}^n, \|\cdot\|_2)$, $n \geq 1$, as defined in Example 1.3.
- The space $L^2(X, \Omega, \mu)$ defined in Example 1.7.
- The space $\ell^2(I)$ defined in Example 1.8.
- The space $H^2(\mathbb{T}, \mu)$ defined in Example 1.9.
- $\mathbb{M}_n(\mathbb{C})$ is a Hilbert space with the inner product $< x, y >= \mathrm{tr}(y^* x)$. Here tr denotes the usual trace functional on $\mathbb{M}_n$, and if $y \in \mathbb{M}_n(\mathbb{C})$, then y^* denotes the conjugate transpose of y.

  It should be noted that the norm induced on $\mathbb{M}_n(\mathbb{C})$ by the inner product is not the operator norm (unless $n = 1$). For $y = [y_{i,j}] \in \mathbb{M}_n(\mathbb{C})$, the induced norm is
$$\|y\|_2 := \left(\sum_{1 \leq i,j \leq n} |y_{i,j}|^2 \right)^{\frac{1}{2}},$$
  and is referred to as the **Fröbenius norm**.

The main theorems

1.16. We now enumerate the main results we shall assume in the sequel. The proofs of all of these results may be found in [**35**].

1.17. Theorem. [**The Hahn-Banach Theorem 01.**] *Suppose $\mathfrak{X}$ is a Banach space, $\mathfrak{M} \subseteq \mathfrak{X}$ is a linear manifold and $f : \mathfrak{M} \to \mathbb{C}$ is a continuous linear functional. Then there exists a functional $g \in \mathfrak{X}^*$ such that $\|g\| = \|f\|$ and $g|_{\mathfrak{M}} = f$.*

1.18. Theorem. [**The Hahn-Banach Theorem 02.**] *Let $\mathfrak{X}$ be a Banach space and suppose $0 \neq x \in \mathfrak{X}$. Then there exists $f \in \mathfrak{X}^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

1.19. Corollary. *Let $\mathfrak{X}$ be a Banach space and suppose that $x \neq y$ are two vectors in $\mathfrak{X}$. Then there exists $f \in \mathfrak{X}^*$ such that $f(x) \neq f(y)$.*

1.20. Theorem. [**The Hahn-Banach Theorem 03.**] *Let $\mathfrak{X}$ be a Banach space, $\mathfrak{M}$ be a closed subspace of $\mathfrak{X}$ and x be a vector in $\mathfrak{X}$ such that $x \notin \mathfrak{M}$. Then there exists $f \in \mathfrak{X}^*$ such that $f \equiv 0$ on $\mathfrak{M}$ and $f(x) \neq 0$.*

1.21. Theorem. [**The Open Mapping Theorem.**] *Let $T : \mathfrak{X} \to \mathfrak{Y}$ be a surjective continuous linear map of a Banach space $\mathfrak{X}$ onto a Banach space $\mathfrak{Y}$. Then T is an open map; that is, $T(V)$ is open in $\mathfrak{Y}$ for all open sets V in $\mathfrak{X}$.*

1.22. Theorem. [**The Banach Isomorphism Theorem.**] *Let $T : \mathfrak{X} \to \mathfrak{Y}$ be a bijective, continuous linear map of a Banach space $\mathfrak{X}$ onto a Banach space $\mathfrak{Y}$. Then T^{-1} is continuous.*

1.23. Definition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and $\mathfrak{M} \subseteq \mathfrak{X}$ be a linear manifold. Then a linear map $T : \mathfrak{M} \to \mathfrak{Y}$ is* ***closed*** *if $x_n \to x$ and $Tx_n \to y$ together imply that $x \in \mathfrak{M}$ and $Tx = y$. This is equivalent to saying that the graph $\mathfrak{G}(T) = \{\, (x, Tx) : x \in \mathfrak{M}\,\}$ is a closed subspace of $\mathfrak{X} \oplus \mathfrak{Y}$.*

1.24. Theorem. [The Closed Graph Theorem.] *If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces and if $T : \mathfrak{X} \to \mathfrak{Y}$ is a closed linear map that is defined everywhere, then T is continuous.*

An alternative formulation reads:

If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, $T : \mathfrak{X} \to \mathfrak{Y}$ is linear, $\{x_n\}_{n=1}^\infty \subseteq \mathfrak{X}$, and if $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Tx_n = b$ together imply that $b = 0$, then T is continuous.

1.25. Theorem. [The Banach-Steinhaus Theorem (aka the Uniform Boundedness Principle).]

Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces and $\mathcal{F} \subseteq \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Suppose that for all $x \in \mathfrak{X}$, $K_x := \sup_{T\in\mathcal{F}} \|Tx\| < \infty$. Then $K := \sup_{T\in\mathcal{F}} \|T\| < \infty$.

1.26. Corollary. *Let $\{T_n\}_{n=1}^\infty$ be a sequence of bounded linear operators in $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ such that $Tx := \lim_{n\to\infty} T_n x$ exists for all $x \in \mathfrak{X}$. Then $\sup_{n\geq 1} \|T_n\| < \infty$ and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$.*

1.27. Theorem. [The Banach-Alaoglu Theorem.] *Let $\mathfrak{X}$ be a Banach space. Then the unit ball $\mathfrak{X}_1^*$ of $\mathfrak{X}^*$ is compact in the* weak**-topology.*

1.28. Of course, this is but a brief outline of some of the major results and definitions which will be relevant to our study of Operator Algebras. For more information, the reader is encouraged to consult the texts of Dunford and Schwarz [**20**], of Bollobás [**7**], and of Pryce [**42**], to name but three. Alternatively, one may choose to save one's hard earned pennies and consult [**35**].

Theorems from measure theory

We shall also require a couple of results from Measure Theory.

1.29. Definition. *A measure μ on the Borel sets of a Hausdorff topological space X is said to be a* ***Radon measure*** *if it satisfies the following two properties.*

(a) *The measure μ is* ***regular****: that is, if $U \subseteq X$ is open, then*

$$\mu(U) = \sup\{\mu(K) : K \subseteq U, K\, compact\}.$$

(b) *The measure μ is* ***locally finite****: that is, $x \in X$ implies that there exists a nbhd U in the neighbourhood base $\mathcal{U}_x$ of x such that $\mu(U) < \infty$¿*

1.30. Theorem. [Lusin's Theorem.] *Let (X, Ω, μ) be a Radon measure space, and let $f : X \to \mathbb{C}$ be a measurable function. Given $\varepsilon > 0$ and $Y \subseteq X$, there exists a closed set $F \subseteq X$ such that $\mu(Y \setminus F) < \varepsilon$ and $f|_F$ is continuous.*

If Y is locally compact, we can choose F to be compact and we can find a continuous function $g_\varepsilon : X \to \mathbb{C}$ with compact support such that $g_\varepsilon|_F = f$ and

$$\sup_{x \in X} |g_\varepsilon(x)| \le \sup_{x \in X} |f(x)|.$$

1.31. Theorem. [The Riesz-Markov Theorem.] *Let X be a locally compact Hausdorff space. If φ is a continuous linear functional on the space $\mathcal{C}_\circ(X)$ of continuous functions on X which vanish at infinity, then there exists a unique regular Borel measure μ on X such that*

$$\varphi(f) = \int_X f d\mu \qquad \textit{for all } f \in \mathcal{C}_\circ(X).$$

If X is compact, then μ is a finite measure, and if $0 \le f \in \mathcal{C}_\circ(X)$ implies that $\varphi(f) \ge 0$, then the measure μ is positive.

Supplementary examples

S1.1. Example. Let $x = (x_n)_n$ be a sequence of complex (or real) numbers. The **total variation** of x is defined by

$$V(x) := \sum_{n=1}^{\infty} |x_{n+1} - x_n|.$$

If $V(x) < \infty$, we say that x has **bounded variation**. The space

$$\mathbf{bv} := \{(x_n)_n : x_n \in \mathbb{K}, n \geq 1, V(x) < \infty\}$$

is called the space of **sequences of bounded variation**. We may define a norm on $\mathbf{bv}$ as follows: for $x \in \mathbf{bv}$, we set

$$\|(x_n)_n\|_{\mathbf{bv}} := |x_1| + V(x) = |x_1| + \sum_{n=1}^{\infty} |x_{n+1} - x_n|.$$

It can be shown that $\mathbf{bv}$ is complete under this norm, and hence that $\mathbf{bv}$ is a Banach space. which is isomorphic to ℓ^1.

If we let $\mathbf{bv}_0 = \{(x_n)_n \in \mathbf{bv} : \lim_n x_n = 0\}$, then

$$\|(x_n)_n\|_{\mathbf{bv}_0} := V((x_n)_n)$$

defines a norm on $\mathbf{bv}_0$, and again, $\mathbf{bv}_0$ is a Banach space with respect to this norm, and that it is isometrically isomorphic to ℓ^1, though not by the identity map. We refer the reader to [**20**] for more details.

S1.2. Example. Given a sequence $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$, we define the extended real number

$$\mu(x) := \sup_n \left| \sum_{k=1}^{n} x_k \right|.$$

We then define the space

$$\mathbf{bs} = \{x = (x_n)_n \in \mathbb{C}^{\mathbb{N}} : \mu(x) < \infty\}.$$

Again, it can be shown that $\|x\|_{\mathbf{bs}} := \mu(x)$ defines a norm on the vector space $\mathbf{bs}$ under which the latter becomes a Banach space. As before, we refer the reader to [**20**] for more details.

S1.3. Example. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ denote the closed unit disk in the complex plane, and $\mathbb{D}^\circ := \{z \in \mathbb{C} : |z| < 1\}$ denote its interior. Then, as we have seen, $\mathcal{C}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} : f \text{ is continuous}\}$ is a Banach space when equipped with the supremum norm $\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{D}\}$.

By the Stone-Weierstraß Theorem, we know that $\mathcal{C}(\mathbb{D})$ is the norm-closure of the set

$$\{p(z, \overline{z}) : p \text{ a polynomial in two commuting variables}\}.$$

The subalgebra $\mathcal{A}(\mathbb{D}) := \{f \in \mathcal{C}(\mathbb{D}, \mathbb{C}) : f|_{\mathbb{D}^\circ} \text{ is holomorphic}\}$ of $\mathcal{C}(\mathbb{D})$ is called the **disk algebra**. It is easily seen to coincide with the norm-closure of the set of all polynomials $\{p(z) : p \text{ a polynomial}\}$ in $\mathcal{C}(\mathbb{D})$, and the function $q^*(z) := \overline{z}$, $z \in \mathbb{D}$ is an example of a function which lies in $\mathcal{C}(\mathbb{D})$ but not in $\mathcal{A}(\mathbb{D})$.

S1.4. Example. Another Banach space of interest to those who study the geometry of said spaces is **James' space**.

For a sequence $(x_n)_n$ of real numbers, consider the following condition, which we shall call **condition** J: for all $k \geq 1$,

$$\sup_{n_1 < n_2 < \cdots < n_k} \left[(x_{n_1} - x_{n_2})^2 + (x_{n_2} - x_{n_3})^2 + \cdots + (x_{n_{k-1}} - x_{n_k})^2 \right] < \infty.$$

The James' space is defined to be:

$$\mathfrak{J} = \{(x_n)_n \in c_0 : (x_n)_n \text{ satisfies condition } J\}.$$

The norm on $\mathfrak{J}$ is defined via:

$$\|(x_n)_n\|_{\mathfrak{J}} := \sup_{n_1 < n_2 < \cdots < n_k} \left[(x_{n_1} - x_{n_2})^2 + (x_{n_2} - x_{n_3})^2 + \cdots + (x_{n_{k-1}} - x_{n_k})^2 \right]^{\frac{1}{2}}.$$

It can be shown that $\mathfrak{J}$ is a Banach space when equipped with this norm.

S1.5. Example. Here's an interesting example we recently found in a paper of Kalton [**32**]. We confess that we are not quite sure what these spaces are used for, but they do appear somewhat exotic, yet tractable.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. A map $f : \mathfrak{X} \to \mathfrak{Y}$ is said to be **positively homogeneous** if for all $x \in \mathfrak{X}$ and $0 < r \in \mathbb{R}$,

$$f(rx) = rf(x).$$

For example, the map $g : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $g(z_1, z_2, z_3) = (|z_3|, |z_2 + z_1|, z_3)$ is positively homogeneous. Observe that g is *not linear*.

We say that a positively homogeneous map is **bounded** if

$$\sup\{\|f(x)\|_{\mathfrak{Y}} : \|x\|_{\mathfrak{X}} \leq 1\} < \infty.$$

We define the space

$$\mathfrak{H}(\mathfrak{X}, \mathfrak{Y}) := \{f : \mathfrak{X} \to \mathfrak{Y} \mid f \text{ is positively homogeneous and bounded}\}.$$

We leave it as a routine exercise for the reader to prove that $\mathfrak{H}(\mathfrak{X}, \mathfrak{Y})$ becomes a Banach space when equipped with the norm

$$\|f\|_{\mathfrak{H}(\mathfrak{X}, \mathfrak{Y})} := \sup\{\|f(x)\|_{\mathfrak{Y}} : \|x\|_{\mathfrak{X}} \leq 1\}.$$

Appendix

A1.1. Topological vector spaces (or TVS's) are a generalisation of normed linear spaces. A vector space $\mathfrak{V}$ over $\mathbb{C}$ is said to be a **topological vector space** if it is equipped with a Hausdorff topology τ under which the vector space operations of summation and scalar multiplication are continuous.

Many introductions to Functional Analysis avoid mention of these, in part because most of their theory can be avoided by simply concentrating on Banach spaces and Banach algebras. The one place where things get "sticky" is when one wishes to consider various topologies on the dual spaces of Banach spaces, such as the *weak operator topology*, the *strong operator topology*, the *weak***-topology*, and so on. In these cases, the resulting topology is not a norm-topology, and it is very fruitful indeed to see that all of these topologies can be grouped under one umbrella, namely they are all locally convex topologies generated by a separating family of semi-norms. For this reason, the reference notes [**35**] introduce just enough topological vector space and locally convex space theory so as to be able to study all of the relevant operator algebra topologies from without having to introduce them *ad hoc*.

Exercises for Chapter 1

Exercise 1.1. Banach spaces of operators

Recall that a normed linear space $\mathfrak{X}$ is complete if and only if every absolutely summable series in $\mathfrak{X}$ is summable; that is, $\mathfrak{X}$ is complete provided that whenever $\{x_n\}_{n=1}^\infty \subseteq \mathfrak{X}$ satisfies $\sum_n \|x_n\| < \infty$, there exists $x \in \mathfrak{X}$ so that $x = \lim_{N\to\infty} \sum_{n=1}^N x_n$.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed linear spaces.

(a) Prove that the *operator norm* $\|T\| := \sup\{\|Tx\| : \|x\| \le 1\}$ is indeed a norm on $\mathcal{B}(\mathfrak{X}, \mathfrak{Y}) := \{T : \mathfrak{X} \to \mathfrak{Y} : \|T\| < \infty\}$;

(b) Prove that $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is complete if and only if $\mathfrak{Y}$ is complete.

Exercise 1.2. ℓ^p-sums and duals

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and $1 < p < \infty$. Let $\mathfrak{Z} = \mathfrak{X} \oplus_p \mathfrak{Y}$, i.e., $\mathfrak{Z} = \{(x, y) : x \in \mathfrak{X}, y \in \mathfrak{Y}\}$ with $\|(x, y)\|_{\mathfrak{Z}} := (\|x\|_{\mathfrak{X}}^p + \|y\|_{\mathfrak{Y}}^p)^{1/p}$.

(a) Prove that $\|\cdot\|_{\mathfrak{Z}}$ is indeed a norm on $\mathfrak{Z}$. You may assume that $(\mathbb{R}^2, \|\cdot\|_p)$ is a Banach space if this helps.

Note: it is understood that for $p = \infty$, $\|(x, y)\|_{\mathfrak{Z}} := \max(\|x\|_{\mathfrak{X}}, \|y\|_{\mathfrak{Y}})$.

(b) Show that $\mathfrak{Z}^* \equiv \mathfrak{X}^* \oplus_q \mathfrak{Y}^*$, where $1/p + 1/q = 1$. That is, prove that there exists a bijective linear isometry from $\mathfrak{Z}^*$ onto $\mathfrak{X}^* \oplus_q \mathfrak{Y}^*$. You may assume that the dual of $(\mathbb{R}^2, \|\cdot\|_p)$ is $(\mathbb{R}^2, \|\cdot\|_q)$ if this helps.

Prove that the same result holds for $p = 1$ and $p = \infty$.

Exercise 1.3. Compact operators between Banach spaces

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Then T is said to be **compact** if $\overline{T(\mathfrak{X}_1)}$ is compact in $\mathfrak{Y}$. The set of compact operators from $\mathfrak{X}$ to $\mathfrak{Y}$ is denoted by $\mathcal{K}(\mathfrak{X}, \mathfrak{Y})$, and if $\mathfrak{Y} = \mathfrak{X}$, we simply write $\mathcal{K}(\mathfrak{X})$.

Recall that a subset K of a metric space L is said to be **totally bounded** if for every $\varepsilon > 0$ there exists a finite cover $\{V_\varepsilon(y_i)\}_{i=1}^n$ of K with $y_i \in K$, $1 \le i \le n$, where $V_\varepsilon(y_i) = \{z \in L : \operatorname{dist}(z, y_i) < \varepsilon\}$.

We remind the reader that if E is a subset of L and E is totally bounded, then so is $\overline{E}$, and that if E is a subset of a metric space, then E is compact if and only if E is complete and totally bounded, if and only if E is sequentially compact (i.e. for each sequence $(x_n)_n$ in E we can find a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_n$ which converges to some element of E).

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Given $r > 0$, we denote by $\mathfrak{X}_r$ the closed ball $\{x \in \mathfrak{X} : \|x\|_{\mathfrak{X}} \le r\}$ of radius r in $\mathfrak{X}$.

Prove that the following are equivalent:

(a) T is compact;

(b) $\overline{T(F)}$ is compact in $\mathfrak{Y}$ for all bounded subsets F of $\mathfrak{X}$;

(c) If $(x_n)_n$ is a bounded sequence in $\mathfrak{X}$, then $(Tx_n)_n$ has a convergent subsequence in $\mathfrak{Y}$;

(d) $T(\mathfrak{X}_1)$ is totally bounded.

Exercise 1.4. TENSOR PRODUCTS OF BANACH SPACES

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be complex Banach spaces and $\mathfrak{X} \odot \mathfrak{Y}$ be the algebraic tensor product of $\mathfrak{X}$ and $\mathfrak{Y}$ as linear spaces over $\mathbb{C}$.

(a) Show that if for $z \in \mathfrak{X} \odot \mathfrak{Y}$ we define

$$\|z\|_\pi = \inf\{\sum_{i=1}^n \|x_i\| \ \|y_i\| : x_1, ..., x_n \in \mathfrak{X}; y_1, ..., y_n \in \mathfrak{Y}; z = \sum_{i=1}^n x_i \otimes y_i\},$$

then $\|\cdot\|_\pi$ is a norm on $\mathfrak{X} \odot \mathfrak{Y}$. The completion of $\mathfrak{X} \odot \mathfrak{Y}$ is called the *projective tensor product* of $\mathfrak{X}$ and $\mathfrak{Y}$ and is denoted by $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$.

(b) Show that if for $z \in \mathfrak{X} \odot \mathfrak{Y}$ we define

$$\begin{aligned} \|z\|_i \quad = \quad & \sup\{\left|\sum_{i=1}^n \varphi(x_i)\, \psi(y_i)\right| : x_1, ..., x_n \in \mathfrak{X}; \\ & \qquad y_1, ..., y_n \in \mathfrak{Y}; \varphi \in \mathfrak{X}^*, \psi \in \mathfrak{Y}^*; \|\varphi\| = \|\psi\| = 1; z = \sum_{i=1}^n x_i \otimes y_i\}, \end{aligned}$$

then $\|\cdot\|_i$ is a norm on $\mathfrak{X} \odot \mathfrak{Y}$. The completion of $\mathfrak{X} \odot \mathfrak{Y}$ with respect to this norm is called the *injective tensor product* of $\mathfrak{X}$ and $\mathfrak{Y}$, and is denoted by $\mathfrak{X} \overline{\otimes} \mathfrak{Y}$.

(c) Show that the identity mapping j on $\mathfrak{X} \odot \mathfrak{Y}$ extends to a contractive mapping from $\mathfrak{X} \widehat{\otimes} \mathfrak{Y}$ to $\mathfrak{X} \overline{\otimes} \mathfrak{Y}$.

Exercise 1.5. WEAK CONVERGENCE

(a) Let (x_n) be a sequence in ℓ_2 converging in the weak topology to $x \in \ell_2$. Prove that if $\|x_n\|_2$ converges to $\|x\|_2$ as $n \to \infty$, then $\|x_n - x\|_2 \to 0$.

(b) Let $\{e_n\}_{n=1}^\infty$ denote the standard ONB of ℓ_2 and let

$$A = \{e_m + me_n : 1 \le m < n\}.$$

Prove that 0 is in the weak-closure of A but that no sequence in A converges weakly to 0.

Exercise 1.6. WEAK CLOSURES OF THE UNIT BALL

Suppose that $\mathfrak{X}$ is an infinite dimensional Banach space. Prove that the weak closure of the unit sphere $S^1 = \{x \in \mathfrak{X} : \|x\| = 1\}$ is the closed unit ball $\mathfrak{X}_1 = \{x \in \mathfrak{X} : \|x\| \le 1\}$.

Exercise 1.7. SCHAUDER BASES

(a) Let $\{e_n\}_{n=1}^\infty$ be a Schauder basis for a separable Banach space $(\mathfrak{X}, \|\cdot\|)$. Define

$$P_n(\sum_i \lambda_i e_i) = \sum_i n\lambda_i e_i$$

and for $x \in \mathfrak{X}$, define $|||x||| = \sup_{n\ge1} \|P_n x\|$. Prove that $|||x|||$ is a norm on $\mathfrak{X}$.

(b) *Assume that $\mathfrak{X}$ is complete with respect to this norm and that $\{e_n\}_{n=1}^\infty$ is also a Schauder basis for $(\mathfrak{X}, |||\cdot|||)$.* (See the addendum at the end of the assignment for a proof.) Prove that the norms $|||\cdot|||$ and $\|\cdot\|$ are equivalent.

(c) Prove that $P_n \in \mathcal{B}(\mathfrak{X})$ for all $n \geq 1$ (where $\mathfrak{X}$ carries its usual norm $\|\cdot\|$), and that

$$\sup_{n\geq 1} \|P_n\| < \infty.$$

Exercise 1.8. Separable Banach spaces

Prove that every separable Banach space $\mathfrak{X}$ is isometrically isomorphic to a quotient space of ℓ^1.

Hint. Let $(x_n)_n$ be a dense subset of $\mathfrak{X}_1$, the closed unit ball of $\mathfrak{X}$. For each $\alpha := (\alpha_n)_n \in \ell^1$, set $y_\alpha := \sum_n \alpha_n x_n$ and consider the map $T(\alpha) := y_\alpha$.

CHAPTER 2

Banach Algebras - an introduction

Who are you going to believe, me or your own eyes?

Groucho Marx

Definitions and Examples

2.1. The principal objects of investigation in this course are Banach algebras. The typical approach to writing a set of notes or a book is to begin with the definition of a Banach algebra (or whatever mathematical structure one wishes to investigate), and to then produce a (hopefully long) list of examples to demonstrate their importance and to justify having made the definition. This process is, however, essentially the inverse of how such a concept develops in the first place. In practice, one normally starts with a long list of examples of objects, each of which has its own intrinsic value. Through inspiration and hard work, one may discover certain commonalities between those objects, and one may observe that any other object which shares that commonality will – by necessity – behave in a certain way as suggested by the initial objects of interest. The advantage of doing this is that if one can prove that certain behaviour is a result of the commonality, as opposed to being specific to one of the examples, then one may deduce that such behaviour will apply to all objects in this collection and beyond.

Granted an unlimited amount of time, it would be instructive to approach an entire course through carefully selected examples which gradually lead the students to discover on their own the desired definitions, propositions and theorems. Over a period of twelve weeks, however, this is not practical, and so we submit to the common, if more prosaic practice of definition, example, theorem and proof.

2.2. Definition. *A* ***Banach algebra*** $\mathcal{A}$ *is a Banach space together with a norm compatible algebra structure, namely: for all* $x, y \in \mathcal{A}$, $\|xy\| \le \|x\| \, \|y\|$. *If* $\mathcal{A}$ *has a multiplicative identity* (*denoted by* e *or* $\mathbb{1}$), *we say that the algebra* $\mathcal{A}$ *is* ***unital***.

Note that when $\mathcal{A}$ is unital,

$$\|\mathbb{1}\| = \|\mathbb{1}^2\| \le \|\mathbb{1}\|^2,$$

and so $\mathbb{1} \neq 0$ implies that $\|\mathbb{1}\| \geq 1$. By scaling the norm if necessary, we may (and do) assume that $\|\mathbb{1}\| = 1$.

2.3. Example. The Banach space $(\mathcal{C}(X), \|\cdot\|_\infty)$ of continuous complex-valued functions on a compact Hausdorff space X introduced in Example 1.5 becomes a Banach algebra under pointwise multiplication of functions. That is, for $f, g \in (\mathcal{C}(X), \|\cdot\|_\infty)$, we set

$$(fg)(x) = f(x)g(x) \quad \text{for all } x \in X.$$

The constant function $\mathbb{1}(x) = 1$, $x \in X$ serves as the multiplicative identity of $\mathcal{C}(X)$.

2.4. Example. The Banach space $(\mathcal{C}_0(X), \|\cdot\|_\infty)$ of continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space as introduced in Example 1.6 also becomes a Banach algebra under pointwise multiplication of functions. Note that this algebra is not unital unless X is compact. In particular, if $X = \mathbb{N}$ with the discrete topology, then $(\mathcal{C}_0(X), \|\cdot\|_\infty) = c_0$ is a non-unital Banach algebra under component-wise multiplication.

2.5. Example. Let us return to Example S1.3. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and let $\mathbb{D}^\circ$ be the interior of $\mathbb{D}$. Also, let $\mathcal{A}(\mathbb{D}) = \{f \in \mathcal{C}(\mathbb{D}) : f \text{ is holomorphic on } \mathbb{D}^\circ\}$. Then $(\mathcal{A}(\mathbb{D}), \|\cdot\|_\infty)$ is a unital Banach algebra under pointwise multiplication of functions, called the **disk algebra**.

The map

$$\begin{array}{rccc} \tau : & \mathcal{A}(\mathbb{D}) & \to & \mathcal{C}(\mathbb{T}) \\ & f & \mapsto & f|_{\mathbb{T}} \end{array}$$

is an isometric embedding, by the Maximum Modulus Principle.

It is often useful to identify the disk algebra with its image under this embedding. Thus, it is not unusual to see the disk algebra defined as

$$\mathcal{A}(\mathbb{D}) = \{f \in \mathcal{C}(\mathbb{T}) : f \text{ can be holomorphically continued to } \mathbb{D}^\circ\}.$$

In general, given a compact subset $X \subseteq \mathbb{C}$, we may define

$$\begin{aligned} \mathcal{A}(X) &= \{f \in \mathcal{C}(X) : f \text{ is holomorphic on int}\,(X)\}, \\ \mathcal{R}(X) &= \{f \in \mathcal{C}(X) : f \text{ is a rational function} \\ &\qquad\qquad \text{with poles outside of } X\}^{-\|\ \|}, \text{ and} \\ \mathcal{P}(X) &= \{f \in \mathcal{C}(X) : f \text{ is a polynomial}\}^{-\|\ \|} \end{aligned}$$

Each of these is a closed subalgebra of $\mathcal{C}(X)$ under the supremum norm. Clearly $\mathcal{P}(X) \subseteq \mathcal{R}(X) \subseteq \mathcal{A}(X) \subseteq \mathcal{C}(X)$, and it is often an interesting and important problem to decide when the inclusions reduce to equalities. That this is the case, for instance, when X is a compact subset of the real line is the content of the Stone-Weierstraß Theorem.

2.6. Example. Let G be a locally compact abelian group, and let ν denote Haar measure on G. Then

$$L^1(G,\nu) = \{f : G \to \mathbb{C} : \int_G |f(x)|\, d\nu(x) < \infty\}.$$

For $f,\, g \in L^1(G,\nu),\ x \in G$, we define the product of f and g via convolution:

$$(f * g)(x) = \int_G f(xy^{-1})\, g(y)\, d\nu(y).$$

We also define $\|f\|_1 = \int_G |f(x)|\, d\nu(x)$.

This is called the **group algebra** of G. It is a standard result (cf. Paragraph 5.32) that $f * g = g * f$ and that $\|f * g\|_1 \leq \|f\|_1\, \|g\|_1$. (We remind the reader that elements of $L^1(G,\nu)$ are really *equivalence classes* of functions, and so the definition of $f * g$ is interpreted as "*almost everywhere*" with respect to ν.)

Writing $\ell^1(\mathbb{Z}) = L^1(\mathbb{Z},\nu)$ where ν represents counting measure on $\mathbb{Z}$, we obtain:

$$\begin{array}{rcl} (f * g)(n) & = & \sum_{k \in \mathbb{Z}} f(n-k) g(k) \\ \|f\|_1 & = & \sum_{k \in \mathbb{Z}} |f(k)|. \end{array}$$

As we shall see in Chapter 5, $\ell^1(\mathbb{Z})$ can be identified with the **Wiener algebra**

$$\mathcal{AC}(\mathbb{T}) = \{f \in \mathcal{C}(\mathbb{T}) : f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta}, \quad \sum_{n \in \mathbb{Z}} |a_n| < \infty\},$$

where $a_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta$.

2.7. Example. The Banach space $L^\infty(X, \Omega, \nu)$ is a unital Banach algebra under pointwise multiplication (almost everywhere).

2.8. Example. The set $\mathcal{C}_b(\Omega)$ continuous, complex-valued, bounded functions on a locally compact space Ω is a unital Banach algebra under the supremum norm and pointwise multiplication.

The above examples are all abelian. The following need not be.

2.9. Example. Let $\mathfrak{X}$ be a Banach space. Then the Banach space $\mathcal{B}(\mathfrak{X})$ from Example 1.13 is a Banach algebra, using the operator norm and composition of linear maps as our product. To verify this, we need only verify that the operator norm is submultiplicative, that is, that $\|AB\| \leq \|A\|\, \|B\|$ for all operators A and B. But indeed, we observe that

$$\begin{array}{rcl} \|AB\| & = & \sup\{\|ABx\| : \|x\| = 1\} \\ & \leq & \sup\{\|A\|\, \|Bx\| : \|x\| = 1\} \\ & \leq & \sup\{\|A\|\, \|B\|\, \|x\| : \|x\| = 1\} \\ & = & \|A\|\, \|B\|. \end{array}$$

In particular, $\mathbb{M}_n$ can be identified with $\mathcal{B}(\mathbb{C}^n)$ by first fixing an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ for $\mathbb{C}^n$, and then identifying a linear map in $\mathcal{B}(\mathbb{C}^n)$ with its matrix representation with respect to this fixed basis.

It is easy to verify that any closed subalgebra of $\mathcal{B}(\mathfrak{X})$ (or indeed, of any Banach algebra) is itself a Banach algebra using the operator norm.

2.10. Example. Let $\mathfrak{X}$ be a Banach space, and let $T \in \mathcal{B}(\mathfrak{X})$. Then

$$\mathrm{Alg}(T) = \{p(T) : p \text{ a polynomial over } \mathbb{C}\}^{-\|\ \|}$$

is a Banach algebra, called the **algebra generated by** T. The norm under consideration is the operator norm.

2.11. Example. Let $\mathcal{T}_n$ denote the set of upper triangular $n \times n$ matrices in $\mathbb{M}_n$, equipped with the operator norm. Then $\mathcal{T}_n$ is a Banach subalgebra of $\mathbb{M}_n$. After fixing an orthonormal basis for the underlying Hilbert space, $\mathcal{T}_n$ can be viewed as a Banach subalgebra of $\mathcal{B}(\mathbb{C}^n)$. In fact, it is the largest subalgebra of $\mathcal{B}(\mathbb{C}^n)$ which leaves each of the subspaces $\mathcal{H}_k = \mathrm{span}\{e_1, e_2, \dots, e_k\}$, $1 \le k \le n$ invariant.

More generally, given a Banach space $\mathfrak{X}$ and a collection $\mathcal{L}$ of closed subspaces $\{L_\alpha\}_{\alpha \in \Lambda}$ of $\mathfrak{X}$, then

$$\mathrm{Alg}(\mathcal{L}) = \{T \in \mathcal{B}(\mathfrak{X}) : T L_\alpha \subseteq L_\alpha \text{ for all } L_\alpha \in \mathcal{L}\}$$

is a Banach algebra. This is closed because if $\lim_{n\to\infty} T_n = T$, then $x \in L_\alpha$ implies $Tx = \lim_{n\to\infty} T_n x \in L_\alpha$ for each α.

2.12. Example. The space $H^\infty(\mathbb{T}, \mu)$ defined in Example 1.9 is a unital Banach algebra under pointwise multiplication of functions (almost everywhere).

2.13. Example. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}(\mathcal{H})$ denote the set of compact operators acting on $\mathcal{H}$. (We shall deal with these in more detail in Chapter 7. Then $\mathcal{K}(\mathcal{H})$ is a Banach subalgebra of $\mathcal{B}(\mathcal{H})$. It is non-unital if $\mathcal{H}$ is infinite-dimensional. In fact, as we shall see, $\mathcal{K}(\mathcal{H})$ is a closed, two-sided ideal of $\mathcal{B}(\mathcal{H})$.

2.14. Example. Let $(\mathcal{A}_\alpha, \|\cdot\|_\alpha)_{\alpha \in \Lambda}$ denote a family of Banach algebras indexed by a set Λ. Then

$$\mathcal{A} = \prod_\alpha \mathcal{A}_\alpha := \{(a_\alpha)_{\alpha \in \Lambda} : a_\alpha \in \mathcal{A}_\alpha, \alpha \in \Lambda, \sup_\alpha \|a_\alpha\| < \infty\}$$

is a Banach algebra when equipped with the norm

$$\|(a_\alpha)_\alpha\| = \sup_\alpha \|a_\alpha\|_\alpha.$$

If each $\mathcal{A}_\alpha$ is unital, then so is $\mathcal{A}$. This is the **direct product** of the algebras $\mathcal{A}_\alpha$.

We may also define

$$\mathcal{B} := \oplus_\alpha \mathcal{A}_\alpha := \{(a_\alpha)_\alpha \in \prod_\alpha \mathcal{A}_\alpha, |\{\alpha \in \Lambda : \|a_\alpha\|_\alpha \ge \varepsilon\}| < \infty \text{ for all } \varepsilon > 0\}.$$

If Λ is infinite, then $\mathcal{B}$ is never unital. We refer to $\mathcal{B}$ as the **direct sum** of the algebras $\mathcal{A}_\alpha$.

2.15. Example. Consider the Hilbert space $\mathfrak{H} = H^2(\mathbb{T}, \mu)$ of Example 1.9. Let $P_{\mathfrak{H}}$ denote the orthogonal projection of $L^2(\mathbb{T}, \mu)$ onto $\mathfrak{H}$. A **Toeplitz operator** with symbol $\varphi \in \mathcal{C}(\mathbb{T})$ is an operator $T_\varphi \in \mathcal{B}(\mathfrak{H})$ of the form

$$T_\varphi(g) = P_{\mathfrak{H}}(\varphi g), \quad g \in \mathfrak{H}.$$

The **Toeplitz algebra** on $\mathfrak{H}$ to be the set of operators in $\mathcal{B}(\mathfrak{H})$ of the form

$$T_\varphi + K,$$

where $\varphi \in \mathcal{C}(\mathbb{T})$ and $K \in \mathcal{K}(\mathfrak{H})$ is a compact operator on $\mathfrak{H}$. This is a Banach subalgebra of $\mathcal{B}(\mathfrak{H})$.

Basic Results

2.16. Now that we have a plentiful supply of Banach algebras at hand, we may begin to prove results about them.

2.17. Proposition. *Let $\mathcal{K}$ be a closed ideal in a Banach algebra $\mathcal{A}$. Then the quotient space $\mathcal{A}/\mathcal{K}$ is a Banach algebra with respect to the quotient norm.*

Proof. That $\mathcal{A}/\mathcal{K}$ is a Banach space follows from Example 1.11. Let π denote the canonical map from $\mathcal{A}$ to $\mathcal{A}/\mathcal{K}$. We must show that

$$\|\pi(x)\pi(y)\| \le \|\pi(x)\| \, \|\pi(y)\|$$

for all $x, y \in \mathcal{A}$.

Suppose $\varepsilon > 0$. By definition of the quotient norm, we can find $m, n \in \mathcal{K}$ such that $\|x + m\| < \|\pi(x)\| + \varepsilon$ and $\|y + n\| < \|\pi(y)\| + \varepsilon$. Then

$$\begin{aligned}
\|\pi(x)\pi(y)\| &= \|\pi(x+m)\pi(y+n)\| \\
&\le \|\pi((x+m)\,(y+n))\| \\
&\le \|(x+m)\,(y+n)\| \\
&\le \|(x+m)\| \, \|(y+n)\| \\
&< (\|\pi(x)\| + \varepsilon)\,(\|\pi(y)\| + \varepsilon).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired result.

□

Recall from Example 2.14 our claim that the set $\mathcal{K}(\mathcal{H})$ of compact operators is a closed, two-sided ideal of $\mathcal{B}(\mathcal{H})$. Using this and the above Proposition, we obtain the following important example.

2.18. Example. Let $\mathcal{H}$ be a Hilbert space. Then the quotient algebra

$$\mathcal{Q}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$$

is a Banach algebra, known as the **Calkin algebra**. The canonical map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{Q}(\mathcal{H})$ is denoted by π.

2.19. Remark. In general, if a Banach algebra $\mathcal{A}$ does not have an identity element, it is possible to append one as follows:

Consider the linear space $\mathcal{A}_u = \mathcal{A} \oplus \mathbb{C}$. We may define a multiplication on $\mathcal{A}_u$ by setting

$$(a, \alpha) \cdot (b, \beta) := (ab + a\beta + b\alpha, \alpha\beta).$$

(This definition is not random – it is helpful to think of the ordered pair (a, α) as "$a + \alpha\mathbb{1}$", where $\mathbb{1} := (0, 1)$ should be the multiplicative identity of $\mathcal{A}_u$. The above equation then writes itself.)

We define a norm on $\mathcal{A}_u$ via $\|(a, \alpha)\| = \|a\| + |\alpha|$. Then we have

$$\begin{aligned} \|(a, \alpha)\,(b, \beta)\| &= \|ab + a\beta + b\alpha\| + |\alpha\beta| \\ &\leq \|a\|\,\|b\| + \|a\|\,|\beta| + \|b\|\,|\alpha| + |\alpha|\,|\beta| \\ &= (\|a\| + |\alpha|)\,(\|b\| + |\beta|) \\ &= (\|(a, \alpha)\|)\,(\|(b, \beta)\|) \end{aligned}$$

It is clear that the embedding of $\mathcal{A}$ into $\mathcal{A}_u$ is linear and isometric, and that $\mathcal{A}$ sits inside of $\mathcal{A}_u$ as a closed ideal. (In fact, as a maximal ideal, given that its co-dimension is equal to 1.) It should be added, however, that this construction is not always natural. The group algebra $L^1(\mathbb{R}, \nu)$ of the real numbers with Lebesgue measure ν is not unital. On the other hand, the most natural candidate for a multiplicative identity here might be the Dirac delta function (corresponding to a discrete measure with mass one at 0 and zero elsewhere), which clearly does not lie in the algebra. Similarly, $\mathcal{C}_0(\mathbb{R})$ is another much studied non-unital algebra. In this case, there is more than one way to embed this algebra into a unital Banach algebra. For instance, one may want to consider the one-point compactification, or the Stone-Čech compactifications of the reals. Each of these induces an imbedding of $\mathcal{C}_0(\mathbb{R})$ into the corresponding unital Banach algebra of continuous functions on these compactifications.

2.20. Proposition. *Every Banach algebra $\mathcal{A}$ embeds isometrically into $\mathcal{B}(\mathfrak{X})$ for some Banach space $\mathfrak{X}$. Here, $\mathcal{A}$ need not have a unit.*

Proof. Consider the map

$$\begin{array}{rccc} \Phi: & \mathcal{A} & \to & \mathcal{B}(\mathcal{A}_u) \\ & a & \mapsto & L_a \end{array}$$

where $L_a(x, \lambda) = (a, 0)\,(x, \lambda)$ is the **left regular representation** of $\mathcal{A}$. That

$$\Phi(\alpha a + b) = L_{\alpha a + b} = \alpha L_a + L_b = \alpha\Phi(a) + \Phi(b)$$

and that

$$\Phi(ab) = L_{ab} = L_a\,L_b = \Phi(a)\Phi(b)$$

for all $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$ are easily verified.

Then

$$\|\Phi(a)\| = \|L_a\| = \sup_{(x,\lambda)\neq(0,0)} \frac{\|(a,0)\,(x,\lambda)\|}{\|(x,\lambda)\|} \leq \|(a,0)\| = \|a\|$$

and

$$\|\Phi(a)\| = \|L_a\| \geq \|(a,0)\,(0,1)\| = \|a\|,$$

so that $\|\Phi(a)\| = \|L_a\| = \|a\|$. In particular, the map is isometric.

□

We recall that a group G equipped with a topology τ is said to be a **topological group** if τ is Hausdorff, and if the maps

$$\begin{array}{rccc} \mu: & G \times G & \to & G \\ & (g,h) & \mapsto & gh \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \iota: & G & \to & G \\ & g & \mapsto & g^{-1} \end{array}$$

are continous.

Our present goal is to show that if $\mathcal{A}$ is a unital Banach algebra, then the group $\mathcal{A}^{-1}$ of invertible elements of $\mathcal{A}$ is a topological group using the norm topology, which is clearly Hausdorff. The next result, while not particularly difficult to prove, is exceedingly useful when studying Banach and C^*-algebras.

2.21. Theorem. *The set $\mathcal{A}^{-1}$ of invertible elements of a unital Banach algebra $\mathcal{A}$ is open in the norm topology. In fact, if $d \in \mathcal{A}^{-1}$, then the open ball of radius $\|d^{-1}\|^{-1}$ centred at d is contained in $\mathcal{A}^{-1}$.*

Proof. If $\|a\| < 1$, then the element $b = \sum_{n=0}^{\infty} a^n$ exists in $\mathcal{A}$ since the defining series is absolutely convergent. As such,

$$\begin{aligned} (\mathbb{1} - a)\, b &= (\mathbb{1} - a) \left(\lim_{k\to\infty} \sum_{n=0}^{k} a^n \right) \\ &= \left(\lim_{k\to\infty} \sum_{n=0}^{k} a^n \right) - \left(\lim_{k\to\infty} \sum_{n=1}^{k+1} a^n \right) \\ &= \lim_{k\to\infty} \mathbb{1} - a^{k+1} \\ &= \mathbb{1} \\ &= \left(\lim_{k\to\infty} \sum_{n=0}^{k} a^n \right) - \left(\lim_{k\to\infty} \sum_{n=1}^{k+1} a^n \right) \\ &= \left(\lim_{k\to\infty} \sum_{n=0}^{k} a^n \right) (\mathbb{1} - a) \\ &= b\,(\mathbb{1} - a), \end{aligned}$$

so that the open ball of radius 1 centred at the identity $\mathbb{1}$ is contained in the set of invertible elements of $\mathcal{A}$.

Now if $d \in \mathcal{A}^{-1}$ and $\|a\| < \|d^{-1}\|^{-1}$, then $(d - a) = d\,(\mathbb{1} - d^{-1}a)$ and $\|d^{-1}a\| < 1$ so that

$$(d-a)^{-1} = (\mathbb{1} - d^{-1}a)^{-1}d^{-1}$$

exists. This means that the open ball of radius $\|d^{-1}\|^{-1}$ centred at d is again contained in $\mathcal{A}^{-1}$. Thus $\mathcal{A}^{-1}$ is open.

□

2.22. Corollary. *If $\mathcal{A}$ is a unital Banach algebra, then the map $\tau : a \mapsto a^{-1}$ is a homeomorphism of $\mathcal{A}^{-1}$ onto itself. It follows that $\mathcal{A}^{-1}$ is a topological group.*

Proof. That multiplication is continuous in $\mathcal{A}^{-1}$ follows from the fact that it is jointly continuous in $\mathcal{A}$. It remains therefore to show that τ is continuous - as it is clearly a bijection which is equal to its own inverse.

Let us first show that τ is continuous at $\mathbb{1}$. If $\|b\| < 1$, then we have just seen that $(\mathbb{1} - b)$ is invertible and

$$\begin{aligned}
\|\mathbb{1} - (1-b)^{-1}\| &= \|\mathbb{1} - \sum_{n=0}^{\infty} b^n\| \\
&= \|\sum_{n=1}^{\infty} b^n\| \\
&\le \sum_{n=1}^{\infty} \|b\|^n \\
&= \|b\|/(\mathbb{1} - \|b\|).
\end{aligned}$$

Thus as $\|b\| \to 0$ (i.e. as $b \to 0$ and hence $(\mathbb{1} - b) \to \mathbb{1}$), we get $(\mathbb{1} - b)^{-1} \to \mathbb{1}$, implying that the map $\tau : a \mapsto a^{-1}$ is continuous at 1, as claimed.

If $a \in \mathcal{A}^{-1}$ and $a_n \to a$, then $a_n\, a^{-1} \to a\, a^{-1} = \mathbb{1}$, and also $a^{-1}\, a_n \to a^{-1}\, a = \mathbb{1}$, so that $a_n^{-1} \to a^{-1}$.

□

2.23. Proposition. *Let G be a locally connected topological group, and let G_0 be the connected component of the identity e in G. Then G_0 is an open and closed normal subgroup of G, the cosets of G_0 are the components of G, and G/G_0 is a discrete group.*

Proof. A component of a topological space is always closed. If $g \in G$, then G locally connected implies that there exists an open connected neighbourhood $\mathcal{O}_g$ of g which clearly lies in the connected component C_g of g. This shows that C_g is open and therefore components of G are both open and closed.

Let $f \in G$. Then the map $L_{f^{-1}} : h \mapsto f^{-1}h$ is a homeomorphism of G, and so $f^{-1}G_0$ is open, closed and connected. If, furthermore, $f \in G_0$ and $g \in G_0$, then $f^{-1}G_0$ is a connected set containing e and $f^{-1}g$, and therefore $f^{-1}g \in G_0$, implying that G_0 is a subgroup of G. Since the map $R_f : h \mapsto h\,f$ is also a homeomorphism of G, it follows that $f^{-1}\,G_0\,f = L_{f^{-1}}(R_f(G_0))$ is an open, closed and connected subset of G containing e, so that $f^{-1}\,G_0\,f = G_0$, and therefore G_0 is normal.

Since $f^{-1}G_0$ is open, closed and connected for all $f \in G$, the cosets of G_0 are precisely the components of G. In particular, in G/G_0, each point is both open and closed in the quotient topology, and thus G/G_0 is discrete.

□

2.24. Definition. *Let $\mathcal{A}$ be a unital Banach algebra. Let $\mathcal{A}_0^{-1}$ denote the connected component of the identity in $\mathcal{A}^{-1}$. Then the* ***abstract index group*** *of $\mathcal{A}$, denoted $\Lambda_{\mathcal{A}}$, is the group $\mathcal{A}^{-1}/\mathcal{A}_0^{-1}$. The* ***abstract index*** *is the canonical homomorphism from $\mathcal{A}^{-1}$ to $\Lambda_{\mathcal{A}}$.*

2.25. Remark. It follows from Proposition 2.23 that the abstract index group of a Banach algebra $\mathcal{A}$ is well-defined, that $\Lambda_{\mathcal{A}}$ is discrete, and that the components of $\mathcal{A}^{-1}$ are the cosets of $\mathcal{A}_0^{-1}$ in $\Lambda_{\mathcal{A}}$.

2.26. Example. We leave it as an exercise for the reader to show that the set $\mathrm{GL}_n(\mathbb{C})$ of invertible $n \times n$ complex matrices is connected. It follows that $\Lambda_{\mathbb{M}_n(\mathbb{C})} = \{0\}$.

2.27. Exercise. More interesting is the fact that $\Lambda_{\mathcal{C}(\mathbb{T})} \simeq \mathbb{Z}$, which we also leave as an exercise.

One of the most important tools to study elements of Banach algebras is their spectrum.

2.28. Definition. *Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. If $\mathcal{A}$ is unital, then the* ***spectrum*** *of a relative to $\mathcal{A}$ is the set*

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : a - \lambda\mathbb{1} \text{ is not invertible in } \mathcal{A}\}.$$

If $\mathcal{A}$ is not unital, then $\sigma_{\mathcal{A}}(a)$ is set to be $\sigma_{\mathcal{A}_u}(a)$.

Observe that since $\mathcal{A}$ is a proper ideal in $\mathcal{A}_u$, 0 always lies in $\sigma_{\mathcal{A}_u}(a)$. When the algebra $\mathcal{A}$ is understood, we generally write $\sigma(a)$. The ***resolvent*** *of a is the set $\rho(a) = \mathbb{C}\backslash\sigma(a)$.*

2.29. Corollary. *Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$. Then $\rho(a)$ is open and $\sigma(a)$ is compact.*

Proof. Clearly $\rho(a) = \{\lambda \in \mathbb{C} : a - \lambda\mathbb{1} \text{ is invertible}\}$ is open, since $\mathcal{A}^{-1}$ is. Indeed, if $a - \lambda_0\mathbb{1}$ is invertible in $\mathcal{A}$, then $\lambda \in \rho(a)$ for all $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| < \|(a - \lambda_0\mathbb{1})^{-1}\|^{-1}$. Thus $\sigma(a)$ is closed.

If $|\lambda| > \|a\|$, then $\lambda\mathbb{1} - a = \lambda(\mathbb{1} - \lambda^{-1}a)$ and $\|\lambda^{-1}a\| < 1$, and so $(\mathbb{1} - \lambda^{-1}a)$ is invertible. This implies

$$(\lambda\mathbb{1} - a)^{-1} = \lambda^{-1}(\mathbb{1} - \lambda^{-1}a)^{-1}.$$

Thus $\sigma(a)$ is contained in the closed disk $\mathbb{D}_{\|a\|} := \{z \in \mathbb{C} : |z| \le \|a\|\}$ of radius $\|a\|$ centred at the origin. Since it both closed and bounded, $\sigma(a)$ is compact.

□

The above proof shows that for all $a \in \mathcal{A}$, $\sigma(a) \subseteq \mathbb{D}_{\|a\|}$. This is a useful fact to keep in mind.

Although we have determined that $\sigma(a)$ is compact for each element a of a unital Banach algebra $\mathcal{A}$, it is still not clear that it is not empty. Indeed, if we were working over the real field $\mathbb{R}$, this could in fact be the case. For example, the matrix $a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{R})$ has no real eigenvalues, and so the set of "real eigenvalues" satisfies

$$\{\lambda \in \mathbb{R} : a - \lambda I_2 \text{ is not invertible}\} = \varnothing.$$

Since spectrum is one of the most important tools we have to study elements of Banach algebras, we concentrate our attention on complex Banach algebras where, as we shall now demonstrate, the spectrum of an element is always non-empty.

2.30. Definition. *Let $\mathfrak{X}$ be a Banach space and $U \subseteq \mathbb{C}$ be an open set. A function $f : U \to \mathfrak{X}$ is said to be **holomorphic** on U if for each $z \in U$,*

$$f'(z) := \lim_{h \to 0} h^{-1}(f(z+h) - f(z))$$

*exists in $\mathfrak{X}$. We say that $f : U \to \mathfrak{X}$ is **weakly holomorphic** if for each $x^* \in \mathfrak{X}^*$, the complex-valued map $x^* \circ f$ is holomorphic on U in the usual sense.*

2.31. Remark. Suppose that $\varnothing \neq U \subseteq \mathbb{C}$ is open, that $\mathfrak{X}$ is a Banach space and that $f : U \to \mathfrak{X}$ is holomorphic on U. Then for $x^* \in \mathfrak{X}^*$, the continuity and linearity of x^* implies that for $z \in U$,

$$\begin{aligned}
(x^* \circ f')(z) &= x^*(f'(z)) \\
&= x^* \left(\lim_{h \to 0} h^{-1}(f(z+h) - f(z)) \right) \\
&= \lim_{h \to 0} h^{-1} x^*(f(z+h) - f(z)) \\
&= \lim_{h \to 0} h^{-1}(x^* \circ f(z+h) - x^* \circ f(z)) \\
&= \lim_{h \to 0} \frac{(x^* \circ f)(z+h) - (x^* \circ f)(z)}{h}
\end{aligned}$$

exists. Thus f is weakly holomorphic and $(x^* \circ f)'(z) = x^* \circ f'(z)$.

If f is weakly holomorphic on $U = \mathbb{C}$, we say that f is **weakly entire**.

2.32. Theorem. [Liouville's Theorem.] *Every bounded, weakly entire function into a Banach space $\mathfrak{X}$ is constant.*

Proof. For each linear functional $x^* \in \mathfrak{X}^*$, $x^* \circ f$ is a bounded, entire function into the complex plane. By the complex-valued version of Liouville's Theorem, it must therefore be constant. Now by the Hahn-Banach Theorem, $\mathfrak{X}^*$ separates the points of $\mathfrak{X}$. So if there exist $z_1, z_2 \in \mathbb{C}$ such that $f(z_1) \neq f(z_2)$, then there must exist $x^* \in \mathfrak{X}^*$ such that $x^*(f(z_1)) \neq x^*(f(z_2))$. This contradiction implies that f is constant.

□

2.33. Definition. *Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \mathcal{A}$. The map*

$$\begin{array}{rccc} R(\cdot, a): & \rho(a) & \to & \mathcal{A} \\ & \lambda & \mapsto & (\lambda \mathbb{1} - a)^{-1} \end{array}$$

is called the ***resolvent function*** *of a.*

2.34. Proposition. [The Common Denominator Formula.] *Let $a \in \mathcal{A}$, a unital Banach algebra. If $\mu, \lambda \in \rho(a)$, then*

$$R(\lambda, a) - R(\mu, a) = (\mu - \lambda)\, R(\lambda, a)\, R(\mu, a).$$

Proof. The proof is transparent if we consider $t \in \mathbb{C}$ and consider the corresponding complex-valued equation:

$$\frac{1}{\lambda - t} - \frac{1}{\mu - t} = \frac{(\mu - t) - (\lambda - t)}{(\lambda - t)\,(\mu - t)} = \frac{(\mu - \lambda)}{(\lambda - t)\,(\mu - t)}.$$

In terms of Banach algebra, we have:

$$\begin{aligned} R(\lambda, a) &= R(\lambda, a)\, R(\mu, a)\, (\mu - a) \\ R(\mu, a) &= R(\mu, a)\, R(\lambda, a)\, (\lambda - a). \end{aligned}$$

Noting that $R(\lambda, a)$ and $R(\mu, a)$ clearly commute, we obtain the desired equation by simply subtracting the second equation from the first.

□

We shall return to this formula when establishing the holomorphic functional calculus in the next Chapter.

2.35. Proposition. *If $a \in \mathcal{A}$, a unital Banach algebra, then $R(\cdot, a)$ is holomorphic on $\rho(a)$.*

Proof. Let $\lambda_0 \in \rho(a)$. Then

$$\begin{aligned} R'(\lambda_0, a) = \lim_{\lambda \to \lambda_0} \frac{R(\lambda, a) - R(\lambda_0, a)}{\lambda - \lambda_0} &= \lim_{\lambda \to \lambda_0} \frac{(\lambda_0 - \lambda)\, R(\lambda, a)\, R(\lambda_0, a)}{\lambda - \lambda_0} \\ &= -R(\lambda_0, a)^2, \end{aligned}$$

since inversion is continuous on $\rho(a)$. Thus the limit of the Newton quotient exists at each point of $\rho(a)$, and so $R(\cdot, a)$ is holomorphic on that set.

□

2.36. Corollary. [Gelfand's Theorem.] *If $a \in \mathcal{A}$, a Banach algebra, then $\sigma(a)$ is non-empty.*

Proof. We may assume that $\mathcal{A}$ is unital, for otherwise $0 \in \sigma(a)$ and we are done. Similarly, if $a = 0$, then $0 \in \sigma(a)$. If $\rho(a) = \mathbb{C}$, then clearly $R(\cdot, a)$ is entire. Now for $|\lambda| > \|a\|$, we have

$$\begin{aligned}(\lambda\mathbb{1} - a)^{-1} &= \big(\lambda(\mathbb{1} - \lambda^{-1}a)\big)^{-1} \\ &= \lambda^{-1}\sum_{n=0}^{\infty}(\lambda^{-1}a)^n \\ &= \sum_{n=0}^{\infty}\lambda^{-n-1}a^n\end{aligned}$$

so that if $|\lambda| \geq 2\|a\|$, then

$$\|(\lambda\mathbb{1} - a)^{-1}\| \leq \sum_{n=0}^{\infty}\frac{\|a\|^n}{(2\|a\|)^{n+1}} \leq \frac{1}{\|a\|}.$$

That is, $\|R(\lambda, a)\| \leq \|a\|^{-1}$ for all $\lambda \geq 2\|a\|$.

Clearly there exists $M < \infty$ such that

$$\max_{|\lambda|\leq 2\|a\|}\|R(\lambda, a)\| \leq M,$$

since $R(\cdot, a)$ is a continuous function on this compact set. The conclusion is that $R(\cdot, a)$ is a bounded, entire function. By Liouville's Theorem 2.32, the resolvent function must be constant. This obvious contradiction implies that $\sigma(a)$ is non-empty.

□

Recall that a ***division algebra*** is an algebra in which each non-zero element is invertible.

2.37. Theorem. [The Gelfand-Mazur Theorem.] *If $\mathcal{A}$ is a Banach algebra and a division algebra, then there is a unique isometric isomorphism of $\mathcal{A}$ onto $\mathbb{C}$.*

Proof. If $b \in \mathcal{A}$, then $\sigma(b)$ is non-empty by Corollary 2.36. Let $\beta \in \sigma(b)$. Then $\beta\mathbb{1} - b$ is not invertible, and since $\mathcal{A}$ is a division algebra, we conclude that $\beta\mathbb{1} = b$; that is to say, that $\sigma(b)$ is a singleton.

Thus: if $a \in \mathcal{A}$, then $\sigma(a)$ is a singleton, say $\{\lambda_a\}$, and $a = \lambda_a\mathbb{1}$. The complex-valued map $\phi : a \mapsto \lambda_a\mathbb{1}$ is then an algebra isomorphism. Moreover,

$$\|a\| = \|\lambda_a\mathbb{1}\| = |\lambda_a| = \|\phi(a)\|,$$

so the map is isometric as well. If $\phi_0 : \mathcal{A} \to \mathbb{C}$ were another such map, then $\phi_0(a) \in \sigma(a)$, implying that

$$\phi_0(a) = \lambda_a = \phi(a).$$

□

2.38. Definition. *Let $a \in \mathcal{A}$, a Banach algebra. The* ***spectral radius*** *of a is*

$$\mathrm{spr}(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

2.39. Lemma. [The Spectral Mapping Theorem : polynomial version.] *Let $a \in \mathcal{A}$, a unital Banach algebra, and suppose $p \in \mathbb{C}[z]$ is a polynomial. Then*

$$\sigma(p(a)) = p(\sigma(a)) := \{p(\lambda) : \lambda \in \sigma(a)\}.$$

Proof. Let $\alpha \in \mathbb{C}$. Then for some $\gamma \in \mathbb{C}$,

$$p(z) - \alpha = \gamma\,(z - \beta_1)\,(z - \beta_2)\cdots(z - \beta_n)$$

and so

$$p(a) - \alpha = \gamma\,(a - \beta_1 \mathbb{1})\,(a - \beta_2 \mathbb{1})\cdots(a - \beta_n \mathbb{1}).$$

Thus (as all of the terms $(a - \beta_i)$ commute),

$$\begin{aligned} \alpha \in \sigma(p(a)) &\iff \beta_i \in \sigma(a) \text{ for some } 1 \le i \le n \\ &\iff p(z) - \alpha = 0 \text{ for some } z \in \sigma(a) \\ &\iff \alpha \in p(\sigma(a)). \end{aligned}$$

□

2.40. Theorem. [Beurling's Theorem : The Spectral Radius Formula.] *If $a \in \mathcal{A}$, a Banach algebra, then*

$$\mathrm{spr}(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. First observe that if $\mathcal{A}$ is not unital, then we can always embed it isometrically into a unital Banach algebra $\mathcal{A}_u$. Since both the left and right hand sides of the above equation remain unchanged when a is considered as an element of $\mathcal{A}_u$, we may (and do) assume that $\mathcal{A}$ is already unital.

Now $\sigma(a^n) = (\sigma(a))^n$, and so $\mathrm{spr}(a^n) = (\mathrm{spr}(a))^n$. Moreover, for all $b \in \mathcal{A}$, the proof of Corollary 2.29 shows that $\mathrm{spr}(b) \le \|b\|$. Thus

$$\mathrm{spr}(a) = (\mathrm{spr}(a^n))^{1/n} \le \|a^n\|^{1/n} \text{ for all } n \ge 1.$$

This tells us that $\mathrm{spr}(a) \le \inf_{n\ge1} \|a^n\|^{1/n}$.

On the other hand, $R(\cdot, a)$ is holomorphic on $\rho(a)$ and hence is holomorphic on $\{\lambda \in \mathbb{C} : |\lambda| > \mathrm{spr}(a)\}$. Furthermore, if $|\lambda| > \|a\|$, then

$$\begin{aligned} R(\lambda, a) &= (\lambda \mathbb{1} - a)^{-1} \\ &= \lambda^{-1}(\mathbb{1} - \lambda^{-1} a)^{-1} \\ &= \sum_{n=0}^{\infty} a^n/\lambda^{n+1}. \end{aligned}$$

Let $x^* \in \mathcal{A}^*$. Then $x^* \circ R(\cdot, a)$ is a holomorphic, complex-valued function,

$$[x^* \circ R(\cdot, a)](\lambda) = \sum_{n=0}^{\infty} x^*(a^n)/\lambda^{n+1}$$

and this Laurent expansion is still valid for $\{\lambda \in \mathbb{C} : |\lambda| > \|a\|\}$, since the series for $R(\cdot, a)$ is absolutely convergent on this set, and applying x^* introduces at most a factor of $\|x^*\|$ to the absolutely convergent sum. Since $[x^* \circ R(\cdot, a)]$ is holomorphic on $\{\lambda \in \mathbb{C} : |\lambda| > \mathrm{spr}(a)\}$, the complex-valued series converges on this larger set.

From this it follows that the sequence $(x^*(a^n)/\lambda^{n+1})_n$ converges to 0 for all $x^* \in \mathcal{A}^*$. In particular, it is bounded for all $x^* \in \mathcal{A}^*$. It is now a consequence of the Uniform Boundedness Principle that for each $\lambda > \mathrm{spr}(a)$, the set

$$\Omega_\lambda := \{a^n/\lambda^{n+1} : n \geq 1\}$$

is bounded in norm, say by $M_\lambda > 0$. That is:

$$\|a^n\| \leq M_\lambda |\lambda^{n+1}|$$

for all $|\lambda| > \mathrm{spr}(a)$. But then, for all $|\lambda| > \mathrm{spr}(a)$,

$$\limsup_{n\geq 1} \|a^n\|^{1/n} \leq \limsup_{n\geq 1} M_\lambda^{1/n} |\lambda^{(n+1)/n}| = |\lambda|,$$

so that

$$\limsup_n \|a^n\|^{1/n} \leq \mathrm{spr}(a).$$

Combining this estimate with the above yields $\mathrm{spr}(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

□

2.41. Remark. One interesting fact about Beurling's Spectral Radius Formula is that it implies that given a Banach algebra $\mathcal{A}$, the limit

$$\lim_{n\to\infty} \|a^n\|^{1/n}$$

exists for any $a \in \mathcal{A}$! This in itself is *a priori* far from obvious.

We also mention that the Spectral Radius Formula is actually a practical tool, as well as a theoretical tool.

Supplementary Examples

S2.1. Example. The following example is closely related to that of Example 2.12. Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space. A **nest** on $\mathcal{H}$ is a chain $\mathfrak{N} := \{N_\alpha : \alpha \in \Lambda\}$ of closed subspaces of $\mathcal{H}$ such that

(a) $\{0\}$ and $\mathcal{H}$ belong to $\mathfrak{N}$; and

(b) $\mathfrak{N}$ is closed under arbitrary intersections and *closed* linear spans. That is, if $\Omega \subseteq \Lambda$, then
$$M_1 := \cap\{N_\lambda : \lambda \in \Omega\} \in \mathfrak{N}$$
and
$$M_2 := \vee\{N_\lambda : \lambda \in \Omega\} \in \mathfrak{N}.$$

Given a nest $\mathfrak{N}$ on $\mathcal{H}$, we define the **nest algebra**
$$\mathcal{T}(\mathfrak{N}) := \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq N \text{ for all } N \in \mathfrak{N}\}.$$
We leave it to the reader to verify that $\mathcal{T}(\mathcal{N})$ is a Banach algebra using the operator norm. In fact, it is closed under the weak operator topology WOT on $\mathcal{B}(\mathcal{H})$. We shall see this topology later in looking at von Neumann algebras.

S2.2. Example. An interesting generalisation of a nest algebra is a so-called **commutative subspace lattice algebra**, or a **CSL algebra** for short. Let $\mathcal{H}$ be a (typically) separable Hilbert space. Recall that an **orthogonal projection** on $\mathcal{H}$ is an idempotent $P = P^2$ of norm one. (There are many equivalent definitions - the most common one being a self-adjoint idempotent. Since we shall introduce adjoints later - we've settled upon this definition.) Given an orthogonal projection $P \in \mathcal{B}(\mathcal{H})$, we denote by M_P the range of P, which is necessarily a closed subspace of $\mathcal{H}$.

Let $\mathcal{P}$ be a complete lattice of commuting orthogonal projections, where we define $P \vee Q := P + Q - PQ$ and $P \wedge Q := PQ$. Corresponding to $\mathcal{P}$ we define a **CSL** (or commutative subspace lattice) $\mathcal{L}(= \mathcal{L}_{\mathcal{P}})$ by
$$\mathfrak{L} := \{\operatorname{ran} P : P \in \mathcal{P}\}.$$
Corresponding to $\mathcal{L}$ we obtain a **CSL algebra**
$$\begin{aligned}\textsc{Alg}(\mathfrak{L}) &:= \{T \in \mathcal{B}(\mathcal{H}) : TM_P \subseteq M_P \text{ for all } M_P \in \mathfrak{L}\}\\ &= \{T \in \mathcal{B}(\mathcal{H}) : TP = PTP \text{ for all } P \in \mathcal{P}\}.\end{aligned}$$
These algebras have a rich theory. We refer the reader to [**18**] for more information regarding both CSL algebras, but especially nest algebras.

S2.3. Example. Let G be a locally compact group with Haar measure μ. A **representation** of G is a pair $(\pi, \mathcal{H})$ where $\mathcal{H}$ is a Hilbert space and $\pi : G \to \mathcal{B}(\mathcal{H})$ is a unital group homomorphism. That is, $\pi(e) = I$ (where e is the identity element of the group and I is the identity operator on $\mathcal{H}$), and $\pi(gh) = \pi(g)\pi(h)$ for all $g, h \in G$.

We say that the representation $(\pi, \mathcal{H})$ is **continuous** if for each $x \in \mathcal{H}$, the map
$$g \mapsto \pi(g)x$$

is continuous. (Another way of expressing this is to say that π is continuous from G with its locally compact topology to $\mathcal{B}(\mathcal{H})$ with the **strong operator topology**. We shall discuss the strong operator topology in greater detail in Chapter 12.) It follows that $\pi(g)$ is invertible for all $g \in G$. If $\pi(g)$ is **unitary** for all $g \in G$ (meaning that $\pi(g)$ is a bijective isometry, or equivalently that $\pi(g)^{-1} = \pi(g)^*$, the Hilbert space adjoint of $\pi(g)$ for each $g \in G$), then we say that the representation is a **unitary representation**.

We define the **Fourier-Stieltjes algebra** $B(G)$ to be set of complex-valued functions on G defined as

$$B(G) := \{g \mapsto \langle \pi(g)\xi, \eta\rangle : \pi : G \to \mathcal{B}(\mathcal{H}) \text{ a unitary representation of } G,\ \xi, \eta \in \mathcal{H}\}.$$

This is an algebra (under point-wise operations), and we endow it with the norm

$$\|\varphi\| := \inf\{\|\xi\|\ \|\eta\| : \varphi(g) = \langle \pi(g)\xi, \eta\rangle \text{ for some unitary representation } \pi\}.$$

This algebra was introduced by Eymard [**22**] and is a particularly important Banach algebra for people studying Abstract Harmonic Analysis.

S2.4. Example. The **Fourier algebra** $A(G)$ – also introduced by Eymard [**22**] – may be described as the norm-closure in $B(G)$ of the algebra of elements with compact support. It admits the following alternate description.

It is known that there exists a group homomorphism

$$\begin{array}{rccc} \lambda : & G & \to & \mathcal{B}(L^2(G,\mu)) \\ & g & \mapsto & \lambda_g \end{array}$$

with the property that for all $y \in G$ and $\xi \in L^2(G,\mu)$, $\lambda_g \xi(x) = \xi(g^{-1}x)$ (a.e. - μ). This is referred to as the **left regular representation** of G, and it is known to be a unitary representation of G, as defined above.

The **Fourier algebra** $A(G)$ is equivalently defined to be the space of functions

$$A(G) := \{y \mapsto \langle \lambda_y \xi, \eta\rangle : \xi, \eta \in L^2(G,\mu)\},$$

equipped with the norm it inherits as a subset of $B(G)$. It is a Banach algebra, being the norm-closure of a subalgebra of a Banach algebra.

Appendix

A2.1. The examples of Banach algebras given in this Chapter are but a tiny fraction of those which are of interest in the theory of Operator Algebras. One particular class which we shall be examining in much greater detail is that of C^*-algebras. Even in this subclass there is a plethora of examples, including $\mathcal{B}(\mathcal{H})$ itself, *von Neumann algebras*, *UHF-algebras* and more generally *AF-algebras*, the *irrational rotation algebras*, *Toeplitz algebras*, *Bunce-Deddens algebras*, *group C^*-algebras*, and many more.

A2.2. Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. For each integer $n \ge 1$, define $\mathcal{H}_n := \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. It is not hard to verify that $\mathcal{N} := \{\{0\}, \mathcal{H}\} \cup \{\mathcal{H}_n : n \ge 1\}$ is a nest on $\mathcal{H}$. Thus

$$\mathcal{T}(\mathcal{N}) := \{T \in \mathcal{B}(\mathcal{H}) : T\mathcal{H}_n \subseteq \mathcal{H}_n, n \ge 1\}$$

is a **nest algebra**, as defined in Example S2.1. It is an immediate generalisation of the finite-dimensional algebra of upper-triangular $n \times n$ matrices relative to a fixed orthonormal basis.

The question of whether or not the invertible group of $\mathcal{T}(\mathcal{N})$ is connected has been open for at least 40 years.

Exercises for Chapter 2

Exercise 2.1. Connected components of invertible groups
Let $n \geq 1$ be an integer and set $\mathrm{GL}_n(\mathbb{C}) := \{T \in \mathbb{M}_n(\mathbb{C}) : T \text{ is invertible}\}$. Prove that $\mathrm{GL}_n(\mathbb{C})$ is connected.

Exercise 2.2. Abstract index groups
Find the abstract index groups of the following Banach algebras:

(a) $\mathcal{A} = \mathcal{C}[0,1]$.
(b) $\mathcal{B} = \mathcal{C}(\mathbb{T})$.
(c) $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$.
(d) $\mathcal{A} = \mathcal{T}_n(\mathbb{C})$.

Exercise 2.3. Alg($\mathcal{L}$)
In Example 2.11, we define $\mathrm{Alg}(\mathcal{L})$, where $\mathcal{L}$ was a collection of closed subspaces of $\mathfrak{X}$. Show by way of example that it is possible to have $\mathcal{L}_1 \neq \mathcal{L}_2$, and yet $\mathrm{Alg}(\mathcal{L}_1) = \mathrm{Alg}(\mathcal{L}_2)$. Is there a way of "correcting this"?

Exercise 2.4. The unitisation of a Banach algebra
Proposition 2.20 shows that if $\mathcal{A}$ is a Banach algebra, then the map

$$\begin{array}{rccc} \Phi: & \mathcal{A} & \to & \mathcal{B}(\mathcal{A}_u) \\ & a & \mapsto & L_a \end{array}$$

is an isometric embedding of $\mathcal{A}$ into $\mathcal{B}(\mathcal{A}_u)$. Let $\mathcal{B} = \{\Phi(a) + \alpha I : a \in \mathcal{A}, \alpha \in \mathbb{C}\}$ be the smallest norm-closed subalgebra of $\mathcal{B}(\mathcal{A}_u)$ which contains $\Phi(\mathcal{A})$ and the identity operator. By identifying $\mathcal{A}$ with $\Phi(\mathcal{A})$, we may view $\mathcal{B}$ as a unitisation of $\mathcal{A}$.

What is the relationship between $\mathcal{B}$ and the unitisation $\mathcal{A}_u$ of $\mathcal{A}$ we defined in Remark 2.19?

Exercise 2.5. An open question
Here is an interesting question to which I would love to see the answer.

Let k and n be positive integers and $\varepsilon > 0$. Suppose that $T \in \mathcal{B}(\mathbb{C}^n) \simeq \mathbb{M}_n(\mathbb{C})$ is a matrix of (operator) norm one, and that $\|T^k\|^{\frac{1}{k}} < \varepsilon$. Does there exist a nilpotent operator $M \in \mathbb{M}_n(\mathbb{C})$ *of order* k (i.e. $M^k = 0$) such that

$$\|M - T\| < f(\varepsilon),$$

where $f : (0,\infty) \to (0,\infty)$ is a function which satisfies $\lim_{x\to 0^+} f(x) = 0$?

The *key* issue here is that the function must be *independent* of n, the dimension of the space upon which T acts. I would be happy if we could choose the order of nilpotence of M to be $2k$, or $3k$, or some (very nice) function of k, as long as this order is independent of n.

Exercise 2.6. The closure of the set of invertibles

Find the norm closure of the invertibles in the following Banach algebras:

(i) $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$.
(ii) $\mathcal{A} = \mathcal{T}_n(\mathbb{C})$.
(iii) $\mathcal{A} = \mathcal{C}([0,1])$.
(iv) ℓ^∞.

Exercise 2.7. Left invertible elements

Show that the set of left invertible elements of a Banach algebra $\mathcal{A}$ is open. (Alternatively, show that the set of right invertible elements of a Banach algebra is open.)

CHAPTER 3

The holomorphic functional calculus

Telegram to a friend who had just become a mother after a prolonged pregnancy: "Good work, Mary. We all knew you had it in you."

Dorothy Parker

Integration in a Banach space

3.1. Let $\alpha \le \beta \in \mathbb{R}$ and let $\mathfrak{X}$ be a Banach space. An **$\mathfrak{X}$-valued step function** f is a function on $[\alpha, \beta]$ for which there exists a partition $P = \{\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta\}$ of $[\alpha, \beta]$ so that

$$f(t) = c_k, \qquad \alpha_{k-1} < t \le \alpha_k,\ 1 \le k \le n \tag{1}$$

for some $c_k \in \mathfrak{X}$, $1 \le k \le n$, and $f(\alpha_0) = c_1$. Given an $\mathfrak{X}$-valued step function f, a partition P satisfying (1) will be referred to as an **admissible partition** for f.

Denote by $S = S([\alpha, \beta], \mathfrak{X})$ the linear manifold of $\mathfrak{X}$-valued step functions in the Banach space $\ell^\infty([\alpha, \beta], \mathfrak{X})$. For each $f \in S$, define

$$\int_\alpha^\beta f = \sum_{k=1}^n (\alpha_k - \alpha_{k-1}) c_k$$

whenever $P = \{\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta\}$ is an admissible partition for f. We remark that this sum is easily seen to be independent of the choice of admissible partitions, and so $\int_\alpha^\beta f$ is well-defined. Moreover, $\| \int_\alpha^\beta f \| \le (\beta - \alpha) \|f\|_\infty$. It follows that the map

$$\begin{array}{rccc} \Phi : & S & \to & \mathfrak{X} \\ & f & \mapsto & \int_\alpha^\beta f \end{array}$$

is continuous.

We may therefore extend Φ to the closure $\overline{S}$ in $\ell^\infty([\alpha, \beta], \mathfrak{X})$ and continue to write $\int_\alpha^\beta f$ or $\int_\alpha^\beta f(t)\, dt$ for $f \in \overline{S}$. Clearly we still have

$$\| \int_\alpha^\beta f \| \le (\beta - \alpha) \|f\|_\infty$$

for all $f \in \overline{S}$.

If $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ for some Banach space $\mathfrak{Y}$, then it is easy to check that $T \circ f \in \overline{S([\alpha, \beta], \mathfrak{Y})}$ for all $f \in \overline{S}$, and

$$T\left(\int_\alpha^\beta f\right) = \int_\alpha^\beta T \circ f.$$

3.2. Proposition. *Let $f \in \mathcal{C}([\alpha, \beta], \mathfrak{X})$ and let $\varepsilon > 0$. Then $f \in \overline{S}$ and there exists $\delta > 0$ such that for every partition $P = \{\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta\}$ of $[\alpha, \beta]$ such that $\|P\| := \max_{1 \le k \le n}(\alpha_k - \alpha_{k-1}) < \delta$, and for all $t_1, t_2, \ldots, t_n$ satisfying $\alpha_{k-1} \le t_k \le \alpha_k$, $1 \le k \le n$, the following statements hold:*

(1) *there exists $g \in S([\alpha, \beta], \mathfrak{X})$ with $g(t) = f(t_k)$, $(\alpha_{k-1} \le t < \alpha_k, 1 \le k \le n)$ and $\|f - g\| \le \varepsilon$.*

(2) $\| \int_\alpha^\beta f - \sum_{k=1}^n (\alpha_k - \alpha_{k-1}) f(t_k) \| \le (\beta - \alpha)\varepsilon$.

Proof. Since f is continuous on the compact set $[\alpha, \beta]$, it is uniformly continuous there, and so we can choose $\delta > 0$ such that $|a - b| < \delta$ implies that $\|f(a) - f(b)\| < \varepsilon$.

Let P be any partition of $[\alpha, \beta]$ with $\|P\| < \delta$, and choose $\{t_k\}_{k=1}^n$ such that $\alpha_{k-1} \le t_k < \alpha_k$, $1 \le k \le n$. Let $g(\alpha_0) = f(t_1)$, and for $1 \le k \le n$, let $g(t) = f(t_k)$, $\alpha_{k-1} < t \le \alpha_k$.

(1) Now

$$\begin{aligned}
\|f - g\|_\infty &= \sup_{t \in [\alpha, \beta]} \|f(t) - g(t)\| \\
&= \max_{1 \le k \le n} \sup_{t \in (\alpha_{k-1}, \alpha_k]} \|f(t) - g(t)\| \\
&= \max_{1 \le k \le n} \sup_{t \in (\alpha_{k-1}, \alpha_k]} \|f(t) - f(t_k)\| \\
&< \varepsilon.
\end{aligned}$$

(2) Secondly,

$$\begin{aligned}
\| \int_\alpha^\beta f - \sum_{k=1}^n (\alpha_k - \alpha_{k-1}) f(t_k) \| &= \| \int_\alpha^\beta f - \int_\alpha^\beta g \| \\
&= \| \int_\alpha^\beta f - g \| \\
&\le \int_\alpha^\beta \|f - g\|_\infty \\
&\le (\beta - \alpha)\varepsilon.
\end{aligned}$$

□

We remark in passing that a minor adaptation of the above proof shows that piecewise continuous functions also lie in $\overline{S}$.

3.3. With $\alpha \le \beta \in \mathbb{R}$ as above, we define a **curve** in $\mathfrak{X}$ to be a continuous function $\tau : [\alpha, \beta] \to \mathfrak{X}$. The interval $[\alpha, \beta]$ is referred to as the **parameter interval** of the curve, and we denote the image of τ in $\mathfrak{X}$ by τ^*. The point $\tau(\alpha)$ is then called the **initial point** of the curve, while $\tau(\beta)$ is referred to as the **final point**.

A **contour** in $\mathfrak{X}$ is a piecewise continuously differentiable curve. That is, there exists a partition $P = \{\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta\}$ of $[\alpha, \beta]$ such that $\tau|_{[\alpha_{i-1}, \alpha_i]}$ is continuously differentiable, $1 \le i \le n$. If $\tau(\alpha) = \tau(\beta)$, we say that the contour τ is **closed**.

Suppose that τ is a contour in $\mathbb{C}$, and that $f : \tau^* \to \mathfrak{X}$ is a continuous function. We can then think of τ as a parametrization of τ^*. We shall define the **integral of** f **over** τ as

$$\int_\tau f(z)\,dz = \int_\alpha^\beta f(\tau(x))\tau'(x)\,dx. \tag{2}$$

Note that the integral on the right hand side exists by the comment following Proposition 3.2.

Suppose next that $\gamma : [\alpha_1, \beta_1] \to [\alpha, \beta]$ is a continuously differentiable bijection with $\gamma(\alpha_1) = \alpha$ and $\gamma(\beta_1) = \beta$. Let $\tau_1 = \tau \circ \gamma$. Then

$$\begin{aligned}
\int_{\tau_1} f(z)\,dz &= \int_{\alpha_1}^{\beta_1} f(\tau_1(x))\tau_1'(x)\,dx \\
&= \int_{\alpha_1}^{\beta_1} f(\tau(\gamma(x))\tau'(\gamma(x))\gamma'(x)\,dx \\
&= \int_\alpha^\beta f(\tau(y))\tau'(y)\,dy \\
&= \int_\tau f(z)\,dz,
\end{aligned}$$

and so the integral is seen to be independent of the parametrization of the contour. Any two such contours τ_1 and τ_2 for which

$$\int_{\tau_1} f(z)\,dz = \int_{\tau_2} f(z)\,dz$$

for all continuous functions $f \in \mathcal{C}(\tau_1^* = \tau_2^*)$ will be considered **equivalent**.

The notion of equivalence of contours allows us to manipulate vector-valued integrals in the standard way. For instance, suppose that the final point of τ_1 equals the initial point of τ_2, and suppose $f \in \mathcal{C}(\tau_1^* \cup \tau_2^*)$. We can "concatenate" the two contours into one longer contour τ satisfying

$$\int_\tau f(z)\,dz = \int_{\tau_1} f(z)\,dz + \int_{\tau_2} f(z)\,dz.$$

Moreover, equation (2) shows that

$$\begin{aligned} \| \int_\tau f(z)\, dz \| &= \| \int_\alpha^\beta f(\tau(x))\tau'(x)\, dx \| \\ &\le \|f\|_\infty \, \| \int_\alpha^\beta \tau'(x)\, dx\| \\ &= \|f\|_\infty \, \|\tau\|, \end{aligned}$$

where $\|f\|_\infty = \max\{\|f(x)\| : x \in \tau^*\}$, while $\|\tau\| = \| \int_\alpha^\beta \tau'(x)\, dx\|$ is (by definition) the **length** of τ^*. Note that this length is finite as τ' is piecewise continuous.

Finally, observe that as before, if $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ for some Banach space $\mathfrak{Y}$, then

$$T\left(\int_\tau f(z)\, dz\right) = \int_\tau (T \circ f)(z)\, dz.$$

3.4. Our present goal is to make sense of expressions of the form $f(a)$, where $a \in \mathcal{A}$, a Banach algebra, and f is a function. An important question in this regard is to find the largest set of functions for which $f(a)$ makes sense. Clearly if $p(z) = \sum_{k=0}^n c_k z^k$ is polynomial over the complex numbers, then

$$p(a) = \sum_{k=0}^n c_k\, a^k$$

can be defined in any unital Banach algebra which contains a. (If we also stipulate that $c_0 = 0$, then $p(a)$ makes sense even if the algebra is not unital.)

Suppose now that the algebra $\mathcal{A}$ is unital, that p and q are polynomials over $\mathbb{C}$, and that $0 \notin q(\sigma(a))$. Then $q(z) = \beta(\Pi_{k=1}^m (z - \lambda_k))$, where $\lambda_k \notin \sigma(a)$ for $1 \le k \le m$, so we can define $r(z) = p(z)/q(z)$ as a holomorphic function on some neighbourhood of $\sigma(a)$ and

$$r(a) = p(a)\beta^{-1}\, (\Pi_{k=1}^m (a - \lambda_k)^{-1}).$$

The question remains: can we do better than rational functions? For general Banach algebras $\mathcal{A}$ and arbitrary elements $a \in \mathcal{A}$, we are now in a position to develop an *holomorphic functional calculus*: that is, we shall make sense of $f(a)$ whenever f is a function which is holomorphic on some neighbourhood of $\sigma(a)$.

This is definitely not the only possible functional calculus that exists. For example, later we shall see that if $\mathbb{A}$ is a C^*-algebra and $n \in \mathbb{A}$ is *normal*, then one can develop a continuous functional calculus for n. As another example, if $T \in \mathcal{B}(\mathcal{H})$ is a contraction, then an H^∞ functional calculus is possible.

Recall from Complex Analysis the following:

3.5. Definition. *If Γ is a finite system of closed contours in $\mathbb{C}$ and $\lambda \notin \Gamma$, then the* ***index*** *or* ***winding number*** *of Γ with respect to λ is*

$$\mathrm{Ind}_\Gamma(\lambda) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{(z - \lambda)}\, dz,$$

and represents the number of times that Γ wraps around λ in the positive (i.e. counterclockwise) direction.

3.6. Theorem. [Cauchy's Theorem] *Let f be holomorphic on an open set $U \subseteq \mathbb{C}$, and let $z_0 \in U$. Let Γ be a finite system of closed contours in U such that $z_0 \notin \Gamma$, $\mathrm{Ind}_\Gamma(z_0) = 1$, and $\{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) \neq 0\} \subseteq U$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)}\, dz.$$

Furthermore,

$$\int_\Gamma f(z)\, dz = 0.$$

3.7. Remarks.

- We shall say that a complex valued function f is ***holomorphic*** on a compact subset K of $\mathbb{C}$ if f is holomorphic on some open subset U of $\mathbb{C}$ which contains K.
- Let $U \subseteq \mathbb{C}$ be open and $K \subseteq U$ be compact. Then there exists a finite system of contours $\Gamma \subseteq U$ such that
  (a) $\mathrm{Ind}_\Gamma(\lambda) \in \{0, 1\}$;
  (b) $\mathrm{Ind}_\Gamma(\lambda) = 1$ for all $\lambda \in K$;
  (c) $\{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) \neq 0\} \subseteq U$.

  The existence of such a system Γ is a relatively standard result from Complex Analysis, and follows from the Jordan Curve Theorem. A proof can be found in [**13**], although (to quote Conway himself [**14**]), "some details are missing".

  In practice, the idea is to cover K by open disks of sufficiently small radius so as to ensure that their closures still lie in U. Since K is presumed to be compact, there will exist a finite subcover V by these disks. Modulo some technicalities, the boundary of V will then yield the desired system Γ of contours.

  In fact, with a bit more work, one can even assume that Γ consists of a finite system of *infinitely differentiable* curves [**14**], Proposition 4.4.

3.8. The Riesz-Dunford Functional Calculus. Let $a \in \mathcal{A}$, a unital Banach algebra, and fix U be an open subset of $\mathbb{C}$ such that $\sigma(a) \subseteq U$. Set

$$\mathcal{F}(a) = \{f : U \to \mathbb{C} : f \text{ is holomorphic}\}.$$

Choose a system $\Gamma \subseteq U$ of closed contours such that

(1) $\mathrm{Ind}_\Gamma(\lambda) = 1$ for all $\lambda \in \sigma(a)$;
(2) $\{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) \neq 0\} \subseteq U$.

We define

$$f(a) = \frac{1}{2\pi i} \int_\Gamma f(z)\, (z - a)^{-1}\, dz.$$

The first question we must ask is whether or not this definition makes sense. As stated, the definition appears to depend upon the choice of the system Γ and of U. The following Theorem addresses this issue.

3.9. Theorem. [The Riesz-Dunford Functional Calculus 01.] *With the above setting, $f(a)$ is well-defined (i.e. independent of the choice of curves Γ), and for $f, g \in \mathcal{F}(a)$, $h \in \mathcal{F}(f(a))$, and $\lambda \in \mathbb{C}$,*

(i) $(f+g)(a) = f(a) + g(a)$;
(ii) $(\lambda f)(a) = \lambda(f(a))$;
(iii) $(fg)(a) = f(a)\, g(a)$;

Proof. First note that if U_1 and U_2 are two open sets containing $\sigma(a)$, then so is $U := U_1 \cap U_2$. If Γ_1 (resp. Γ_2) is an eligible system of contours in U_1 (resp. U_2), then it suffices to show that the integral along each of Γ_1 and Γ_2 agrees with the integral along an eligible system of contours Γ contained in U. By symmetry, it suffices to show that the integral along Γ_1 agrees with the integral along Γ. Since $U \subseteq U_1$, this implies that the problem reduces to the case where Γ_1 and Γ_2 sit inside the same open set U.

Let Γ_1 and Γ_2 be two eligible systems of contours. We must show that

$$b = \frac{1}{2\pi i}\int_{\Gamma_1} f(z)(z-a)^{-1}\, dz - \frac{1}{2\pi i}\int_{\Gamma_2} f(z)(z-a)^{-1}\, dz = 0.$$

By the Corollary to the Hahn-Banach Theorem [Corollary 1.19], it suffices to show that $x^*(b) = 0$ for all $x^* \in \mathcal{A}^*$. Now

$$x^*(b) = \frac{1}{2\pi i}\int_{\Gamma_1 - \Gamma_2} f(z)\, x^*(z-a)^{-1}\, dz.$$

Also, $f(z)$ is holomorphic on U, $R(z,a) = (z-a)^{-1}$ is holomorphic on $\rho(a) \supseteq \Gamma_1, \Gamma_2$ and so $x^*((z-a)^{-1})$ is holomorphic on $\rho(a)$ for all $x^* \in \mathcal{A}^*$. So the integrand is holomorphic on the open set $U \cap \rho(a)$. To apply Cauchy's Theorem above, we need only verify the index conditions.

If $\lambda \notin U$, then we have $\mathrm{Ind}_{\Gamma_1}(\lambda) = \mathrm{Ind}_{\Gamma_2}(\lambda) = 0$, and therefore

$$\mathrm{Ind}_{\Gamma_1 - \Gamma_2}(\lambda) = \mathrm{Ind}_{\Gamma_1}(\lambda) - \mathrm{Ind}_{\Gamma_2}(\lambda) = 0.$$

If $\lambda \in \sigma(a)$, then $\mathrm{Ind}_{\Gamma_1}(\lambda) = \mathrm{Ind}_{\Gamma_2}(\lambda) = 1$, therefore $\mathrm{Ind}_{\Gamma_1 - \Gamma_2}(\lambda) = 0$.

Thus $\{z \in \mathbb{C} : \mathrm{Ind}_{\Gamma_1 - \Gamma_2}(\lambda) \neq 0\} \subseteq U \cap \rho(a)$ and so Cauchy's Theorem applies, namely:

$$x^*(b) = \frac{1}{2\pi i}\int_{\Gamma_1 - \Gamma_2} f(z)\, x^*(z-a)^{-1}\, dz = 0 \text{ for all } x^* \in \mathcal{A}^*.$$

Thus $b = 0$ and so $f(a)$ is indeed well-defined.

(i) $(f+g)(a) = f(a) + g(a)$:
This follows for the linearity of the integral, and is left as an exercise.

(ii) $(\lambda f)(a) = \lambda(f(a))$:
Again, this follows from the linearity of the integral.

(iii) $(fg)(a) = f(a)\, g(a)$:
Now f and g are both holomorphic on some open set $U \supseteq \sigma(a)$. Choose two systems of contours Γ_1 and Γ_2 such that
(a) $\{z \in \mathbb{C} : \mathrm{Ind}_{\Gamma_i}(z) \neq 0\} \subseteq U$, $i = 1, 2$;
(b) $\mathrm{Ind}_{\Gamma_i}(z) = 1$ for all $z \in \sigma(a)$, $i = 1, 2$;

(c) $\mathrm{Ind}_{\Gamma_1}(z) = 1$ for all $z \in \Gamma_2$.

To get part (c), we choose Γ_2 first and then choose Γ_1 to lie "outside" of Γ_2. Then

$$\begin{aligned}
f(a)\,g(a) &= \frac{1}{2\pi i}\int_{\Gamma_1} f(z)(z-a)^{-1}\,dz \;\; \frac{1}{2\pi i}\int_{\Gamma_2} g(w)(w-a)^{-1}\,dw \\
&= (\frac{1}{2\pi i})^2 \int_{\Gamma_1}\int_{\Gamma_2} f(z)g(w)(z-a)^{-1}(w-a)^{-1}\,dw\,dz \\
&= (\frac{1}{2\pi i})^2 \int_{\Gamma_1}\int_{\Gamma_2} f(z)g(w)(\frac{1}{w-z})[(z-a)^{-1}-(w-a)^{-1}]\,dz\,dw \\
&= (\frac{1}{2\pi i})^2 \int_{\Gamma_1} f(z)(z-a)^{-1}\int_{\Gamma_2} g(w)(w-z)^{-1}\,dw\,dz - \qquad (3) \\
&\quad (\frac{1}{2\pi i})^2 \int_{\Gamma_2} g(w)(w-a)^{-1}\int_{\Gamma_1} f(z)(w-z)^{-1}\,dz\,dw \\
&= \frac{1}{2\pi i}\int_{\Gamma_2} g(w)(w-a)^{-1}f(w)\,dw \\
&= (fg)(a).
\end{aligned}$$

where the first integral in equation (3) is zero since z lies "outside" of Γ_2 and g is holomorphic.

□

3.10. Remark. Let a be an element of a unital Banach algebra $\mathcal{A}$ and let U be an open set in the complex plane such that $\sigma(a) \subseteq U$. Let

$$H(U) = \{f : U \to \mathbb{C} : \; f \text{ is holomorphic}\}.$$

From (i), (ii) and (iii) above, we conclude that the map:

$$\begin{array}{rccc} \Phi: & H(U) & \to & \mathcal{A} \\ & f & \mapsto & f(a) \end{array}$$

is an algebra homomorphism. Moreover, for all $a \in \mathcal{A}$ and $f, g \in H(U)$, we have $f(a)g(a) = g(a)f(a)$ since $f(z)g(z) = g(z)f(z)$.

3.11. Proposition. *Suppose $\mathcal{A}$ is a unital Banach algebra and that $a \in \mathcal{A}$. Let $U \subseteq \mathbb{C}$ be an open set containing $\sigma(a)$, and let $(f_n)_{n=0}^{\infty}$ be a sequence of holomorphic functions on U converging uniformly to f on compact subsets of U. Then f is also holomorphic on U and*

$$\lim_{n\to\infty} \|f_n(a) - f(a)\| = 0.$$

Proof. Choose an open set V with boundary Γ consisting of a finite number of piecewise smooth curves such that $\sigma(a) \subseteq V \subseteq \overline{V} \subseteq U$.

Since $(f_n)_{n=0}^\infty$ converges uniformly on compact subsets of U, f is holomorphic on U. Thus $f \in H(U)$ and $(f_n)_{n=0}^\infty$ converges uniformly to f on Γ. It follows that

$$\begin{aligned}\|f_n(a) - f(a)\| &= \|(1/2\pi i)\int_\Gamma [f_n(z) - f(z)](z-a)^{-1}\,dz\| \\ &\leq (1/2\pi)K\,\|\Gamma\|\,\|f_n - f\|_\Gamma,\end{aligned}$$

where $K = \sup\{\|(z-a)^{-1}\| : z \in \Gamma\}$, $\|\Gamma\|$ represents the arclength of the contour, and $\|f_n - f\|_\Gamma = \sup\{|f_n(z) - f(z)| : z \in \Gamma\}$. Since both K and $\|\Gamma\|$ are fixed and $\|f_n - f\|_\Gamma$ tends to zero as $n \to \infty$, we obtain the desired conclusion.

□

3.12. Theorem. [The Riesz-Dunford Functional Calculus 02.] *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. If $f(z) = \sum_{n=0}^\infty c_n z^n$ converges to a function holomorphic in a neighbourhood of $\sigma(a)$, then $f(a) = \sum_{n=0}^\infty c_n a^n$.*

Proof. Suppose $f(z) = \sum_{n=0}^\infty c_n z^n$ converges in $D_R(\{0\}) \supseteq \sigma(a)$. Then consider the curve $\Gamma = \{re^{i\theta} : 0 \leq \theta \leq 2\pi\}$ for some r, $\mathrm{spr}(a) < r \leq R$ and consider

$$\begin{aligned}f(a) &= \frac{1}{2\pi i}\int_\Gamma \left(\sum_{n=0}^\infty c_n z^n\right)(z-a)^{-1}\,dz \\ &= \frac{1}{2\pi i}\sum_{n=0}^\infty c_n \int_\Gamma z^n\,(z-a)^{-1}\,dz \qquad (4)\\ &= \sum_{n=0}^\infty c_n z^n(a),\end{aligned}$$

where $z(a) = \frac{1}{2\pi i}\int_\Gamma z(z-a)^{-1}\,dz$ is the identity function evaluated at a. Note that (4) uses the uniform convergence of the series on Γ.

But

$$\begin{aligned}z(a) &= \frac{1}{2\pi i}\int_\Gamma z(z-a)^{-1}\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma \sum_{n=0}^\infty z^{-n} a^n\,dz \\ &= \sum_{n=0}^\infty a^n \frac{1}{2\pi i}\int_\Gamma z^{-n}\,dz \qquad (5)\\ &= \sum_{n=0}^\infty a^n(\delta_{n1}) \qquad (6)\\ &= a.\end{aligned}$$

Here (5) uses the uniform convergence of the series when $|z| = r > \mathrm{spr}(a)$, and (6) uses the Residue Theorem. We can now apply induction on part (iii) of Theorem 3.9 to get $z^n(a) = (z(a))^n = a^n$, and so

$$f(a) = \sum_{n=0}^\infty c_n a^n,$$

as desired.

□

3.13. Corollary. **[Dunford: The Spectral Mapping Theorem.]** *Let $a \in \mathcal{A}$, a unital Banach algebra and suppose that f is holomorphic on $\sigma(a)$. Then*

$$\sigma(f(a)) = f(\sigma(a)).$$

Proof. If $\lambda \notin f(\sigma(a))$, then $g(z) = (\lambda - f(z))^{-1}$ is holomorphic on $\sigma(a)$. From the functional calculus,

$$\begin{aligned} g(a)\,(\lambda - f(a)) &= (g(z)\,(\lambda - f(z)))\,(a) \\ &= 1(a) \\ &= 1 \\ &= (\lambda - f(a))\,g(a), \end{aligned}$$

since everything commutes. Thus $\lambda \notin \sigma(f(a))$.

If $\lambda \in f(\sigma(a))$, then $\lambda - f(z)$ has a zero on $\sigma(a)$, say at z_0. As such,

$$\lambda - f(z) = (z_0 - z)g(z)$$

for some function g which is holomorphic on $\sigma(a)$. Via the functional calculus, we obtain

$$\lambda - f(a) = (z_0 - a)g(a),$$

and since $(z_0 - a)$ is not invertible and $(z_0 - a)$ commutes with $g(a)$, we conclude that $\lambda - f(a)$ is not invertible either. Thus $\lambda \in \sigma(f(a))$.

Combining the two results, $f(\sigma(a)) = \sigma(f(a))$.

□

3.14. Theorem. **[The Riesz-Dunford Functional Calculus 03.]** *Suppose that $\mathcal{A}$ is a unital Banach algebra, and that g is a complex-valued function which is holomorphic on $\sigma(a)$ while f is a complex-valued function which is holomorphic on $g(\sigma(a))$. Then $(f \circ g)(a) = f(g(a))$.*

Proof. Let V be an open neighbourhood of $g(\sigma(a))$ upon which f is holomorphic and consider $U = g^{-1}(V)$, an open neighbourhood of $\sigma(a)$. Let Γ_1 be a system of closed contours in U such that

(a) $\mathrm{Ind}_{\Gamma_1}(\lambda) = 1$ for all $\lambda \in \sigma(a)$;
(b) $\mathrm{Ind}_{\Gamma_1}(\lambda) \neq 0$ implies that $\lambda \in U$.

Let Γ_2 be a system of closed contours in V such that

(A) $\mathrm{Ind}_{\Gamma_2}(\beta) = 1$ for all $\beta \in g(\sigma(a))$;
(B) $\mathrm{Ind}_{\Gamma_2}(\beta) = 1$ for all $\beta \in g(\Gamma_1)$;
(C) $\mathrm{Ind}_{\Gamma_1}(\beta) \neq 0$ implies that $\beta \in V$.

(One can view Γ_2 as lying "outside" of $g(\Gamma_1)$ in V.)

Then

$$
\begin{aligned}
(f \circ g)(a) &= \frac{1}{2\pi i} \int_{\Gamma_1} (f \circ g)(z)(z-a)^{-1}\, dz \\
&= \frac{1}{2\pi i} \int_{\Gamma_1} f(g(z))(z-a)^{-1}\, dz \\
&= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{2\pi i} \int_{\Gamma_2} f(w)(w-g(z))^{-1}\, dw\, (z-a)^{-1}\, dz \\
&= \frac{1}{2\pi i} \int_{\Gamma_2} f(w) \frac{1}{2\pi i} \int_{\Gamma_1} (w-g(z))^{-1}(z-a)^{-1}\, dz\, dw \\
&= \frac{1}{2\pi i} \int_{\Gamma_2} f(w)(w-g(a))^{-1}\, dw \\
&= f(g(a)).
\end{aligned}
$$

□

3.15. Corollary. [The Riesz Decomposition Theorem.] *Let $a \in \mathcal{A}$, a unital Banach algebra, and suppose that Δ is a non-trivial, relatively closed and open subset of $\sigma(a)$.*

(i) *There exists a non-trivial idempotent $E(\Delta)$ in $\mathcal{A}$ which commutes with a;*
(ii) *If $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{X})$ for some Banach space $\mathfrak{X}$, then $E(\Delta)\mathfrak{X}$ and $(I - E(\Delta))\mathfrak{X}$ are complementary subspaces invariant under a.*
(iii) *Let $a_\Delta = a|_{E(\Delta)\mathfrak{X}}$. Then $\sigma(a_\Delta) = \Delta$. Moreover, for any function f which is holomorphic on $\sigma(a)$, we have $f(a_\Delta) = f(a)|_{E(\Delta)\mathfrak{X}}$.*

Proof.

(i) Consider an holomorphic function g such that $g \equiv 1$ on Δ and $g \equiv 0$ on $\sigma(a) \setminus \Delta$. Let $E(\Delta) = g(a)$. Then $g^2 = g$ and so $E^2(\Delta) = g^2(a) = g(a) = E(\Delta)$ is an idempotent. Note that $E(\Delta) \neq 0$ since $1 \in \sigma(g(a)) = g(\sigma(a))$. Similarly, $E(\Delta) \neq I$ as $0 \in \sigma(g(a)) = g(\sigma(a))$.

Since $zg(z) = g(z)z$, $E(\Delta)$ commutes with a.

(ii) Let $\mathfrak{X}$ be a Banach space and assume that $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{X})$. Then $E(\Delta)\mathfrak{X} = \ker(I - E(\Delta))$ is closed, as is $(I - E(\Delta))\mathfrak{X} = \ker E(\Delta)$. Clearly

$$\mathfrak{X} = E(\Delta)\mathfrak{X} + (I - E(\Delta))\mathfrak{X}.$$

Moreover, if $y \in E(\Delta)\mathfrak{X} \cap (I - E(\Delta))\mathfrak{X}$, then

$$y = E(\Delta)y = E(\Delta)\,(I - E(\Delta))y = 0.$$

Thus $E(\Delta)\mathfrak{X}$ and $(I - E(\Delta))\mathfrak{X}$ are complementary. Finally, let $x \in E(\Delta)\mathfrak{X}$. Then $ax = aE(\Delta)x = E(\Delta)ax \in E(\Delta)\mathfrak{X}$. Therefore $E(\Delta)\mathfrak{X}$ is invariant under a, as is $(I - E(\Delta))\mathfrak{X}$.

(iii) First we show that $\sigma(a_\Delta) \subseteq \Delta$.

If $\lambda \notin \Delta$, let

$$h(z) = \begin{cases} (\lambda - z)^{-1} & \text{for } z \text{ in a neighbourhood of } \Delta \\ 0 & \text{for } z \text{ in a neighbourhood of } \sigma(a) \setminus \Delta. \end{cases}$$

Then $h(z)(\lambda - z) = g(z)$. Thus $h(a)(\lambda - a) = g(a) = E(\Delta)$. Now $h(a)$ leaves $E(\Delta)\mathfrak{X}$ and $(I - E(\Delta))\mathfrak{X}$ invariant (since $h(a)$ commutes with $g(a)$). If $R_\lambda := h(a)|_{E(\Delta)\mathfrak{X}}$, then

$$R_\lambda(\lambda - a_\Delta) = (\lambda - a_\Delta)R_\lambda = I_{E(\Delta)\mathfrak{X}},$$

so that $\lambda \in \rho(a_\Delta)$, i.e. $\sigma(a_\Delta) \subseteq \Delta$.

Suppose now that $\lambda \in \Delta \cap \rho(a_\Delta)$, so that for some $b \in \mathcal{B}(E(\Delta)\mathfrak{X})$, we have

$$b\,(\lambda - a_\Delta) = (\lambda - a_\Delta)\,b = I_{E(\Delta)\mathfrak{X}}.$$

Let

$$k(z) = \begin{cases} (\lambda - z)^{-1} & \text{for } z \text{ in a neighbourhood of } \sigma(a) \setminus \Delta \\ 0 & \text{for } z \text{ in a neighbourhood of } \Delta. \end{cases}$$

Then

$$k(a)(\lambda - a) = (\lambda - a)k(a) = I - E(\Delta).$$

Define $r = k(a) + bE(\Delta)$. Then

$$\begin{aligned} r\,(\lambda - a) &= k(a)\,(\lambda - a) + b\,E(\Delta)\,(\lambda - a) \\ &= (I - E(\Delta)) + b\,(\lambda - a_\Delta)\,E(\Delta) \\ &= (I - E(\Delta)) + E(\Delta) \\ &= I. \end{aligned}$$

Similarly, $(\lambda - a)\,r = I$, and so $\lambda \in \rho(a)$, a contradiction. We conclude that $\sigma(a_\Delta) = \Delta$.

Finally, suppose that f is holomorphic on $\sigma(a)$. Then for an eligible system Γ of contours we obtain

$$\begin{aligned} f(a_\Delta) &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - a_\Delta)^{-1}\,d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - a)^{-1}|_{E(\Delta)\mathfrak{X}}\,d\lambda \\ &= \frac{1}{2\pi i}\left(\int_\Gamma f(\lambda)(\lambda - a)^{-1}\,d\lambda\right)|_{E(\Delta)\mathfrak{X}} \\ &= f(a)|_{E(\Delta)\mathfrak{X}}. \end{aligned}$$

□

Supplementary Examples

S3.1. Example. One of the most important questions in operator theory is the question of whether every continuous linear operator $T \in \mathcal{B}(\mathcal{H})$ acting on an infinite-dimensional, separable Hilbert space admits a non-trivial **invariant subspace**, that is, a closed subspace $\mathcal{M} \subseteq \mathcal{H}$ (other than the trivial spaces $\{0\}$ and $\mathcal{H}$ itself) for which $m \in \mathcal{M}$ implies that $Tm \in \mathcal{M}$.

The Riesz Decomposition Theorem 3.15 shows that if $T \in \mathcal{B}(\mathcal{H})$ and $\sigma(T)$ is not connected, then T immediately admits a non-trivial invariant subspace, namely the range of the Riesz idempotent determined by that Theorem.

Thus it suffices to consider operators with connected spectrum. Even in the case where $\sigma(T) = \{0\}$ is a singleton set, the answer is not known.

S3.2. Example. Let $n \in \mathbb{N}$ and suppose that $J \in \mathbb{M}_n(\mathbb{C}) \simeq \mathcal{B}(\mathbb{C}^n)$ is a **Jordan cell** relative to the standard orthonormal basis for $\mathbb{C}^n$. Then $J^n = 0$, from which we deduce that $\sigma(J) = 0$. Let $\alpha \in \mathbb{C}$ and set $T := \alpha I_n + J$, so that $\sigma(T) = \{\alpha\}$.

Suppose now that U is an open nbhd of $\sigma(T)$ and that $f : U \to \mathbb{C}$ is a holomorphic function. Then f is analytic on U, implying that for some $\delta > 0$, f admits a power series representation centred at α on the disc $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\}$, say

$$f(z) = \sum_k a_k (z - \alpha)^k \qquad \text{for all } z \in B(\lambda, \delta).$$

It follows that

$$f(T) = \sum_k a_k (T - \alpha I_n)^k = \sum_k a_k J^k = a_0 I_n + a_1 J + a_2 J^2 + \cdots + a_{n-1} J^{n-1}.$$

We leave it to the interested reader to ponder how one goes from calculating $f(T)$ for operators $T = \alpha I_n + J$ as above to calculating $f(X)$ for general elements of $X \in \mathcal{B}(\mathbb{C}^n)$.

S3.3. Example. Suppose that $T \in \mathbb{M}_n(\mathbb{C})$ is a *diagonalisable operator*; that is, suppose that there exists an invertible operator $S \in \mathrm{GL}_n(\mathbb{C})$ such that

$$D = S^{-1} T S$$

admits a diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ relative to the standard orthonormal basis for $\mathbb{C}^n$. Let U be an open nbhd of $\sigma(T) = \sigma(D) = \{d_k\}_{k=1}^n$.

An innocuous-looking comment at the end of Paragraph 3.3 shows that

$$\begin{aligned} f(T) &= \frac{1}{2\pi i} \int_\Gamma f(z)(zI - T)^{-1} dz \\ &= S \left(\frac{1}{2\pi i} \int_\Gamma f(z)(zI - D)^{-1} dz \right) S^{-1} \\ &= S f(D) S^{-1}. \end{aligned}$$

The calculation of $f(D)$ is now simple; $f(D) = \mathrm{diag}(f(d_1), f(d_2), \ldots, f(d_n))$. There are details to fill in to justify the above statements, and these are left to the reader. Once we have $f(D)$, we clearly have $f(T)$.

Appendix

A3.1. The Riesz-Dunford functional calculus made its first appearance in a paper of Riesz [**47**]. In his case, he studied only compact operators acting on a Hilbert space $\mathcal{H}$, and then the only functions he considered were the characteristic functions of an isolated point of the spectrum of the given operator. Indeed, alongside a number of related results by a number of authors, it was Dunford who presented the work in its most complete form. Recently, Conway and Morrel [**17**] and again Conway, Herrero and Morrel [**16**] have considered what might be termed a "converse" to the Riesz-Dunford functional calculus.

A3.2. As we have seen, in the Riesz-Dunford functional calculus, one begins with an element a of a unital Banach algebra $\mathcal{A}$ and considers the class $\mathcal{F}(a)$ of functions f which are analytic on some open neighbourhood of the spectrum of a. One then obtains an algebra homomorphism

$$\begin{array}{rccc} \tau: & \mathcal{F}(a) & \to & \mathcal{A} \\ & f & \mapsto & f(a). \end{array}$$

In the Conway, Herrero and Morrel approach, one begins with a subset Δ of the complex plane $\mathbb{C}$, and the class $\mathcal{S}(\Delta)$ of operators T acting on a separable Hilbert space $\mathcal{H}$ and satisfying $\sigma(T) \subseteq \Delta$.

The aim of their program is to determine $f(\mathcal{S}(\Delta)) = \{f(T) : T \in \mathcal{S}(\Delta)\}$, where $f : \Delta \to \mathbb{C}$ is a *fixed* analytic function. As an example, suppose $\Delta = \mathbb{D}$ so that $\mathcal{S}(\Delta)$ contains an appropriate scalar multiple of every bounded linear operator on $\mathcal{H}$. If $f(z) = z^2$, then $\mathbb{C}f(\mathcal{S}(\Delta))$ coincides with the set of all operators possessing a square root. However, as noted in the Conway and Morrel paper [**17**], this proves beyond the scope of present day operator theory, even for such simple functions as $f(z) = z^p$ or $f(z) = e^z$. Because of this, they study the norm closure in $\mathcal{B}(\mathcal{H})$ of the set $\mathcal{S}(\Delta)$. This allows them to employ the elaborate machinery of the *Similarity Theorem* for Hilbert space operators, developed by Apostol, Herrero, and Voiculescu [**2**]. This theorem and its many consequences detail the structure of the closure of many similarity invariant subsets of $\mathcal{B}(\mathcal{H})$. In particular, much of the analysis may be applied to $\overline{f(\mathcal{S}(\Delta))}$, which is itself similarity invariant.

Examples of results found in [**16**] are:

- If $\overline{f(\mathcal{S}(\Delta))} = \mathcal{B}(\mathcal{H})$, then $f(\mathcal{S}(\Delta)) = \mathcal{B}(\mathcal{H})$.
- If $\Delta = \mathbb{C}$ and $f(z) = z \sin z$ or $f(z) = \Pi_{n=1}^{\infty}(1 - a/n^2)$, then $f(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$.
- Let $\Delta = \{z \in \mathbb{C} : z \neq 2\}$ and $f(z) = z^2(2 - z)$. If $U(= S^*)$ is the unilateral forward shift operator (cf. Example 7.9 below), then $U \oplus U \in f(\mathcal{S}(\Delta))$, but $U \notin \overline{f(\mathcal{S}(\Delta))}$. On the other hand, $U \oplus 0 \in f(\mathcal{S}(\Delta))$.

Exercises for Chapter 3

Exercise 3.1. EXPONENTIALS

Let $\mathcal{A}$ be a unital Banach algebra, and let $a, b \in \mathcal{A}$, and $\exp(z) = e^z$, $z \in \mathbb{C}$.

(a) If $ab = ba$, prove that $\exp(a)\exp(b) = \exp(a+b)$.

(b) Does this necessarily hold if $ab \neq ba$?

Exercise 3.2. THE FUNCTIONAL CALCULUS - AN EXAMPLE

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ be two elements of $\mathbb{M}_3(\mathbb{C})$.

Let

$$T = \begin{bmatrix} 4 & 1 & 0 & -4 & 0 & 0 \\ 0 & 4 & -1 & 0 & -4 & 2 \\ 0 & 0 & 6 & 0 & 0 & -6 \\ 2 & 0 & 0 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 & -2 & 2 \\ 0 & 0 & 3 & 0 & 0 & -3 \end{bmatrix} \in \mathbb{M}_6(\mathbb{C}).$$

(a) Let $\exp(z) = e^z$, $z \in \mathbb{C}$. Find $\exp(T)$.

HINT. $S := \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ is invertible and

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -x+2y & 2x-2y \\ -x+y & 2x-y \end{bmatrix}.$$

(b) Let $g(z) = \begin{cases} 1 & \text{if } |z| < \frac{1}{2} \\ 0 & \text{if } |z-2| < \frac{1}{2} \\ 0 & \text{if } |z-3| < \frac{1}{2} \end{cases}$. Find $g(T)$.

Exercise 3.3. IDEMPOTENTS FROM THE FUNCTIONAL CALCULUS

Let a and b be two elements of a unital Banach algebra $\mathcal{A}$. Suppose that $\sigma(a)$ is a disjoint union of two non-empty compact sets σ_0 and σ_1. Let Γ be a system of closed contours in $\mathbb{C}$ such that $\sigma_k \cap \Gamma^* = \varnothing$, $k = 1, 2$ and $\mathrm{Ind}_\Gamma(z) = 1$ for all $z \in \sigma_1$; $\mathrm{Ind}_\Gamma(z) = 0$ for all $z \in \sigma_0$.

Prove that

$$f := \frac{1}{2\pi i} \int_\Gamma (z1 - a)^{-1} dz$$

is an idempotent in $\mathcal{A}$. That is, $f = f^2$.

Exercise 3.4. SQUARE ROOTS

If $a \in \mathbb{M}_2(\mathbb{C})$, does there exist $b \in \mathbb{M}_2(\mathbb{C})$ such that $b^2 = a$? More generally, under what circumstances does $a \in \mathbb{M}_n(\mathbb{C})$ have a square root?

Exercise 3.5. More square roots

Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let S be the operator satisfying $Se_n = e_{n+1}$ for all $n \geq 1$. (Extend S by linearity and continuity to all of $\mathcal{H}$.) Does there exist $B \in \mathcal{B}(\mathcal{H})$ such that $B^2 = S$?

Exercise 3.6. Laurent™ series expansions

Let $\mathfrak{X}$ be a Banach space and let $G \subseteq \mathbb{C}$ be an open set which contains an open annulus

$$A_{r_1,r_2}(\lambda_0) := \{z \in \mathbb{C} : r_1 < |z - \lambda_0| < r_2\}.$$

Prove that there exists a unique sequence $(y_n)_{n\in\mathbb{Z}} \subseteq \mathfrak{X}$ such that for all $z \in A_{r_1,r_2}(\lambda_0)$,

$$f(z) = \sum_{n\in\mathbb{Z}} (z - \lambda_0)^n y_n.$$

(On $A_{r_1,r_2}(\lambda_0)$, the series is norm-convergent.)

Hint. Let $\rho \in (r_1, r_2)$ be a real number and set $\Gamma(\theta) := \lambda_0 + \rho e^{2\pi i\theta}$, $\theta \in [0,1]$. Show that one can take

$$y_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - \lambda_0)^{n+1}} dz, \quad n \in \mathbb{Z}.$$

CHAPTER 4

The spectrum

Ordinarily he is insane. But he has lucid moments when he is only stupid.

Heinrich Heine

Basic theory

4.1. Spectrum relative to a subalgebra. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras with $1 \in \mathcal{A} \subseteq \mathcal{B}$. For $a \in \mathcal{A}$, we have have two notions of spectrum, namely:

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : (\lambda 1 - a)^{-1} \notin \mathcal{A}\}$$

and

$$\sigma_{\mathcal{B}}(a) = \{\lambda \in \mathbb{C} : (\lambda 1 - a)^{-1} \notin \mathcal{B}\}.$$

In general, it is clear that $\sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{A}}(a)$. Our present intention is to exhibit a closure relation between the two spectra.

4.2. Example. Let $\mathcal{B} = \mathcal{C}(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in the complex plane. Let $\mathcal{A} = \mathbb{A}(\mathbb{D})$ be the disk algebra defined in Example 2.5. By the same Example, $\mathcal{A} \subseteq \mathcal{B}$.

Let f be the identity function $f(z) = z$, so that clearly $f \in \mathcal{A}$. Then $\|f\| = 1$, so that $\sigma_{\mathcal{A}}(f)$, $\sigma_{\mathcal{B}}(f) \subseteq \mathbb{D}$. Now if $|\lambda| < 1$, then the function $g_\lambda(z) = \frac{1}{\lambda - z} \notin \mathcal{A}(\mathbb{D})$, and so $\lambda \in \sigma_{\mathcal{A}}(f)$. Since the spectrum of an element is always compact and hence closed, $\sigma_{\mathcal{A}}(f) = \mathbb{D}$.

In contrast, $g_\lambda \in \mathcal{B} = \mathcal{C}(\mathbb{T})$, so that $\sigma_{\mathcal{B}}(f) \subseteq \mathbb{T}$. If $|\lambda| = 1$, then g_λ is clearly not continuous on the circle, so that $\lambda \in \sigma_{\mathcal{B}}(f)$. We conclude that $\sigma_{\mathcal{B}}(f) = \mathbb{T}$.

This example proves to be prototypical of the phenomenon we wish to explore.

4.3. Definition. *Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ is said to be a* ***right topological divisor of zero*** *if there exists a sequence $(x_n)_n \subseteq \mathcal{A}$, $\|x_n\| = 1$ for all $n \geq 1$ such that*

$$\lim_n x_n a = 0.$$

Similarly, we say that $a \in \mathcal{A}$ is a ***left topological divisor of zero*** *if there exists a sequence $(x_n)_n \subseteq \mathcal{A}$, $\|x_n\| = 1$ for all $n \geq 1$ such that*

$$\lim_n ax_n = 0.$$

Finally, we say that $a \in \mathcal{A}$ is a ***joint topological divisor of zero*** *if there exists a sequence $(x_n)_n \subseteq \mathcal{A}$, $\|x_n\| = 1$ for all $n \geq 1$ such that*

$$\lim_n (ax_n + x_n a) = 0.$$

4.4. Theorem. *Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \partial(\mathcal{A}^{-1})$. Then a is a joint topological divisor of zero.*

Proof. Since $a \in \partial(\mathcal{A}^{-1})$, there exists a sequence $(b_n)_n \subseteq \mathcal{A}^{-1}$ such that $\lim_n b_n = a$. Now we claim that the set $\{\|b_n^{-1}\|\}_{n=1}^{\infty}$ is unbounded, for if $\|b_n^{-1}\| \leq M$ for some $M > 0$ and for all $n \geq 1$, then

$$\begin{aligned} \|b_n^{-1} - b_m^{-1}\| &= \|b_n^{-1}(b_m - b_n)b_m^{-1}\| \\ &\leq M^2 \|b_m - b_n\|. \end{aligned}$$

Thus $(b_n^{-1})_n$ is a Cauchy sequence. Let $c = \lim_n b_n^{-1}$. Then by the continuity of inversion, $c = a^{-1}$ and so $a \in \mathcal{A}^{-1}$. But $\mathcal{A}^{-1}$ is open, which contradicts the fact that $a \in \partial(\mathcal{A}^{-1})$.

Next, by choosing a suitable subsequence of $(b_n)_n$ and reindexing if necessary, we may assume that $\|b_n^{-1}\| \geq n,\ n \geq 1$. Let $x_n = b_n^{-1}/\|b_n^{-1}\|$ for each n, and

$$\begin{aligned} \|ax_n\| &= \|(a - b_n)x_n + b_n x_n\| \\ &\leq \|(a - b_n)x_n\| + \|b_n^{-1}\|^{-1}. \end{aligned}$$

Thus $\lim_n ax_n = 0$, and similarly, $\lim_n x_n a = 0$.

□

As an immediate Corollary to this, we obtain the following.

4.5. Corollary. *Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. If $\lambda \in \partial(\sigma(a))$, then $(a - \lambda)$ is a joint topological divisor of 0.*

4.6. Proposition. *Let $\mathcal{A}$ be a Banach algebra and suppose that $a \in \mathcal{A}$ is a joint topological divisor of 0 in $\mathcal{A}$. Then $0 \in \sigma_{\mathcal{A}}(a)$.*

Proof. Suppose that there exists $b = a^{-1} \in \mathcal{A}$. Take $(x_n)_n \in \mathcal{A}^{\mathbb{N}}$, $\|x_n\| = 1$ for all $n \geq 1$, such that $\lim_n ax_n = 0$. Then

$$\|x_n\| = \|b\, a\, x_n\| \leq \|b\|\, \|a\, x_n\|$$

so that $\lim_n \|x_n\| = 0$, a contradiction.

□

We note that if $a \in \mathcal{A}$ is a joint topological divisor of 0 in $\mathcal{A}$, and if $\mathcal{B}$ is a Banach algebra containing $\mathcal{A}$, then a is a joint topological divisor of 0 in $\mathcal{B}$, and so $0 \in \sigma_{\mathcal{B}}(a)$ as well.

4.7. Proposition. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and suppose $a \in \mathcal{A} \subseteq \mathcal{B}$. Then*

(i) $\sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{A}}(a)$; *and*
(ii) $\partial(\sigma_{\mathcal{A}}(a)) \subseteq \sigma_{\mathcal{B}}(a)$.

Proof.

(i) Immediate.
(ii) If $\lambda \in \partial(\sigma_{\mathcal{A}}(a))$, then $a - \lambda$ is a topological divisor of 0 in $\mathcal{A}$ and so $a - \lambda$ is not invertible in $\mathcal{B}$, by Proposition 4.6.

□

4.8. Remark. The conclusion of Proposition 4.7 is that the most that can happen to the spectrum of an element a when passing to a subalgebra that contains a is that we "fill in" the "holes" of the spectrum, that is, the bounded components of the resolvent of a in the larger algebra.

4.9. Theorem. *Let $\mathcal{B}$ be a Banach algebra and $a \in \mathcal{B}$. Let Ω be a subset of $\rho_{\mathcal{B}}(a)$ which has non-empty intersection with each bounded component of $\rho_{\mathcal{B}}(a)$. Finally, let $\mathcal{A}$ be the smallest closed subalgebra of $\mathcal{B}$ containing* 1, a, *and* $(\lambda - a)^{-1}$ *for each* $\lambda \in \Omega$. *Then* $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$.

Proof. Choose $\phi \in \mathcal{B}^*$ so that $\phi(x) = 0$ for all $x \in \mathcal{A}$. Define the function

$$\begin{array}{rccc} h_\phi : & \rho_{\mathcal{B}}(a) & \to & \mathbb{C} \\ & z & \mapsto & \phi((z-a)^{-1}) \end{array}$$

so that h_ϕ is holomorphic on its domain. We shall now show that $h_\phi \equiv 0$. Since this is true for all $\phi \in \mathcal{B}^*$ that annihilates $\mathcal{A}$, we can then invoke Corollary 1.19 to obtain the desired result.

Now if $|z| > \operatorname{spr}(a)$, then

$$(z-a)^{-1} = \sum_{n=0}^{\infty} z^{-n-1} a^n$$

converges uniformly and thus $(z-a)^{-1} \in \mathcal{A}$. Hence $h_\phi(z) \equiv 0$ for all z, $|z| > \operatorname{spr}(a)$. Thus $h_\phi \equiv 0$ on the unbounded component of $\rho_{\mathcal{B}}(a)$.

If $\lambda \in \Omega$ lies in a bounded component of $\rho_{\mathcal{B}}(a)$, then note that

$$(z-a) = (\lambda - a)\,(1 - (\lambda - z)\,(\lambda - a)^{-1}).$$

Thus if $|\lambda - z| < \|(\lambda - a)^{-1}\|^{-1}$, we have

$$(z-a)^{-1} = \sum_{n=0}^{\infty} (\lambda - z)^n\, (\lambda - a)^{-n-1}$$

which converges in norm and therefore lies in $\mathcal{A}$. As such, $h_\phi \equiv 0$ on an open neighbourhood of λ and so $h_\phi \equiv 0$ on the entire component of $\rho_\mathcal{B}(a)$ containing λ.

Since Ω intersects every bounded component of $\rho_\mathcal{B}(a)$, $h_\phi \equiv 0$ on $\rho_\mathcal{B}(a)$. As this is true for all $\phi \in \mathcal{B}^*$ which annihilates the closed subspace $\mathcal{A}$, we conclude that $(z-a)^{-1} \in \mathcal{A}$ for all $z \in \rho_\mathcal{B}(a)$. That is, $\rho_\mathcal{A}(a) = \rho_\mathcal{B}(a)$, or equivalently, $\sigma_\mathcal{A}(a) = \sigma_\mathcal{B}(a)$.

□

It is worth pointing out that what we have shown is that $\mathcal{A}$ coincides with the closed algebra generated by 1, a, and $(z-a)^{-1}$ for all $z \in \rho_\mathcal{B}(a)$; in other words, the algebra generated by the rational functions with poles outside of $\sigma_\mathcal{B}(a)$. This algebra is often denoted by $\mathrm{Rat}(a)$ in the literature.

4.10. Definition. *If $\mathcal{B}$ is a Banach algebra, then a subalgebra $\mathcal{A}$ of $\mathcal{B}$ is said to be a **maximal abelian subalgebra** if it is commutative and it is not properly contained in any commutative subalgebra of $\mathcal{B}$.*

4.11. Example. Let $2 \le n \in \mathbb{N}$ and $\mathcal{H} = \mathbb{C}^n$. Fix an orthonormal basis $\{e_k\}_{k=1}^n$ for $\mathcal{H}$ and let $J \in \mathcal{B}(\mathbb{C}^n) \simeq \mathbb{M}_n(\mathbb{C})$ be the Jordan cell relative to this basis; i.e. J is the unique element of $\mathcal{B}(\mathbb{C}^n)$ defined by $Je_1 = 0$ and $Je_k = e_{k-1}$, $2 \le k \le n$.

Then $\mathcal{A} := \mathrm{Alg}(J) = \{\sum_{k=0}^{n-1} \alpha_k J^k : \alpha_k \in \mathbb{C}, 0 \le k \le n-1\}$ is a maximal abelian subalgebra of $\mathcal{B}(\mathbb{C}^n)$. The verification of this is left to the reader.

4.12. Proposition. *Let $\mathcal{B}$ be a unital Banach algebra, and suppose that $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{B}$. Then $\sigma_\mathcal{A}(a) = \sigma_\mathcal{B}(a)$ for all $a \in \mathcal{A}$.*

Proof. First observe that $1 \in \mathcal{A}$, for otherwise the algebra generated by 1 and $\mathcal{A}$ is abelian and properly contains $\mathcal{A}$, a contradiction.

Clearly $\sigma_\mathcal{B}(a) \subseteq \sigma_\mathcal{A}(a)$. Suppose that $\lambda \in \rho_\mathcal{B}(a)$. Then for all $c \in \mathcal{A}$, $c\,(a-\lambda 1) = (a-\lambda 1)\,c$. If we let $b = (a-\lambda 1)^{-1} \in \mathcal{B}$, then multiplying this equation on the left and the right by b yields $b\,c = c\,b$ for all $c \in \mathcal{A}$. Thus $b \in \mathcal{A}$, as $\mathcal{A}$ is maximal abelian. In other words, $\lambda \in \rho_\mathcal{A}(a)$, and we are done.

□

The upper-semicontinuity of the spectrum

4.13. We now turn to the question of determining in what sense the map that sends an element a of a Banach algebra $\mathcal{A}$ to its spectrum $\sigma(a) \subseteq \mathbb{C}$ is continuous.

To do this, we shall first define a new metric on the collection of compact subsets of $\mathbb{C}$, called the *Hausdorff metric*. Our usual notion of distance between two compact sets A and B is

$$\mathrm{dist}(A, B) := \inf\{|a-b| \ : a \in A,\, b \in B\}.$$

Of course, if $A = \{a\}$ is a singleton , we simply write $\mathrm{dist}(a, B)$.

The problem (for our purposes) with this distance is the following. If we let $A = \{0\}$ and $B = \mathbb{D}$, the closed unit disk, then $\mathrm{dist}(A, B) = 0$. We are looking for a notion of distance that indicates how far two subsets of $\mathbb{C}$ are from being identical.

4.14. Definition. *Given two compact subsets A and B of $\mathbb{C}$, we define the* ***Hausdorff distance*** *between A and B to be*

$$d_{\mathrm{H}}(A,B) = \max\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\}.$$

We remark that the Hausdorff distance between $\{0\}$ and $\mathbb{D}$ is 1.

4.15. Definition. *Let X and Y be topological spaces and let $\Phi : X \to \mathcal{P}(Y)$ be a function, where $\mathcal{P}(Y)$ denotes the power set of Y. The mapping Φ is said to be* ***upper-semicontinuous*** *if for every $x_0 \in X$ and every neighbourhood U of $\Phi(x_0)$ in Y, there exists a neighbourhood V of x_0 such that $\Phi(x) \subseteq U$ for all $x \in V$.*

4.16. Theorem. [The upper-semicontinuity of the spectrum.] *Let $\mathcal{A}$ be a Banach algebra. Then the mapping*

$$\begin{array}{rccc} \Phi : & \mathcal{A} & \to & \mathcal{P}(\mathbb{C}) \\ & a & \mapsto & \sigma(a) \end{array}$$

is upper-semicontinuous.

Proof. We must show that if U is an open set in $\mathbb{C}$ containing $\sigma(a)$, then there exists $\delta > 0$ such that $\|x - a\| < \delta$ implies $\sigma(x) \subseteq U$.

Suppose otherwise. Then by choosing $\delta_n = 1/n$, $n \geq 1$, we can find $x_n \in \mathcal{A}$ with $\|x_n - a\| < \delta_n$ and $\lambda_n \in \sigma(x_n) \cap (\mathbb{C} \setminus U)$. Since $|\lambda_n| \leq \operatorname{spr}(x_n) \leq \|x_n\| \leq \|a\| + 1/n \leq \|a\| + 1$, we know that the sequence $(\lambda_n)_n$ is bounded, and so by the Bolzano-Weierstraß Theorem (by dropping to a subsequence if necessary), we may assume that $\lambda = \lim_n \lambda_n$ exists.

Clearly $\lambda \notin U$ as $\lambda_n \notin U$, $n \geq 1$, and $\mathbb{C} \setminus U$ is closed. Thus $\lambda - a \in \mathcal{A}^{-1}$. Since $\lambda - a = \lim_{n\to\infty} \lambda_n - x_n$ and $\mathcal{A}^{-1}$ is open, we must have $\lambda_n - x_n \in \mathcal{A}^{-1}$ for some $n \geq 1$, a contradiction.

This completes the proof.

□

It is worth noting that the map Φ above need not in general be continuous. For example, it is possible to find a sequence $(Q_n)_{n=1}^{\infty}$ of Hilbert space operators such that $\sigma(Q_n) = \{0\}$ for each $n \geq 1$, converging to an operator $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) = \{z \in \mathbb{C} : |z| \leq 1\}$.

The above theorem, while basic, is of extreme importance in the theory of approximation of Hilbert space operators. While this result in itself is sufficient for a large number of applications, sometimes we require a stronger result; one which implies the upper-semicontinuity of the "parts" or components of the spectrum.

The theorem we have in mind is due to Newburgh (see Theorem 4.18 below), and as a corollary we obtain a class of elements for which the spectrum is continuous, as opposed to just semi-continuous. We begin with the following proposition.

4.17. Proposition. *Let $a \in \mathcal{A}$, a unital Banach algebra, and let $(a_n)_n \subseteq \mathcal{A}$ be a sequence such that $a = \lim_n a_n$. Let $U \supseteq \sigma(a)$ be open and suppose*

(i) *$\sigma(a_n) \subseteq U$ for all $n \geq 1$;*
(ii) *$f : U \to \mathbb{C}$ is analytic.*

Then $\lim_n f(a_n) = f(a)$.
Note: Condition (i) can always be obtained simply by applying Theorem 4.16 and dropping to an appropriate subsequence.
Proof. Let $V \subseteq \mathbb{C}$ be an open subset satisfying $\sigma(a) \subseteq V \subseteq \overline{V} \subseteq U$. Without loss of generality, we may assume $\sigma(a_n) \subseteq V$ for all $n \geq 1$. Let Γ be a finite system of closed contours satisfying

(a) $\mathrm{Ind}_\Gamma(\lambda) = 1$ for all $\lambda \in V$;
(b) $\{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) \neq 0\} \subseteq U$.

Then $f(a)$, $f(a_n)$ are all well-defined. Moreover,

$$\begin{aligned}
\|f(a) - f(a_n)\| &= \|\frac{1}{2\pi i}\int_\Gamma f(z)(z-a)^{-1} - f(z)(z-a_n)^{-1}\,dz\| \\
&= \frac{1}{2\pi}\|\int_\Gamma f(z)\left((z-a)^{-1} - (z-a_n)^{-1}\right)\,dz\| \\
&\leq \frac{1}{2\pi}\|\Gamma\|\,\|f\|_\Gamma \sup_{z\in\Gamma}\|(z-a)^{-1} - (z-a_n)^{-1}\|,
\end{aligned}$$

where $\|\Gamma\|$ denotes the arclength of Γ, and $\|f\|_\Gamma = \sup\{|f(z)| : z \in \Gamma\}$.

Since inversion is continuous and Γ is compact, the latter quantity tends to 0 as n tends to infinity, and so we obtain

$$\lim_{n\to\infty}\|f(a) - f(a_n)\| = 0.$$

□

4.18. Theorem. [Newburgh.] *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Suppose that U and V are two disjoint open sets such that $\sigma(a) \subseteq U \cup V$ and that $\sigma(a) \cap U \neq \varnothing$. Then there exists $\delta > 0$ such that $\|x - a\| < \delta$ implies $\sigma(x) \cap U \neq \varnothing$.*
Proof. By the upper-semicontinuity of the spectrum, there exists $\varepsilon > 0$ such that $\|x - a\| < \varepsilon$ implies $\sigma(x) \subseteq U \cup V$. Suppose that our assertion is false. Then there exists a sequence $(x_n)_n \subseteq \mathcal{A}$ satisfying

(a) $\lim_{n\to\infty} x_n = a$; and
(b) $\sigma(x_n) \subseteq V$.

Consider the function $f : U \cup V \to \mathbb{C}$ defined to be 1 on U and 0 on V. Then f is clearly analytic on $U \cup V$, and so by Proposition 4.17, $\lim_n \|f(a) - f(x_n)\| = 0$. But $f(x_n) = 0$ for all $n \geq 1$, and $1 \in f(\sigma(a)) = \sigma(f(a))$. Thus $f(a) \neq 0$, a contradiction. We conclude that the assertion holds.

□

It follows that if $(a_n)_{n=1}$ is a sequence in a Banach algebra $\mathcal{A}$ converging to an element $a \in \mathcal{A}$, and if $\sigma(a_n)$ is connected for each $n \geq 1$, then $\sigma(a)$ is connected. While this is an easy consequence of Newburgh's Theorem, it is a useful one.

4.19. Corollary. [Newburgh.] *Suppose that $\mathcal{A}$ is a unital Banach algebra and that $\sigma(a)$ is totally disconnected. Then the map $a \mapsto \sigma(a)$ is continuous at a.*

Proof. Let $\varepsilon > 0$. Since $\sigma(a)$ is totally disconnected, we can find a cover of $\sigma(a)$ consisting of disjoint open sets $U_1, U_2, \dots, U_n$, each of which intersects $\sigma(a)$ non-trivially and has diameter less than ε. By the upper-semicontinuity of the spectrum, there exists $\delta_1 > 0$ such that $\|x - a\| < \delta_1$ implies $\sigma(x) \subseteq \cup_{j=1}^{n} U_j$.

By Newburgh's Theorem 4.18, there exists $\delta_2 > 0$ such that $\|x - a\| < \delta_2$ implies that $\sigma(x) \cap U_j \neq \varnothing$, $1 \leq j \leq n$. Thus the Hausdorff distance

$$d_{\mathrm{H}}(\sigma(a), \sigma(x)) < \varepsilon$$

for all $x \in \mathcal{A}$, $\|x - a\| < \min(\delta_1, \delta_2)$, implying that the map $a \mapsto \sigma(a)$ is indeed continuous at a.

□

Supplementary Examples

S4.1. Example. Let $\mathcal{H} = \ell^2$ with orthonormal basis $\{e_n\}_n$. Let $\{d_n\}_n$ be a dense subset of $\mathbb{D}$, and define an operator $D \in \mathcal{B}(\mathcal{H}))$ via $De_n = d_n e_n$, $n \geq 1$. (We let the reader verify that such an operator exists and is unique.) We invite the reader to check that $\sigma(D) = \mathbb{D}$. (Unless we explicitly specify otherwise, when considering an operator $T \in \mathcal{B}(\mathcal{H})$, the spectrum $\sigma(T)$ will always refer to the spectrum relative to $\mathcal{B}(\mathcal{H})$.)

It can be shown that there exists a sequence $(M_n)_n$ of nilpotent operators (i.e. $M_n^{k_n} = 0$ for an appropriate choice of $k_n \in \mathbb{N}$) such that $N = \lim_n M_n$. By the Spectral Mapping Theorem, $\sigma(M_n) = \{0\}$ for all $n \geq 1$.

From this we see that the map $T \mapsto \sigma(T)$ is far from continuous. We should definitely learn to appreciate the upper semicontinuity of the spectrum a bit more.

S4.2. Example. More generally, let $K \subseteq \mathbb{C}$ be an arbitrary non-empty compact set, and let $\{d_n\}_n$ be a countable (possibly finite if K is) dense subset of K. As in the above example, let $\mathcal{H} = \ell^2$ with orthonormal basis $\{e_n\}_n$. Define an operator $D \in \mathcal{B}(\mathcal{H}))$ via $De_n = d_n e_n$, $n \geq 1$. (Once again, e let the reader verify that such an operator exists and is unique.) We invite the reader to check that $\sigma(D) = K$.

In other words, *any* compact subset of $\mathbb{C}$ serves as the spectrum of *some* operator on $\mathcal{H}$.

S4.3 Example. As we shall see in greater detail below, the set $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space forms the only non-trivial ideal of $\mathcal{B}(\mathcal{H})$. Ideals are generally thought of as being "small" in some sense. In this case, we shall also discover that $\mathcal{K}(\mathcal{H})$ is the closure of the set $\mathcal{F}(\mathcal{H})$ of finite-rank operators on $\mathcal{H}$, and as such, this gives a measure of how small $\mathcal{K}(\mathcal{H})$ is.

Given $T \in \mathcal{B}(\mathcal{H})$, therefore, an operator $T + K$ is referred to as a **compact perturbation** of T, and is thought of as "affecting T mostly on a finite-dimensional space". The impact upon the spectrum of T, however, can still be huge. For example, let $\mathcal{H} = \ell^2(\mathbb{Z})$ with orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$. The map U defined by setting $Ue_n = e_{n+1}$ for all n extends by linearity and continuity to a bijective isometry (i.e. a *unitary operator*) on $\mathcal{H}$. We invite the reader to verify that $\sigma(U) = \mathbb{T}$.

If we set $K = e_1 \otimes e_0^*$, where for $x, y \in \mathcal{H}$ we define $x \otimes y^*(z) = \langle z, y\rangle x$, then K is a rank-one operator and $U - K$ can be shown to have spectrum equal to the unit disc $\mathbb{D}$. (In fact, $U - K$ is unitarily equivalent to $S \oplus S^*$, where S is the unilateral forward shift we shall encounter later.)

Appendix

A4.1. Analysis of the spectrum and the functional calculus are key ingredients in Single Operator Theory, where one is often interested in studying a class of operators which may or may not possess an algebraic structure. For instance, on may begin with the set of **algebraic operators** on $\mathcal{H}$,

$$\text{ALG} := \{T \in \mathcal{B}(\mathcal{H}) : p(T) = 0 \text{ for some polynomial } p\}.$$

The characterisation of the norm-closure of this set in $\mathcal{B}(\mathcal{H})$ was obtained in the 1970s by D. Voiculescu [**52**] in terms of spectral conditions. More precisely, he showed that

$$\overline{\text{ALG}} = \{T \in \mathcal{B}(\mathcal{H}) : \dim \ker(T - \lambda) = \operatorname{codim} \operatorname{ran}(T - \lambda) \ \forall \lambda \in \rho_{\text{sF}}(T)\}$$

Here, $\rho_{\text{sF}}(T)$ denotes the **semi-Fredholm domain** of T. It is defined as the set of complex numbers for which the range of T is closed, and at least one of $\dim \ker T$ or $\operatorname{codim} \operatorname{ran} T$ is finite.

A4.2. Another important notion of relative spectrum is that of the spectrum of the image of an element in a quotient algebra. As we have seen in Proposition 2.17, if $\mathcal{K}$ is a closed ideal of a Banach algebra $\mathcal{A}$, then $\mathcal{A}/\mathcal{K}$ is a Banach algebra with respect to the quotient norm. Letting π denote the canonical projection map, it is clear that if $a \in \mathcal{A}$, then $\sigma_{\mathcal{A}/\mathcal{K}}(\pi(a)) \subseteq \sigma_{\mathcal{A}}(a)$.

One particular instance of quotient algebras deserves special mention. Recall from Example 2.18 that the quotient algebra $\mathcal{Q}$ of $\mathcal{B}(\mathcal{H})$ by $\mathcal{K}(\mathcal{H})$ is referred to as the **Calkin algebra**. If $T \in \mathcal{B}(\mathcal{H})$, and π is the canonical homomorphism from $\mathcal{B}(\mathcal{H})$ to $\mathcal{Q}$, then the spectrum of $\pi(T)$ is called the **essential spectrum** of T, and is often denoted by $\sigma_e(T)$. In this connection, two of the most important results concerning the spectrum are:

Theorem. **[The Putnam-Schechter Theorem]** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Suppose $\lambda \in \partial(\sigma(T))$. Then either λ is isolated, or $\lambda \in \sigma_e(T)$.*

Corollary. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma(T) = \sigma_e(T) \cup \Omega$, where Ω consists of some bounded components of the resolvent of $\pi(T)$, and a sequence of isolated points in $\rho(\pi(T))$ converging to $\sigma_e(T)$.*

Proofs of the above results appear in Appendix A.

A4.3. The upper-semicontinuity of the spectrum and Newburgh's Theorem are very powerful and useful tools, especially in the theory of approximation of Hilbert space operators. For example, in the 1970's Paul Halmos asked: what is the norm-closure in $\mathcal{B}(\mathcal{H})$ of the set

$$\text{NIL} := \{M \in \mathcal{B}(\mathcal{H}) : M^k = 0 \text{ for some } k \geq 1\}$$

of all nilpotent operators in $\mathcal{B}(\mathcal{H})$?

Using Newburgh's Theorem, one sees that the spectrum of an operator $T \in \overline{\text{NIL}}$ must be connected. The fact that the invertible operators in $\mathcal{B}(\mathcal{H})$ form an open set implies that $0 \in \sigma(T)$.

If $T = \lim_n M_n$, where $M_n \in \text{NIL}$ for all $n \geq 1$ (say $M_n^{k_n} = 0$ for appropriately chosen $k_n \in \mathbb{N}$) and if $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ is the canonical quotient map of $\mathcal{B}(\mathcal{H})$ into the Calkin algebra, then $\pi(T) = \lim_n \pi(M_n)$, and $\pi(M_n)^{k_n} = \pi(M_n^{k_n}) = \pi(0) = 0$, and so $\pi(T)$ is a limit of nilpotent elements of the Calkin algebra. Applying the above analysis to $\pi(T)$ shows that $\sigma_e(T) := \sigma(\pi(T))$ is connected and contains the origin as well.

The question was finally answered in 1976 by C. Apostol, C. Foiaş, and D. Voiculescu [**4**]. They obtained the result:

Theorem. [**Apostol, Foiaş and Voiculescu**] *An operator $T \in \mathcal{B}(\mathcal{H})$ belongs to the closure of the set of nilpotent operators if and only if*

(a) *$\sigma(T)$ is connected and contains $\{0\}$;*
(b) *$\sigma_e(T)$ is connected and contains $\{0\}$; and*
(c) *dim ker$(T - \lambda)$ = codim ran$(T - \lambda)$ $\forall \lambda \in \rho_{\mathrm{sF}}(T)$.*

An earlier result due to D.A. Herrero [**27**] showed that a **normal operator** N is a limit of nilpotent operators if and only if $\sigma(N)$ is connected and contains $\{0\}$.

A4.4. There are other results concerning the continuity of the spectrum and of the spectral radius of Banach algebra elements. In particular, Murphy [**37**] has obtained the following results.

Suppose that $K \subseteq \mathbb{C}$ is compact, and $\mathcal{A}$ is a unital Banach algebra. Define $\alpha(K) = \sup\{\inf_{\lambda \in C} |\lambda| : C \text{ a component of } K\}$ and $r(K) = \sup_{\lambda \in K} |\lambda|$. Then $\alpha(K) \leq r(K)$. Let D be a *diagonal* operator on $\mathcal{B}(\ell^2(\mathbb{N}))$; that is, if $\{e_n\}_{n=1}^\infty$ denotes an orthonormal basis for $\ell^2(\mathbb{N})$, then $De_n = d_n e_n$ for some bounded sequence $\{d_n\}_{n=1}^\infty$ of complex numbers.

Proposition. [**Murphy**] *The following statements are equivalent:*

(i) *Every element of every unital Banach algebra $\mathcal{A}$ with spectrum K is a point of continuity of the function $a \mapsto \sigma_{\mathcal{A}}(a)$;*
(ii) *$\alpha(K) = r(K)$;*
(iii) *D is a point of continuity of the function $T \mapsto \sigma_{\mathcal{B}(\mathcal{H})}(T)$.*

As for the spectral radius, let

$$K_0 = \{\lambda \in K : \text{ the component of } \lambda \text{ in } K \text{ is } \{\lambda\}\}.$$

Thus $K = K_0$ if and only if K_0 is totally disconnected.

Proposition. **[Murphy]** *The following statements are equivalent:*

(i) *Every element of every unital Banach algebra* $\mathcal{A}$ *with spectrum* K *is a point of continuity of the function* $a \mapsto \operatorname{spr}_{\mathcal{A}}(a)$*;*

(ii) $K = \overline{K_0}$*;*

(iii) D *is a point of continuity of the function* $T \mapsto \operatorname{spr}_{\mathcal{B}(\mathcal{H})}(T)$*;*

(iv) *For each* $\varepsilon > 0$ *and for each* $\lambda \in K$, $B(\lambda, \varepsilon) = \{\mu \in \mathbb{C} : |\mu - \lambda| < \varepsilon\}$ *contains a component of* K.

Exercises for Chapter 4

Exercise 4.1. Joint topological divisors of zero

Here's a question to which I do not know the answer (in part because it's just occurred to me and I haven't given it much thought). Suppose that $\mathcal{A}$ is a Banach algebra and $a \in \mathcal{A}$ is both a left- and a right-topological divisor of zero. Is a a joint topological divisor of zero?

The issue is that the sequences $(x_n)_n$ and $(y_n)_n$ from $\mathcal{A}$ such that

$$\lim_n x_n a = 0 = \lim_n a y_n$$

could potentially be different.

Exercise 4.2. The spectrum is not continuous

Find an example of a Banach algebra $\mathcal{A}$ and a sequence $(a_n)_n \in \mathcal{A}^{\mathbb{N}}$ converging to $a \in \mathcal{A}$ such that

- $\sigma(a_n) = \{0\}$ for all $n \geq 1$, and
- $\sigma(a) = \mathbb{D}$.

Hint. There may be several such examples, but one standard example consists of choosing $\mathcal{A} = \mathcal{B}(\mathcal{H})$ for $\mathcal{H} \simeq \ell^2$, and letting each a_n be a **weighted shift** (with a cleverly chosen set of weights!). We shall define weighted shifts in Chapter XX, and as such, it might be easier to wait until then before attempting this question.

Exercise 4.3. Continuity of the spectrum in $\mathbb{M}_n(\mathbb{C})$

Let $(T_k)_k$ be a sequence in $\mathbb{M}_n(\mathbb{C})$ and suppose that $T = \lim_k T_k$. Prove that $\sigma(T) = \lim_k \sigma(T_k)$, where the limit is taken with respect to the Hausdorff metric.

What can you say about the algebraic multiplicity of the eigenvalues of T relative to $\sigma(T_k)$?

Exercise 4.4. The density of the set of invertible elements of $\mathbb{M}_n(\mathbb{C})$

Prove that $\mathrm{GL}_n(\mathbb{C})$ is dense in $\mathbb{M}_n(\mathbb{C})$. (C^*-algebraists sometimes refer to this property by saying that $\mathbb{M}_n(\mathbb{C})$ has **topological stable rank** equal to 1.

Exercise 4.5. Components of the spectrum

Find a Banach algebra $\mathcal{A}$ such that for each $\varepsilon > 0$, there exist elements $a, b \in \mathcal{A}$ such that $\|a - b\| < \varepsilon$ and

(a) $\sigma(a)$ has only one component;
(b) $\sigma(b)$ has infinitely many components.

Now find a Banach algebra $\mathcal{B}$ for which given *any* $\varepsilon > 0$, there *do not* exist elements a and b of $\mathcal{B}$ satisfying the above two conditions.

Exercise 4.6. SPECTRUM AND DIRECT SUMS

Let $A_n \in \mathbb{M}_{k_n}(\mathbb{C})$, $n \geq 1$. Suppose that there exists $M > 0$ such that $\|A_n\| \leq M$ for all $n \geq 1$.

(a) Is it true that $\sigma(\oplus_{n=1}^{\infty} A_n) = \overline{\cup_{n=1}^{\infty} \sigma(A_n)}$?

(b) Now suppose that the sequence $(k_n)_n$ is bounded above. Does this make any difference to the solution of (a)?

CHAPTER 5

Abelian Banach algebras

You can observe a lot by just watching.

Yogi Berra

The Gelfand Transform

5.1. In this chapter we focus our attention on those Banach algebras which are abelian. In any algebra (normed, abelian or otherwise) it is of interest to study the ideal structure. Banach algebras are no exception.

5.2. Definition. *Let $\mathcal{A}$ be an abelian Banach algebra. An ideal $\mathcal{J}$ of $\mathcal{A}$ is said to be* ***modular*** *(also called* ***regular****) if we can find an element $e \in \mathcal{A}$ such that $ex - x \in \mathcal{J}$ for all $x \in \mathcal{A}$.*

Recall that given a Banach algebra $\mathcal{A}$ and an ideal $\mathcal{I}$ of $\mathcal{A}$, we use $\pi_{\mathcal{J}}$ to denote the canonical algebra map from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{J}$, and that when the ideal $\mathcal{J}$ is understood, we abbreviate this to π.

Armed with this notation, the statement that $\mathcal{J}$ is a modular ideal of $\mathcal{A}$ is readily seen to be equivalent to saying that the quotient algebra $\mathcal{A}/\mathcal{J}$ admits an identity element, namely

$$\pi(e) = e + \mathcal{J}.$$

Clearly every proper ideal in a unital Banach algebra is modular.

5.3. Example. Let $\mathcal{A} = \mathcal{C}_0(\mathbb{R})$, the set of complex-valued continuous functions on $\mathbb{R}$ vanishing at infinity. Define $\mathcal{M} = \{f \in \mathcal{A} : f(x) = 0 \text{ if } x \in [-1,1]\}$. It is readily seen that $\mathcal{A}$ is a non-unital Banach algebra and $\mathcal{M}$ is an ideal of $\mathcal{A}$.

Let $e \in \mathcal{A}$ be the function $e(x) = \begin{cases} 0 & \text{if } 2 < |x| \\ 2 - |x| & \text{if } 1 \le |x| \le 2 \\ 1 & \text{if } |x| \le 1. \end{cases}$

We leave it to the reader to verify that e is an identity for $\mathcal{A}/\mathcal{M}$, and that $\mathcal{M}$ is therefore a regular ideal of $\mathcal{A}$.

5.4. Proposition. *Let $\mathcal{J}$ be a proper regular ideal of an abelian Banach algebra $\mathcal{A}$. If e is an identity modulo $\mathcal{J}$, then*

$$\inf_{m\in\mathcal{J}} \|e - m\| \geq 1.$$

Proof. First note that if $\mathcal{J}$ is closed, then $\mathcal{A}/\mathcal{J}$ is a Banach algebra by Proposition 2.17. But then $\|\pi_{\mathcal{J}}(e) = \inf_{m\in\mathcal{J}} \|e - m\| \geq 1$ by the submultiplicativity of the quotient norm.

Now consider the case where $\mathcal{J}$ is not closed. Suppose $\|e - m\| < 1$ for some $m \in \mathcal{J}$. Then $x = \sum_{n=1}^{\infty}(e - m)^n$ converges in $\mathcal{A}$. But $(e - m)x = \sum_{n=2}^{\infty}(e - m)^n$, so

$$\begin{aligned} x &= (e - m)x + (e - m) \\ &= ex - mx + e - m; \end{aligned}$$

thus $e = x - ex + mx - m \in \mathcal{J}$. Since $ea - a \in \mathcal{J}$ for all $a \in \mathcal{A}$, we conclude that $a \in \mathcal{J}$ for all $a \in \mathcal{A}$, i.e. $\mathcal{A} \subseteq \mathcal{J}$, a contradiction.

Thus $\inf_{m\in\mathcal{J}} \|e - m\| \geq 1$.

□

5.5. Definition. *A proper ideal $\mathcal{M}$ of an algebra $\mathcal{A}$ is said to be* ***maximal*** *if it is not contained in any ideal of $\mathcal{A}$ except itself, and the entire algebra $\mathcal{A}$.*

In other words, $\mathcal{M}$ is a maximal ideal of $\mathcal{A}$ if $\mathcal{M}$ is a maximal element of the set $\mathfrak{J} := \{\mathcal{J} \subseteq \mathcal{A} : \mathcal{J} \text{ is an ideal}\}$, partially ordered with respect to inclusion.

5.6. Examples.

(a) Let $\mathcal{A} = \mathcal{C}_0(\mathbb{R})$, and set $\mathcal{M} = \{f \in \mathcal{A} : f(0) = 0\}$. Then $\mathcal{M}$ is a maximal ideal of $\mathcal{A}$. This is clear since $\dim \mathcal{A}/\mathcal{M} = 1$.

(b) Let k, n be integers satisfying $1 \leq k \leq n$. Set $\mathcal{K} = \{T = [t_{ij}] \in \mathcal{T}_n(\mathbb{C}) : t_{kk} = 0\}$. Then $\mathcal{K}$ is a maximal ideal of $\mathcal{A}$, again, because $\dim \mathcal{A}/\mathcal{K} = 1$.

(c) The ideal $\mathcal{K} = \{0\}$ of $\mathbb{M}_n(\mathbb{C})$ is maximal because $\mathbb{M}_n(\mathbb{C})$ is simple.

(d) Culture. Although we do not yet have the tools to prove this, we mention that if $\mathcal{H}$ is an infinite-dimensional, separable Hilbert space, then $\mathcal{K}(\mathcal{H})$ is a maximal ideal of $\mathcal{B}(\mathcal{H})$. Indeed, it is the only non-zero, proper closed ideal of $\mathcal{B}(\mathcal{H})$.

(e) Let

$$J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{M}_n(\mathbb{C}).$$

Set $\mathcal{A} = \text{Alg}(J) = \{p_0 I + p_1 J + p_2 J^2 + \cdots + p_{n-1} J^{n-1} : p_k \in \mathbb{C}, 0 \leq k \leq n-1\}$.

Let $\mathcal{K} := \{p_1 J + p_2 J^2 + \cdots + p_{n-1} J^{n-1} : p_k \in \mathbb{C}, 1 \leq k \leq n-1\}$. Then $\mathcal{A}$ is commutative and $\mathcal{K}$ is a maximal ideal of $\mathcal{A}$.

5.7. Corollary. *Let $\mathcal{A}$ be an abelian Banach algebra. If $\mathcal{J}$ is a proper modular ideal of $\mathcal{A}$, then $\mathcal{J}$ is contained in some maximal (modular) ideal $\mathcal{M}$ of $\mathcal{A}$. Furthermore, all maximal modular ideals of $\mathcal{A}$ are closed.*

Proof. Let $\mathcal{J} \triangleleft \mathcal{A}$ be a proper modular ideal, and let $e \in \mathcal{A}$ denote the identity modulo $\mathcal{J}$. First we observe that if $\mathcal{K}$ is any proper ideal of $\mathcal{A}$ containing $\mathcal{J}$, then $\mathcal{K}$ is also modular. Indeed, clearly e also serves as an identity modulo $\mathcal{K}$.

Consider the set

$$\mathfrak{K} = \{\mathcal{K} \triangleleft \mathcal{A} : \mathcal{J} \subseteq \mathcal{K} \text{ and } e \notin \mathcal{K}\},$$

partially ordered with respect to inclusion. Then $\mathcal{J} \in \mathfrak{K}$, so $\mathfrak{K} \neq \varnothing$. Choose an increasing chain $\mathfrak{C}$ in $\mathfrak{J}$, say

$$\mathfrak{C} = \{\mathcal{K}_\alpha\}_{\alpha \in \Lambda}.$$

Let $\mathcal{K} = \cup_{\alpha\in\Lambda}\mathcal{K}_\alpha$. It is routine to verify that $\mathcal{K} \triangleleft \mathcal{A}$, and of course $e \notin \mathcal{K}$, since $e \notin \mathcal{K}_\alpha$ for all α.

It follows that $\mathcal{K}$ is an upper bound for $\mathfrak{C}$. By Zorn's Lemma, there exists a maximal element $\mathcal{M}$ in $\mathfrak{K}$, and $\mathcal{J} \subseteq \mathcal{M}$. Clearly $e \notin \mathcal{M}$ since $e \notin \mathcal{K}$ for any $\mathcal{K} \in \mathfrak{J}$. Thus $\mathcal{M}$ is a proper maximal ideal of $\mathcal{A}$ containing $\mathcal{J}$.

As for the last statement, suppose that $\mathcal{L}$ is a maximal ideal of $\mathcal{A}$, and let $e_{\mathcal{L}}$ be an identity modulo $\mathcal{L}$. Then the norm closure of $\mathcal{L}$ is also seen to be an ideal of $\mathcal{A}$. By maximality, $\mathcal{L} = \overline{\mathcal{L}}$ or $\mathcal{L} = \mathcal{A}$. But by Proposition 5.4, $\inf_{m\in\mathcal{L}} \|e_{\mathcal{L}} - m\| \geq 1$, and so $e_{\mathcal{L}} \notin \overline{\mathcal{L}}$. Thus $\mathcal{L} = \overline{\mathcal{L}}$ is closed.

□

5.8. Proposition. *Let $\mathcal{A}$ be a commutative, unital Banach algebra and let $a \in \mathcal{A}$. If a is not invertible, then a is an element of some maximal ideal $\mathcal{M}$ of $\mathcal{A}$.*

Proof. Now $\langle a\rangle = a\mathcal{A}$ is an ideal of $\mathcal{A}$. Since a is not invertible, $\langle a\rangle \neq \mathcal{A}$. By Corollary 5.7, $a \in \langle a\rangle \subseteq \mathcal{M}$ for some maximal ideal $\mathcal{M}$.

□

5.9. Definition. *Let $\mathcal{A}$ be a Banach algebra. A non-zero complex linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is said to be* ***multiplicative*** *if $\varphi(ab) = \varphi(a)\,\varphi(b)$ for all $a, b \in \mathcal{A}$. In other words, φ is a non-zero algebra homomorphism from $\mathcal{A}$ into $\mathbb{C}$. The set of all non-zero multiplicative linear functionals on $\mathcal{A}$ is denoted by $\Sigma_{\mathcal{A}}$, and is called the* ***spectrum*** *of $\mathcal{A}$.*

Let $\varphi^\circ = 0$ denote the zero functional on $\mathcal{A}$, so that $\varphi^\circ(a) = 0$ for all $a \in \mathcal{A}$. We shall also require the notation $\Sigma^\circ_{\mathcal{A}} = \Sigma_{\mathcal{A}} \cup \{\varphi^\circ\}$.

5.10. Remark. Note that if $\mathcal{A}$ is a unital Banach algebra and $\varphi \in \Sigma_{\mathcal{A}}$, then $\varphi(1) = \varphi(1^2) = \varphi(1)^2$, and so $\varphi(1) \in \{0, 1\}$. If $\varphi(1) = 0$, then $\varphi(a) = \varphi(1a) = \varphi(1)\,\varphi(a) = 0$ for all $a \in \mathcal{A}$, contradicting the fact that $\varphi \neq 0$. Thus $\varphi(1) = 1$.

5.11. Proposition. *Let $\mathcal{A}$ be a Banach algebra and $0 \ne \varphi$ be a multiplicative linear functional on $\mathcal{A}$. Then φ is bounded; in fact, $\|\varphi\| = 1$.*

Proof. If $1 \notin \mathcal{A}$, then we may consider

$$\begin{array}{rccl} \varphi^+ : & \mathcal{A}^+ & \to & \mathbb{C} \\ & (\lambda, a) & \mapsto & \lambda + \varphi(a), \end{array}$$

which is a linear functional on $\mathcal{A}^+$, the unitization of $\mathcal{A}$ as defined in Remark 2.19. It is not hard to verify that φ is bounded if and only if φ^+ is. As such, we may assume that $1 \in \mathcal{A}$.

Let $\mathcal{M} = \ker \varphi$ and $a \in \mathcal{A}$. Since $\varphi \ne 0$, $\mathcal{M}$ is a proper ideal of $\mathcal{A}$. Then $\varphi(a - \varphi(a)1) = 0$, and $a = \varphi(a)1 + (a - \varphi(a)1)$. Write $\lambda = \varphi(a)$ and $b = (a - \varphi(a)1)$, so that $\lambda \in \mathbb{C}$, $b \in \mathcal{M}$. Then

$$\begin{aligned} \|\varphi\| &= \sup\{\frac{|\varphi(x)|}{\|x\|} : \|x\| \ne 0\} \\ &= \sup\{\frac{|\varphi(\lambda + b)|}{\|\lambda + b\|} : \lambda \ne 0, b \in \ker \varphi\} \\ &= \sup\{\frac{|\lambda|}{\|\lambda + b\|} : \lambda \ne 0, b \in \ker \varphi\} \\ &= \sup\{\frac{1}{\|1 + b'\|} : b' \in \ker \varphi\} \\ &= 1, \end{aligned}$$

since otherwise $\|1 + b'\| < 1$ would imply that b' is invertible, contradicting the fact that $b' \in \mathcal{M}$, a proper ideal of $\mathcal{A}$.

□

5.12. Proposition. *Let $\mathcal{A}$ be an abelian Banach algebra. Then there is a one-to-one correspondence between the spectrum $\Sigma_{\mathcal{A}}$ of $\mathcal{A}$, and the set of maximal modular ideals of $\mathcal{A}$.*

Proof. Let $\mathcal{M}$ be a maximal modular ideal of $\mathcal{A}$. Then $\mathcal{A}/\mathcal{M}$ is a unital Banach algebra with no proper ideals. Thus every non-zero element of $\mathcal{A}/\mathcal{M}$ is invertible, by Proposition 5.8. By the Gelfand-Mazur Theorem 2.37, there exists a unique isometric isomorphism $\tau : \mathcal{A}/\mathcal{M} \to \mathbb{C}$. The map

$$\begin{array}{rccl} \varphi_{\mathcal{M}} : & \mathcal{A} & \to & \mathbb{C} \\ & a & \mapsto & \tau(\pi_{\mathcal{M}}(a)) \end{array}$$

is easily seen to be a multiplicative linear functional, and $\ker \varphi_{\mathcal{M}} = \mathcal{M}$. Moreover, if $\mathcal{M}_1 \ne \mathcal{M}_2$ are two maximal modular ideals of $\mathcal{A}$, then $\varphi_{\mathcal{M}_1} \ne \varphi_{\mathcal{M}_2}$, since their kernels are distinct.

Conversely, if $\varphi \in \sum_{\mathcal{A}}$, let $\mathcal{M} = \ker \varphi$. Then $\mathbb{C} \simeq \varphi(\mathcal{A}) \simeq \mathcal{A}/\ker \varphi = \mathcal{A}/\mathcal{M}$, so $\mathcal{M}$ is a maximal regular ideal, as $\mathbb{C}$ is unital and has no non-trivial ideals. Consider $\varphi_{\mathcal{M}}$ defined as above. Since the isomorphism between $\mathcal{A}/\mathcal{M}$ and $\mathbb{C}$ is unique, $\varphi_{\mathcal{M}} = \varphi$.

□

Because of this result, $\Sigma_{\mathcal{A}}$ is also referred to as the **maximal ideal space** of $\mathcal{A}$.

5.13. Proposition. *Let $\mathcal{A}$ be an abelian Banach algebra. Then $\Sigma_{\mathcal{A}}$ is locally compact in the weak*-topology on the unit ball of $\mathcal{A}^*$. If $\mathcal{A}$ is unital, then $\Sigma_{\mathcal{A}}$ is in fact compact.*

Proof. Recall that $\Sigma^\circ_{\mathcal{A}} = \Sigma_{\mathcal{A}} \cup \{\varphi^\circ\}$, where φ° denotes the zero functional. Now $\Sigma^\circ_{\mathcal{A}}$ is clearly contained in the unit ball of $\mathcal{A}^*$. Let $(\varphi_\alpha)_{\alpha\in\Lambda}$ be a net in $\Sigma^\circ_{\mathcal{A}}$ such that $w^* - \lim_{\alpha\in\Lambda} \varphi_\alpha = \varphi \in \mathcal{A}^*$.

Then for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned} \varphi(\lambda x + y) &= \lim_\alpha \varphi_\alpha(\lambda x + y) \\ &= \lim_\alpha \lambda\varphi_\alpha(x) + \varphi_\alpha(y) \\ &= \lambda\varphi(x) + \varphi(y) \end{aligned}$$

and

$$\begin{aligned} \varphi(xy) &= \lim_\alpha \varphi_\alpha(xy) \\ &= \lim_\alpha \varphi_\alpha(x)\,\varphi_\alpha(y) \\ &= \varphi(x)\,\varphi(y). \end{aligned}$$

Thus $\varphi \in \Sigma^\circ_{\mathcal{A}}$. In particular, therefore, $\Sigma^\circ_{\mathcal{A}}$ is compact, being a closed subset of the weak*-compact unit ball of $\mathcal{A}^*$. Since the unit ball of $\mathcal{A}^*$ is Hausdorff in the weak*-topology, $\{\varphi^\circ\}$ is closed in $\Sigma^\circ_{\mathcal{A}}$. Thus $\Sigma_{\mathcal{A}}$ is a relatively open subset of the weak*-compact set $\Sigma^\circ_{\mathcal{A}}$, and so $\Sigma_{\mathcal{A}}$ is locally compact in the weak*-topology.

If $\mathcal{A}$ is unital, then $\{\varphi^\circ\}$ is isolated in $\Sigma^\circ_{\mathcal{A}}$ since $\varphi(1) = 1$ for all $\varphi \in \Sigma_{\mathcal{A}}$ while $\varphi^\circ(1) = 0$. Thus $\Sigma_{\mathcal{A}}$ is closed in $\Sigma^\circ_{\mathcal{A}}$, and thus is weak*-compact itself.

□

We shall need the following Proposition to help prove Theorem 5.17 below.

5.14. Proposition. *Let $\mathcal{A}$ be a non-unital, abelian Banach algebra, and let $\mathcal{A}^+$ denote its unitisation, as defined in Remark 2.19. Then there exists a bijective correspondence between $\Sigma_{\mathcal{A}^+}$ and $\Sigma^\circ_{\mathcal{A}}$.*

Proof. First observe that the map

$$\begin{array}{rccc} \varphi_\beta : & \mathcal{A} & \to & \mathbb{C} \\ & a & \mapsto & \beta((a,0)) \end{array}$$

is a multiplicative linear functional. That is, $\varphi_\beta \in \Sigma^\circ_{\mathcal{A}}$. Let

$$\begin{array}{rccc} \Theta : & \Sigma_{\mathcal{A}^+} & \to & \Sigma^\circ_{\mathcal{A}} \\ & \beta & \mapsto & \varphi_\beta. \end{array}$$

We claim that Θ is a bijection.

Now, if $\beta \in \Sigma_{\mathcal{A}^+}$, then $\beta(0,1) = 1$, by Remark 5.10. Thus, if $\beta, \gamma \in \Sigma_{\mathcal{A}^+}$ and $\varphi_\beta = \Theta(\beta) = \Theta(\gamma) = \varphi_\gamma$, then for all $(a, \lambda) \in \mathcal{A}^+$, we have

$$\begin{aligned}\beta((a,\lambda)) &= \beta((a,0) + (0,\lambda)) \\ &= \varphi_\beta(a) + \lambda + \varphi_\gamma(a) + \lambda \\ &= \gamma((a,0) + (0,\lambda)) \\ &= \gamma(a,\lambda).\end{aligned}$$

In other words, $\beta = \gamma$ and Θ is injective.

If $\varphi \in \Sigma_{\mathcal{A}}^\circ$, we may define $\beta_\varphi \in \Sigma_{\mathcal{A}^+}$ by

$$\beta_\varphi(a, \lambda) = \varphi(a) + \lambda.$$

(That β_φ is a non-zero multiplicative linear functional is left as an exercise for the reader.) Since $\Theta(\beta_\varphi) = \varphi$, we see that Θ is surjective, which completes the proof.

□

5.15. Definition. *Let $\mathcal{A}$ be an abelian Banach algebra. Given $a \in \mathcal{A}$, we define the **Gelfand Transform** $\hat{a}$ of a as follows:*

$$\begin{array}{rccc} \hat{a}: & \Sigma_{\mathcal{A}} & \to & \mathbb{C} \\ & \varphi & \mapsto & \varphi(a). \end{array}$$

By definition of the weak*-topology on $\Sigma_{\mathcal{A}}$, we have that $\hat{a} \in \mathcal{C}(\Sigma_{\mathcal{A}})$. If $\epsilon > 0$, then $\{\varphi \in \Sigma_{\mathcal{A}} : |\hat{a}(\varphi)| \geq \epsilon\}$ is closed in $\Sigma_{\mathcal{A}}^\circ$, and hence it is compact. Thus $\hat{a} \in \mathcal{C}_0(\Sigma_{\mathcal{A}})$.

5.16. Theorem. **[The Gelfand Transform]** *Let $\mathcal{A}$ be an abelian Banach algebra.*

(a) *The map*

$$\begin{array}{rccc} \Gamma: & \mathcal{A} & \to & \mathcal{C}_0(\Sigma_{\mathcal{A}}) \\ & a & \mapsto & \hat{a} \end{array}$$

is a contractive homomorphism, and

(b) $\hat{\mathcal{A}} = \operatorname{ran} \Gamma$ *separates the points of* $\Sigma_{\mathcal{A}}$.

Proof.

(i) We have seen that $\hat{a}$ is continuous and vanishes at infinity. Now

$$\begin{aligned}\|\Gamma(a)\| &= \|\hat{a}\| \\ &= \sup_{\varphi \in \Sigma_{\mathcal{A}}} |\hat{a}(\varphi)| \\ &= \sup_{\varphi \in \Sigma_{\mathcal{A}}} |\varphi(a)| \\ &\leq \|a\|.\end{aligned}$$

Thus $\|\Gamma\| \leq 1$. That Γ is indeed a homomorphism follows immediately from the fact that each $\varphi \in \Sigma_{\mathcal{A}}$ is linear and multiplicative.

(ii) If $\varphi \neq \psi \in \Sigma_{\mathcal{A}}$, then there exists $a \in \mathcal{A}$ such that $\varphi(a) \neq \psi(a)$. But then $\hat{a}(\varphi) \neq \hat{a}(\psi)$, and so $\hat{\mathcal{A}}$ indeed separates the points of $\Sigma_{\mathcal{A}}$, as claimed.

□

5.17. Theorem. *Let $\mathcal{A}$ be an abelian Banach algebra, and let $\Sigma_{\mathcal{A}}$ be its spectrum. Let $\Gamma : \mathcal{A} \to \mathcal{C}_0(\Sigma_{\mathcal{A}})$ be the Gelfand transform of $\mathcal{A}$.*

(a) *If $\mathcal{A}$ is unital, then $\sigma_{\mathcal{A}}(a) = \operatorname{ran} \Gamma_{\mathcal{A}}(a)$.*
(b) *If $\mathcal{A}$ is non-unital, then $\sigma_{\mathcal{A}}(a) = \operatorname{ran} \Gamma_{\mathcal{A}}(a) \cup \{0\}$.*
(c) *In either case,* $\operatorname{spr}(a) = \|\Gamma_{\mathcal{A}}(a)\|$.

Proof.

(a) If $\mathcal{A}$ is unital, then $\mathcal{C}_0(\Sigma_{\mathcal{A}}) = \mathcal{C}(\Sigma_{\mathcal{A}})$. Thus

$$\begin{aligned} \lambda \in \sigma_{\mathcal{A}}(a) \quad &\iff \quad (\lambda - a) \notin \mathcal{A}^{-1} \\ &\iff \quad (\lambda - a) \text{ lies in a maximal ideal } \mathcal{M} \text{ of } \mathcal{A} \\ &\iff \quad \varphi_{\mathcal{M}}(\lambda - a) = 0 \text{ where } \mathcal{M} \text{ is a maximal ideal of } \mathcal{A} \\ &\iff \quad \lambda - \varphi_{\mathcal{M}}(a) = 0 \text{ where } \mathcal{M} \text{ is a maximal ideal of } \mathcal{A} \\ &\iff \quad \lambda - \hat{a}(\varphi_{\mathcal{M}}) = 0 \text{ where } \mathcal{M} \text{ is a maximal ideal of } \mathcal{A} \\ &\iff \quad \lambda \in \operatorname{ran} \hat{a}. \end{aligned}$$

(b) Suppose that $\mathcal{A}$ is non-unital, and let $\mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}$ denote its unitisation, as defined in Remark 2.19. By definition, $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{A}^+}((a, 0))$.

By Proposition 5.14 and part (a) above,

$$\begin{aligned} \sigma_{\mathcal{A}^+}((a,0)) &= \operatorname{ran} \Gamma_{\mathcal{A}^+}((a,0)) \\ &= \{\varphi_+((a,0)) : \varphi_+ \in \Sigma_{\mathcal{A}^+}\} \\ &= \{\varphi(a) : \varphi \in \Sigma_{\mathcal{A}}^{\circ}\} \\ &= \operatorname{ran} \Gamma_{\mathcal{A}}(a) \cup \{0\}. \end{aligned}$$

(c) In either case, $\Gamma_{\mathcal{A}}(a) \in \mathcal{C}_0(\Sigma_{\mathcal{A}})$, and so

$$\begin{aligned} \|\Gamma_{\mathcal{A}}(a)\| &= \operatorname{spr}(\Gamma_{\mathcal{A}}(a)) \\ &= \sup\{|\lambda| : \lambda \in \sigma(\Gamma_{\mathcal{A}}(a))\} \\ &= \sup\{|\lambda| : \lambda \in \sigma(\Gamma_{\mathcal{A}}(a)) \cup \{0\}\} \\ &= \sup(\{|\lambda| : \lambda \in \operatorname{ran}(\Gamma_{\mathcal{A}}(a)) \cup \{0\}\} \\ &= \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a) \cup \{0\}\} \\ &= \operatorname{spr}(a). \end{aligned}$$

□

The radical

5.18. The kernel of the Gelfand transform plays a particular role in the study of homomorphims between Banach algebras.

5.19. Definition. *Let $\mathcal{A}$ be a commutative Banach algebra. Then the **Jacobson radical** of $\mathcal{A}$ is the kernel of the Gelfand transform. As such,*

$$\begin{aligned}\operatorname{rad}\mathcal{A} &= \cap\{\ker\varphi : \varphi \in \Sigma_{\mathcal{A}}\}\\ &= \cap\{\mathcal{M} : \mathcal{M} \textit{ a maximal modular ideal of } \mathcal{A}\}.\end{aligned}$$

*We say that $\mathcal{A}$ is **semisimple** if* $\operatorname{rad}\mathcal{A} = \{0\}$.

5.20. Proposition. *Let $\mathcal{A}$ be an abelian Banach algebra. Then* $\operatorname{rad}\mathcal{A} = \{a \in \mathcal{A} : \operatorname{spr}(a) = 0\}$ *and the following are equivalent:*

(a) *$\mathcal{A}$ is semisimple, i.e. the Gelfand transform $\Gamma : \mathcal{A} \to \mathcal{C}_0(\Sigma_{\mathcal{A}})$ is injective;*
(b) *$\Sigma_{\mathcal{A}}$ separates the points of $\mathcal{A}$;*
(c) *the spectral radius is a norm on $\mathcal{A}$.*

Proof. First note that $a \in \operatorname{rad}\mathcal{A}$ if and only if $\Gamma(a) = 0$. But $0 = \Gamma(a) \in \mathcal{C}_0(\Sigma_{\mathcal{A}})$ if and only if $\operatorname{spr}(\hat{a}) = 0$, i.e. if and only if $\operatorname{spr}(a) = 0$.

(a) implies (b). Suppose $\mathcal{A}$ is semisimple. Let $a_1 \ne a_2 \in \mathcal{A}$. Then $0 \ne a_1 - a_2$, and so $\operatorname{spr}(a_1 - a_2) \ne 0$ from above. Thus there exists $0 \ne \lambda \in \operatorname{ran}(\Gamma(a_1 - a_2))$. Let $\varphi \in \Sigma_{\mathcal{A}}$ such that $\Gamma(a_1 - a_2)(\varphi) = \lambda$. Then $\Gamma(a_1)(\varphi) - \Gamma(a_2)(\varphi) = \varphi(a_1 - a_2) = \lambda \ne 0$, so that $\Sigma_{\mathcal{A}}$ separates points.
(b) implies (a). Suppose that $\Sigma_{\mathcal{A}}$ separates the points of $\mathcal{A}$. Let $a_1 \ne a_2 \in \mathcal{A}$ and choose $\varphi \in \Sigma_{\mathcal{A}}$ such that $\varphi(a_1) \ne \varphi(a_2)$. Then $\Gamma(a_1)(\varphi) \ne \Gamma(a_2)(\varphi)$, so that $\Gamma(a_1 \ne \Gamma(a_2)$, and the Gelfand transform is injective.
(a) implies (c). Suppose that the Gelfand transform Γ is injective. In general, we have $\|\hat{a}\| = \operatorname{spr}(\hat{a}) = \operatorname{spr}(a)$. Then for all $a, b \in \mathcal{A}$,
- $\operatorname{spr}(\lambda a + b) = \|\Gamma(\lambda a + b\| \le |\lambda|\|\Gamma(a)\| + \|\Gamma(b)\| = |\lambda|\operatorname{spr}(a) + \operatorname{spr}(b)$.
- $\operatorname{spr}(ab) = \|\Gamma(ab)\| \le \|\Gamma(a)\|\,\|\Gamma(b)\| = \operatorname{spr}(a)\operatorname{spr}(b)$.
- $\operatorname{spr}(a) = \|\Gamma(a)\| \ge 0$.
- Finally, $\operatorname{spr}(a) = 0$ if and only if $\|\Gamma(a)\| = 0$. But since Γ is injective, this happens if and only if $a = 0$.

It follows that $\operatorname{spr}(\cdot)$ is a norm on $\mathcal{A}$.
(c) implies (a). Finally, suppose $\operatorname{spr}(\cdot)$ is a norm on $\mathcal{A}$. Then $\operatorname{spr}(a) = 0$ implies $a = 0$, so that $\operatorname{rad}\mathcal{A} = \{0\}$, and $\mathcal{A}$ is semisimple.

□

5.21. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian Banach algebras and suppose $\mathcal{B}$ is semisimple. Let $\tau : \mathcal{A} \to \mathcal{B}$ be an algebra homomorphism. Then τ is continuous.*
Proof. Let $\varphi \in \sum_{\mathcal{B}}$, the maximal ideal space of $\mathcal{B}$. Then $\varphi \circ \tau$ is a multiplicative linear functional on $\mathcal{A}$, and so $\|\varphi \circ \tau\| = 1$, implying that $\varphi \circ \tau$ is continuous.

The Closed Graph Theorem tells us that if $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces and $T : \mathfrak{X} \to \mathfrak{Y}$ is a linear map such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Tx_n = y$ together imply $y = 0$, then T is continuous.

Suppose therefore that we are given a sequence $(a_n)_n \in \mathcal{A}^{\mathbb{N}}$, that $\lim_{n\to\infty} a_n = 0$, and that $\lim_{n\to\infty} \tau(a_n) = b$. Then for $\varphi \in \sum_{\mathcal{B}}$,

$$\begin{aligned} \varphi(b) &= \varphi(\lim_{n\to\infty} \tau(a_n)) \\ &= \lim_{n\to\infty} \varphi \circ \tau(a_n) \\ &= (\varphi \circ \tau)(\lim_{n\to\infty} a_n) \\ &= (\varphi \circ \tau)(0) \\ &= 0. \end{aligned}$$

Thus $b \in \operatorname{rad} \mathcal{B} = \{0\}$. By the Closed Graph Theorem, τ is continuous.

□

5.22. Definition. *A Banach algebra $\mathcal{A}$ has* ***uniqueness of norm*** *if all norms on $\mathcal{A}$ making it into a Banach algebra are equivalent.*

5.23. Theorem. *Let $\mathcal{A}$ be an abelian Banach algebra. If $\mathcal{A}$ is semisimple, then $\mathcal{A}$ has uniqueness of norm.*

Proof. With $\mathcal{A}$ abelian and semisimple, let $\|\cdot\|_1$ and $\|\cdot\|_2$ denote two Banach algebra norms on $\mathcal{A}$. Consider the natural injection

$$\begin{array}{rccc} \iota: & (\mathcal{A}, \|\cdot\|_1) & \to & (\mathcal{A}, \|\cdot\|_2) \\ & a & \mapsto & a. \end{array}$$

Then clearly ι is an algebra isomorphism, and hence from Theorem 5.21, ι is continuous. By the Banach Isomorphism Theorem, ι is a topological isomorphism, and so the two norms are equivalent.

□

5.24. Corollary. *Let $\mathcal{A}$ be a semi-simple abelian Banach algebra and $\alpha : \mathcal{A} \to \mathcal{A}$ be an algebra automorphism. Then α is also a homeomorphism.*

Proof. Theorem 5.21 implies that both α and α^{-1} are continuous.

□

Examples

5.25. Depending upon how big the radical of an abelian Banach algebra is compared to the algebra itself, the Gelfand Transform might not yield very much information at all.

5.26. Example. Let $n \in \mathbb{N}$, and consider the algebra $\mathcal{A} \subseteq \mathbb{M}_{2n}(\mathbb{C})$, where

$$A = \left\{ \begin{bmatrix} \lambda I_n & B \\ 0 & \lambda I_n \end{bmatrix} : B \in \mathbb{M}_n(\mathbb{C}), \lambda \in \mathbb{C} \right\}.$$

Then $\mathcal{A}$ is commutative. Let $\varphi \in \Sigma_{\mathcal{A}}$. Then $\varphi(I_{2n}) = 1$, and so $\varphi(\lambda I_{2n}) = \lambda,\ \lambda \in \mathbb{C}$.

Also,

$$\begin{aligned} 0 &= \varphi(0) \\ &= \varphi(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}^2) \\ &= \varphi(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix})^2 \end{aligned}$$

and so $\varphi(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}) = 0$.

Thus $\varphi(\begin{bmatrix} \lambda I_n & B \\ 0 & \lambda I_n \end{bmatrix}) = \lambda$ for all $\lambda \in \mathbb{C}$ and $B \in \mathbb{M}_n(\mathbb{C})$. In other words, φ is completely determined by the above conditions, and $\Sigma_{\mathcal{A}} = \{\varphi\}$, a singleton.

5.27. Let X be a compact, Hausdorff space. We wish to consider the spectrum of the algebra $\mathcal{C}(X)$ of continuous functions on X. To do this, we first recall a preliminary result from topology.

5.28. Proposition. *Let X be a compact space and Y be a Hausdorff space. Suppose that $\tau : X \to Y$ is a bijective, continuous map. Then τ is a homeomorphism, i.e., τ^{-1} is also continuous.*

5.29. Theorem. *Let X be a compact, Hausdorff space. Then $\sum_{\mathcal{C}(X)}$ equipped with its weak*-topology as a subset of $\mathcal{C}(X)^*$ is homeomorphic to X.*

Proof. Let $x \in X$, and consider the map

$$\begin{array}{rccc} \delta_x : & \mathcal{C}(X) & \to & \mathbb{C} \\ & f & \mapsto & f(x). \end{array}$$

It is easy to see that $\delta_x \in \sum_{\mathcal{C}(X)}$. Such maps are called **evaluation functionals**. Note that the corresponding maximal ideal is $\mathcal{M}_x = \ker \delta_x = \{f \in \mathcal{C}(X) : f(x) = 0\}$. It is clear that given $x \neq y \in X$, $\delta_y \neq \delta_x$ since $\mathcal{C}(X)$ separates the points of X. Thus the map $\Delta : x \mapsto \delta_x$ is injective. Our next goal is to show that it is surjective.

Let $\mathcal{M}$ be a maximal ideal of $\mathcal{C}(X)$. We shall show that there exists $x \in X$ such that $\mathcal{M} = \mathcal{M}_x$, where $\mathcal{M}_x$ is defined as above.

Suppose that for any $x \in X$, there exists $f_x \in \mathcal{M}$ such that $f_x(x) \neq 0$. Since f is continuous, we can find an open neighbourhood $\mathcal{O}_x$ of x such that $y \in \mathcal{O}_x$ implies $f_x(y) \neq 0$. Then the family $\{\mathcal{O}_x : x \in X\}$ is an open cover of the compact space X, and as such, we can find a finite subcover $\{\mathcal{O}_{x_i} : 1 \le i \le n\}$. Consider the function $g := \sum_{i=1}^n f_{x_i} \overline{f_{x_i}} \in \mathcal{M}$. Then clearly $g \ge 0$ and for any $x \in X$, there exists x_i such that

$f_{x_i}(x) \neq 0$. Thus $g(x) \geq |f_{x_i}(x)|^2 > 0$, and so g is in fact invertible! This contradicts the fact that $\mathcal{M}$ is a maximal ideal, and thus is proper. It follows that there exists $x \in X$ such that $f(x) = 0$ for all $f \in \mathcal{M}$. But then $\mathcal{M} \subseteq \mathcal{M}_x$, and so by maximality, we conclude that $\mathcal{M} = \mathcal{M}_x$, and hence the map $\Delta : x \mapsto \delta_x$ is surjective.

By Proposition 5.28, there remains only to show that Δ is continuous. Let $(x_\alpha)_{\alpha \in \Lambda}$ be a net in X converging to the element x. Then $(f(x_\alpha))_\alpha$ converges to $f(x)$ for each $f \in \mathcal{C}(X)$. But then $(\delta_{x_\alpha}(f))_\alpha$ converges to $\delta_x(f)$ for all $f \in \mathcal{C}(X)$, and so $(\delta_{x_\alpha})_\alpha$ converges to δ_x in the weak*-topology on $\mathcal{C}(X)^*$. That is, $\Delta : x \mapsto \delta_x$ is continuous, and our result is proved.

□

5.30. Let X be a compact, Hausdorff space. In light of the homeomorphism $\Delta : X \to \Sigma_{\mathcal{C}(X)}$ defined above, we typically view the Gelfand map on $\mathcal{C}(X)$ as the identity map and suppress the map Δ from our notation.

That is, we identify $\Gamma : \mathcal{C}(X) \to \mathcal{C}(\Sigma_{\mathcal{C}(X)})$ with the map $\Phi_\Delta \circ \Gamma$, where $\Phi_\Delta(g) = g \circ \Delta$ for all $g \in \mathcal{C}(\Sigma_{\mathcal{C}(X)})$. Then, for $f \in \mathcal{C}(X)$ and $x \in X$,

$$(\Phi_\Delta \circ \Gamma(f))(x) = \Phi_\Delta(\delta_x(f)) = (\Phi_\Delta(\delta_x))(f) = f(x),$$

so that $\Phi_\Delta \circ \Gamma$ is the identity function.

5.31. Corollary. *Let X be a compact, Hausdorff space. Then $\mathcal{C}(X)$ has uniqueness of norm.*

Proof. The Gelfand map Γ is the identity map, so it is injective, and thus $\mathcal{C}(X)$ is semisimple. We now apply Theorem 5.23.

□

5.32. Let G be a locally compact abelian group equipped with a Haar measure μ. It is well-known that if λ is any other Haar measure on G, then λ is a positive multiple of μ. (See, for example, the book of Folland [**23**, Theorem 2.10, Theorem 2.20].) Moreover, since G is abelian, it is **unimodular**, from which it follows that $d\mu(x^{-1}) = d\mu(x)$, as measures on G. Consider $f, g \in L^1(G, \mu)$. Then for $x \in G$, we have (a.e. - μ)

$$\begin{aligned}
(f * g)(x) &= \int f(y) g(y^{-1}x) d\mu(y) \\
&= \int f(xv) g(v^{-1}) d\mu(v) \quad (v = x^{-1}y) \\
&= \int f(xz^{-1}) g(z) d\mu(z) \quad (z = v^{-1}) \\
&= \int g(z) f(z^{-1}x) d\mu(z) \quad (xz^{-1} = z^{-1}x) \\
&= (g * f)(x).
\end{aligned}$$

Thus $L^1(G, \mu)$ is abelian.

To verify that the norm on $L^1(G,\mu)$ is indeed a Banach algebra norm, consider

$$\begin{aligned}
\|f\|_1\,\|g\|_1 &= \int |f(y)|\,\|g\|_1 dy \\
&\geq \int |f(y)| \int |g(y^{-1}x)| dx\, dy \\
&\geq \int \int |f(y)\, g(y^{-1}x)| dx\, dy \\
&= \int \int |f(y)\, g(y^{-1}x)| dy\, dx \\
&\geq \int | \int f(y) g(y^{-1}x) dy | dx \\
&\geq \int |(f * g)(x)| dx \\
&= \|f * g\|_1.
\end{aligned}$$

5.33. Definition. *Given a locally compact abelian group G, we consider the set $\hat{G}$ of continuous homomorphisms of G into $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Such homomorphisms are called* ***characters*** *of G, and $\hat{G}$ is referred to as the* ***dual group*** *of G.*

5.34. We leave it to the reader to verify that $\hat{G}$ is indeed a group. In fact, $\hat{G}$ corresponds to the set of irreducible representations of G, which are always one dimensional when G is abelian.

For the sake of convenience, let us write Σ_G for $\Sigma_{L^1(G,\mu)}$, and dx for $d\mu(x)$. Given $\varphi \in \hat{G}$, we can define an element $\varphi \in \Sigma_G$ via

$$\varphi(f) = \int_G \varphi(x)\, f(x)\, d\mu(x).$$

Indeed, for each f, $g \in L^1(G,\mu)$,

$$\begin{aligned}
\varphi(f * g) &= \int \varphi(x)\,(f * g)(x) dx \\
&= \int \varphi(x) \int f(y)\, g(y^{-1}x) dy\, dx \\
&= \int \int f(y) g(z) \varphi(yz) dz\, dy \qquad (z = y^{-1}x) \\
&= \int \int f(y)\varphi(y)\varphi(z) g(z) dz\, dy \\
&= (\int f(y)\varphi(y) dy)\, (\int \varphi(z) g(z) dz) \\
&= \varphi(f)\,\varphi(g).
\end{aligned}$$

If $\varphi_1 \neq \varphi_2 \in \hat{G}$, then $0 \neq \varphi_1 - \varphi_2 \in \mathcal{C}_0(G) \subseteq L^\infty(G,\mu)$. Thus there exists $g \in L^1(G,\mu)$ such that $\int g(x)(\varphi_1 - \varphi_2)(x) dx \neq 0$. In particular, therefore, if φ_1 (resp. φ_2) is the element of Σ_G corresponding to φ_1 (resp. φ_2) as above, then $\varphi_1(g) \neq \varphi_2(g)$, so that the map $\varphi \mapsto \varphi$ is injective.

5.35. Theorem. *Let G be a locally compact abelian group with Haar measure μ. Then $\Sigma_G \simeq \hat{G}$.*

Proof. From above, we see that $\hat{G}$ embeds injectively into Σ_G. Next suppose that $\varphi \in \Sigma_G$. Since $\Phi \in L^1(G,\mu)^* \simeq L^\infty(G,\mu)$, there exists $\varphi \in L^\infty(G,\mu)$ such that

$$\Phi(f) = \int f(x)\varphi(x)dx \text{ for all } f \in L^1(G,\mu).$$

Choose $f \in L^1(G,\mu)$ such that $0 \neq \Phi(f)$. Then for any $g \in L^1(G,\mu)$,

$$\begin{aligned}\Phi(f)\int \varphi(y)g(y)dy &= \Phi(f)\,\Phi(g)\\ &= \Phi(f * g)\\ &= \int\int \varphi(x)f(xy^{-1})g(y)dy\,dx\\ &= \int \Phi(L_y f)g(y)dy.\end{aligned}$$

Thus $\varphi(y) = \Phi(L_y f)/\Phi(f)$ a.e. . Redefine $\varphi(y) = \Phi(L_y f)/\Phi(f)$ for every y, so that φ is continuous. Then

$$\begin{aligned}\varphi(xy)\Phi(f) &= \Phi(L_{xy}f)\\ &= \Phi(L_x L_y f)\\ &= \varphi(x)\Phi(L_y f)\\ &= \varphi(x)\varphi(y)\Phi(f),\end{aligned}$$

and hence $\varphi(xy) = \varphi(x)\,\varphi(y)$.

Finally, $\varphi(x^n) = \varphi(x)^n$ for every $n \geq 1$, and φ bounded implies that $|\varphi(x)| \leq 1$, while $\varphi(x^{-n})$ bounded implies that $|\varphi(x)| = 1$ for all $x \in G$. Thus $\varphi \in \hat{G}$, and so the map $\varphi \mapsto \Phi$ is onto, as claimed.

The topology we consider on $\hat{G}$ is that of uniform convergence on compact sets. Since $\hat{G}$ consists of continuous functions, this is the same as pointwise convergence, under which the operations of multiplication and inversion are clearly continuous. *Although we shall not show it here, it can be demonstrated that this topology coincides with the weak*-topology on $\hat{G}$ inherited from $L^\infty(G,\mu)$.*

But $\hat{G} \cup \{0\}$ is the set of all homomorphisms from $L^1(G,\mu)$ into $\mathbb{C}$, which is closed in the unit ball of $L^\infty(G,\mu)$, and hence is weak*-compact, by Alaoglu's Theorem.

Thus $\hat{G}$ must be locally compact, as $\{0\}$ is closed.

□

5.36. Theorem.

(a) $\hat{\mathbb{Z}} \simeq \mathbb{T}$, *and thus* $\sum_{\ell^1(\mathbb{Z})} \simeq \mathbb{T}$;

(b) $\hat{\mathbb{R}} \simeq \mathbb{R}$, *and thus* $\sum_{L^1(\mathbb{R},dx)} \simeq \mathbb{R}$;

(c) $\hat{\mathbb{T}} \simeq \mathbb{Z}$, *and thus* $\sum_{L^1(\mathbb{T},dm)} \simeq \mathbb{Z}$, *where dm represents normalised Lebesgue measure on the unit circle.*

Remark: We shall content ourselves here with the algebraic calculation, and omit the explicit determination of the underlying topologies, which are the natural topologies on the spaces involved.

Proof.

(a) For each $\alpha \in \mathbb{T}$, define $\varphi_\alpha \in \hat{\mathbb{Z}}$ via $\varphi_\alpha(1) = \alpha$. Suppose $\varphi \in \hat{\mathbb{Z}}$. If $\alpha = \varphi(1)$, then $\alpha \in \mathbb{T}$, and $\varphi(n) = \varphi(1)^n = \alpha^n$ for all $n \in \mathbb{Z}$. Thus $\varphi = \varphi_\alpha$. It follows that the map $\alpha \mapsto \varphi_\alpha$ is surjective. That it is injective is trivial.

(b) If $\varphi \in \hat{\mathbb{R}}$, then we have $\varphi(0) = 1$, so there exists $a > 0$ so that $\int_0^a \varphi(t)dt \neq 0$. Setting $\kappa = \int_0^a \varphi(t)dt$, we have

$$\begin{aligned} \kappa\varphi(x) &= \int_0^a \varphi(t)dt\varphi(x) \\ &= \int_0^a \varphi(x)\varphi(t)dt \\ &= \int_0^a \varphi(x+t)dt \\ &= \int_x^{x+a} \varphi(t)dt. \end{aligned}$$

It follows that φ is differentiable and

$$\begin{aligned} \varphi'(x) &= \lim_{h\to 0} \frac{\varphi(x+h) - \varphi(x)}{h} \\ &= \lim_{h\to 0} \frac{\varphi(x)(\varphi(h) - 1)}{h} \\ &= \varphi(x)\varphi'(0). \end{aligned}$$

Thus $\varphi(x) = e^{cx}$, where $c = \varphi'(0)$, and since $|\varphi(x)| = 1$ for all x, we find that $c = 2\pi ib$ for some $b \in \mathbb{R}$.

Conversely, for any $b \in \mathbb{R}$, $\varphi_b(x) = e^{(2\pi ib)x}$ determines an element of $\hat{\mathbb{R}}$. Clearly the map $b \mapsto \varphi_b$ is injective.

(c) Since $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$ via the identification of $x \in \mathbb{R}/\mathbb{Z}$ with $\alpha = e^{(2\pi i)x}$, the characters of $\mathbb{T}$ are just the characters of $\mathbb{R}$ that vanish on $\mathbb{Z}$. But $\varphi_b(1) = 1$ implies that $e^{2\pi ib} = 1$, and so $b \in \mathbb{Z}$. Thus $\hat{\mathbb{T}} \simeq \mathbb{Z}$.

□

5.37. Definition. *Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. Then a is said to* ***generate*** *$\mathcal{A}$ if the smallest closed subalgebra of $\mathcal{A}$ containing a is $\mathcal{A}$ itself.*

The next theorem provides some justification for the term *spectrum* when referring to the set of non-zero multiplicative linear functionals on a Banach algebra.

5.38. Theorem. *Let $\mathcal{A}$ be a commutative unital Banach algebra and let a be a generator for $\mathcal{A}$. Then the mapping $\Gamma(a) : \Sigma_{\mathcal{A}} \mapsto \sigma(a)$ is a homeomorphism.*

Proof. We already know that $\Gamma(a) \in \mathcal{C}(\Sigma_{\mathcal{A}})$ and that $\operatorname{ran}\Gamma(a) = \sigma(a)$. Since both $\Sigma_{\mathcal{A}}$ and $\sigma(a)$ are compact and Hausdorff, it suffices to show that $\Gamma(a)$ is injective. We can then apply Proposition 5.28 to obtain the desired result.

Suppose that $\varphi_1, \varphi_2 \in \Sigma_{\mathcal{A}}$ and that $\Gamma(a)(\varphi_1) = \Gamma(a)(\varphi_2)$. Then $\varphi_1(a) = \varphi_2(a)$. Let $\mathcal{B} = \{x \in \mathcal{A} : \varphi_1(x) = \varphi_2(x)\}$. Since φ_1, φ_2 are continuous, multiplicative and linear, $\mathcal{B}$ is an algebra that contains 1 and a, and $\mathcal{B}$ is closed. Thus $\mathcal{B} = \mathcal{A}$ and so $\varphi_1 = \varphi_2$, proving that $\Gamma(a)$ is injective, as required.

□

5.39. Example. Consider the disk algebra $\mathcal{A}(\mathbb{D})$. Now it is a classical result that $\mathcal{A}(\mathbb{D})$ is generated by 1 and f, where $f(z) = z$ for all $z \in \mathbb{D}$. (Indeed, this is the solution to the Dirichlet Problem for the circle.) By Theorem 5.38, $\Sigma_{\mathcal{A}(\mathbb{D})}$ is homeomorphic to $\sigma_{\mathcal{A}(\mathbb{D})}(f)$. But as we have seen in Example 4.2, $\sigma_{\mathcal{A}(\mathbb{D})}(f) = \{z \in \mathbb{C} : |z| \le 1\}$. We conclude that $\sum_{\mathcal{A}(\mathbb{D})} = \mathbb{D}$ (up to homeomorphism). Unsurprisingly, the multiplicative linear functionals are the evaluation functionals at each point of the disk.

5.40. Example. Let us revisit $\ell^1(\mathbb{Z})$. For a function $f \in \mathcal{C}(\mathbb{T})$, consider the sequence $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ of **Fourier coefficients** of f given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta} d\theta.$$

Define the **Wiener algebra**

$$\mathcal{AC}(\mathbb{T}) = \{f \in \mathcal{C}(\mathbb{T}) : (\hat{f}(n))_n \in \ell^1(\mathbb{Z})\},$$

equipped with the norm $\|f\| = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|$.

Clearly $\mathcal{AC}(\mathbb{T})$ is abelian. Let f and g lie in $\mathcal{AC}(\mathbb{T})$, so that

$$f(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta} \text{ and } g(\theta) = \sum_{n\in\mathbb{Z}} b_n e^{in\theta}.$$

Then $(\hat{fg})(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, g(\theta)\, e^{-in\theta}\, d\theta$. Next,

$$\begin{aligned} f(\theta)\, g(\theta) &= \left(\sum_{k\in\mathbb{Z}} a_k e^{ik\theta}\right)\left(\sum_{n\in\mathbb{Z}} b_n e^{in\theta}\right) \\ &= \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a_k b_n e^{i(k+n)\theta} \\ &= \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_k b_{m-k} e^{im\theta} \quad (m = n+k) \end{aligned}$$

Thus

$$(\hat{fg})(n) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_k b_{m-k} e^{i(m-n)\theta} d\theta.$$

If $m \neq n$, we get 0, and so

$$\begin{aligned}(\hat{fg})(n) &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k\in\mathbb{Z}} a_k b_{n-k} e^{i0} d\theta \\ &= \sum_{k\in\mathbb{Z}} a_k b_{n-k} \\ &= (ab)_n,\end{aligned}$$

where $a = (a_n)_n$ and $b = (b_n)_n$ lie in $\ell^1(\mathbb{Z})$. It follow that the map

$$\begin{aligned}\tau : \quad \ell^1(\mathbb{Z}) &\to \mathcal{AC}(\mathbb{T}) \\ (a_n)_n &\mapsto \textstyle\sum_{n\in\mathbb{Z}} a_n e^{in\theta}\end{aligned}$$

is an isometric algebra isomorphism.

Suppose that φ is a non-zero multiplicative linear functional on $\mathcal{AC}(\mathbb{T})$. If $\varphi(e^{i\theta}) = \lambda$, then $|\lambda| = \varphi(e^{i\theta})| \le \|\varphi\| \, \|e^{i\theta}\|_1 = 1$. Also, $\varphi(e^{-i\theta}) = \varphi((e^{i\theta})^{-1}) = \frac{1}{\lambda}$, and $|\frac{1}{\lambda}| = |\varphi(e^{-i\theta})| \le \|\varphi\| \, \|e^{-i\theta}\|_1 = 1$. Thus $|\lambda| = 1$.

Conversely, if $|\lambda| = 1$, then

$$\varphi(\sum_{n\in\mathbb{Z}} a_n e^{in\theta}) = \sum_{n\in\mathbb{Z}} a_n \lambda^n$$

is an absolutely convergent, multiplicative evaluation functional, and $\varphi(1) = 1$.

We conclude again that $\sum_{\mathcal{AC}(\mathbb{T})} = \sum_{\mathbb{Z}} = \mathbb{T}$. The argument with regards to the topology follows as in Theorem 5.29. Namely, let $\{\lambda_\alpha\}_\alpha$ be a net in $\mathbb{T}$ with $\lim_\alpha \lambda_\alpha = \lambda \in \mathbb{T}$. Let φ_{λ_α}, φ_α be the associated multiplicative linear functionals with $\varphi_{\alpha_\lambda}(e^{i\theta}) = \lambda_\alpha$, $\varphi_\lambda(e^{i\theta}) = \lambda$. Then $\lim_\alpha \lambda_\alpha = \lambda$ implies $\lim_\alpha f(\lambda_\alpha) = f(\lambda)$ for all $f \in \mathcal{C}(\mathbb{T})$, hence $\lim_\alpha \varphi_{\lambda_\alpha}(f) = \varphi_\lambda(f)$ for all $f \in \mathcal{AC}(\mathbb{T})$. Thus $\lim_\alpha \varphi_{\lambda_\alpha} = \varphi_\lambda$ in the weak*-topology on $\sum_{\mathcal{AC}(\mathbb{T})}$.

As an application of this result, we obtain the following:

5.41. Theorem. **[Wiener's Tauberian Theorem]** *If $f \in \mathcal{AC}(\mathbb{T})$ and $f(z) \neq 0$ for all $z \in \mathbb{T}$, then $\dfrac{1}{f}$ has an absolutely convergent Fourier series.*

Proof. By Theorem 5.17, $\sigma(f) = \sigma(\Gamma(f)) = \operatorname{ran}\Gamma(f)$. But if $\varphi \in \sum_{\mathcal{AC}(\mathbb{T})}$, then $\varphi = \varphi_\lambda$ for some $\lambda \in \mathbb{T}$, where $\varphi_\lambda(f) = f(\lambda)$ is the evaluation functional corresponding to λ. Thus

$$\begin{aligned}\operatorname{ran}\Gamma(f) &= \{\Gamma(f)(\varphi_\lambda) : \varphi_\lambda \in \Sigma_{\mathcal{AC}(\mathbb{T})}\} \\ &= \{\Gamma(f)(\varphi_\lambda) : \lambda \in \mathbb{T}\} \\ &= \{\varphi_\lambda(f) : \lambda \in \mathbb{T}\} \\ &= \{f(\lambda) : \lambda \in \mathbb{T}\} \\ &= \operatorname{ran} f.\end{aligned}$$

Since $0 \notin \operatorname{ran} f$, we get $0 \notin \sigma_{\mathcal{AC}(\mathbb{T})}(f)$, so $1/f$ has an absolutely convergent Fourier series.

□

Supplementary Examples

S5.1. Example. Not every abelian Banach algebra admits maximal modular ideals. For example, let J_n denote the Jordan cell in $\mathbb{M}_n(\mathbb{C}) \simeq \mathcal{B}(\mathbb{C}^n)$, and let $\mathcal{A} = \{\sum_{k=1}^{n-1} \alpha_k J_n^k : \alpha_k \in \mathbb{C}, 1 \le k \le n\}$. Then $\mathcal{A}$ is a (non-unital) abelian Banach algebra.

The space $\mathcal{J} := J_n\mathcal{A} = \{J_n A : A \in \mathcal{A}\}$ is a maximal ideal of $\mathcal{A}$ (it has co-dimension one), but it is not modular. Indeed, if $E \in \mathcal{A}$, then $EJ_n - J_n \notin \mathcal{J}$.

For example, if $n = 3$, then $E \in \mathcal{A}$ implies that there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ such that

$$E = \begin{bmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & \alpha_1 \\ 0 & 0 & 0 \end{bmatrix},$$

while $K \in \mathcal{J}$ implies that there exists $\beta \in \mathbb{C}$ such that

$$K = \begin{bmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

But then

$$EJ_3 - J_3 = \begin{bmatrix} 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \notin \mathcal{J}.$$

S5.2. Example. Here's an alternative proof of the above result.

If $\mathcal{A}$ were to admit modular ideals, then it would also admit maximal modular ideals, by Corollary 5.7. By Proposition 5.12, $\mathcal{A}$ would admit a non-zero multiplicative linear functional.

Let $\varphi \in \Sigma_{\mathcal{A}}$. Then $\varphi(J_n) \in \sigma(J_n) = \{0\}$, so $\varphi(J_n) = 0$. Thus for $A \in \mathcal{A}$, say $A = \sum_{k=1}^{n-1} \alpha_k J_n^k$, we find that

$$\varphi(A) = \sum_{k=1}^{n-1} \alpha_k \varphi(J_n)^k = 0,$$

contradicting the fact that $\varphi \ne 0$.

S5.3. Example. Essentially the same example may be dressed up to impress people at a fancy dinner party as follows.

Let $\mathcal{A}(\mathbb{D})$ denote the disk algebra (thought of as functions on $\mathbb{D}$). Let $\kappa(z) = z$, $z \in \mathbb{D}$, and define

$$\mathcal{B} := \{\kappa f : f \in \mathcal{A}(\mathbb{D})\}.$$

It is reasonable straightforward to show that $\mathcal{J}$ is a closed ideal, hence a closed subalgebra of $\mathcal{A}(\mathbb{D})$. (It corresponds to the set of functions $g \in \mathcal{A}(\mathbb{D})$ such that $g(0) = 0$.) The set

$$\mathcal{J} := \{\kappa g : g \in \mathcal{B}\}$$

is a closed ideal of $\mathcal{B}$, but it is not modular. (In fact, $\mathcal{J}$ corresponds to those elements of $\mathcal{A}(\mathbb{D})$ such that $f(0) = f'(0) = 0$.) We leave it to the reader to show that for any function $e \in \mathcal{B}$,

$$e\kappa - \kappa \notin \mathcal{J}.$$

Appendix

A5.1. The notion of the *Jacobson radical* comes from Ring Theory. In this general context, it is defined as follows.

Definition. *Let R be a ring.*

- *A left R-module M is said to be **simple** if $R \cdot M \neq \{0\}$ and M has no non-trivial submodules. Let $\mathfrak{M}$ denote the set of simple left R-modules.*
- *The **Jacobson radical** $J(R)$ of R is the set*

$$J(R) := \cap\{r \in R : \ r \cdot M = 0 \text{ for all } M \in \mathfrak{M}\}.$$

Thus $J(R)$ is the set of all elements of R that annihilate all simple left R-modules.

A5.2. Theorem. *If R is a ring, then $J(R)$ is a two-sided ideal of R.*

Proof. Suppose that M is a simple left R-module and that $j, j_1, j_2 \in J(R)$. Clearly $0 \in J(R) \neq \varnothing$, and for all $m \in M$,

$$(j_1 - j_2) \cdot m = j_1 \cdot m - j_2 \cdot m = 0 - 0 = 0.$$

Hence $j_1 - j_2 \in J(R)$.

Let $r \in R$. Given $m \in M$, note that $m' := r \cdot m \in M$ and thus

$$(rj) \cdot m = r \cdot (j \cdot m) = r \cdot 0 = 0,$$

while

$$(jr) \cdot m = j \cdot (r \cdot m) = j \cdot m' = 0.$$

Hence $rj, jr \in J(R)$.

By the Ideal Test, $J(R)$ is a two-sided ideal of R.

□

A5.3. Let R be a ring and M be a simple left R-module. Then there exists $m \in M$ and $r \in R$ such that $r \cdot m \neq 0$. Thus $R \cdot m$ is a non-zero submodule of M, whence $R \cdot m = M$. We say that m is **cyclic** element for R. In fact, *every* non-zero element of M is cyclic for R. Indeed, let $n \in M$. If there exists $r \in R$ such that $r \cdot n \neq 0$, then the above argument shows that $R \cdot n = M$, and thus n is cyclic.

Let $Z := \{z \in R : r \cdot z = 0 \text{ for all } r \in R\}$. Then Z is easily seen to be a left R-module. By simplicity of M, $Z \neq M$ and hence $Z = \{0\}$. In other words, every non-zero element of M is cyclic for R.

To see the connection between the above definition of the Jacobson radical and the "Jacobson radical" we defined in Definition 5.19, we first extend our definition of modularity to include one-sided ideals.

A5.4. Definition. *Let R be a ring. A left ideal M of R is said to be **modular** if there exists $e \in R$ such that $re - r \in M$ for all $r \in R$.*

A5.5. Proposition.

(i) *If R is a ring and M is a simple left R-module, then there exists a maximal modular left ideal N of R such that M is isomorphic to R/N.*

(ii) *Conversely, if N is a maximal modular left ideal of R, then R/N is a simple left R-module.*

Proof.

(i) Note that if $M \in \mathfrak{M}$ is a simple left R-module, then there exists $m \in M$ and $r \in R$ such that $r \cdot m \neq 0$. Thus $R \cdot m$ is a non-zero submodule of M, whence $R \cdot m = M$.

Let $N := \{s \in R : s \cdot m = 0\}$. Then N is a left-ideal of R, and so R/N is a left R-module. Consider the left R-module map

$$\begin{array}{rccc} \Theta : & R/N & \to & M \\ & r+N & \mapsto & r \cdot m. \end{array}$$

(That this is indeed a well-defined left R-module map is left as an exercise for the reader.) It is easy to see that $\ker \Theta = \{0 + N\}$ is trivial, and that Θ is surjective, proving that $M \simeq R/N$ as left R-modules. If N is not maximal, say $N \subset L \subset R$ for some left ideal L of R (recall that $\subset$ denotes *proper* containment), then L/N is a non-trivial left ideal of R/N, and thus $Lx = \Theta(L/N)$ is a non-trivial left R-submodule of M, contradicting the simplicity of M.

To see that N is a *modular* left ideal, note that $M = R \cdot m$, and thus there exists $e \in R$ such that $e \cdot m = m$. But then for all $r \in R$,

$$r \cdot m = r \cdot (e \cdot m) = (re) \cdot m,$$

so that $(r - re) \in N$.

(ii) Conversely, if N is a maximal modular left ideal of R, then clearly R/N is a left R-module. Note that $R \cdot (R/N) \neq \{0 + N\}$, since if $e \in R$ is the element for which $re - r \in N$ for all $r \in R$, then $r \cdot (e + N) = re + N = r + N \neq 0 + N$ whenever $r \in R \setminus N$. Furthermore, the left submodules of R/N correspond to left ideals of R/N, which in turn correspond to left ideals of R which contain N, of which there are only two, namely N, and R itself. Thus R/N is a simple left R-module.

□

A5.6. Our present goal is to show that if R is a ring, then $J(R)$ is the intersection of all maximal modular left ideals of R. To do this, we need another definition, which also provides a useful characterisation of $J(R)$.

Definition. *An element q of a ring R is said to be **left quasi-regular** if there exists $w \in R$ such that*

$$w \diamond q := w + q + wq = 0.$$

*The element w is said to be a **left quasi-inverse of** q. A left ideal L of R is said to be **left quasi-regular** if every element of L is left quasi-regular.*

A5.7. Such a definition may look rather arcane and mysterious, but the following result tells us the concept we are trying to generalise from unital to non-unital rings.

Proposition. *Let R be a unital ring. An element $q \in R$ is left quasi-regular if and only if $1 + q$ is left-invertible in R.*

Proof.

- Suppose that q is left quasi-invertible with left quasi-inverse w. Then

$$(1 + w)(1 + q) = 1 + w + q + wq = 1 + w \diamond q = 1.$$

Thus $(1 + q)$ is left-invertible in R.

- Suppose that $b \in R$ satisfies $b(1 + q) = 1$. Let $w = b - 1$. Then

$$w \diamond q = w + q + wq = b - 1 + q + bq - q = b(1 + q) - 1 = 0.$$

Thus q is left quasi-regular.

□

The next two lemmas will give us the extra characterisation of $J(R)$ we shall need.

A5.8. Lemma. *Let R be a ring, and define*

$$\mathfrak{J} := \cap\{N \subseteq R : N \textit{ is a maximal modular left ideal}\}.$$

Then $\mathfrak{J}$ is a left quasi-regular left ideal of R.

Proof. First note that if R does not admit any maximal modular left ideals, then $\mathfrak{J} = R$ is clearly a left quasi-regular left ideal of R.

Suppose therefore, that R *does* admit maximal modular left ideals.

That $\mathfrak{J}$ is a left-ideal is clear, since it is the intersection of left-ideals. Let $q \in \mathfrak{J}$, and consider the left ideal $K_q := \{r + rq : r \in R\}$. Our goal is to show that $K_q = R$. For then there exists $w \in R$ such that $w + wq = -q$, or equivalently, $w \diamond q = 0$, showing that q is left quasi-regular. Since $q \in \mathfrak{J}$ was arbitrary, we conclude that $\mathfrak{J}$ is left quasi-regular.

Suppose to the contrary that $K_q \neq R$. Now K_q is not only a left-ideal of R, but in fact it is a modular left-ideal of R, since for any $s \in R$, $s - s(-q) = s + sq \in K_q$. By an easy application of Zorn's Lemma, K_q is contained in a maximal modular left-ideal N of R. Now $q \in \mathfrak{J} \subseteq N$, and thus $rq \in N$ for all $r \in R$. But $r + rq \in K_q \subseteq N$, and therefore $r \in N$ for all $r \in R$, implying that $R = N$, contradicting the maximality of N. Hence $K_q = R$.

□

A5.9. Lemma. *Let R be a ring. If V is a left quasi-regular left ideal of R, then*

$$V \subseteq J(R).$$

Proof. If R does not admit any simple left R-modules, then $J(R) = R$ and the result is trivial. Suppose, therefore, that R does admit a simple left R-module.

We argue by contradiction. Suppose that there exists a simple left R-module M such that $V \cdot M \neq 0$. Choose $m_0 \in M$ such that $V \cdot m_0 \neq 0$. Obviously $m_0 \neq 0$. Then $V \cdot m_0$ is a non-zero submodule of M, and by the simplicity of M, we must have $V \cdot m_0 = M$. In particular, $-m_0 = v \cdot m_0$ for some $v \in V$. Since V is left quasi-regular, there exists $w \in R$ such that

$$w \diamond v = 0.$$

Thus

$$0 = 0 \cdot m_0 = (w \diamond v) \cdot m_0 = (w + v + wv) \cdot m_0 = w \cdot m_0 + (-m_0) + w(-m_0) = -m_0,$$

a contradiction. This concludes the proof.

□

A5.10. Theorem. *Let R be a ring. Then*

$$J(R) = \mathfrak{J} := \cap\{N : N \textit{ is a maximal modular left ideal of } R\}.$$

Proof. As in Lemma A5.8, we set

$$\mathfrak{J} := \cap\{N : N \text{ is a maximal modular left ideal of } R\}.$$

By that Lemma, $\mathfrak{J}$ is a left quasi-regular left ideal of R. By Lemma A5.9, $\mathfrak{J} \subseteq J(R)$.

Conversely, suppose that $k \in J(R)$. Let N be a maximal modular left-ideal of R, with distinguished element $e \in R$ such that $re - r \in N$ for all $r \in R$. By Proposition A5.5, R/N is a simple left R-module. Now $k \cdot R/N = \{0 + N\}$, implying in particular that

$$0 + N = k \cdot (e + N) = ke + N = k + N.$$

Thus $k \in N$. This shows that $J(R) \subseteq \mathfrak{J}$, completing the proof.

□

A5.10. Theorem. *Let R be a unital ring. Then*

$$J(R) = \{q \in R : 1 + xqy \textit{ is invertible for all } x, y \in R\}.$$

Proof. As we have just seen, $J(R) = \mathfrak{J}$ is a left quasi-regular ideal of R.

Fix $q \in J(R)$. Since $q \in J(R)$ implies that $xqy \in R$ for all $x, y \in R$, to show that $J(R)$ is contained in the set on the right-hand side of the stated equation, it suffices to show that $q \in J(R)$ implies that $1 + q$ is invertible.

By Proposition A5.7, we see that $1 + q$ is left-invertible. Choose $w \in R$ such that $w(1 + q) = 1$. Then $w = 1 - wq$. But $-wq \in J(R)$, because $J(R)$ is an ideal of R, and thus $w = 1 - wq$ is left-invertible, again by Proposition A5.7. But w is right-invertible (with right-inverse $1 + q$), and thus w is invertible. Hence $1 + q = w^{-1}$ is invertible as well.

Conversely, suppose that $q \in R$ and $1 + xqy$ is invertible for all $x, y \in R$. Then $V := Rq$ is a left quasi-regular left ideal of R, and thus by Lemma A5.9, $V \subseteq J(R)$. In particular, $q \in J(R)$.

This completes the proof.

$\square$

A5.11. There is an apparent lack of symmetry in our choice of the definition of $J(R)$, namely: why did we choose *left* R-modules? As it turns out, we could just have easily chosen right R-modules. The "right" Jacobson radical we would have obtained would be the same as the "left" Jacobson radical we did obtain. This is left to the exercises.

A5.12. Of course, if R is commutative, then left-ideals coincide with ideals, and we see that

$$J(R) = \cap\{N : N \text{ is a maximal modular ideal of } R\}.$$

Applying this to (commutative) Banach algebras $\mathcal{A}$, we see that

$$J(\mathcal{A}) = \mathrm{rad}\,(\mathcal{A}).$$

A5.13. Theorem A5.10 is very useful in the Banach algebra context. If $\mathcal{A}$ is a unital Banach algebra, then

$$J(\mathcal{A}) = \{q \in \mathcal{A} : 1 + xqy \text{ is invertible for all } x, y \in \mathcal{A}\}.$$

It follows that for all $x, y \in \mathcal{A}$, $-1 \notin \sigma(xqy)$. In particular, if $0 \neq \alpha \in \mathbb{C}$, then $-1 \notin \sigma(-\alpha^{-1}q)$, whence $\alpha \notin \sigma(q)$. Thus $q \in J(\mathcal{A})$ implies that $\sigma(q) = \{0\}$, i.e. that q is quasinilpotent.

It is not true, however, that every quasinilpotent element of $\mathcal{A}$ lies in the Jacobson radical of $\mathcal{A}$. An easy computation shows that if $\mathcal{A} = \mathbb{M}_2(\mathbb{C}) \simeq \mathcal{B}(\mathbb{C}^2)$, then $J(\mathcal{A}) = \{0_2\}$. Certainly $\mathcal{A}$ admits non-trivial quasinilpotent elements (for example - an off-diagonal matrix unit will do).

Exercises for Chapter 5

Exercise 5.1. MULTIPLICATIVE LINEAR FUNCTIONALS

(a) Determine all of the multiplicative linear functionals on ℓ_n^∞.
(b) Determine all of the multiplicative linear functionals on c_0.

Exercise 5.2. MULTIPLICATIVE LINEAR FUNCTIONALS II

Consider the disk algebra $\mathcal{A}(\mathbb{D})$.

(i) Show that $f^*(z) = \overline{f(\overline{z})}$ defines an isometric involution on $\mathcal{A}(\mathbb{D})$.
(ii) Show that not every multiplicative linear functional on $\mathcal{A}(\mathbb{D})$ is self-adjoint.

Exercise 5.3. FINITELY-GENERATED UNITAL, ABELIAN BANACH ALGEBRAS

Let $\mathcal{A}$ be a unital, abelian Banach algebra. Given $a_1, a_2, \dots, a_n \in \mathcal{A}$, we define the **joint spectrum** of the n-tuple $(a_1, a_2, \dots, a_n) \in \mathcal{A}^(n)$ to be

$$\sigma(a_1, a_2, \dots, a_n) := \{\alpha := (\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n)) : \varphi \in \Sigma_{\mathcal{A}}\}.$$

(a) Prove that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \sigma(a_1, a_2, \dots, a_n)$ if and only if there exist $b_k \in \mathcal{A}$, $1 \le k \le n$ such that

$$\sum_{k=1}^{n} (\lambda_k \mathbb{1} - a_k) b_k = \mathbb{1}.$$

(b) Suppose that $\{a_1, a_2, \dots, a_n\}$ **generates** $\mathcal{A}$; that is, suppose that $\mathcal{A}$ is the smallest unital subalgebra (of itself) that contains all of these elements. Prove that the map

$$\begin{array}{cccc} \Delta: & \Sigma_{\mathcal{A}} & \to & \sigma(a_1, a_2, \dots, a_n) \\ & \varphi & \mapsto & (\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n)) \end{array}$$

is a homeomorphism.

(c) A compact subset $K \subseteq \mathbb{C}^n$ is said to be **polynomially convex** if for all polynomials $p \in \mathbb{C}[x_1, x_2, \dots, x_n]$ in n commuting variables and for all $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n$,

$$|p(\lambda_1, \lambda_2, \dots, \lambda_n)| \le \sup\{|p(k_1, k_2, \dots, k_n)| : (k_1, k_2, \dots, k_n) \in K\}$$

implies that $(\lambda_1, \lambda_2, \dots, \lambda_n) \in K$.

In the case where $n = 1$, this is equivalent to say that K admits no "holes"; that is, $\mathbb{C} \setminus K$ has no bounded components.

Prove that if $\{a_1, a_2, \dots, a_n\}$ generates $\mathcal{A}$, then $\sigma(a_1, a_2, \dots, a_n)$ is **polynomially convex**.

Hint. Suppose that $(\lambda_1, \lambda_2, \dots, \lambda_n) \notin K$. By part (a) above, one can find b_k, $1 \le k \le n$ such that $\sum_{k=1}^n (\lambda_k \mathbb{1} - a_k) b_k = \mathbb{1}$. Note that $\sum_{k=1}^n (\lambda_k \mathbb{1} - a_k) b_k \in \mathcal{A}$, and thus can be approximated by polynomials in $(a_1, a_2, \dots, a_n)$.

Exercise 5.4. Closed ideals and the radical

Let $\mathcal{A}$ be a unital, non-commutative Banach algebra.

(a) Prove that $\mathcal{A}/\mathrm{rad}\,(\mathcal{A})$ is semisimple.

(b) Suppose that $\mathcal{J}$ is a closed ideal of $\mathcal{A}$. Prove that $\mathrm{rad}\,(\mathcal{J}) = \mathcal{J} \cap \mathrm{rad}\,(\mathcal{A})$.

(c) Would part (a) have worked with any ring, or did we need to consider a Banach algebra? Similarly, if R is any ring and $K \triangleleft R$ is an ideal of R, is

$$J(K) = K \cap J(R)?$$

Exercise 5.5. The "right" Jacobson radical of a ring R

In analogy to the discussion in the Appendix above, given a ring R, a right R-module M is said to be **simple** if $M \cdot R \neq \{0\}$ and M has no non-trivial submodules. We will (very temporarily) define the **"right" Jacobson radical** K(R) to be the set

$$K(R) := \cap\{r \in R : M \cdot r = 0 \text{ for all simple right } R\text{-modules}\}.$$

Prove that

$$K(R) = J(R).$$

For this reason, we do not talk of "left-" or "right-"Jacobson radicals, but only of the **Jacobson radical**.

CHAPTER 6

The algebra of Banach space operators

Before I speak, I have something important to say.

Groucho Marx

Introduction.

6.1. As we have already seen there are myriads of examples of Banach algebras. We begin our study with a very important subclass, namely the class of *operator algebras.* We shall divide our analysis into the study of operators on general Banach spaces and on Hilbert spaces. The loss of generality in specifying the underlying space is made up for in the strength of the results we can obtain. We begin by recalling a definition.

6.2. Definition. *Let $\mathfrak{X}$ be a Banach space. Then $\mathcal{B}(\mathfrak{X})$ consists of those linear maps $T : \mathfrak{X} \to \mathfrak{X}$ which are continuous in the norm topology. Given $T \in \mathcal{B}(\mathfrak{X})$, we define the* ***norm*** *of T to be*

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}.$$

It follows immediately from the definition that $\|Tx\| \le \|T\| \, \|x\|$ for all $x \in \mathfrak{X}$, and that $\|T\|$ is the smallest non-negative number with this property.

6.3. Remark. We assume that the reader is familiar with the fact that $\mathcal{B}(\mathfrak{X})$ is a Banach space. To verify that it is indeed a Banach algebra, we need only verify that the operator norm is submultiplicative, that is, that $\|AB\| \le \|A\| \, \|B\|$ for all operators A and B.

But

$$\begin{aligned} \|AB\| &= \sup\{\|ABx\| : \|x\| = 1\} \\ &\le \sup\{\|A\| \, \|Bx\| : \|x\| = 1\} \\ &\le \sup\{\|A\| \, \|B\| \, \|x\| : \|x\| = 1\} \\ &= \|A\| \, \|B\|. \end{aligned}$$

Since $\mathcal{B}(\mathfrak{X})$ is a Banach algebra, all of the results from Chapters 2, 3, and 4 apply. In particular, for $T \in \mathcal{B}(\mathfrak{X})$, the spectrum of T is a non-empty, compact subset of $\mathbb{C}$. The function $R(\lambda, T) = (\lambda I - T)^{-1}$ is analytic on $\rho(T)$, and we can (and do!) define the operator $f(T)$ when f is analytic on a neighbourhood of $\sigma(T)$.

What we shall find, however, is that if $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{X})$ is a Banach algebra, then we can draw more information about $\mathcal{A}$ and its elements thanks to the nature of bounded linear maps. The next proposition is a case in point: as opposed to merely having an abstract definition of invertibility, we can provide alternate and at times more useful characterisations of this property.

6.4. Proposition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. The following are equivalent:*

(a) *T is invertible.*
(b) *T is bounded below and has dense range.*
(b) *T is a bijection.*

Proof.

(a) implies (b). Suppose T is invertible. Let $x \in \mathfrak{X}$. Then $x = T^{-1}Tx$, and so $\|x\| \le \|T^{-1}\| \, \|Tx\|$, i.e. $\|Tx\| \ge \|T^{-1}\|^{-1} \|x\|$ and T is bounded below. Since T is onto, its range is trivially dense.

(b) implies (c). Suppose T is bounded below by, say, $\delta > 0$. We shall first show that in this case, the range of T is closed.

Indeed, suppose that there exists a sequence $y_n = Tx_n$, $n \ge 1$ and y such that $\lim_{n\to\infty} y_n = y$. Then $\delta\|x_m - x_n\| \le \|y_m - y_n\|$, forcing $\{x_n\}_{n=1}^\infty$ to be a Cauchy sequence. Let $x = \lim_{n\to\infty} x_n$. By the continuity of T, we have $Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y$. We have shown that $y \in \operatorname{ran} T$, and hence that $\operatorname{ran} T$ is closed.

It follows that if T has dense range, as per our hypothesis, then T is surjective.

As well, suppose that $x \in \ker T$. Then $\delta\|x\| \le \|Tx\| = 0$, forcing x to be zero, and T to be injective.

(c) implies (a). Suppose that T is a bijection. The Open Mapping Theorem 1.21 then implies that the inverse image map T^{-1} is continuous, and thus that T is invertible.

□

In general, for $T \in \mathcal{B}(\mathfrak{X})$, there are many subclassifications of the spectrum of T. Condition (b) above leads to the following obvious ones.

6.5. Definition. *Let $\mathfrak{X}$ be a Banach space and $T \in \mathcal{B}(\mathfrak{X})$. Then the* ***point spectrum*** *of T is*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injective}\}.$$

These are the eigenvalues of T. The ***approximate point spectrum*** *of T is the set*

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

The ***compression spectrum*** *of* T *is*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not have dense range}\}.$$

6.6. Remarks.

- If $\dim \mathfrak{X} < \infty$, then $\sigma_p(T) = \sigma_a(T) = \sigma_c(T) = \sigma(T)$.
- If $\lambda \in \sigma_a(T)$, then for all $n \geq 1$, there exists $0 \neq x_n \in \mathfrak{X}$ such that $\|(T - \lambda I)x_n\| \leq \frac{1}{n}\|x_n\|$. Let $y_n = x_n/\|x_n\|, n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$, and $(T - \lambda I)y_n \to 0$.
- $\sigma(T) = \sigma_a(T) \cup \sigma_c(T)$ and in general, $\sigma_p(T) \subseteq \sigma_a(T)$.

6.7. Example. Let μ be a finite, positive, regular Borel measure on a non-empty set X, and suppose that $f \in L^\infty(X, \mu)$. (Here, as is common, we use function notation despite the fact that elements of $L^\infty(X, \mu)$ are equivalence classes of functions.)

Consider the linear map

$$\begin{array}{cccc} M_f : & L^2(X,\mu) & \to & L^2(X,\mu) \\ & g & \mapsto & fg. \end{array}$$

(That M_f is well-defined is left as a routine exercise.) We refer to the operator M_f as a **multiplication operator with symbol** f. Observe that

$$\|M_f g\|_2^2 = \|fg\|_2^2 = \int_X |fg|^2 \mu \leq \|f\|_\infty^2 \int_X |g|^2 d\mu = \|f\|_\infty^2 \ \|g\|_2^2,$$

whence $\|M_f\| \leq \|f\|_\infty$. For each $n \geq 1$, set $E_n := \{x \in X : |f(x)| > \|f\|_\infty - \frac{1}{n}\}$. (The set E_n is of course defined up to a set of measure zero.) Then the characteristic function χ_{E_n} of E_n lies in $L^2(X, \mu)$, and

$$\|M_f \chi_{E_n}\|_2^2 = \int_X |f\chi_{E_n}|^2 d\mu = \int_{E_n} |f|^2 d\mu \geq \int_{E_n} (\|f\|_\infty^2) d\mu \geq (\|f\|_\infty^2) \ \|\chi_{E_n}\|_2^2,$$

and thus $\|M_f\| \geq \|f\|_\infty$. Combined with the above estimate, we see that

$$\|M_f\| = \|f\|_\infty.$$

That $\lambda I - M_f = M_{\lambda 1 - f}$ for all $\lambda \in \mathbb{C}$ is easy to verify.

Claim: $\sigma(M_f) = \text{ess ran} f$, the **essential range** of f.

Indeed, if $\lambda \in \text{ess ran} f$, then for each $\varepsilon > 0$, the set $E_\varepsilon := \{x \in X : f(x) \in B(\lambda, \varepsilon)\}$ has positive measure. As such, $\chi_{E_\varepsilon} \neq 0$ in $L^2(X, \mu)$ and

$$\|(\lambda I - M_f)\chi_{E_\varepsilon}\|_2^2 = \int_{E_\varepsilon} |\lambda - f|^2 d\mu < \varepsilon^2 \|\chi_{E_\varepsilon}\|_2^2.$$

Since $\lambda I - M_f$ is not bounded below, $\lambda \in \sigma_a(M_f) \subseteq \sigma(M_f)$.

Conversely, if $\lambda \notin \text{ess ran} f$, then there exists $\delta > 0$ such that E_δ defined as above has measure equal to zero. If follows that $|\lambda - f(x)| \geq \delta$ a.e.-μ. But then $h = (\lambda 1 - f)^{-1} \in L^\infty(X, \mu)$, and

$$M_h(\lambda I - M_f) = M_h M_{\lambda 1 - f} = M_{h(\lambda 1 - f)} = M_1 = I = (\lambda 1 - M_f)M_h.$$

That is, $\lambda \notin \sigma(M_f)$.

From this we see that $\sigma_a(M_f) = \sigma(M_f)$. (As we shall see later, M_f is an example of a **normal operator** in $\mathcal{B}(L^2(X,\mu))$, and for any normal operator N acting on a Hilbert space, $\sigma_a(N) = \sigma(N)$.)

Suppose that $\lambda \in \sigma_p(M_f)$. Then there exists a non-zero element $g \in L^2(X,\mu)$ such that

$$(\lambda I - M_f)g = (\lambda 1 - f)g = 0 \text{ a.e.} - \mu.$$

It then follows that the set $E := \{x \in X : f(x) = \lambda\}$ (defined a.e.-μ) has positive measure. That is, f is constant (a.e.-μ) on set of positive measure.

For example, the multiplication operator $M_f : L^2([0,1], dm) \to L^2([0,1], dm)$ where $f(x) = x$ (a.e.-m), and where dm represents Lebesgue measure, has no eigenvalues, given that the function $f(x) = x$ is not constant on any set of positive measure in $[0,1]$.

Banach space adjoints

6.8. Definition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Let $T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y})$. We shall now define an operator $T^* \in \mathcal{B}(\mathfrak{Y}^*,\mathfrak{X}^*)$, called the* ***Banach space adjoint*** *of T.*

First, for $x^* \in \mathfrak{X}^*$, we adopt the notation $\langle x, x^* \rangle = x^*(x)$. Then for $y^* \in \mathfrak{Y}^*$, define T^* so that

$$\langle x, T^*(y^*) \rangle = \langle Tx, y^* \rangle.$$

That is, $(T^* y^*)(x) = y^*(Tx)$ for all $x \in \mathfrak{X},\ y^* \in \mathfrak{Y}^*$. It is not hard to verify that T^* is linear.

6.9. Proposition. *Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ be Banach spaces, $S, T \in \mathcal{B}(\mathfrak{X},\mathfrak{Y})$, and let $R \in \mathcal{B}(\mathfrak{Y},\mathfrak{Z})$. Then*

(a) *for all α, $\beta \in \mathbb{C}$, we have $(\alpha S + \beta T)^* = \alpha S^* + \beta T^*$;*
(b) $(R \circ T)^* = T^* \circ R^*$.

Proof. Let $x \in \mathfrak{X},\ y^* \in \mathfrak{Y}^*$, and $z^* \in \mathfrak{Z}^*$. Then

(a)

$$\begin{aligned}
\langle x, (\alpha S + \beta T)^* y^* \rangle &= \langle (\alpha S + \beta T)x,\ y^* \rangle \\
&= y^*((\alpha S + \beta T)x) \\
&= \alpha y^*(Sx) + \beta y^*(Tx) \\
&= \alpha \langle Sx,\ y^* \rangle + \beta \langle Tx,\ y^* \rangle \\
&= \alpha \langle x,\ S^* y^* \rangle + \beta \langle x, T^* y^* \rangle.
\end{aligned}$$

Since this is true for all $x \in \mathfrak{X}$ and $y^* \in \mathfrak{Y}^*$, we conclude that $(\alpha S + \beta T)^* = \alpha S^* + \beta T^*$.

(b)

$$\begin{aligned}\langle x, (R\circ T)^* z^*\rangle &= \langle (R\circ T)x, z^*\rangle \\ &= \langle R(Tx), z^*\rangle \\ &= \langle Tx,\, R^* z^*\rangle \\ &= \langle x, T^*(R^* z^*)\rangle \\ &= \langle T^*\circ R^* z^*\rangle.\end{aligned}$$

Again, this shows that $(R\circ T)^* = T^*\circ R^*$.

□

6.10. Theorem. *Let $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces. Then $\|T^*\| = \|T\|$.*

Proof.

For any $y^* \in \mathfrak{Y}^*$, we have

$$\begin{aligned}\|T^* y^*\| &= \sup\{|T^* y^*(x)| : x \in \mathfrak{X},\ \|x\| = 1\} \\ &= \sup\{|y^*(Tx)| : x \in \mathfrak{X},\ \|x\| = 1\} \\ &\le \sup\{\|y^*\|\,\|Tx\| : x \in \mathfrak{X},\ \|x\| = 1\} \\ &= \|y^*\|\,\|T\|.\end{aligned}$$

Thus we see that $\|T^*\| \le \|T\|$.

Next, let $x \in \mathfrak{X}$. By the Hahn-Banach Theorem, we can choose $y^* \in \mathfrak{Y}^*$ such that $y^*(Tx) = \|Tx\|$ and $\|y^*\| = 1$. Then

$$\begin{aligned}\|Tx\| &= y^*(Tx) \\ &= \langle Tx, y^*\rangle \\ &= \langle x, T^* y^*\rangle \\ &= (T^* y^*)(x) \\ &\le \|T^* y^*\|\,\|x\| \\ &\le \|T^*\|\|x\|.\end{aligned}$$

Thus $\|T\| \le \|T^*\|$.

Combining this with the previous estimate, we have that $\|T^*\| = \|T\|$.

□

6.11. Proposition. *Let $\mathfrak{X} = \mathbb{C}^n$ and $A \in \mathcal{B}(\mathfrak{X}) \simeq \mathbb{M}_n$. Then the matrix of the Banach space adjoint A^* of A with respect to the dual basis coincides with A^t, the transpose of A.*

Proof. Recall that $\mathfrak{X}^* \simeq \mathfrak{X}$. We then let $\{e_i\}_{i=1}^n$ be a basis for $\mathfrak{X}$ and let $\{f_j\}_{j=1}^n$ be the corresponding dual basis; that is, $f_j(e_i) = \delta_{ij}$, where δ_{ij} is the Dirac delta function. Let $x \in \mathfrak{X}$. Define $\lambda_j = f_j(x)$.

Writing the matrix of $A \in \mathcal{B}(\mathfrak{X})$ as $[a_{ij}]$, we have

$$Ae_j = [a_{ij}] \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{j-1\,j} \\ a_{jj} \\ a_{j+1\,j} \\ . \\ . \\ a_{nj} \end{bmatrix} = \sum_{k=1}^{n} a_{kj} e_k.$$

Thus $a_{ij} = f_i(Ae_j)$.

Now $A^* \in \mathcal{B}(\mathfrak{X}^*) \simeq \mathbb{M}_n$, and so we can also write the matrix for A^* with respect to $\{f_j\}_{j=1}^n$. As above, we have

$$A^* f_j = \sum_{k=1}^{n} \alpha_{kj} f_k.$$

Thus

$$\alpha_{ij} = (A^* f_j)(e_i) = f_j(Ae_i) = a_{ji}.$$

In particular, the matrix for A^* with respect to $\{f_j\}_{j=1}^n$ is simply the transpose of the matrix for A with respect to $\{e_j\}_{j=1}^n$.

□

Keeping in mind that the Banach space adjoint generalises the notion of the transpose of a matrix $A \in \mathbb{M}_n(\mathbb{C})$, and that the spectrum of the transpose of A agrees with the spectrum of A, the next result is perhaps less surprising than it otherwise could be. The proof, however, is not completely straightforward.

6.12. Proposition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and let $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Then T is invertible if and only if T^* is invertible.*

Proof. First assume that T is invertible, i.e., that $T^{-1} \in \mathcal{B}(\mathfrak{Y}, \mathfrak{X})$. Then $I_{\mathfrak{X}^*} = (I_{\mathfrak{X}})^* = (T^{-1} \circ T)^* = T^* \circ (T^{-1})^*$.

Also, $I_{\mathfrak{Y}^*} = (I_{\mathfrak{Y}})^* = (T \circ T^{-1})^* = (T^{-1})^* \circ T^*$. Thus T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Now assume that T^* is invertible. Then $\operatorname{ran} T$ is dense, for otherwise by the Hahn-Banach Theorem we can take $y^* \in \mathfrak{Y}^*$ such that $\|y^*\| = 1$ and $y^*|_{\overline{(\operatorname{ran} T)}} = 0$. Then

$$(T^* y^*)(x) = y^*(Tx) = 0$$

for all $x \in \mathfrak{X}$. Thus $T^* y^* = 0$ but $y^* \neq 0$, implying that T^* is not injective, a contradiction.

Moreover, T is bounded below. For consider: T^* invertible implies that $T^{**} = (T^*)^*$ is invertible from above. Thus T^{**} is bounded below. Recall that $\mathfrak{X}$ embeds isometrically isomorphically into $\mathfrak{X}^{**}$ via the map

$$\begin{array}{ccc} \mathfrak{X} & \simeq & \widehat{\mathfrak{X}} \subseteq \mathfrak{X}^{**} \\ x & \mapsto & \hat{x} \end{array}$$

where $\hat{x}(x^*) = x^*(x)$ for all $x^* \in \mathfrak{X}^*$. (Recall that $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ and that $T^{**} \in \mathcal{B}(\mathfrak{X}^{**}, \mathfrak{Y}^{**})$.)

Now $T^{**}(\hat{x}) \in \mathfrak{Y}^{**}$, and

$$\begin{aligned} ((T^*)^*(\hat{x}))(y^*) &= \hat{x}(T^* y^*) \\ &= (T^* y^*)(x) \\ &= y^*(Tx) \text{ for all } y^* \in \mathfrak{Y}^*. \end{aligned}$$

Thus

$$\sup\{|(T^{**}\hat{x})(y^*)| : y^* \in \mathfrak{Y}^*, \ \|y^*\| = 1\} = \sup\{|y^*(Tx)| : y^* \in \mathfrak{Y}^*, \ \|y^*\| = 1\}.$$

In other words, $\|T^{**}\hat{x}\| = \|Tx\|$. Since T^{**} is bounded below, say by $\delta > 0$,

$$\delta\|x\| = \delta\|\hat{x}\| \leq \|T^{**}\hat{x}\| = \|Tx\|.$$

In other words, T is also bounded below.

Finally, T bounded below and $\operatorname{ran} T$ dense together imply that T is invertible, by Proposition 6.4.

□

6.13. Corollary. *Let $\mathfrak{X}$ be a Banach space and $T \in \mathcal{B}(\mathfrak{X})$. Then $\sigma(T) = \sigma(T^*)$.*

Compact operators acting on Banach spaces

6.14. Definition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Then T is said to be **compact** if $\overline{T(\mathfrak{X}_1)}$ is compact in $\mathfrak{Y}$. The set of compact operators from $\mathfrak{X}$ to $\mathfrak{Y}$ is denoted by $\mathcal{K}(\mathfrak{X}, \mathfrak{Y})$, and if $\mathfrak{Y} = \mathfrak{X}$, we simply write $\mathcal{K}(\mathfrak{X})$.*

Recall that a subset K of a metric space L is said to be **totally bounded** if for every $\varepsilon > 0$ there exists a finite cover $\{B_\varepsilon(y_i)\}_{i=1}^n$ of K with $y_i \in K$, $1 \leq i \leq n$, where $B_\varepsilon(y_i) = \{z \in L : \operatorname{dist}(z, y_i) < \varepsilon\}$.

6.15. Proposition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. The following are equivalent:*

(i) *T is compact;*
(ii) *$\overline{T(F)}$ is compact in $\mathfrak{Y}$ for all bounded subsets F of $\mathfrak{X}$;*
(iii) *If $(x_n)_n$ is a bounded sequence in $\mathfrak{X}$, then $(Tx_n)_n$ has a convergent subsequence;*
(iv) *$T(\mathfrak{X}_1)$ is totally bounded.*

Proof. Exercise.

□

6.16. Definition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Then $F \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is said to be a* ***finite-rank operator*** *if* $\dim F(\mathfrak{X})$ *is finite. The set of finite rank operators from $\mathfrak{X}$ to $\mathfrak{Y}$ is denoted by $\mathcal{F}(\mathfrak{X}, \mathfrak{Y})$.*

6.17. Proposition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Then $\mathcal{F}(\mathfrak{X}, \mathfrak{Y}) \subseteq \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$.*

Proof. Suppose $F \in \mathcal{F}(\mathfrak{X}, \mathfrak{Y})$. Then $\overline{F\mathfrak{X}_1}$ is closed and bounded in $\operatorname{ran} F$, but $\operatorname{ran} F$ is finite-dimensional in $\mathfrak{Y}$, as F is finite rank. Thus $\overline{F\mathfrak{X}_1}$ is compact in $\operatorname{ran} F$, and thus compact in $\mathfrak{Y}$ as well. That is, F is a compact operator.

□

6.18. Proposition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and suppose that $K \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$. The following are equivalent.*

(i) *$K(\mathfrak{X})$ is closed in $\mathfrak{Y}$, and*
(ii) *K is a finite-rank operator.*

Proof.

(i) implies (ii).

Suppose that $K(\mathfrak{X})$ is closed. Then $K(\mathfrak{X})$ is a Banach space and the map

$$\begin{array}{rccc} K_0 : & \mathfrak{X} & \to & K(\mathfrak{X}) \\ & x & \mapsto & Kx \end{array}$$

is a surjection. By the Open Mapping Theorem 1.21, it is also an open map. In particular, $K_0(\mathfrak{X}_1^\circ)$ is open in $K(\mathfrak{X})$ and $0 \in K_0(\mathfrak{X}_1^\circ)$. Let G be an open ball in $K(\mathfrak{X})$ centred at 0 and contained in $K_0(\mathfrak{X}_1^\circ)$. Then $\overline{K_0(\mathfrak{X}_1)} = \overline{K(\mathfrak{X}_1)}$ is compact, and also contains $\overline{G}$. Thus $\overline{G}$ is compact in $K(\mathfrak{X})$ and so $\dim K(\mathfrak{X})$ is finite; i.e. K is a finite-rank operator.

(ii) implies (i).

$K(\mathfrak{X})$ is easily seen to be a submanifold of $\mathfrak{Y}$. Since finite-dimensional manifolds are always closed, we find that $\dim K(\mathfrak{X}) < \infty$ implies $K(\mathfrak{X})$ is closed in $\mathfrak{Y}$.

□

6.19. Proposition. *Let $\mathfrak{X}$ be a Banach space. Then $\mathcal{K}(\mathfrak{X}) = \mathcal{B}(\mathfrak{X})$ if and only if $\mathfrak{X}$ is finite dimensional.*

Proof. If $\dim \mathfrak{X} < \infty$, then $\mathcal{B}(\mathfrak{X}) = \mathcal{F}(\mathfrak{X}) \subseteq \mathcal{K}(\mathfrak{X}) \subseteq \mathcal{B}(\mathfrak{X})$, and equality follows.

If $\mathcal{K}(\mathfrak{X}) = \mathcal{B}(\mathfrak{X})$, then $I \in \mathcal{K}(\mathfrak{X})$, so $\overline{I(\mathfrak{X}_1)} = I(\mathfrak{X}_1) = \mathfrak{X}_1$ is compact. In particular, $\mathfrak{X}$ is finite-dimensional.

□

6.20. Theorem. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and suppose $K \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$. Then $K^* \in \mathcal{K}(\mathfrak{Y}^*, \mathfrak{X}^*)$.*

Proof. Let $\epsilon > 0$. Then $K(\mathfrak{X}_1)$ is totally bounded, so we can find $x_1, x_2, \ldots, x_n \in \mathfrak{X}_1$ such that if $x \in \mathfrak{X}_1$, then $\|Kx - Kx_i\| < \epsilon/3$ for some $1 \le i \le n$. Let

$$\begin{array}{rccc} R: & \mathfrak{Y}^* & \to & \mathbb{C}^n \\ & \phi & \mapsto & (\phi(K(x_1)), \phi(K(x_2)), \ldots, \phi(K(x_n))). \end{array}$$

Then $R \in \mathcal{F}(\mathfrak{Y}^*, \mathbb{C}^n) \subseteq \mathcal{K}(\mathfrak{Y}^*, \mathbb{C}^n)$, and so $R(\mathfrak{Y}_1^*)$ is totally bounded, where $\mathfrak{Y}_1^*$ is the unit ball of $\mathfrak{Y}^*$. Thus we can find $y_1^*, y_2^*, \ldots, y_m^* \in \mathfrak{Y}_1^*$ such that if $y^* \in \mathfrak{Y}_1^*$, then $\|Ry^* - Ry_j^*\| < \epsilon/3$ for some $1 \le j \le m$. Now

$$\begin{aligned} \|Ry^* - Ry_j^*\| &= \max_{1 \le i \le n} |y^*(K(x_i)) - y_j^*(K(x_i))| \\ &= \max_{1 \le i \le n} |K^*(y^*)(x_i) - K^*(y_j^*)(x_i)|. \end{aligned}$$

Suppose $x \in \mathfrak{X}_1$. Then $\|Kx - Kx_i\| < \epsilon/3$ for some $1 \le i \le n$, and $|K^*(y^*)(x_i) - K^*(y_j^*)(x_i)| < \epsilon/3$ for some $1 \le j \le m$, so

$$\begin{aligned} |K^*(y^*)(x) - K^*(y_j^*)(x)| &\le |K^*(y^*)(x) - K^*(y^*)(x_i)| + \\ &\qquad |K^*(y^*)(x_i) - K^*(y_j^*)(x_i)| + \\ &\qquad\quad |K^*(y_j^*)(x_i) - K^*(y_j^*)(x)| \\ &\le \|y^*\| \, \|Kx - Kx_i\| + \epsilon/3 + \|y_j^*\| \, \|Kx - Kx_i\| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Thus $\|K^*y^* - K^*y_j^*\| \le \epsilon$ and so $K^*(\mathfrak{Y}_1^*)$ is totally bounded. We conclude that $K^* \in \mathcal{K}(\mathfrak{Y}^*, \mathfrak{X}^*)$.

□

For the remainder of this section, unless explicitly stated otherwise, $\mathfrak{X}$ will denote an *infinite-dimensional* Banach space.

6.21. Theorem. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Then $\mathcal{K}(\mathfrak{X},\mathfrak{Y})$ is a closed subspace of $\mathcal{B}(\mathfrak{X},\mathfrak{Y})$.*

Proof. Let $\alpha, \beta \in \mathbb{C}$ and let $K_1, K_2 \in \mathcal{K}(\mathfrak{X},\mathfrak{Y})$. Let $(x_n)_n$ be a bounded sequence in $\mathfrak{X}$. Then K_1 generates a convergent subsequence, say $(K_1(x_{n(j)}))_j$. Similarly, K_2 generates a convergent subsequence from $(x_{n(j)})_j$, say $(K_2(x_{n(j(i))}))_i$.

Then $((\alpha K_1 + \beta K_2)(x_{n(j(i))}))_i$ is a convergent subsequence in $\mathfrak{Y}$. From part (c) of Proposition 6.15, $\alpha K_1 + \beta K_2 \in \mathcal{K}(\mathfrak{X},\mathfrak{Y})$.

Now we show that $\mathcal{K}(\mathfrak{X},\mathfrak{Y})$ is closed. Suppose $K_n \in \mathcal{K}(\mathfrak{X},\mathfrak{Y})$ for $n \geq 1$ and $\lim_{n\to\infty} K_n = K \in \mathcal{B}(\mathfrak{X},\mathfrak{Y})$. We shall show that $K(\mathfrak{X}_1)$ is totally bounded. First let $\varepsilon > 0$, and choose $N > 0$ such that $\|K_N - K\| < \varepsilon/3$.

Since $K_N(\mathfrak{X}_1)$ is totally bounded, we can find $\{y_i = K_N(x_i)\}_{i=1}^M$ such that $\{B_{\varepsilon/3}(y_i)\}_{i=1}^M$ is a finite cover of $K_N(\mathfrak{X}_1)$. Thus for all $x \in \mathfrak{X}_1$,

$$\|K_N(x) - K_N(x_j)\| < \varepsilon/3 \text{ for some } 1 \leq j = j(x) \leq M.$$

Then

$$\begin{aligned}
\|K(x) - K(x_j)\| &= \|K(x) - K_N(x) + K_N(x) - K_N(x_j) + K_N(x_j) - K(x_j)\| \\
&\leq \|K - K_N\|\,\|x\| + \|K_N(x) - K_N(x_j)\| + \|K_N - K\|\,\|x_j\| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Thus $K(\mathfrak{X}_1)$ is totally bounded and so $K \in \mathcal{K}(\mathfrak{X},\mathfrak{Y})$.

□

6.22. Theorem. *Let $\mathfrak{W}$, $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be Banach spaces. Suppose $R \in \mathcal{B}(\mathfrak{W},\mathfrak{X})$, $K \in \mathcal{K}(\mathfrak{X},\mathfrak{Y})$, and $T \in \mathcal{B}(\mathfrak{Y},\mathfrak{Z})$. Then $TK \in \mathcal{K}(\mathfrak{X},\mathfrak{Z})$ and $KR \in \mathcal{K}(\mathfrak{W},\mathfrak{Y})$.*

Proof. Now

$$\begin{aligned}
\overline{T \circ K(\mathfrak{X}_1)} &= \overline{T(K(\mathfrak{X}_1))} \\
&\subseteq \overline{T(\overline{K(\mathfrak{X}_1)})}.
\end{aligned}$$

Since $\overline{K(\mathfrak{X}_1)}$ is compact and T is continuous, $\overline{T \circ K(\mathfrak{X}_1)}$ is a closed subset of the compact set $\overline{T(\overline{K(\mathfrak{X}_1)})} = T(\overline{K(\mathfrak{X}_1)})$, and so it is compact as well. Thus $TK \in \mathcal{K}(\mathfrak{X},\mathfrak{Z})$.

Next, note that

$$\overline{KR(\mathfrak{W}_1)} = \overline{K(R(\mathfrak{W}_1))}.$$

But $R(\mathfrak{W}_1)$ is bounded since R is, and so by Proposition 6.15, $\overline{KR(\mathfrak{W}_1)}$ is compact. Thus $KR \in \mathcal{K}(\mathfrak{W},\mathfrak{Y})$.

□

6.23. Corollary. *If $\mathfrak{X}$ is a Banach space, then $\mathcal{K}(\mathfrak{X})$ is a closed, two-sided ideal of $\mathcal{B}(\mathfrak{X})$.*

The Fredholm Alternative

6.24. A very familiar result from linear algebra states that every linear map T on a complex vector space of dimension $n \geq 1$ admits n eigenvalues (counted according to algebraic multiplicity), and that for $\alpha \in \mathbb{C}$, $T - \alpha I$ is injective if and only if it is surjective, in which case it is invertible.

As we have seen, the multiplication operator M_f on $L^2([0,1], dx)$, where $f(x) = x$ is injective, but it is not surjective and thus not invertible. Similarly, if $\mathcal{H} = \ell^2$ and $\{e_n\}_n$ is an orthonormal basis for $\mathcal{H}$, then the operator $S \in \mathcal{B}(\mathcal{H})$ defined by $Se_1 = 0$ and $Se_n = n-1$ if $n \geq 2$ is easily seen to be surjective but not injective.

The Fredholm Alternative (Theorem 6.32 below) shows that if $K \in \mathcal{K}(\mathfrak{X})$ and if $0 \neq \lambda \in \mathbb{C}$, then $\lambda I - K$ is injective if and only if $\lambda I - K$ is surjective. In other words, we recover an analogue of the finite-dimensional result for all *non-zero* complex numbers. The technique used to prove the Fredholm Alternative is also the key to establishing the wonderful result that the spectrum of a compact operator acting on a Banach space is a sequence (potentially finite) converging to zero.

6.25. Lemma. *Let $\mathfrak{X}$ be a Banach space and $\mathfrak{M}$ be a finite dimensional subspace of $\mathfrak{X}$. Then there exists a closed subspace $\mathfrak{N}$ of $\mathfrak{X}$ such that $\mathfrak{M} \oplus \mathfrak{N} = \mathfrak{X}$.*

Proof. Let $\{e_i\}_{i=1}^n$ be a basis for $\mathfrak{M}$ and let $\{f_i\}_{i=1}^n$ be the dual basis to $\{e_i\}_{i=1}^n$ (cf. Proposition 6.11). Then we can extend $\{f_i\}_{i=1}^n$ to $\{\phi_i\}_{i=1}^n \subseteq \mathfrak{X}^*$ by the Hahn-Banach Theorem. We then let $\mathfrak{N} = \cap_{i=1}^n \ker \phi_i$. It remains to check that $\mathfrak{N}$ is the desired space. Clearly it is closed.

If $x \in \mathfrak{X}$, then let $\lambda_i = \phi_i(x)$, $1 \leq i \leq n$, and set $y = \sum_{i=1}^n \lambda_i e_i \in \mathfrak{M}$. Let $z = x - y$ so that $x = y + z$. Then $\phi_i(z) = \phi_i(x) - \phi_i(y) = \lambda_i - \lambda_i = 0$, $1 \leq i \leq n$. Hence $z \in \mathfrak{N}$, which shows that $\mathfrak{X} = \mathfrak{M} + \mathfrak{N}$.

If $x \in \mathfrak{M} \cap \mathfrak{N}$, write $x = \sum_{i=1}^n \lambda_i e_i$. Since $x \in \mathfrak{N}$, we have $0 = \phi_j(x) = \sum_{i=1}^n \lambda_i \phi_j(e_i) = \sum_{i=1}^n \lambda_i \delta_{ij} = \lambda_j$, $1 \leq j \leq n$. Thus $x = 0$; that is, $\mathfrak{M} \cap \mathfrak{N} = \{0\}$, and so $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}$.

□

6.26. Proposition. *Let $\mathfrak{X}$ be a Banach space, and $K \in \mathcal{K}(\mathfrak{X})$. Suppose $0 \neq \lambda \in \mathbb{C}$. Then*

(i) $\mathfrak{M} = \ker(\lambda I - K)$ *is finite dimensional;*
(ii) $\mathfrak{R} = \operatorname{ran}(\lambda I - K)$ *is a closed subspace of $\mathfrak{X}$;*
(iii) $\dim(\mathfrak{X}/\mathfrak{R}) = \dim \ker(\lambda I - K^*) < \infty$.

Proof.

(i) Clearly $\mathfrak{M}$ is a closed subspace of $\mathfrak{X}$, and hence a Banach space itself. Consider
$$\begin{array}{rccc} K_0 : & \mathfrak{M} & \to & \mathfrak{X} \\ & x & \mapsto & Kx \ (= \lambda x). \end{array}$$
Then K_0 is compact. Moreover, $K_0(\mathfrak{M}) = \mathfrak{M}$ is closed. By Proposition 6.18, $\mathfrak{M}$ is finite-dimensional.

(ii) From above, $\mathfrak{M}$ is closed and finite dimensional, and so we can find $\mathfrak{N} \subseteq \mathfrak{X}$, a closed subspace such that $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}$. Consider

$$\begin{array}{rccc} T: & \mathfrak{N} & \to & \mathfrak{X} \\ & y & \mapsto & (\lambda I - K)y, \end{array}$$

(i.e. $T = (\lambda I - K)|_{\mathfrak{N}}$).

We claim that T is bounded below, for otherwise, there exists a sequence $(y_n)_n$ of norm-one vectors such that $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty}(\lambda I - K)y_n = 0$.

Moreover, since K is compact, there exists a subsequence $(y_{n(j)})_j$ such that $\lim_{j\to\infty} Ky_{n(j)} = z \in \mathfrak{X}$ exists. But then

$$\begin{aligned} \lim_{j\to\infty} (\lambda I - K)y_{n(j)} &= \lim_{j\to\infty} \lambda y_{n(j)} - \lim_{j\to\infty} Ky_{n(j)} \\ &= \lim_{j\to\infty} \lambda y_{n(j)} - z \\ &= 0, \end{aligned}$$

and so $\lambda^{-1}z = \lim_{j\to\infty} y_{n(j)}$. Moreover, $z \in \mathfrak{N}$, since $\mathfrak{N}$ is closed. Then

$$\begin{aligned} (\lambda I - K)\lambda^{-1}z &= \lambda(\lambda^{-1}z) - K(\lambda^{-1})z \\ &= z - K(\lim_{j\to\infty} y_{n(j)}) \\ &= z - z \\ &= 0, \end{aligned}$$

so that $\lambda^{-1}z$ and hence $z \in \ker(\lambda I - K) = \mathfrak{M}$. But $z \in \mathfrak{M} \cap \mathfrak{N}$ implies $z = 0$, i.e.

$$0 = \|\lambda^{-1}z\| = \lim_{j\to\infty} y_{n(j)}.$$

This contradicts the fact that $\|y_{n(j)}\| = 1$ for all $j \geq 1$. The conclusion must be that T is bounded below.

As in the proof of Proposition 6.4, we find that $\operatorname{ran} T$ is closed. But $\operatorname{ran} T = \operatorname{ran}(\lambda I - K)$, so that the latter is closed as well.

(iii) First note that

$$\begin{aligned} \tau \in \ker(\lambda I - K^*) &\iff (\lambda I - K^*)(\tau)x = 0 \text{ for all } x \in \mathfrak{X} \\ &\iff \tau((\lambda I - K)x) = 0 \text{ for all } x \in \mathfrak{X} \\ &\iff \tau|_{\mathfrak{R}} = 0. \end{aligned}$$

We define the map

$$\begin{array}{rccc} \Phi: & \ker(\lambda I - K^*) & \to & (\mathfrak{X}/\mathfrak{R})^* \\ & \tau & \mapsto & \Phi(\tau), \end{array}$$

where $\Phi(\tau)(x + \mathfrak{R}) := \tau(x)$.

If $x + \mathfrak{R} = y + \mathfrak{R}$, then

$$\begin{aligned} \Phi(\tau)(x + \mathfrak{R}) - \Phi(\tau)(y + \mathfrak{R}) &= \tau(x) - \tau(y) \\ &= \tau(x - y) \qquad (\text{but } x - y \in \mathfrak{R}) \\ &= 0, \end{aligned}$$

so that $\Phi(\tau)$ is well-defined.

We wish to show that Φ is an isomorphism of $\ker(\lambda I - K^*)$ onto $(\mathfrak{X}/\mathfrak{R})^*$. Since K is compact implies that K^* is compact, using (i) above for K^* will then imply that $\ker(\lambda I - K^*)$ is finite dimensional, so that $(\mathfrak{X}/\mathfrak{R})^*$ will be as well.

Suppose $0 \neq \tau \in \ker(\lambda I - K^*)$. Then, since $\tau|_{\mathfrak{R}} = 0$ and $\tau \neq 0$, we can find $x \in \mathfrak{X} \backslash \mathfrak{R}$ such that $\tau(x) \neq 0$. Then $\Phi(\tau)(x + \mathfrak{R}) = \tau(x) \neq 0$, so that $\Phi(\tau) \neq 0$. In particular, Φ is injective.

If $\overline{\phi} \in (\mathfrak{X}/\mathfrak{R})^*$ and $\pi : \mathfrak{X} \to (\mathfrak{X}/\mathfrak{R})$ is the canonical map, then define $\phi \in \mathfrak{X}^*$ via $\phi = \overline{\phi} \circ \pi$. Clearly $\phi|_{\mathfrak{R}} = 0$, so that $\phi \in \ker(\lambda I - K^*)$. Finally,

$$\begin{aligned}\Phi(\phi)(x + \mathfrak{R}) &= \phi(x) \\ &= \overline{\phi}(\pi(x)) \\ &= \overline{\phi}(x + \mathfrak{R})\end{aligned}$$

so that $\Phi(\phi) = \overline{\phi}$, and hence Φ is surjective. Thus Φ is an isomorphism, and so from above, we conclude that

$$\dim(\mathfrak{X}/\mathfrak{R}) = \dim(\mathfrak{X}/\mathfrak{R})^* = \dim\ker(\lambda I - K^*)$$

is finite.

$\square$

6.27. Remark. The above proof actually shows that if $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, and $\operatorname{ran} T$ is closed, then

$$\ker T^* \simeq (\mathfrak{Y}/\operatorname{ran} T)^*.$$

Compactness was only used to show that this was finite in the case we were considering.

6.28. Let $\mathfrak{X}$ be a Banach space and $T \in \mathcal{B}(\mathfrak{X})$. Then associated to T are two linearly ordered sequences of linear manifolds:

$$\mathcal{C}_a = \{\ker T^n\}_{n=1}^{\infty} \text{ and } \mathcal{C}_d = \{\operatorname{ran} T^n\}_{n=1}^{\infty}.$$

Definition. *Let $\mathfrak{X}$ be a Banach space and $T \in \mathcal{B}(\mathfrak{X})$. If $\ker T^n \neq \ker T^{n+1}$ for all $n \geq 0$, then T is said to have **infinite ascent**, and we write $\operatorname{asc} T = \infty$. Otherwise, we set $\operatorname{asc} T = p$, where p is the least non-negative integer such that $\ker T^p = \ker T^n$, $n \geq p$.*

*If $\operatorname{ran} T^n \neq \operatorname{ran} T^{n+1}$ for all $n \geq 0$, then T is said to have **infinite descent**, and we write $\operatorname{desc} T = \infty$. Otherwise, we set $\operatorname{desc} T = q$, where q is the least non-negative integer such that $\operatorname{ran} T^q = \operatorname{ran} T^n$, $n \geq q$.*

6.29. Lemma. *Let $\mathfrak{X}$ be a Banach space and $K \in \mathcal{K}(\mathfrak{X})$. Suppose we can find $(\lambda_n)_n \subseteq \mathbb{C}$ and a sequence $(V_n)_n$ of closed subspaces of $\mathfrak{X}$ satisfying:*

(i) $V_n \subset V_{n+1}$ *for all $n \geq 1$, where $'' \subset ''$ denotes* proper *containment;*
(ii) $KV_n \subseteq V_n$ *for all $n \geq 1$;*
(iii) $(K - \lambda_n)V_n \subseteq V_{n-1}$, *for all $n \geq 1$.*

Then $\lim_{n\to\infty} \lambda_n = 0$. Moreover, the same conclusion holds if $(W_n)_n$ is a sequence of closed subspaces of $\mathfrak{X}$ satisfying:

(iv) $W_n \supset W_{n+1}$ *for all $n \geq 1$, where $'' \supset ''$ denotes* proper *containment;*
(v) $KW_n \subseteq W_n$ *for all $n \geq 1$;*
(vi) $(K - \lambda_n)W_n \subseteq W_{n+1}$, *for all $n \geq 1$.*

Proof. Let $\overline{z_n} \in V_n/V_{n-1}$, $\|\overline{z_n}\| = 1/2$ and choose $x_n \in V_n$ such that $\overline{x_n} = \overline{z_n}$. Since $\|\overline{z_n}\| = \inf\{\|x_n + y\| : y \in V_{n-1}\}$, we can find $y_n \in V_{n-1}$ such that if we let $w_n = x_n + y_n$, then $\overline{w_n} = \overline{z_n}$ and $1/2 \leq \|w_n\| < 1$.

Then $w_n \in V_n$, so $(K - \lambda_n)w_n \in V_{n-1}$. That is, $Kw_n = \lambda_n w_n + v_{n-1}$ for some $v_{n-1} \in V_{n-1}$.

If $m < n$,

$$\begin{aligned}\|Kw_n - Kw_m\| &= \|\lambda_n w_n + (v_{n-1} - \lambda_m w_m - v_{m-1})\| \\ &\geq \inf\{\|\lambda_n w_n + y\| : y \in V_{n-1}\} \\ &= |\lambda_n|\,\|\overline{w_n}\| \\ &= |\lambda_n|/2.\end{aligned}$$

Suppose $\lim_{n\to\infty} \lambda_n \neq 0$. Find $(\lambda_{n(j)})_j$ such that $\inf\{|\lambda_{n(j)}| : j \geq 1\} = \delta > 0$. Then $(Kw_{n(j)})_j$ has no convergent subsequence, although $(w_{n(j)})_j$ is bounded. This contradicts the compactness of K. Thus $\lim_{n\to\infty} \lambda_n = 0$.

The second statement is proven in a similar fashion.

□

6.30. Lemma. *Let $\mathfrak{X}$ be a Banach space, $K \in \mathcal{K}(\mathfrak{X})$, and $0 \neq \lambda \in \mathbb{C}$. Then* $\operatorname{ran}(\lambda I - K)^n$ *is closed for all $n \geq 0$.*

Proof. Exercise.

□

6.31. Theorem. *Let K be a compact operator on a Banach space $\mathfrak{X}$ and suppose $0 \neq \lambda \in \mathbb{C}$. Then $(\lambda I - K)$ has finite ascent and finite descent.*

Proof. Suppose that $(K - \lambda I)$ has infinite ascent. Then we can apply Lemma 6.29 with $\lambda_n = \lambda$ for all $n \geq 1$ and $V_n = \ker(K - \lambda I)^n$, $n \geq 1$ to conclude that $\lim_{n\to\infty} \lambda_n = \lambda = 0$, a contradiction. Thus $(K - \lambda I)$ has finite ascent.

Similarly, if $(K - \lambda I)$ has infinite descent, then putting $\lambda_n = \lambda$ and putting $W_n = \operatorname{ran}(K - \lambda I)^n$ for all $n \geq 1$ again implies that $\lim_{n\to\infty} \lambda_n = \lambda = 0$, a contradiction. Thus $(K - \lambda I)$ has finite descent.

□

6.32. Theorem. [**The Fredholm Alternative**] *Let $\mathfrak{X}$ be a Banach space and let $K \in \mathcal{K}(\mathfrak{X})$. Suppose $0 \neq \lambda \in \mathbb{C}$. Then $(\lambda I - K)$ is injective if and only if it is surjective.*

Proof. First assume that $(\lambda I - K)$ is surjective, and suppose that it is not injective. Let $V_n = \ker(\lambda I - K)^n$ for each $n \geq 1$. Each $(\lambda I - K)^n$ is onto. Let $0 \neq y \in \ker(\lambda I - K)$ and let $x \in \mathfrak{X}$ such that $y = (\lambda I - K)^n x$. Then $x \in V_{n+1}$ but $x \notin V_i$, $1 \leq i \leq n$. That is, V_n is a proper subset of V_{n+1} for all $n \geq 1$. But $(\lambda I - K)$ has finite ascent, by Theorem 6.31, a contradiction. Thus $(\lambda I - K)$ is injective.

Now assume that $(\lambda I - K)$ is injective. Let $\mathfrak{M} = \operatorname{ran}(\lambda I - K)$. By Proposition 6.26, $\mathfrak{M}$ is closed. Consider the operator

$$\begin{array}{rccc} R: & \mathfrak{X} & \to & \mathfrak{M} \\ & x & \mapsto & (\lambda I - K)x. \end{array}$$

Then R is bijective and so by Proposition 6.4, R is invertible. Moreover, $R^* : \mathfrak{M}^* \to \mathfrak{X}^*$ is invertible, and hence surjective.

Take $x^* \in \mathfrak{X}^*$ and choose $m^* \in \mathfrak{M}^*$ such that $R^* m^* = m^* \circ R = x^*$. We can extend m^* to a functional $x_m^* \in \mathfrak{X}^*$ by the Hahn-Banach Theorem. Then for all $x \in \mathfrak{X}$,

$$\begin{aligned} ((\lambda I - K)^* x_m^*)(x) &= x_m^*((\lambda I - K)x) \\ &= (m^* \circ R)x \\ &= x^*(x). \end{aligned}$$

Thus $(\lambda I - K)^* x_m^* = x^*$, showing that $(\lambda I - K)^*$ is surjective. From the first half of the proof, it follows that $(\lambda I - K)^*$ is injective, and hence invertible. But then $(\lambda I - K)$ is invertible, and therefore surjective.

□

6.33. Corollary. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space and $K \in \mathcal{K}(\mathfrak{X})$. Then $\sigma(K) = \{0\} \cup \sigma_p(K)$.*

Note that the presence of 0 in the spectrum of K is unavoidable, since K lies in the proper ideal $\mathcal{K}(\mathfrak{X})$, and hence can not be invertible. Also, recall that eigenvectors corresponding to distinct eigenvalues of a linear operator $T \in \mathcal{B}(\mathfrak{X})$ are linearly independent.

6.34. Theorem. *Let $\mathfrak{X}$ be an infinite dimensional Banach space and $K \in \mathcal{K}(\mathfrak{X})$. Then for all $\epsilon > 0$, $\sigma(K) \cap \{z \in \mathbb{C} : |z| > \epsilon\}$ is finite. In other words, $\sigma(K)$ is a sequence of eigenvalues of finite multiplicity, and this sequence must converge to* 0.

Proof. Let $\epsilon > 0$. Suppose $\sigma(K) \cap \{z \in \mathbb{C} : |z| > \epsilon\}$ contains a sequence $(\lambda_n)_n$ with $\lambda_i \neq \lambda_j$, $1 \leq i \neq j < \infty$. Let $(v_n)_n$ be eigenvectors corresponding to $(\lambda_n)_n$ and for each $n \geq 1$, let $V_n = \operatorname{span}_{1 \leq k \leq n}\{v_k\}$.

Then $(V_n)_n$ and $(\lambda_n)_n$ satisfy the conditions of Lemma 6.29. We conclude from that Lemma that $\lim_{n\to\infty} \lambda_n = 0$, a contradiction.

Thus $\sigma(K) = \{0\} \cup \{\lambda_n\}_{n=1}^r$, where r is either finite or $\aleph_0$. Moreover, each λ_n is an eigenvalue of K, and $\lim_{n\to\infty} \lambda_n = 0$ when r is not finite.

□

Supplementary Examples

S6.1. Example. Let $\mathfrak{X} = \mathcal{C}([0,1])$, and consider $V \in \mathcal{B}(\mathfrak{X})$ given by

$$(Vf)(x) = \int_0^1 k(x,y) f(y) dy,$$

where

$$k(x,y) = \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x \geq y. \end{cases}$$

Then $(Vf)(x) = \int_0^x f(y)dy$. This is an example of a **Volterra operator**. The function $k(x,y)$ is referred to as the **kernel** of the integral operator. This should not be confused with the notion of a null space, also referred to as a *kernel*.

We wish to determine the spectrum of V. Now

$$\begin{aligned} (V^2 f)(x) &= (V(Vf))(x) \\ &= \int_0^1 k(x,t)\,(Vf)(t)dt \\ &= \int_0^1 k(x,t) \int_0^1 k(t,y)\, f(y) dy\, dt \\ &= \int_0^1 f(y) \int_0^1 k(x,t)\, k(t,y) dt\, dy \\ &= \int_0^1 f(y)\, k_2(x,y) dy, \end{aligned}$$

where $k_2(x,y) = \int_0^1 k(x,t)\, k(t,y) dt$ is a new kernel. Note that

$$\begin{aligned} |k_2(x,y)| &= |\int_0^1 k(x,t)\, k(t,y) dt| \\ &= |\int_y^x k(x,t)\, k(t,y) dt| \\ &= (x-y) \text{ for } x > y, \end{aligned}$$

while for $x < y$, $k_2(x,y) = 0$.

In general, since $x - y < 1 - 0 = 1$, we get

$$\begin{aligned} (V^n f)(x) &= \int_0^1 f(y)\, k_n(x,y) dy \\ k_n(x,y) &= \int_0^1 k(x,t)\, k_{n-1}(t,y) dt \\ |k_n(x,y)| &\leq \frac{1}{(n-1)!}(x-y)^{n-1} \leq \frac{1}{(n-1)!}. \end{aligned}$$

Thus if we take $\|f\| \leq 1$, then

$$\begin{aligned} \|V^n\| &= \sup_{\|f\|=1} \|V^n f\| \\ &= \sup_{\|f\|=1} \| \int_0^1 f(y)\, k_n(x,y) dy \| \\ &\leq \sup_{\|f\|=1} \|f\| \, |k_n(x,y)| \\ &\leq 1/(n-1)!. \end{aligned}$$

Thus $\operatorname{spr}(V) = \lim_{n\to\infty} \|V^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} (1/n!)^{\frac{1}{n}} = 0$. In particular, $\sigma(V) = \{0\}$.

Now let $f_n(x) = x^n$, $0 \leq x \leq 1$. Then $\|f_n\|_\infty = 1$. Also,

$$\begin{aligned} (Vf_n)(x) &= \int_0^x f_n(y) dy \\ &= \int_0^x y^n dy \\ &= \frac{y^{n+1}}{(n+1)}\Big|_0^x \\ &= \frac{x^{n+1}}{n+1}. \end{aligned}$$

As such, $\|Vf_n\| = \frac{1}{n+1}$, and so V is not bounded below. Hence $0 \in \sigma_a(V)$.

Finally, let $f \in \mathcal{C}([0,1],\mathbb{C})$ be arbitrary. Then $(Vf)(0) = 0$, and so

$$\|1 - Vf\|_\infty \geq |1(0) - Vf(0)| = 1.$$

Thus $0 \in \sigma_c(V)$. It is a standard result that $0 \notin \sigma_p(V)$.

S6.2. Example. The following example is very similar to Example 6.7.

Let $\mathfrak{X} = \mathcal{C}([0,1],\mathbb{C})$. Let $f \in \mathfrak{X}$, and consider the bounded linear operator M_f given by

$$\begin{aligned} M_f : \ \mathcal{C}([0,1],\mathbb{C}) &\to \mathcal{C}([0,1],\mathbb{C}) \\ g &\mapsto fg. \end{aligned}$$

M_f is referred to as "multiplication by f". We leave it to the reader to verify that

- $\lambda I - M_f = M_{\lambda 1 - f}$ for all $\lambda \in \mathbb{C}$, and
- $\|M_f\| = \|f\|$.

Claim: $\sigma(M_f) = \operatorname{ran} f = f([0,1])$.

For if $\lambda \notin f([0,1])$, then $h = (\lambda 1 - f)^{-1}$ is continuous and $M_h(\lambda I - M_f) = M_h M_{\lambda 1 - f} = M_{h(\lambda 1 - f)} = M_1 = I = (\lambda I - M_f) M_h$. In particular, $\lambda \notin \sigma(M_f)$.

Now suppose $\lambda = f(t_0)$ for some $t_0 \in [0,1]$. Take

$$g_n(t) = \begin{cases} 0 & \text{if } |t - t_0| > \frac{1}{n}, \\ 1 - n|t - t_0| & \text{if } t \in [t_0 - \frac{1}{n}, t_0 + \frac{1}{n}]. \end{cases}$$

(See Figure 1 below.)

Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|f(t) - \lambda| < \varepsilon$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Then, when $\frac{1}{n} < \delta$,

$$\begin{aligned}
\|M_{\lambda 1 - f} g_n\| &= \|(\lambda 1 - f) g_n\| \\
&\leq \sup_{|t - t_0| < \frac{1}{n}} |\lambda g_n(t) - f(t) g_n(t)| \\
&\leq \sup_{|t - t_0| < \delta} |\lambda 1 - f(t)| \, |g_n(t)| \\
&\leq \varepsilon \|g_n(t)\|_\infty \\
&= \varepsilon.
\end{aligned}$$

Since $\|g_n\|_\infty = 1$ for $n \geq 1$, we see that $\lambda I - M_f$ is not bounded below. In other words, $\lambda \in \sigma_a(M_f)$.

Moreover, if $\lambda = f(t_0)$ for some $t_0 \in [0, 1]$, then

$$\begin{aligned}
\|1 - (\lambda I - M_f) g\| &= \|1 - (\lambda 1 - f) g\| \\
&\geq |1(t_0) - (\lambda - f(t_0)) g(t_0)| \\
&= 1.
\end{aligned}$$

Thus the range of $\lambda I - M_f$ is not dense; i.e. $\lambda \in \sigma_c(M_f)$.

Suppose now that $\lambda \in \sigma_p(M_f)$. Then $(\lambda I - M_f) g = M_{\lambda 1 - f} g = 0$ for some non-zero continuous function g. It follows that

$$(\lambda - f(t)) g(t) = 0 \text{ for all } t \in [0, 1].$$

Since $g \neq 0$, we can choose $t_0 \in [0, 1]$ such that $g(t_0) \neq 0$. Since g is continuous, there exists an open neighbourhood U of t_0 such that $g(t) \neq 0$ for all $t \in U$. But then $\lambda - f(t) = 0$ for all $t \in U$. We conclude that if $\lambda \in \sigma_p(M_f)$, then f must be constant on some interval. We leave it to the reader to check that the converse is also true.

In particular, if we choose $f(x) = x$ for all $x \in [0, 1]$ and write M_x for M_f (as is usually done), then we see that

$$M_x : \mathcal{C}([0, 1]) \to \mathcal{C}([0, 1])$$

has no eigenvalues!

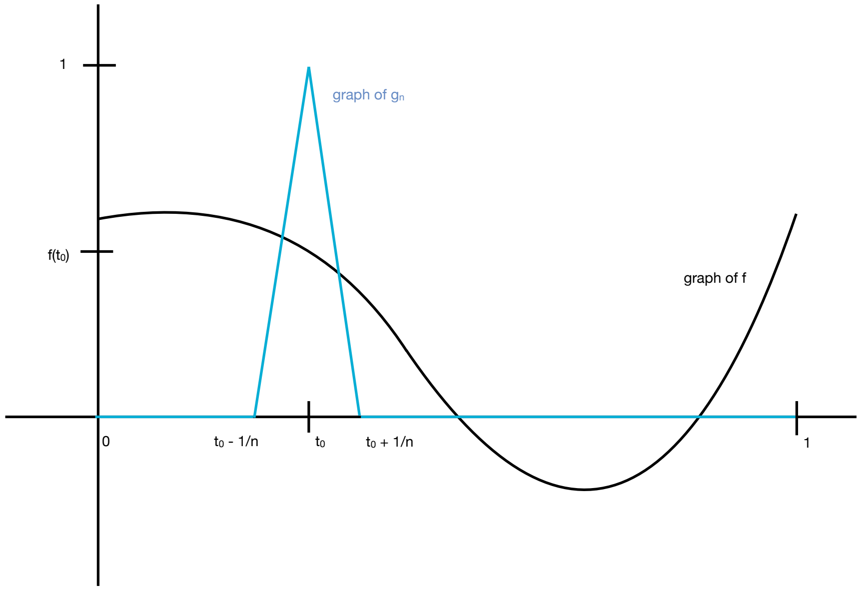

Figure 1. The graphs of g_n and of f.

Appendix

A6.1. In Chapter 2, we defined the notion of an abstract index group corresponding to a Banach algebra $\mathcal{A}$. The notion of the Fredholm index is very closely related to this, although we shall not have the time to delve into this in these notes.

A6.2. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. An operator $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is said to be* ***Fredholm*** *if:*

(i) $\operatorname{ran} T$ *is closed;*
(ii) $\operatorname{nul} T = \dim \ker T$ *is finite; and*
(iii) $\operatorname{nul} T^* = \operatorname{codim} \operatorname{ran} T$ *is finite.*

Given $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ a Fredholm operator, we define the ***Fredholm index*** *of T as follows:*

$$\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*.$$

A6.3. Example. Let $\mathfrak{X}$ be a Banach space, $K \in \mathcal{K}(\mathfrak{X})$ and $0 \neq \lambda \in \mathbb{C}$. Then $\lambda I - K$ is Fredholm.

In fact, we shall now show that $\operatorname{ind}(\lambda I - K) = 0$. We shall then return to Fredholm operators when we study $\mathcal{K}(\mathcal{H})$, the set of compact operators on a Hilbert space $\mathcal{H}$.

A6.4. Example. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Define an operator $U \in \mathcal{B}(\mathcal{H})$ via $Ue_n = e_{n+1}$ for all $n \geq 1$. (We extend this definition by linearity and continuity to all of $\mathcal{H}$. U is referred to as *the unilateral forward shift.* Then U is an isometry with range equal to the span of $\{e_n\}_{n=2}^\infty$. As such, the range of U is closed, the nullity of U is zero, and the codimension of the range of U is 1. Hence U is a Fredholm operator of index -1. We shall return to this example later.

A6.5. Lemma. *Let $\mathfrak{X}$ be a Banach space and $\mathfrak{M}$ be a finite codimensional subspace of $\mathfrak{X}$. Then there exists a finite dimensional subspace $\mathfrak{N}$ of $\mathfrak{X}$ such that $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}$. Moreover,* $\dim \mathfrak{N} = \dim(\mathfrak{X}/\mathfrak{M})$.

Proof. Let $\{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\}$ be a basis for $\mathfrak{X}/\mathfrak{M}$, and choose $\{x_1, x_2, \ldots, x_n\} \subseteq \mathfrak{X}$ such that $\pi(x_j) = \overline{x_j}$, $1 \leq j \leq n$, where $\pi : \mathfrak{X} \to \mathfrak{X}/\mathfrak{M}$ is the canonical map.

Let $\mathfrak{N} = \overline{\operatorname{span}}\{x_1, x_2, \ldots, x_n\} = \operatorname{span}\{x_1, x_2, \ldots, x_n\}$. If $z \in \mathfrak{M} \cap \mathfrak{N}$, then $z \in \mathfrak{N}$ so that $z = \sum_{i=1}^n \lambda_i x_i$. But $z \in \mathfrak{M}$ and so $\overline{z} = 0 = \sum_{i=1}^n \lambda_i \overline{x_i}$. Thus $\lambda_i = 0$ for all i and hence $z = 0$. In other words, $\mathfrak{M} \cap \mathfrak{N} = \{0\}$.

Now let $x \in \mathfrak{X}$. Then $\overline{x} = \sum_{i=1}^n \lambda_i \overline{x_i}$ and so $x = \sum_{i=1}^n \lambda_i x_i + y$ for some $y \in \mathfrak{M}$. Therefore $\mathfrak{X} = \mathfrak{M} + \mathfrak{N}$, so that $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}$.

□

A6.6. If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is Fredholm, then there exists a closed subspace $\mathfrak{N}$ of $\mathfrak{Y}$ such that

$$\mathfrak{Y} = \operatorname{ran} T \oplus \mathfrak{N}.$$

Moreover, $\dim \mathfrak{N} = \operatorname{codim}\ \mathrm{ran\,T} < \infty$.

A6.7. Theorem. *Let $\mathfrak{X}$ be a Banach space, $K \in \mathcal{K}(\mathfrak{X})$, and $0 \neq \lambda \in \mathbb{C}$. Then*

$$\operatorname{ind}(\lambda I - K) = 0.$$

Proof. Let $\mathfrak{M} = \ker(\lambda I - K)$. Then $\dim \mathfrak{M} < \infty$ and so $\mathfrak{M}$ has a closed complement $\mathfrak{N} \subseteq \mathfrak{X}$ such that $\mathfrak{M} \oplus \mathfrak{N} = \mathfrak{X}$. Let $\mathfrak{R} = \operatorname{ran}(\lambda I - K)$. Then $\mathfrak{R}$ is closed and finite codimensional, so by Lemma A6.5, $\mathfrak{R}$ has a closed complement $\mathfrak{S} \subseteq \mathfrak{X}$ satisfying $\mathfrak{R} \oplus \mathfrak{S} = \mathfrak{X}$. Let $n = \min(\dim \mathfrak{M}, \dim \mathfrak{S})$.

Choose $\overline{\phi_1}, \overline{\phi_2}, \ldots, \overline{\phi_n}$ linearly independent in $(\mathfrak{X}/\mathfrak{N})^*$. Let $\pi : \mathfrak{X} \to \mathfrak{X}/\mathfrak{N}$ be the canonical map and define $\phi_i = \overline{\phi_i} \circ \pi$ so that $\phi_i \in \mathfrak{X}^*$, $1 \leq i \leq n$. Choose $\{f_i\}_{i=1}^n$ linearly independent in $\mathfrak{S}$. We shall define $Q \in \mathcal{K}(\mathfrak{X})$ via $Qx = \sum_{i=1}^n \phi_i(x) f_i$, $x \in \mathfrak{X}$.

Then

$$K - Q \in \mathcal{K}(\mathfrak{X}) \text{ and } \ (\lambda I - (K - Q)) = (\lambda I - K) + Q$$

is either surjective (if $n = \dim \mathfrak{S}$), or injective (if $n = \dim \mathfrak{M}$). In either case, by the Fredholm Alternative, it is bijective.

We conclude that $\dim \mathfrak{M} = \dim \mathfrak{S}$, so that $\operatorname{nul}(\lambda I - K) = \operatorname{codim} \operatorname{ran}(\lambda I - K)$, or equivalently,

$$\operatorname{ind}(\lambda I - K) = 0.$$

□

A6.8. We conclude this section by showing that although not all finite-co-dimensional subspaces of a Banach space are closed, nevertheless, this is true for operator ranges. In particular, this means that in order to know if an operator T is Fredholm, one need only verify that the nullity and the co-dimension of the range are finite.

A6.9. Proposition. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and let $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Suppose that $\mathfrak{N}$ is a closed subspace such that $\operatorname{ran} T \oplus \mathfrak{N}$ is closed in $\mathfrak{Y}$. Then $\operatorname{ran} T$ is closed.

Proof. Define a norm on the space $(\mathfrak{X}/\ker T) \times \mathfrak{N}$ by $\|(\overline{x}, n)\| = \|\overline{x}\| + \|n\|$. Let T_0 denote the operator

$$\begin{aligned} T_0 : \quad (\mathfrak{X}/\ker T) \times \mathfrak{N} &\to \operatorname{ran} T \oplus \mathfrak{N} \\ (\overline{x}, n) &\mapsto Tx + n. \end{aligned}$$

It is easy to check that T_0 is well-defined, continuous, injective and that $\operatorname{ran} T_0$ is $\operatorname{ran} T \oplus \mathfrak{N}$. Since $\operatorname{ran} T \oplus \mathfrak{N}$ is closed, T_0 is invertible. This means that we can find $\delta > 0$ such that $\|Tx + n\| \geq \delta \|(\overline{x}, n)\|$. That is, $\|Tx + n\| \geq \delta (\|\overline{x}\| + \|n\|)$.

If $n = 0$, we get $\|Tx\| \geq \delta \|\overline{x}\|$, and so the restriction T_1 of T_0 to $(\mathfrak{X}/\ker T)$ is bounded below. But then $\operatorname{ran} T_1 = \operatorname{ran} T$ is closed, as pointed out in Proposition 6.4.

□

A6.10. Corollary. If $\mathfrak{X}$ is a Banach space, $T \in \mathcal{B}(\mathfrak{X})$, and $\mathfrak{X}/\operatorname{ran} T$ is finite dimensional, then $\operatorname{ran} T$ is closed.

Proof. By Lemma A6.5, there exists a finite-dimensional (and therefore closed) subspace of $\mathfrak{X}$ such that $\operatorname{ran} T \oplus \mathfrak{N} = \mathfrak{X}$. Since $\mathfrak{X}$ is obviously closed, we may now apply the above Proposition A6.9 to conclude that $\operatorname{ran} T$ is closed, as desired.

$\square$

Exercises for Chapter 6

Exercise 6.1. The spectral radius

Let $\mathfrak{X}$ be a Banach space, and suppose that $A, B \in \mathcal{B}(\mathfrak{X})$ commute. Prove that

$$\operatorname{spr}(AB) \le \operatorname{spr}(A) \operatorname{spr}(B).$$

Exercise 6.2. Cesaro operators

Let $\mathfrak{X}$ be a Banach space. An invertible operator $T \in \mathcal{B}(\mathfrak{X})$ is said to be a **Cesaro operator** if there exists a real number $M > 0$ such that

$$\|T^n\| \le M \text{ for all } n \in \mathbb{Z}.$$

Prove that if T is a Cesaro operator, then $\sigma(T) \subseteq \mathbb{T}$.

Exercise 6.3. Nilpotent and quasinilpotent operators

Let $\mathfrak{X}$ be a Banach space and $Q \in \mathcal{B}(\mathfrak{X})$. Prove that the following are equivalent:

(a) Q is nilpotent; that is, there exists $k \ge 1$ such that $Q^k = 0$; and

(b) Q is quasinilpotent and has finite descent.

Exercise 6.4. The holomorphic functional calculus and invariant subspaces

Let $\mathfrak{X}$ be a Banach space, and $T \in \mathcal{B}(\mathfrak{X})$. Let Ω denote the (unique) unbounded component of the resolvent $\rho(T)$ of T. Recall that a (closed) subspace $\mathfrak{Y}$ of $\mathfrak{X}$ is said to be **invariant** for T if $Ty \in \mathfrak{Y}$ for all $y \in \mathfrak{Y}$. We normally abbreviate this to $T\mathfrak{Y} \subseteq \mathfrak{Y}$.

(a) Prove that $\mathfrak{Y}$ is invariant for T if and only if $\mathfrak{Y}$ is invariant for $R(z, T)$ for all $z \in \Omega$.

(b) Let $\mathfrak{Y}$ be invariant for T, and denote by $T|_{\mathfrak{Y}}$ the restriction of T to $\mathfrak{Y}$. Prove that $\Omega \subseteq \rho(T|_{\mathfrak{Y}})$, and that $R(z, T|_{\mathfrak{Y}}) = R(z, T)|_{\mathfrak{Y}}$ for all $z \in \Omega$.

(c) If $\Omega = \rho(T)$, prove that $\sigma(T|_{\mathfrak{Y}}) \subseteq \sigma(T)$.

Exercise 6.5. The holomorphic functional calculus and invariant subspaces

Using the notation from Exercise 6.4 above, suppose that Ω contains the open annulus $A_{r_1,r_2}(\lambda_0)$, and that

$$R(z, T) = \sum_{n \in \mathbb{Z}} (z - \lambda_0)^n T_n$$

is the Laurent™ series expansion of the resolvent function (see Exercise 3.6). Prove that if the closed subspace $\mathfrak{Y}$ is invariant for T, then $\mathfrak{Y}$ is invariant for each T_n, $n \in \mathbb{Z}$.

CHAPTER 7

The algebra of Hilbert space operators

He's very clever, but sometimes his brains go to his head.

Margot Asquith

Introduction.

7.1. We now consider the special case where the Banach space under consideration is in fact a Hilbert space, which we shall always typically denote by $\mathcal{H}$. The inner product on $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle$.

All of the results from the previous chapter of course apply to Hilbert space operators. On the other hand, the identification of a Hilbert space with its dual space (an anti-isomorphism) allows us to consider a new version of adjoints, based upon the Riesz Representation Theorem.

7.2. Theorem. **[The Riesz Representation Theorem]** *Let $\mathcal{H}$ be a Hilbert space and $\varphi \in \mathcal{H}^*$. Then there exists a vector $y \in \mathcal{H}$ such that $\varphi(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$.*

7.3. Theorem. *Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. Then there exists a unique operator $T^* \in \mathcal{B}(\mathcal{H})$, called the **Hilbert space adjoint** of T, satisfying*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for all $x, y \in \mathcal{H}$.

Proof. Fix $y \in \mathcal{H}$. Then the map

$$\begin{array}{rccc} \varphi_y : & \mathcal{H} & \to & \mathbb{C} \\ & x & \mapsto & \langle Tx, y \rangle \end{array}$$

is a linear functional and so there exists a vector $z_y \in \mathcal{H}$ such that

$$\varphi_y(x) = \langle Tx, y \rangle = \langle x, z_y \rangle$$

for all $x \in \mathcal{H}$. Define a map $T^* : \mathcal{H} \to \mathcal{H}$ by $T^* y = z_y$. We leave it to the reader to verify that T^* is in fact linear, and we concentrate on showing that it is bounded.

To see that T^* is bounded, consider the following. Let $y \in \mathcal{H}$, $\|y\| = 1$. Then $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for all $x \in \mathcal{H}$, so

$$\begin{aligned}\|T^* y\|^2 &= \langle T^* y, T^* y\rangle \\ &= \langle T T^* y, y\rangle \\ &\leq \|T\| \, \|T^* y\| \, \|y\|.\end{aligned}$$

Thus $\|T^* y\| \leq \|T\|$, and so $\|T^*\| \leq \|T\|$.

T^* is unique, for if there exists $A \in \mathcal{B}(\mathcal{H})$ such that $\langle Tx, y\rangle = \langle x, T^* y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathcal{H}$, then $\langle x, (T^* - A)y\rangle = 0$ for all $x, y \in \mathcal{H}$, and so $(T^* - A)y = 0$ for all $y \in \mathcal{H}$, i.e. $T^* = A$.

□

7.4. Corollary. *Let $T \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Then $(T^*)^* = T$. It follows that $\|T\| = \|T^*\|$.*
Proof. For all $x, y \in \mathcal{H}$, we get

$$\begin{aligned}\langle x, (T^*)^* y\rangle &= \langle T^* x, y\rangle \\ &= \overline{\langle y, T^* x\rangle} \\ &= \overline{\langle Ty, x\rangle} \\ &= \langle x, Ty\rangle,\end{aligned}$$

and so $(T^*)^* = T$. Applying Theorem 7.3, we get

$$\|T\| = \|(T^*)^*\| \leq \|T^*\| \leq \|T\|,$$

and so $\|T\| = \|T^*\|$.

□

7.5. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma(T) = \sigma(T^*)^* := \{\overline{\lambda} : \lambda \in \sigma(T)\}$.*
Proof. If $\lambda \notin \sigma(T)$, let $R = (\lambda - T)^{-1}$. For all $x, y \in \mathcal{H}$,

$$\begin{aligned}\langle x, y\rangle &= \langle R(\lambda - T)x, y\rangle \\ &= \langle (\lambda - T)x, R^* y\rangle \\ &= \langle x, (\lambda - T)^* R^* y\rangle.\end{aligned}$$

Thus $(\lambda - T)^* R^* = I$, and similarly, $R^* (\lambda - T)^* = I$. But $(\lambda - T)^* = \overline{\lambda} - T^*$, so that $R^* = (\overline{\lambda} - T^*)^{-1} = [(\lambda - T)^*]^{-1}$. Thus $\rho(T)^* \subseteq \rho(T^*)$.

Moreover, $\rho(T^*)^* \subseteq \rho(T^{**}) = \rho(T)$. In other words, $\rho(T^*) \subseteq \rho(T)^*$. We conclude that $\sigma(T) = \sigma(T^*)^*$.

□

7.6. Remark. The above proof also shows that for a Hilbert space $\mathcal{H}$ and $A, B \in \mathcal{B}(\mathcal{H})$, we have $(AB)^* = B^* A^*$. The adjoint operator

$$* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

is an example of an *involution* on a Banach algebra. Namely, for all $\alpha, \beta \in \mathbb{C}$ and $A, B \in \mathcal{B}(\mathcal{H})$, we obtain

(i) $(\alpha A)^* = \overline{\alpha} A^*$;
(ii) $(A + B)^* = A^* + B^*$; and
(iii) $(AB)^* = B^* A^*$.
(iv) $(A^*)^* = A$.

Involutions will appear again in our study of C*-algebras.

7.7. Proposition. *Let $\mathcal{H} = \mathbb{C}^n$ and $T \in \mathcal{B}(\mathcal{H}) \simeq \mathbb{M}_n$. Then the matrix of T^* with respect to any orthonormal basis is the conjugate transpose of that of T.*

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $\mathcal{H}$. With respect to this basis, T has a matrix $[t_{ij}]_{1 \le i,j \le n}$ and T^* has a matrix $[r_{ij}]_{1 \le i,j \le n}$.

But $t_{ij} = \langle Te_j, e_i \rangle = \langle e_j, T^* e_i \rangle = \overline{\langle T^* e_i, e_j \rangle} = \overline{r_{ji}}$, completing the proof.

□

7.8. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then $(\operatorname{ran} T)^\perp = \ker T^*$. In particular, therefore:*

(i) $\overline{\operatorname{ran} T} = (\ker T^*)^\perp$;
(ii) *for* $\lambda \in \mathbb{C}$, $\lambda \in \sigma_c(T)$ *if and only if* $\overline{\lambda} \in \sigma_p(T^*)$;
(iii) $\operatorname{ran} T$ *is not dense in* $\mathcal{H}$ *if and only if* $\ker T^* \ne 0$.

Proof. Let $y \in \mathcal{H}$. Then

$$\begin{aligned} y \in \ker T^* &\iff \text{for all } x \in \mathcal{H},\ 0 = \langle x, T^* y \rangle \\ &\iff \text{for all } x \in \mathcal{H},\ 0 = \langle Tx, y \rangle \\ &\iff y \in (\operatorname{ran} T)^\perp. \end{aligned}$$

□

7.9. Example. Let $\mathcal{H}$ be a Hilbert space with orthomormal basis $\{e_n\}_{n=1}^\infty$. Define the operator $S \in \mathcal{B}(\mathcal{H})$ by first setting $Se_{n+1} = e_n$ for all $n \ge 1$ and $Se_1 = 0$, and then extending S by linearity and continuity to all of $\mathcal{H}$.

S is then called the **unilateral (backward) shift**, and with respect to the above basis for $\mathcal{H}$, the matrix for S is:

$$[S] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

It is easily verified that $\|S\| = 1$. As for the spectral radius of S, note that $\|S^n\| \le \|S\|^n \le 1$, while $\|S^n e_{n+1}\| = \|e_1\| = 1$, so that $\|S^n\| \ge 1$. Hence $\operatorname{spr}(S) = \lim_{n\to\infty} \|S^n\|^{\frac{1}{n}} = 1$.

Let $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Consider $(\lambda I - S)$. Let $x_n = (1/\sqrt{n}) \sum_{i=1}^n \lambda^i e_i$. Then $\|x_n\| = 1$ for all $n \ge 1$, and

$$\|(\lambda I - S)x_n\| = \|(1/\sqrt{n})\lambda^{n+1} e_n\| = 1/\sqrt{n}.$$

Letting n tend to ∞ yields $\lambda \in \sigma_a(S)$.

Now let $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$, and $y = \sum_{i=1}^\infty \lambda^i e_i$. Then

$$\begin{aligned}(\lambda I - S)y &= \sum_{i=1}^\infty \lambda^{i+1} e_i - \sum_{i=1}^\infty \lambda^{i+1} e_i \\ &= 0.\end{aligned}$$

As for $\lambda = 0$, e_1 lies in the kernel of S, and hence of $0 - S$. Hence $\sigma_p(S) \supseteq \{z \in \mathbb{C} : |z| < 1\}$. In particular, $\overline{\lambda} \in \sigma_c(S^*)$, i.e. $\sigma_c(S^*) \supseteq \{z \in \mathbb{C} : |z| < 1\}$.

Note: An easy calculation which is left as an exercise shows that $S^* e_n = e_{n+1}$, $n \ge 1$, and hence

$$[S^*] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Clearly, $\operatorname{ran} S = \mathcal{H}$, so $\ker S^* = \{0\}$. In fact, S^* is an isometry! Finally, $\sigma(S) = \{z \in \mathbb{C} : |z| \le 1\} = \sigma(S^*)$.

7.10. Definition. *Given an infinite dimensional, separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n\}_{n=1}^\infty$, a **unilateral (forward) weighted shift** W on $\mathcal{H}$ is an operator satisfying $We_n = w_n e_{n+1}$, $n \ge 1$, where $\{w_n\}_{n=1}^\infty \in \ell^\infty(\mathbb{N})$ is called the sequence of **weights** of W. The adjoint of a unilateral forward weighted shift is referred to as a **unilateral backward weighted shift**.*

*A **bilateral weighted shift** is an operator $V \in \mathcal{B}(\mathcal{H})$ such that $Vf_n = v_n f_{n+1}$ for all $n \in \mathbb{Z}$, where $\{f_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}$ and $\{v_n\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$.*

Weighted shifts are of interest because they provide one of the few tractable classes of operators which exhibit a reasonably wide variety of phenomena typical of more general operators. As such, they are an excellent test case for conjectures about general operators.

In the case where all of the weights are constant and equal to 1, the shift in question is referred to as *the* forward (backward, bilateral) shift. Technically speaking, this is a misnomer, since one should also specify the orthonormal basis with respect to which the shift acts. However, all such shifts are **unitarily equivalent**, a concept we shall identify below, and as such are considered in some sense “equal”. The terms **unilateral shift** and **unweighted shift** are also used, it usually being clear from the context whether the shift is forward or backward.

7.11. Definition. *Let $\mathcal{H}$ be a Hilbert space, and $N \in \mathcal{B}(\mathcal{H})$. Then*

(i) *If $N = N^*$, N is said to be* ***self-adjoint****, or* ***hermitian****;*
(ii) *if $N = N^*$ and $\langle Nx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, then N is said to be* ***positive****;*
(iii) *if $N\,N^* = N^*\,N$, then N is said to be* ***normal****;*
(iv) *if $N^* = N^{-1}$, then N is said to be* ***unitary****; observe that all unitary operators are automatically normal.*
(v) *if $N = N^* = N^2$, then N is called an* ***(orthogonal) projection****.*

7.12. Remark. Suppose that $U \in \mathcal{B}(\mathcal{H})$ is unitary. Since U is invertible, it must be bijective. Moreover, given x and y in $\mathcal{H}$, we find that

$$\langle Ux, Uy\rangle = \langle U^*\,Ux, y\rangle = \langle U^{-1}Ux, y\rangle = \langle Ix, y\rangle = \langle x, y\rangle.$$

In particular, unitaries preserve inner products, and therefore preserve both angles and lengths. Indeed, *they serve as the isomorphisms in the category of Hilbert spaces.*

7.13. Example. Let $\mathcal{H}$ be a Hilbert space and let B denote the unweighted bilateral shift. It is straightforward to verify that B is unitary. On the other hand, if S denotes the backward shift, then $S\,S^* - S^*\,S = P$, where P is a rank one projection. Thus S is not normal.

7.14. Example. Let us return for a moment to Example 6.7. Let μ be a finite, positive, regular Borel measure on a non-empty set X, and suppose that $f \in L^\infty(X, \mu)$. Recall that we defined the corresponding multiplication operator M_f via:

$$\begin{array}{rccc} M_f: & L^2(X,\mu) & \to & L^2(X,\mu) \\ & g & \mapsto & fg. \end{array}$$

Given $g, h \in L^2(X, \mu)$, we have

$$\begin{aligned} \langle g, (M_f)^* h\rangle &= \langle M_f g, h\rangle \\ &= \int_X (M_f g)\overline{h}d\mu \\ &= \int_X (gf)\overline{h}d\mu \\ &= \int_X g\overline{(\overline{f}h)}d\mu \\ &= \langle g, M_{\overline{f}}h\rangle. \end{aligned}$$

Thus $M_f^* = M_{\overline{f}}$. But the $M_f M_f^* = M_f M_{\overline{f}} = M_{|f|^2} = M_{\overline{f}} M_f = M_f^* M_f$, implying that M_f is normal.

M_f will be self-adjoint precisely if $M_f = M_{\overline{f}}$, and it is readily seen that this happens if and only if $\overline{f} = f$; namely if f is real-valued. (All such statements are meant to hold "almost-everywhere"-μ.)

M_f will be positive if and only if $\langle M_f g, g\rangle \geq 0$ for all $g \in \mathcal{H}$. But this happens precisely when

$$\int_X f(x)\,|g(x)|^2 d\mu \geq 0$$

for all $g \in L^2(X,\mu)$, which in turn is equivalent to the condition that $f(x) \geq 0$ almost everywhere in X.

Finally, M_f will be unitary if $M_{\overline{f}} = {M_f}^{-1}$, which is equivalent to $\overline{f} = f^{-1}$. In other words, $|f(x)| = 1$ almost everywhere in X.

7.15. Example. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and let $D = \operatorname{diag}\{d_n\}_{n=1}^\infty$, where $\{d_n\}_{n=1}^\infty \in \ell^\infty(\mathbb{N})$. Suppose that $\{e_n\}_{n=1}^\infty$ is the standard orthonormal basis for $\mathcal{H}$, and that $De_n = d_n\,e_n$ for all $n \geq 1$. Then $\|D\| = \sup_{n\geq 1}|d_n|$. Then $D^* = \operatorname{diag}\{\overline{d_n}\}_{n=1}^\infty$, and it is not hard to check that D is normal. In fact, D can be thought of as a multiplication operator on an L^2-space with respect to counting measure.

Furthermore, $\sigma_p(D) = \{d_n\}_{n=1}^\infty$, while $\sigma_a(D) = \sigma(D) = \overline{\{d_n\}_{n=1}^\infty}$. Finally, $\sigma_c(D) = \overline{\sigma_p(D^*)} = \{d_n\}_{n=1}^\infty$.

Again, D is self-adjoint precisely when $d_n \in \mathbb{R}$ for all $n \geq 1$, D is positive if and only if $d_n \geq 0$ for all $n \geq 1$, and D is unitary if and only if $|d_n| = 1$ for all $n \geq 1$.

7.16. Lemma. *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. If N is normal, then $\|Nx\| = \|N^*x\|$ for all $x \in \mathcal{H}$. In particular, therefore,* $\ker N = \ker N^*$.

Proof. Let $x \in \mathcal{H}$. Then

$$\begin{aligned} \|Nx\|^2 &= \langle Nx, Nx\rangle \\ &= \langle N^*Nx, x\rangle \\ &= \langle N\,N^*x, x\rangle \\ &= \langle N^*x, N^*x\rangle \\ &= \|N^*x\|^2. \end{aligned}$$

That is, $\|Nx\| = \|N^*x\|$.

□

7.17. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. If N is normal, then $\sigma(N) = \sigma_a(N)$.*

Proof. Clearly $\sigma_a(N) \subseteq \sigma(N) = \sigma_a(N) \cup \sigma_c(N)$. Assume $\lambda \in \sigma_c(N)$. Then $\overline{\lambda} \in \sigma_p(N^*)$, by Proposition 7.8. Let $0 \neq x \in \ker(N^* - \overline{\lambda})$. Then $x \in \ker(N^* - \overline{\lambda})^* = \ker(N - \lambda I)$ by the above Lemma. This means that $\lambda \in \sigma_p(N) \subseteq \sigma_a(N)$. We conclude that $\sigma(N) \subseteq \sigma_a(N)$.

□

Compact operators acting on Hilbert spaces.

7.18. The set of compact operators acting on a Hilbert space is more tractable in general than the set of compact operators acting on an arbitrary Banach space. One of the reasons for this is the characterization given below. Recall that the set of finite rank operators acting on a Banach space $\mathfrak{X}$ is denoted by $\mathcal{F}(\mathfrak{X})$.

7.19. Theorem. *Let $\mathcal{H}$ be a Hilbert space and let $K \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(i) *K is compact;*
(ii) *K^* is compact;*
(iii) *There exists a sequence $\{F_n\}_{n=1}^\infty \subseteq \mathcal{F}(\mathcal{H})$ such that $K = \lim_{n\to\infty} F_n$.*

Proof.

(i) $\Rightarrow$ (iii) Let B_1 denote the unit ball of $\mathcal{H}$, and let $\varepsilon > 0$. Since $\overline{K(B_1)}$ is compact, it must be separable (i.e. it is totally bounded). Thus $\mathcal{M} = \overline{\operatorname{ran} K}$ is a separable subspace of $\mathcal{H}$, and thus possesses an orthonormal basis $\{e_n\}_{n=1}^\infty$.

Let P_n denote the orthogonal projection of $\mathcal{H}$ onto $\operatorname{span}\{e_k\}_{k=1}^n$. Set $F_n = P_n K$, noting that each F_n is finite rank. We now show that $K = \lim_{n\to\infty} F_n$.

Let $x \in \mathcal{H}$ and consider $y = Kx \in \mathcal{M}$, so that $\lim_{n\to\infty} \|P_n y - y\| = 0$. Thus $\lim_{n\to\infty} \|F_n x - Kx\| = \lim_{n\to\infty} \|P_n y - y\| = 0$. Since K is compact, $K(B_1)$ is totally bounded, so we can choose $\{x_k\}_{k=1}^m \subseteq B_1$ such that $K(B_1) \subseteq \cup_{k=1}^m B(Kx_k, \varepsilon/3)$, where given $z \in \mathcal{H}$ and $\delta > 0$, $B(z,\delta) = \{w \in \mathcal{H} : \|w - z\| < \delta\}$.

If $\|x\| \le 1$, choose i such that $\|Kx_i - Kx\| < \varepsilon/3$. Then for any $n > 0$,

$$\begin{aligned}
\|Kx - F_n x\| &\le \|Kx - Kx_i\| + \|Kx_i - F_n x_i\| + \|F_n x_i - F_n x\| \\
&< \varepsilon/3 + \|Kx_i - F_n x_i\| + \|P_n\| \, \|Kx_i - Kx\| \\
&< 2\varepsilon/3 + \|Kx_i - F_n x_i\|.
\end{aligned}$$

Choose $N > 0$ such that $\|Kx_i - F_n x_i\| < \varepsilon/3$, $1 \le i \le m$ for all $n > N$. Then $\|Kx - F_n x\| \le 2\varepsilon/3 + \varepsilon/3 = \varepsilon$. Thus $\|K - F_n\| \le \varepsilon$ for all $n > N$. Since $\varepsilon > 0$ was arbitrary, $K = \lim_{n\to\infty} F_n$.

(iii) $\Rightarrow$ (ii) Suppose $K = \lim_{n\to\infty} F_n$, where F_n is finite rank for all $n \ge 1$. Note that F_n^* is also finite rank (why?), and that $\|K^* - F_n^*\| = \|K - F_n\|$ for all $n \ge 1$, which clearly implies that $K^* = \lim_{n\to\infty} F_n^*$, and hence that K^* is compact.

(ii) $\Rightarrow$(i) Since K compact implies K^* is compact from above, we deduce that K^* compact implies $(K^*)^* = K$ is compact, completing the proof.

□

We can restate the above Theorem more succinctly by saying that $\mathcal{K}(\mathcal{H})$ is the norm closure of the set of finite rank operators on $\mathcal{H}$. This is an extraordinarily useful result.

7.20. Remark. Contained in the above proof is the following interesting observation. If K is a compact operator acting on a separable Hilbert space $\mathcal{H}$, then for *any* sequence $\{P_n\}_{n=1}^\infty$ of finite rank projections tending strongly (i.e. pointwise) to the identity, $\|K - P_n K\|$ tends to zero. By considering adjoints, we find that $\|K - KP_n\|$ also tends to zero.

Let $\varepsilon > 0$, and choose $N > 0$ such that $n \ge N$ implies $\|K - KP_n\| < \varepsilon/2$ and $\|K - P_n K\| < \varepsilon/2$. Then for all $n \ge N$ we get

$$\begin{aligned} \|K - P_n K P_n\| &\le \|K - KP_n\| + \|KP_n - P_n K P_n\| \\ &\le \|K - KP_n\| + \|K - P_n K\| \, \|P_n\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

It follows that if $\mathcal{H}$ has an orthonormal basis indexed by the natural numbers, say $\{e_n\}_{n=1}^\infty$, then the matrix for K with respect to this basis comes within ε of the matrix for $P_N K P_N$. In other words, K "virtually lives" on the "top left-hand corner".

Alternatively, if $\mathcal{H}$ has an orthonormal basis indexed by the integers, say $\{f_n\}_{n\in\mathbb{Z}}$, and we let P_n denote the orthogonal projection onto span $\{e_k\}_{k=-n}^n$, then the matrix for K with respect to this basis can be arbitrarily well estimated by a sufficiently large but finite "central block".

7.21. Recall from Theorem 6.34 that if $K \in \mathcal{B}(\mathcal{H})$ is compact, then for all $\varepsilon > 0$, the set $\{\lambda \in \sigma(K) : |\lambda| \ge \varepsilon\}$ is finite. From this the next example easily follows.

7.22. Example. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Let $\{d_n\}_{n=1}^\infty$ be a bounded sequence and consider the diagonal operator $D \in \mathcal{B}(\mathcal{H})$ defined locally by $De_n = d_n e_n$ and extended to all of $\mathcal{H}$ by linearity and continuity.

Then $D \in \mathcal{K}(\mathcal{H})$ if and only if $\lim_{n\to\infty} d_n = 0$.

7.23. Example. Let $\mathcal{H} = L^2([0,1], dx)$, and consider the function $k(x,t) \in L^2([0,1] \times [0,1], dm)$, where dm represents Lebesgue planar measure. Then we define a Volterra operator

$$\begin{array}{rccl} V: & L^2([0,1], dx) & \to & L^2([0,1], dx) \\ & (Vf)(x) & = & \int_0^1 f(t)\, k(x,t)\, dt. \end{array}$$

(The classical Volterra operator has $k(x,t) = 1$ if $x \ge t$, and $k(x,t) = 0$ if $x < t$.)

Now for $f \in L^2([0,1], dx)$ we have

$$\begin{aligned}
\|Vf\|^2 &= \int_0^1 |Vf(x)|^2 dx \\
&= \int_0^1 \left| \int_0^1 f(t)\, k(x,t) dt \right|^2 dx \\
&\le \int_0^1 \left(\int_0^1 |f(t)\, k(x,t)| dt \right)^2 dx \\
&\le \int_0^1 \|f\|_2^2 \int_0^1 |k(x,t)|^2 dt dx \text{ by the Cauchy-Schwartz Inequality} \\
&= \|f\|_2^2\, \|k\|_2^2,
\end{aligned}$$

so that $\|V\| \le \|k\|_2$.

Let $\mathcal{A}$ denote the algebra of continuous functions on $[0,1] \times [0,1]$ which can be resolved as $g(x,t) = \sum_{i=1}^n u_i(x)\, w_i(t)$. Then $\mathcal{A}$ is an algebra which separates points, contains the constant functions, and is closed under complex conjugation. By the Stone-Weierstraß Theorem, given $\varepsilon > 0$ and $h \in \mathcal{C}([0,1] \times [0,1])$, there exists $g \in \mathcal{A}$ such that $\|h - g\|_2 \le \|h - g\|_\infty < \varepsilon$. But since $\mathcal{C}([0,1] \times [0,1])$ is dense (in the L^2-topology) in $L^2([0,1] \times [0,1], dm)$, $\mathcal{A}$ must also be dense (in the L^2-topology) in $L^2([0,1] \times [0,1], dm)$.

Let $\varepsilon > 0$. For k as above, choose $g \in \mathcal{A}$ such that $\|k - g\|_2 < \varepsilon$. Define

$$\begin{array}{rlcl}
V_0 : & L^2([0,1], dx) & \to & L^2([0,1], dx) \\
& V_0 f(x) & = & \int_0^1 f(t)\, g(x,t) dt.
\end{array}$$

From above, we find that $\|V - V_0\| \le \|k - g\|_2 < \varepsilon$.

To see that V_0 is finite rank, consider the following; first, $g(x,t) = \sum_{i=1}^n u_i(x)\, w_i(t)$. If we set $\mathcal{M} = \mathrm{span}_{1 \le i \le n}\{u_i\}$, then $\mathcal{M}$ is a finite dimensional subspace of $L^2([0,1], dx)$. Moreover,

$$\begin{aligned}
V_0 f(x) &= \int_0^1 f(t)\, g(x,t) dt \\
&= \sum_{i=1}^n \left(\int_0^1 f(t)\, w_i(t) dt \right) u_i(x),
\end{aligned}$$

so that $V_0 f \in \mathcal{M}$.

Thus V can be approximated arbitrarily well by elements of the form $V_0 \in \mathcal{F}(L^2([0,1], dx)$, and so V is compact.

7.24. Definition. *Let $\mathfrak{X}$ be a Banach space and $T \in \mathcal{B}(\mathfrak{X})$. Then T is said to be* ***quasinilpotent*** *if $\sigma(T) = 0$. By the spectral mapping theorem* **??**, *it is easily seen that every nilpotent operator is automatically quasinilpotent.*

7.25. Example. Let V denote the classical Volterra operator defined in Example 7.23 above. We shall show that V is quasinilpotent. (Note that we have seen that the Volterra operator acting in $\mathcal{B}(\mathcal{C}[0,1])$ is quasinilpotent in Example S6.1.)

Since $V \in \mathcal{K}(\mathcal{H})$, we know that $\sigma(V) = \{0\} \cup \sigma_p(V)$. Suppose $0 \ne \lambda \in \sigma_p(V)$, and that $f \in \ker(\lambda - V)$. Then

$$\begin{aligned} |\lambda|\,|f(x)| &= \left|\int_0^x f(t)dt\right| \\ &\le \int_0^x |f(t)|dt \\ &\le \int_0^1 |f(t)|dt \\ &\le \|f\|_2\,\|1\|_2 \\ &= \|f\|_2. \end{aligned}$$

Then for $0 \le x \le 1$,

$$\begin{aligned} |f(x)| &\le (1/|\lambda|)\int_0^x |f(t_1)|dt_1 \\ &\le (1/|\lambda|)\int_0^x (1/|\lambda|)\int_0^{t_1} |f(t_2)|dt_2dt_1 \\ &\le \dots \\ &\le (1/|\lambda|^{n+1})\int_0^x\int_0^{t_1}\dots\int_0^{t_n} |f(t_{n+1})|dt_{n+1}dt_n\dots dt_1 \\ &\le (1/|\lambda|^{n+1})\int_0^x\int_0^{t_1}\dots\int_0^{t_n} (\|f\|_2/|\lambda|)dt_{n+1}dt_n\dots dt_1 \\ &= (1/|\lambda|^{n+2})\,\|f\|_2\int_0^x\int_0^{t_1}\dots\int_0^{t_n} 1\,dt_{n+1}dt_n\dots dt_1 \\ &\le (1/|\lambda|^{n+2})\,\|f\|_2\,x^{n+1}/(n+1)! \text{ for all } n \ge 1. \end{aligned}$$

Thus $f(x) = 0$ for all $x \in [0,1]$, and hence $f = 0$. But then $\lambda \notin \sigma_p(V)$. Therefore $\sigma(V) = \{0\}$, and so V is quasinilpotent as claimed.

The spectral theorem for compact, normal operators.

7.26. We now turn our attention to the first of three "spectral theorems" that we shall prove for normal operators acting on a Hilbert space. The version that we do now is the one which is most closely related to the Spectral Theorem for normal matrices, which states that any normal matrix N acting on $\mathbb{C}^n$ can be **diagonalised** with respect to some orthonormal basis. That is, there exists an orthonormal basis $\{e_k\}_{k=1}^n$ for $\mathbb{C}^n$ with respect to which the matrix $[N] = [n_{i,j}] = [\langle Ne_j, e_i\rangle]$ is diagonal.

7.27. Definition. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ be a subspace of $\mathcal{H}$, and suppose that $T \in \mathcal{B}(\mathcal{H})$. Recall that $\mathcal{M}$ is called **invariant** for T provided that $T\mathcal{M} \subseteq \mathcal{M}$. We say that $\mathcal{M}$ is **reducing** for T if $\mathcal{M}$ is invariant both for T and for T^*.*

7.28. Notation. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a subspace of $\mathcal{H}$. By $P_{\mathcal{M}}$ we shall denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. We leave it as an exercise for the reader to verify that $\mathcal{M}$ is invariant for $T \in \mathcal{B}(\mathcal{H})$ if and only if $(I - P_{\mathcal{M}})TP_{\mathcal{M}} = 0$.

7.29. Proposition. *Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$, and $\mathcal{M}$ be a subspace of $\mathcal{H}$. Then $\mathcal{M}$ is reducing for T if and only if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for T. When this is the case, we can write*

$$T = T_1 \oplus T_2 = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$.

Furthermore, T is compact if and only if both T_1 and T_2 are compact, and T is normal if and only if T_1 and T_2 are.

Proof. First suppose that $\mathcal{M}$ is reducing for T. Then $(I - P_{\mathcal{M}})\,T\,P_{\mathcal{M}} = 0$. Since $T^*\mathcal{M} \subseteq \mathcal{M}$, we also get $(I - P_{\mathcal{M}})\,T^*\,P_{\mathcal{M}} = 0$, and so after taking adjoints, $P_{\mathcal{M}}\,T\,(I - P_{\mathcal{M}}) = 0$. (Note that $P_{\mathcal{M}}$ is self-adjoint.) It follows that both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for T.

Now suppose that $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for T, so that

$$(I - P_{\mathcal{M}})\,T\,P_{\mathcal{M}} = 0 = P_{\mathcal{M}}\,T\,(I - P_{\mathcal{M}}).$$

By taking adjoints once more, $(I - P_{\mathcal{M}})\,T^*\,P_{\mathcal{M}} = 0$, and so $\mathcal{M}$ is reducing for T. The matrix form for T follows directly from these observations.

If T_1 and T_2 are compact, then they are limits of finite rank operators F_n and G_n respectively, from which we conclude that T is a limit of the finite rank operators $F_n \oplus G_n$. Thus T is compact.

If T is compact, then the compression of T to any subspace is compact, and so both T_1 and T_2 are compact.

We leave it as an exercise to the reader to show that $T^* = T_1^* \oplus T_2^*$. Given this, it is easy to see that T is normal if and only if $0 = [T, T^*] = [T_1, T_1^*] \oplus [T_2, T_2^*]$, which is equivalent to the simultaneous normality of T_1 and T_2.

□

7.30. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal. Then* $\ker N = \ker N^*$ *is reducing for N.*

Proof. That $\ker N = \ker N^*$ is the second half of Lemma 7.16. Now let $x \in \ker N$. Then $N^2 x = N(Nx) = 0$, and $NN^*x = N^*Nx = 0$. Thus $\ker N$ is invariant for both N and N^*, and hence is reducing for N.

□

7.31. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. If N is normal and $\lambda \neq \mu \in \sigma_p(N)$, then* $\ker(N - \lambda I)$ *is orthogonal to* $\ker(N - \mu I)$.
Proof. Let $x \in \ker(N - \lambda I)$ and $y \in \ker(N - \mu I)$. Then

$$\lambda\langle x, y\rangle = \langle Nx, y\rangle = \langle x, N^* y\rangle = \langle x, \overline{\mu} y\rangle = \mu\langle x, y\rangle.$$

Thus $(\lambda - \mu)\langle x, y\rangle = 0$. Since $\lambda - \mu \neq 0$, we must have $x \perp y$.

□

7.32. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. If N is normal, then* $\operatorname{spr}(N) = \|N\|$.
Proof. Consider first:

$$\begin{aligned}
\|N^2\| &= \sup_{\|x\|=1} \|N^2 x\| \\
&= \sup_{\|x\|=1} \|N^* N x\| \\
&\geq \sup_{\|x\|=1} |\langle N^* N x, x\rangle| \\
&= \sup_{\|x\|=1} \langle Nx, Nx\rangle \\
&= \sup_{\|x\|=1} \|Nx\|^2 \\
&= \|N\|^2.
\end{aligned}$$

By induction, $\|N^{2^n}\| \geq \|N\|^{2^n}$ for all $n \geq 1$. The reverse inequality follows immediately from the submultiplicativity of the norm in a Banach algebra. Thus $\|N^{2^n}\| = \|N\|^{2^n}$ for all $n \geq 1$. By Beurling's Spectral Radius Formula, Theorem 2.40,

$$\operatorname{spr}(N) = \lim_{n\to\infty} \|N^{2^n}\|^{1/2^n} = \|N\|.$$

□

7.33. Corollary. *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. If N is normal and $\sigma(N) = \{\lambda\}$, then $N = \lambda I$.*
Proof. Now $\sigma(N - \lambda I) = \{0\}$ by the Spectral Mapping Theorem. Since $N - \lambda I$ is also normal, $\|N - \lambda I\| = \operatorname{spr}(N - \lambda I) = 0$.

□

7.34. Lemma. *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. Suppose N is compact and normal and that $\{\lambda_i\}_{i=1}^n \subseteq \sigma_p(N)$. Let $\mathcal{M} = \oplus_{i=1}^n \ker(N - \lambda_i I)$. Then $\mathcal{M}$ is a reducing subspace for N and if $N_1 = (I - P_{\mathcal{M}})N|_{\mathcal{M}^\perp} \in \mathcal{B}(\mathcal{M}^\perp)$, then $\sigma_p(N_1) = \sigma_p(N)\backslash\{\lambda_i\}_{i=1}^n$.*
Proof.
That $\mathcal{M}$ is reducing for N follows from the fact that each $\ker(N - \lambda_i I)$ is reducing for N, $1 \leq i \leq n$. Now N_1 is both compact and normal by Proposition 7.29.

Suppose $\lambda \in \rho(N)$. Then $(N_1 - \lambda I)^{-1} = (I - P_{\mathcal{M}})(N - \lambda I)^{-1}|_{\mathcal{M}^\perp}$, so that $\lambda \in \rho(N_1)$.

Let $\lambda \in \{\lambda_i\}_{i=1}^n$. Then $\ker(N - \lambda I) \subseteq \mathcal{M}$ by definition. Thus $N_1 - \lambda I$ is injective, so that $\lambda \notin \sigma_p(N_1)$. If $\lambda \in \sigma_p(N) \backslash \{\lambda_i\}_{i=1}^n$, then $\ker(N - \lambda I)$ is orthogonal to $\mathcal{M}$, so there exists $0 \neq x \in \ker(N - \lambda I)$ and then $(N_1 - \lambda I)x = (N - \lambda I)x = 0$. Hence $\lambda \in \sigma_p(N_1)$.

We now have $\sigma_p(N) \backslash \{\lambda_i\}_{i=1}^n \subseteq \sigma_p(N_1) \subseteq \sigma(N) \backslash \{\lambda_i\}_{i=1}^n$.

Finally, if $\lambda \in \sigma_p(N_1)$, then there exists $0 \neq x \in \mathcal{M}^\perp$ such that $(N_1 - \lambda I)x = 0$. But then $(N - \lambda I)x = 0$, so that $\lambda \in \sigma_p(N)$, completing the proof.

□

7.35. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. Suppose N is compact and normal and that $\sigma_p(N) = \{\lambda_i\}_{i=1}^\infty$. Then $\mathcal{H} = \oplus_{n=1}^\infty \ker(N - \lambda_n I)$.*

Proof. Let $\mathcal{M} = \oplus_{n=1}^\infty \ker(N - \lambda_n I)$. As above, $\mathcal{M}$ is reducing for N. Let $N_1 = P_{\mathcal{M}^\perp} N|_{\mathcal{M}^\perp}$, viewed as an element of $\mathcal{B}(\mathcal{M}^\perp)$. Then $\sigma_p(N_1)$ is empty, for if $\lambda \in \sigma_p(N_1)$, then as in the previous lemma, we see that $\lambda \in \sigma_p(N)$, and hence $\ker(N_1 - \lambda I) \subseteq \ker(N - \lambda I) \subseteq \mathcal{M}$, a contradiction.

Since N_1 is compact, $\sigma(N_1) = \{0\}$, but $0 \notin \sigma_p(N_1)$ implies that N_1 is injective. On the other hand, N_1 is also normal, so $\|N_1\| = \operatorname{spr}(N_1) = 0$, and hence $N_1 = 0$. Since it is injective, we are forced to conclude that $\mathcal{M}^\perp = \{0\}$, completing the proof.

□

.

7.36. Theorem. The spectral theorem for compact normal operators. *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be a compact, normal operator. Suppose $\{\lambda_k\}_{k=1}^\infty$ are the distinct eigenvalues of N and that $P_{\mathcal{M}_k}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}_k := \ker(N - \lambda_k I)$. Then $P_{\mathcal{M}_k} P_{\mathcal{M}_j} = 0 = P_{\mathcal{M}_j} P_{\mathcal{M}_k}$ if $j \neq k$, and*

$$N = \sum_{k=1}^\infty \lambda_k P_{\mathcal{M}_k},$$

where the series converges in the norm topology in $\mathcal{B}(\mathcal{H})$.

Proof. That $P_{\mathcal{M}_k} P_{\mathcal{M}_j} = 0 = P_{\mathcal{M}_j} P_{\mathcal{M}_k}$ if $j \neq k$ is simply the statement that $\mathcal{M}_k$ is orthogonal to $\mathcal{M}_j$ for $j \neq k$, and this we saw in Proposition 7.31.

Recall also that $\lim_{k\to\infty} \lambda_k = 0$, by Theorem 6.34.

Consider $n > 0$, and $N - \sum_{k=1}^n \lambda_k P_{\mathcal{M}_k}$. If $x \in \mathcal{M}_j$ for some $1 \le j \le n$, then

$$(N - \sum_{k=1}^n \lambda_k P_{\mathcal{M}_k})x = Nx - \lambda_j x = 0.$$

Thus $\oplus_{k=1}^n \mathcal{M}_k \subseteq \ker(N - \sum_{k=1}^n \lambda_k P_{\mathcal{M}_k})$. If x is orthogonal to $\oplus_{k=1}^n \mathcal{M}_k$, then $P_{\mathcal{M}_k} x = 0$, $1 \le k \le n$, so that $(N - \sum_{k=1}^n \lambda_k P_{\mathcal{M}_k})x = Nx$. Moreover, $\oplus_{k=1}^n \mathcal{M}_k$ reduces N, so we let $N_n = P((\oplus_{k=1}^n \mathcal{M}_k)^\perp) N|_{(\oplus_{k=1}^n \mathcal{M}_k)^\perp}$.

Then $\|N - \sum_{k=1}^n \lambda_k P_{\mathcal{M}_k}\| = \|N_n\|$. Also, N_n is compact and normal by Proposition 7.29, and from Lemma 7.34,

$$\sigma_p(N_n) = \{\lambda_{n+1}, \lambda_{n+2}, \lambda_{n+3}, \dots\}.$$

Thus $\|N_n\| = \mathrm{spr}\,(N_n) = \sup_{k>n} |\lambda_k|$. In particular, $\lim_{n\to\infty} \|N_n\| = 0$, so that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k P_{\mathcal{M}_k} = \sum_{k=1}^{\infty} \lambda_k P_{\mathcal{M}_k} = N.$$

□

7.37. Corollary. *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be a compact, normal operator. Then there exists an orthonormal basis $\{e_\alpha\}_{\alpha\in\Lambda}$ for $\mathcal{H}$ such that each e_α is an eigenvector for N.*

Proof. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the set of eigenvalues of N. For each $n \ge 1$, choose an orthonormal basis $\{e_{(n,\beta)}\}_{\beta\in\Lambda_n}$ for $\ker\,(N - \lambda_n I)$. (Note that if $\lambda_n \ne 0$, then the cardinality of Λ_n is finite.) Then each $e_{(n,\beta)}$, $\beta \in \Lambda_n$, $n \ge 1$ is an eigenvector for N corresponding to λ_n, the $e_{(n,\beta)}$'s are all orthogonal since all of the $\ker\,(N - \lambda_n I)$'s are. Finally, $\overline{\mathrm{span}}\{e_{(n,\beta)}\}_{\beta\in\Lambda_n,\, n\ge 1} = \oplus_{n=1}^{\infty} \ker\,(N - \lambda_n I) = \mathcal{H}$ by Proposition 7.35. Let $\{e_\alpha\}_{\alpha\in\Lambda} = \{e_{(n,\beta)}\}_{\beta\in\Lambda_n,\, n\ge 1}$.

□

7.38. Corollary. *Let $\mathcal{H}$ be a Hilbert space and let $N \in \mathcal{B}(\mathcal{H})$. Then N is compact and normal if and only if there exist an orthonormal set $\{f_n\}_{n=1}^{\infty}$ and a sequence of scalars $\{\beta_n\}_{n=1}^{\infty}$ such that*

(i) $\lim_{n\to\infty} \beta_n = 0$*;*
(ii) $N f_n = \beta_n f_n$, $n \ge 1$*;*
(iii) $Nx = 0$ *if* $x \in \mathcal{H}$, x *orthogonal to* $\overline{\mathrm{span}}\{f_n\}_{n=1}^{\infty}$.

Proof. Suppose the sets $\{f_n\}_{n=1}^{\infty}$ and $\{\beta\}_{n=1}^{\infty}$ as above exist. Then N is seen to be compact, using the arguments of Theorem 7.2. We leave it as an exercise for the reader to prove that N is normal.

Now if N is normal and compact, let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of N, the existence of which is guaranteed by the preceding Corollary. Let $\{f_n\}_{n=1}^{\infty}$ be the subset of $\{e_\alpha\}_{\alpha\in\Lambda}$ comprised of those vectors whose corresponding eigenvalues $\{\beta_n\}_{n\ge 1}$ are different from zero. That $\{f_n\}_{n\ge 1}$ is at most countable follows from the fact that $\sigma_p(N)$ is countable, and $\mathrm{nul}\,(N - \lambda_n I) < \infty$ for all $0 \ne \lambda_n \in \sigma_p(N)$.

Clearly $N f_n = \beta_n f_n$ for all $n \ge 1$, and $\lim_{n\to\infty} \beta_n = 0$ from the argument above combined with the fact that $\sigma_p(N)$ is a sequence tending to zero when N is compact. Finally, $(\overline{\mathrm{span}}\{f_n\}_{n=1}^{\infty})^{\perp} = \ker\,(N - 0I) = \ker N$, from which condition (iii) also follows.

□

Supplementary Examples

S7.1. Despite there being a plethora of Hilbert space operators (the space $\mathcal{B}(\mathcal{H})$ is non-separable if $\mathcal{H}$ is infinite-dimensional), nevertheless, very few classes of operators are well-understood. The best understood class (other than scalar operators) is the class of normal operators. We have established the Spectral Theorem for compact, normal operators. In Chapter 13.21, we shall establish a more general version of the Spectral Theorem that holds for arbitrary normal operators acting on a separable Hilbert space.

Let us describe a (small) selection of classes of Hilbert space operators whose existence has attracted the attention of researchers.

S7.3. Example. An operator $R \in \mathcal{B}(\mathcal{H})$ is said to be **subnormal** if it is the restriction of a normal operator N to an invariant subspace. That is, there exists a Hilbert space $\mathcal{K}$ which contains $\mathcal{H}$, and a normal operator $N \in \mathcal{B}(\mathcal{K})$ such that $\mathcal{H}$ is invariant for N and $R = N|_{\mathcal{H}}$. In operator-matrix notation, relative to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$, we have

$$N = \begin{bmatrix} R & N_2 \\ 0 & N_4 \end{bmatrix}.$$

As a concrete example, the unilateral forward shift S is subnormal, being (unitarily equivalent to) the compression of the bilateral shift B (which satisfies $Be_n = e_{n+1}$ for all $n \in \mathbb{Z}$, where $\{e_n\}_{n\in\mathbb{Z}}$ is an ONB for $\mathcal{K} = \ell^2(\mathbb{Z})$) to the invariant subspace $\mathcal{H} = \overline{\text{span}}\{e_n\}_{n\geq 1}$.

We mention only one result about this class, namely: S. Brown has shown [**9**] that every subnormal operator admits a non-trivial invariant subspace. We refer the interested reader to the monograph [**15**] for an introduction to subnormal operators.

S7.3. Example. Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space. An operator $B \in \mathcal{B}(\mathcal{H})$ is said to be **block-diagonal** if there exists an increasing sequence $(P_n)_n$ of finite-rank projections which converges **strongly** to the identity (i.e. for all $x \in \mathcal{H}$, $x = \lim_n P_n x$) and which satisfies $P_n B = B P_n$ for each $n \geq 1$.

If we set $P_0 = 0$ and $\mathcal{H}_n := \text{ran}\,(P_n - P_{n-1})$, $n \geq 1$, then each $\mathcal{H}_n$ is finite-dimensional $\mathcal{H} = \oplus_n \mathcal{H}_n$ and relative to this decomposition, $B = \oplus_n B_n$, where $B_n \in \mathcal{B}(\mathcal{H}_n)$.

We say that an operator $Q \in \mathcal{B}(\mathcal{H})$ is **quasidiagonal** if there exists a sequence $(P_n)_n$ as above such that

$$\lim_n \|P_n B - B P_n\| = 0.$$

It was shown by Halmos [**26**] that the set (QD) of quasidiagonal operators is the norm-closure in $\mathcal{B}(\mathcal{H})$ of the set of block-diagonal operators.

There are still a number of interesting open questions about quasidiagonal operators. One which is dear to your humble author's heart is the following: suppose that $Q \in \mathcal{B}(\mathcal{H})$ is quasidiagonal and quasinilpotent. Is Q a limit of block-diagonal nilpotent operators? It is known that Q is a limit of nilpotent operators (due to the large body of work by C. Apostol, C. Foiaş and D. Voicluescu), and as just noted,

Q is a limit of block-diagonal operators. What is not known is whether one can choose the sequence to consist of operators which are *simultaneously* block-diagonal and nilpotent. (We mention in passing that Herrero [**28**] has provided an example of a block-diagonal operator B which is a limit of nilpotent operators in $\mathcal{B}(\mathcal{H})$, but which is not a limit of block-diagonal nilpotent operators.)

S7.4. Example. Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **triangular** if there exists an ONB $\mathfrak{B} = \{e_n\}_{n\geq 1}$ for $\mathcal{H}$ such that the matrix of T relative to $\mathfrak{B}$ is upper-triangular. In other words, $\langle Te_j, e_i\rangle = 0$ if $i > j$. Equivalently, T is triangular if there exists an increasing sequence $(P_n)_n$ of finite-rank projections which converges **strongly** to the identity in the sense above and which satisfies $TP_n = P_nTP_n$ for each $n \geq 1$.

We say that $R \in \mathcal{B}(\mathcal{H})$ is **quasitriangular** if there exists a sequence $(P_n)_n$ as above such that

$$\lim_n \|TP_n - P_nTP_n\| = 0.$$

Once again, it was shown by Halmos [**26**] that the set (QT) of quasitriangular operators is the norm-closure in $\mathcal{B}(\mathcal{H})$ of the set of triangular operators.

Of course, when $\mathcal{H}$ is finite-dimensional, every operator in $\mathcal{B}(\mathcal{H})$ is upper-triangularisable with respect to *some* ONB for $\mathcal{H}$. As it happens, the set of quasitriangular operators is nowhere dense in $\mathcal{B}(\mathcal{H})$.

The set of quasitriangular operators was crucial to the characterisation by Apostol, Foiaş and Voiculescu of the norm-closure of the set of nilpotent operators in $\mathcal{B}(\mathcal{H})$. (See Section A4.3.) Indeed, the following deep result gives a wonderfully useful way of verifying whether or not an operator is quasitriangular. We first remind the reader that if $T \in \mathcal{B}(\mathcal{H})$, then

$$\rho_{sF}(T) = \{z \in \mathbb{C} : \pi(T - \lambda I) \text{ is invertible in the Calkin algebra}\}$$

is the **semi-Fredholm domain** of T.

Theorem. [**Apostol, Foiaş, and Voiculescu**] [**3**] *Let $\mathcal{H}$ be a separable, infinite-dimensional Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(i) *T is quasitriangular.*
(ii) $\operatorname{ind}(T - \alpha I) \geq 0$ *for all $\alpha \in \rho_{sF}(T)$.*

Appendix

A7.1. The notion of a Fredholm operator was introduced in Appendix 6.28, where it was shown that if K is a compact operator acting on a Banach space $\mathfrak{X}$ and if λ is a non-zero scalar, then $\lambda I - K$ is Fredholm of index zero. We now wish to consider Fredholm operators acting on a Hilbert space. We shall establish the fact that the Fredholm operators are precisely the operators which are invertible modulo the compact operators, and that the index function serves to classify components of the set of invertible elements in the Calkin algebra .

A7.2. Recall that an operator T acting on a Hilbert space $\mathcal{H}$ is said to be *Fredholm* if

(i) $\operatorname{ran} T$ is closed;
(ii) $\operatorname{nul} T$ is finite; and
(iii) $\operatorname{codim} \operatorname{ran} T$ is finite.

As before, when T is Fredholm we may define the *Fredholm index* of T to be

$$\operatorname{ind} T = \operatorname{nul} T - \operatorname{codim} \operatorname{ran} T.$$

From Remark 6.27, we see that when T is Fredholm, we may replace $\operatorname{codim} \operatorname{ran} T$ by $\operatorname{nul} T^\dagger$, where $T^\dagger$ now denotes the *Banach space adjoint* of T, as opposed to the *Hilbert space adjoint* of T, which we denote by T^*. The distinction is important, since it is not *a priori* obvious that we may replace $\operatorname{codim} \operatorname{ran} T$ by $\operatorname{nul} T^*$. On the other hand, since $\operatorname{ran} T$ is closed, we obtain the decomposition $\mathcal{H} = \operatorname{ran} T \oplus (\operatorname{ran} T)^\perp$, and so

$$\operatorname{codim} \operatorname{ran} T = \dim (\mathcal{H}/\operatorname{ran} T) = \dim (\operatorname{ran} T)^\perp.$$

Since $(\operatorname{ran} T)^\perp = \ker T^*$, it follows that $\operatorname{nul} T^* = \operatorname{codim} \operatorname{ran} T = \operatorname{nul} T^\dagger$, and so, as in the Banach space setting, we retrieve the equation

$$\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*.$$

A7.3. Let $\mathcal{H}$ be a separable Hilbert space and let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. Let $S \in \mathcal{B}(\mathcal{H})$ denote the unilateral shift operator acting on this basis as defined in Example 7.9. That is, $Se_{n+1} = e_n$ if $n \geq 1$, and $Se_1 = 0$.

Then $\operatorname{ran} S = \mathcal{H}$, so that $\operatorname{ran} S$ is closed. Also, $\ker S = \operatorname{span}\{e_1\}$, so that $\operatorname{nul} S = 1$. Finally, $\ker S^* = \{0\}$, so that $\operatorname{nul} S^* = 0$. Thus S is Fredholm and

$$\operatorname{ind} S = \operatorname{nul} S - \operatorname{nul} S^* = 1 - 0 = 1.$$

Note also that S^* is Fredholm as well, and that

$$\operatorname{ind} S^* = \operatorname{nul} S^* - \operatorname{nul} S^{**} = \operatorname{nul} S^* - \operatorname{nul} S = -\operatorname{ind} S = -1.$$

Finally, S^n and $(S^*)^n$ are both Fredholm as well, and $\operatorname{ind} S^n = n = -\operatorname{ind} (S^*)^n$ for each $n \geq 1$.

A7.4. Let $\mathcal{H}$ be a Hilbert space and $K \in \mathcal{K}(\mathcal{H})$. As we have seen, if $0 \neq \lambda \in \mathbb{C}$, then $\lambda I - K$ is Fredholm of index zero. It follows that so is any operator of the form $T + L$ where T is invertible and L is compact. Indeed, $T + L = T(I - (-T^{-1}L))$. The verification of the index is left to the reader.

It follows that if S is the unilateral forward shift, then S is not of the form $T + L$ for any T invertible and L compact.

A7.5. Proposition. *Suppose $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ is Fredholm. Then $T|_{(\ker T)^\perp}$ is bounded below.*

Proof. Suppose $x, y \in (\ker T)^\perp$. Then $0 = Tx - Ty = T(x - y)$ implies $x - y \in \ker T$ and hence $x = y$. In particular, the map

$$\begin{array}{rlll} T_0 : & (\ker T)^\perp & \to & \operatorname{ran} T \\ & x & \mapsto & Tx \end{array}$$

is a 1-1, onto map, and thus it is invertible. Let $R : \operatorname{ran} T \to (\ker T)^\perp$ denote the inverse of T_0. Then for $x \in (\ker T)^\perp$,

$$\begin{array}{rcl} \|x\| = \|RT_0x\| & = & \|RTx\| \\ & \le & \|R\| \, \|Tx\| \end{array}$$

and so $\|Tx\| \ge \|R\|^{-1} \|x\|$.

Thus T is bounded below on $(\ker T)^\perp$, as claimed.

□

A7.6. Let $\mathcal{H}$ be a Hilbert space. Recall that the Calkin algebra $\mathcal{Q}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the quotient of $\mathcal{B}(\mathcal{H})$ by the closed, two-sided ideal of compact operators, and as such, $\mathcal{Q}(\mathcal{H})$ is a Banach algebra.

A7.7. Our present goal is to establish a relationship between the set of Fredholm operators acting on a Hilbert space $\mathcal{H}$, and the set of invertible elements in the Calkin algebra. In fact, the relation we wish to establish is *equality*!

We record here a couple of facts which will prove useful:

- $(\mathcal{A}(\mathcal{H}))^{-1}$ is open in $\mathcal{A}(\mathcal{H})$.
- The involution on $\mathcal{B}(\mathcal{H})$ naturally gives rise to an involution in the Calkin algebra. Given $t \in \mathcal{A}(\mathcal{H})$, $t = \pi(T)$ for some $T \in \mathcal{B}(\mathcal{H})$. We then set $t^* = \pi(T^*)$. If $R \in \mathcal{B}(\mathcal{H})$ and $\pi(R) = t$, then $K = R - T \in \mathcal{K}(\mathcal{H})$. Thus $K^* = R^* - T^* \in \mathcal{K}(\mathcal{H})$, and so $\pi(R^*) = \pi(T^*)$, from which it follows that our involution is indeed well-defined. We then have that $\mathcal{A}(\mathcal{H})$ and $(\mathcal{A}(\mathcal{H}))^{-1}$ are self-adjoint. Indeed, for $t \in (\mathcal{A}(\mathcal{H}))^{-1}$, $(t^*)^{-1} = (t^{-1})^*$.

A7.8. Theorem. **[Atkinson's Theorem.]** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ denote the canonical quotient map from $\mathcal{B}(\mathcal{H})$ into the Calkin algebra. The following are equivalent.*

(i) *$T \in \mathcal{B}(\mathcal{H})$ is Fredholm.*
(ii) *$t := \pi(T)$ is invertible in $\mathcal{Q}(\mathcal{H})$.*

Proof. Suppose T is Fredholm. Then $\operatorname{ran} T$ is closed, $\operatorname{nul} T$ is finite and $\operatorname{nul} T^* = \operatorname{codim} \operatorname{ran} T$ is finite. Let us once again decompose

$$\begin{aligned} \mathcal{H} &= \ker T \oplus (\ker T)^\perp \\ &= \operatorname{ran} T \oplus (\operatorname{ran} T)^\perp. \end{aligned}$$

As in the proof of Proposition A7.5, we see that

$$\begin{aligned} T_0 : (\ker T)^\perp &\to \operatorname{ran} T \\ x &\mapsto Tx \end{aligned}$$

is invertible. Let $R_0 : \operatorname{ran} T \to (\ker T)^\perp$ denote the inverse of T_0, and define $R \in \mathcal{B}(\mathcal{H})$ via $Rx = \begin{cases} R_0 x & \text{if } x \in \operatorname{ran} T \\ 0 & \text{if } x \in (\operatorname{ran} T)^\perp. \end{cases}$

Then $RTx = (I - P(\mathcal{M}))x$, where $\mathcal{M} = \ker T$, and $TRx = (I - P(\mathcal{N}))x$, where $\mathcal{N} = (\operatorname{ran} T)^\perp$. Since both $P(\mathcal{M})$ and $P(\mathcal{N})$ are finite rank, we obtain:

$$\pi(R)\,\pi(T) = \pi(RT) = \pi(I) = \pi(TR) = \pi(T)\,\pi(R),$$

and so $t = \pi(T)$ is invertible with inverse $r = \pi(R)$.

Next, suppose that $t = \pi(T) \in \mathcal{A}(\mathcal{H})$ is invertible. Then there exists $r \in \mathcal{A}(\mathcal{H})$ with $rt = 1 = \pi(I) = tr$, and choosing $R \in \mathcal{B}(\mathcal{H})$ with $\pi(R) = r$, we get

$$RT = I + K_1, \qquad TR = I + K_2$$

for some $K_1, K_2 \in \mathcal{K}(\mathcal{H})$.

Since $\operatorname{nul}(I + K_1) < \infty$ by Proposition 6.26, $\operatorname{nul} T < \infty$. Since $\operatorname{ran} T \supseteq \operatorname{ran}(I + K_2)$ and $\operatorname{codim} \operatorname{ran}(I + K_2) < \infty$ by Proposition 6.26, $\operatorname{codim} \operatorname{ran} T < \infty$.

By Corollary A6.10, $\operatorname{ran} T$ is closed, and so we are done.

□

A7.9. We now wish to consider some of the stability properties of Fredholm operators and the index function. We mention that most, if not all, of the following results are true for Fredholm operators acting on a Banach space. On the other hand, certain arguments simplify when looking at Hilbert spaces, and we have made use of these simplifications. For the most general results, we refer the reader to the book of Caradus, Pfaffenberger and Yood [**11**].

A7.10. Lemma. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be Fredholm. If R is invertible, then*

$$\operatorname{ind} TR = \operatorname{ind} RT = \operatorname{ind} T.$$

Proof. Exercise.

□

A7.11. Lemma. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. If T is Fredholm and $\operatorname{ind} T = 0$, then there exists a finite rank operator F such that $T+F$ is invertible.*

Proof. As we saw in Theorem A7.8, we can decompose $\mathcal{H}$ in two ways, namely:

$$\begin{aligned} \mathcal{H} &= \ker T \oplus (\ker T)^{\perp} \\ &= \operatorname{ran} T \oplus (\operatorname{ran} T)^{\perp}. \end{aligned}$$

Since $\operatorname{ind} T = 0$ by hypothesis, $\operatorname{nul} T = \operatorname{codim} \operatorname{ran} T$. Let $\{e_k\}_{k=1}^n$ and $\{f_k\}_{k=1}^n$ be orthonormal bases for $\ker T$ and $(\operatorname{ran} T)^{\perp}$ respective and let $F \in \mathcal{B}(\mathcal{H})$ be defined via $Fe_k = f_k$, $1 \le k \le n$, $Fz = 0$ if z is orthogonal to $\ker T$. Then F is clearly finite rank. We claim that $T+F$ is bijective, and hence invertible.

If $0 \ne x \in \mathcal{H}$, then $x = x_1 + x_2$, where $x_1 \in \ker T$, $x_2 \in (\ker T)^{\perp}$, and $\|x_1\| + \|x_2\| \ne 0$. If $x_1 \ne 0$, then

$$\begin{aligned} (T+F)x &= Tx + Fx \\ &= Tx_2 + Fx_1 \end{aligned}$$

and $0 \ne Fx_1 \in (\operatorname{ran} T)^{\perp}$ forces $(T+F)x \ne 0$. If $x_2 \ne 0$, then $(T+F)x = Tx_2 + Fx_1$ and $0 \ne Tx_2 \in (\operatorname{ran} F)^{\perp}$ forces $(T+F)x \ne 0$.

In either case, we see that $T+F$ is injective.

Now choose $y \in \mathcal{H}$ and decompose y as $y = y_1 + y_2$ where $y_1 \in \operatorname{ran} T$ and $y_2 \in (\operatorname{ran} T)^{\perp}$. Choose $x_1 \in (\ker T)^{\perp}$ such that $Tx_1 = y_1$ and $x_2 \in \ker T$ such that $Fx_2 = y_2$. Then

$$\begin{aligned} (T+F)(x_1+x_2) &= T(x_1+x_2) + F(x_1+x_2) \\ &= Tx_1 + Fx_2 \\ &= y_1 + y_2 \\ &= y. \end{aligned}$$

Thus T is surjective, and therefore bijective, completing the proof.

□

A7.12. Theorem. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be Fredholm. If $K \in \mathcal{K}(\mathcal{H})$, then*

$$\operatorname{ind}(T+K) = \operatorname{ind} T.$$

Proof. Suppose $\operatorname{ind} T = 0$. Then there exists F finite rank such that $T+F$ is invertible. Moreover, $T+K = (T+F) + (K-F)$ and $K-F \in \mathcal{K}(\mathcal{H})$. Thus

$$T+K = (T+F)(I + (T+F)^{-1}(K-F)),$$

and so by Lemma A7.10, $\operatorname{ind}(T+K) = 0 = \operatorname{ind} T$.

Suppose next that $\operatorname{ind} T = n > 0$. Letting S denote the forward unilateral shift, $\operatorname{ind}(T \oplus S^n) = \operatorname{ind} T + \operatorname{ind} S^n = 0$. If $K \in \mathcal{K}(\mathcal{H})$, then $K \oplus 0 \in \mathcal{K}(\mathcal{H} \oplus \mathcal{H})$, and

$$\begin{bmatrix} T & 0 \\ 0 & S^n \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T+K & 0 \\ 0 & S^n \end{bmatrix}.$$

From above, $\operatorname{ind}((T+K) \oplus S^n) = 0 = \operatorname{ind}(T+K) + \operatorname{ind} S^n = \operatorname{ind}(T+K) - n$. Thus $\operatorname{ind}(T+K) = n = \operatorname{ind} T$.

If $\operatorname{ind} T = n < 0$, then $\operatorname{ind} T^* = -n > 0$. From above, for all $K \in \mathcal{K}(\mathcal{H})$,

$$\operatorname{ind}(T^* + K^*) = -n = -\operatorname{ind}(T+K),$$

and so $\operatorname{ind}(T+K) = n = \operatorname{ind} T$.

$\square$

A7.13. Theorem. *Let $\mathcal{H}$ be a Hilbert space and suppose that T, $R \in \mathcal{B}(\mathcal{H})$ are Fredholm. Then*

$$\operatorname{ind} TR = \operatorname{ind} T + \operatorname{ind} R.$$

Proof. First suppose that $\operatorname{ind} T = n > 0$ and $\operatorname{ind} R = m > 0$. Let S denote the unilateral forward shift. Then

$$\operatorname{ind} \begin{bmatrix} T & 0 \\ 0 & S^n \end{bmatrix} = 0 = \operatorname{ind} \begin{bmatrix} R & 0 \\ 0 & S^m \end{bmatrix}.$$

Thus there exists $K \in \mathcal{K}(\mathcal{H} \oplus \mathcal{H})$ such that $\begin{bmatrix} T & 0 \\ 0 & S^n \end{bmatrix} + K$ is invertible.

By Lemma A7.10,

$$0 = \operatorname{ind} \begin{bmatrix} R & 0 \\ 0 & S^m \end{bmatrix} = \operatorname{ind} \left(\begin{bmatrix} T & 0 \\ 0 & S^n \end{bmatrix} + K \right) \left(\begin{bmatrix} R & 0 \\ 0 & S^m \end{bmatrix} \right),$$

and by Theorem A7.13, this is equal to

$$\operatorname{ind} \begin{bmatrix} T & 0 \\ 0 & S^n \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & S^m \end{bmatrix} = \operatorname{ind} \begin{bmatrix} TR & 0 \\ 0 & S^{n+m} \end{bmatrix}.$$

Thus $0 = \operatorname{ind}(TR) + \operatorname{ind} S^{n+m} = \operatorname{ind} TR + (-n-m)$, and so $\operatorname{ind}(TR) = n + m = \operatorname{ind} T + \operatorname{ind} R$.

The cases where $n < 0$ (resp. $m < 0$) are handled similarly using $(S^*)^n$ (resp. $(S^*)^m$) instead of S^n (resp. S^m), and Theorem A7.13 if necessary.

$\square$

A7.14. Notation. Let $\operatorname{Fred}(\mathcal{H}) = \pi^{-1}(\mathcal{A}(\mathcal{H})^{-1})$ denote the set of Fredholm operators, and for each $n \in \mathbb{Z}$, set

$$\operatorname{Fred}_n(\mathcal{H}) = \{T \in \operatorname{Fred}(\mathcal{H}) : \operatorname{ind} T = n\}.$$

A7.15. Theorem. *Let $\mathcal{H}$ be a Hilbert space. Then for each $n \in \mathbb{Z}$, $\mathrm{Fred}_n(\mathcal{H})$ is open. In particular, therefore,* $\mathrm{ind}(\cdot)$ *is a continuous function on* $\mathrm{Fred}(\mathcal{H})$.

Proof. Of course, since $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{A}(\mathcal{H})$ is continuous, we see that $\mathrm{Fred}(\mathcal{H}) = \pi^{-1}((\mathcal{A}(\mathcal{H})^{-1})$ is open. Suppose $n \in \mathbb{Z}$.

Let $T \in \mathrm{Fred}_n(\mathcal{H})$. Since $\mathrm{Fred}(\mathcal{H})$ is open, there exists $\varepsilon_1 > 0$ such that $\|U\| < \varepsilon_1$ implies $T + U \in \mathrm{Fred}(\mathcal{H})$. Moreover, by Atkinson's Theorem A7.8, there exists $R \in \mathcal{B}(\mathcal{H})$ (in fact, $R \in \mathrm{Fred}_{-n}(\mathcal{H})$) such that $TR = I + K$ for some $K \in \mathcal{K}(\mathcal{H})$. Note that

$$\begin{aligned}(T+U)R &= TR + UR \\ &= (I+K) + UR \\ &= (I+UR) + K.\end{aligned}$$

Now take $\varepsilon_2 = 1/\|R\|$. If $\|U\| < \varepsilon_2$, then $I + UR$ is invertible in $\mathcal{B}(\mathcal{H})$. By Theorem A7.12, we conclude that if $\|U\| < \min(\varepsilon_1, \varepsilon_2)$, then $(T+U)R = (I+UR) + K$ satisfies

$$\begin{aligned}\mathrm{ind}\,(T+U)R &= \mathrm{ind}\,(I+UR) + K \\ &= \mathrm{ind}\,(I+UR) \\ &= 0 \\ &= \mathrm{ind}\,(T+U) + \mathrm{ind}\,R \\ &= \mathrm{ind}\,T + \mathrm{ind}\,R.\end{aligned}$$

Thus $\mathrm{ind}\,(T+U) = \mathrm{ind}\,T$ and so $T + U \in \mathrm{Fred}_n(\mathcal{H})$. In other words, $\mathrm{Fred}_n(\mathcal{H})$ is open.

□

A7.16. The astute reader may recall that the same word *index* appeared when we discussed the *abstract index group* $\mathcal{A}^{-1}/\mathcal{A}_0^{-1}$ of a Banach algebra $\mathcal{A}$. The same reader might now be asking if there is any relationship between the two notions of index. In fact, there is.

Let $\mathcal{A} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denote the Calkin algebra. We defined the abstract index group of $\mathcal{A}$ to be

$$\Gamma_{\mathcal{A}} = \mathcal{A}^{-1}/\mathcal{A}_0^{-1},$$

where $\mathcal{A}_0^{-1}$ denotes the connected component of the identity in $\mathcal{A}^{-1}$.

Consider the map

$$\begin{aligned}\gamma : \quad \mathcal{A}^{-1} &\to (\mathbb{Z}, +) \\ \pi(T) &\mapsto \mathrm{ind}(T).\end{aligned}$$

That γ is well-defined follows from Theorem A7.12, for if $\pi(T) = \pi(R) \in \mathcal{A}^{-1}$, then T and R are Fredholm operators and $T = R + K$ for some $K \in \mathcal{K}(\mathcal{H})$. Applying that Theorem shows that

$$\gamma(\pi(T)) = \mathrm{ind}(T) = \mathrm{ind}(R+K) = \mathrm{ind}(R) = \gamma(\pi(R)).$$

Now by Theorem A7.13,

$$\gamma(\pi(R)\pi(T)) = \gamma(\pi(RT)) = \operatorname{ind} RT = \operatorname{ind} R + \operatorname{ind} T = \gamma(\pi(R)) + \gamma(\pi(T)),$$

proving that γ is a group homomorphism. As noted in Example A7.3, $\operatorname{ind} S^n = -n = -\operatorname{ind}(S^*)^n$ for all $n \geq 1$, and $\operatorname{ind} I = 0$, so that γ is a surjective map. Thus $\mathbb{Z} \simeq \mathcal{A}^{-1}/\ker\gamma$, where $\ker\gamma = \{\pi(T) : T \in \mathrm{Fred}_0(\mathcal{H})\}$.

The last piece in the puzzle consists of showing that $\ker\gamma = \mathcal{A}_0^{-1}$. Unfortunately, we do not yet have the tools to do this. But by the end of the course, we shall.

A7.17. Theorem 7.19 shows us that in a Hilbert space $\mathcal{H}$, every compact operator K is a norm limit of finite rank operators F_n. Since $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$, it follows that $\mathcal{K}(\mathcal{H}) = \overline{\mathcal{F}(\mathcal{H})}$.

In the Banach space setting, the inclusion $\mathcal{F}(\mathfrak{X}) \subseteq \mathcal{K}(\mathfrak{X})$ remains valid. The question of whether the reverse inclusion holds remained open for some time, and was referred to as the Finite Approximation Problem. In 1973, Per Enflo [**21**] resolved this question by constructing an example of a Banach space $\mathfrak{X}$ and a compact operator on $\mathfrak{X}$ which cannot be approximated by finite rank operators.

One of the most famous open problems in Operator Theory today is the Invariant Subspace Problem.

- Given $\mathcal{H}$, a Hilbert space, and $T \in \mathcal{B}(\mathcal{H})$, does there exist a closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that $\mathcal{M} \neq \{0\}, \mathcal{H}$ and $T\mathcal{M} \subseteq \mathcal{M}$?

Such a space is called a *non-trivial invariant subspace for* T. It is a standard exercise that if $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{H}$ is *not* separable, then we can decompose $\mathcal{H} = \oplus_{\alpha\in\Lambda}\mathcal{H}_\alpha$, where each $\mathcal{H}_\alpha$ is a separable, reducing subspace for T. Also, if $\mathcal{H}$ is finite dimensional, every operator can be upper triangularised, and thus has invariant subspaces. As such, the proper context in which to examine the Invariant Subspace Problem is in separable, infinite dimensional Hilbert spaces.

While the answer is not known in general, many results have been obtained. One of the strongest results is a generalisation of a result of Lomonosov [**34**] from 1973.

A7.18. Theorem. [Lomonosov's Theorem.] *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be a non-scalar operator. Suppose there exists $0 \neq K \in \mathcal{K}(\mathcal{H})$ such that $TK = KT$. Then there exists a closed subspace $\mathcal{M}$ of $\mathcal{H}$ which is **hyperinvariant** for T, that is: $\mathcal{M}$ is a non-trivial invariant subspace for every operator that commutes with T.*

A7.19. Corollary. *Every compact operator on $\mathcal{H}$ has a non-trivial hyperinvariant subspace.*

A7.20. A natural question that arises from this theorem is whether or not every operator in $\mathcal{B}(\mathcal{H})$ commutes with a non-scalar operator which in turn commutes with a non-zero compact operator. In other words, does Lomonosov's Theorem solve the Invariant Subspace Problem? That the answer is *no* was first shown by D.H. Hadwin, E.A. Nordgren, H. Radjavi, and P. Rosenthal [**24**].

Results are known for other classes of operators as well. For example, we saw the next result of S. Brown [**9**] in Example S7.3.

A7.21. Theorem. **[Brown.]** *Every subnormal operator possesses a non-trivial invariant subspace.*

A slightly more recent result is the following.

A7.22. Theorem. **[Brown, Chevreau, Pearcy [10].]** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Suppose that*

- $\|T\| \le 1$*; and*
- $\sigma(T) \supseteq \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Then T has a non-trivial invariant subspace.

A7.23. The corresponding question has been answered (negatively) for Banach spaces. In particular, in 1984, C.J. Read [**45**] gave an example of a Banach space $\mathfrak{X}$ and a bounded linear operator T on $\mathfrak{X}$ such that $\mathfrak{X}$ and $\{0\}$ are the only closed subspaces of $\mathfrak{X}$ which are invariant for T. In the paper [**46**] from 1985, he modified the construction to produce a bounded linear operator $T \in \mathcal{B}(\ell^1)$ such that T does *not* have any non-trivial invariant subspace. The question remains open for reflexive Banach spaces.

A7.24. As we have seen, the Invariant Subspace Problem for Hilbert spaces has remained open for over 80 years. In the past 10 years, however, an entirely new and exciting approach has been put forward with the work of A. Popov and A. Tcaciuc, as well as further work by A. Tcaciuc on his own.

A7.25. Definition. *Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and $T \in \mathcal{B}(\mathcal{H})$. A subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be* ***almost-invariant*** *for T if there exists a finite-dimensional subspace $\mathcal{F}$ of $\mathcal{H}$ such that $T\mathcal{M} \subseteq \mathcal{M} + \mathcal{F}$. If $\mathcal{M}$ is almost-invariant for T, then the minimum dimension of a space $\mathcal{F}$ for which $T\mathcal{M} \subseteq \mathcal{M} + \mathcal{F}$ is called the* ***defect*** *of $\mathcal{M}$. We say that $\mathcal{M}$ is a* ***half-space*** *if* $\dim \mathcal{M} = \dim \mathcal{M}^\perp = \infty$.

(One can extend the notion of a half-space to the Banach space setting as follows: a subspace $\mathfrak{Y}$ of a Banach space $\mathcal{X}$ is a half-space if $\dim \mathfrak{Y} = \dim \mathfrak{X}/\mathfrak{Y} = \infty$.)

Consider the following absolutely beautiful results. They come as close to solving the Invariant Subspace Problem as one can come without actually solving it. What is most amazing is that (in the first instance) it applies to *all* Hilbert space operators. So far, all of the positive results (other than Lomonosov's) applies to comparitively small classes of operators.

A7.26. Theorem. **[Popov and Tcaciuc [41].]** *Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and $T \in \mathcal{B}(\mathcal{H})$. Then T admits an almost-invariant half-space of defect equal to* 1.

A7.27. It is not hard to see that a half-space $\mathcal{M}$ is invariant for T if and only if there exists a finite-rank operator F such that $\mathcal{M}$ is invariant for $T + F$. Moreover, one can choose F to have rank equal to the defect of $\mathcal{M}$, and thus the above result states that every operator admits a rank-one perturbation with an invariant half-space. (That every operator admits a rank-one perturbation with an invariant subspace is trivial; the issue here is that the invariant subspace in question has infinite dimension and co-dimension!) In some cases, Popov and Tcaciuc were even able to control the norm of the operator F. A new and quite spectacular result of Tcaciuc extends this to all Banach spaces and all operators.

Theorem. **[Tcaciuc [51].]** *Let $\mathfrak{X}$ be a complex Banach space and $T \in \mathcal{B}(\mathfrak{X})$. Given $\varepsilon > 0$, there exists $F \in \mathcal{B}(\mathfrak{X})$ of rank at most one and $\|F\| < \varepsilon$ such that $T + F$ admits an invariant half-space.*

A7.28. If one considers *reducing* rather than invariant subspaces, then one can also obtain *approximate* results. A major result of D. Voiculescu's [**53**] known as his non-commutative Weyl-von Neumann Theorem implies that given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exist an isometric, involution preserving map ρ from $C^*(\pi(T))$, the closed Banach algebra generated by $\pi(T)$ and $\pi(T^*)$ in the Calkin algebra, into some $\mathcal{B}(\mathcal{H}_\rho)$, a unitary operator $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}_\rho^{(\infty)})$ and $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that

$$T = U^*(T \oplus \rho(\pi(T))^{(\infty)})U + K.$$

It follows that every operator is a *limit* of operators with non-trivial reducing subspaces.

On the other hand, P. Halmos [**25**] has shown that the set of *irreducible operators* (i.e. those with no non-trivial reducing subspaces) is dense in $\mathcal{B}(\mathcal{H})$.

What is not known, and what remains a *very interesting* open problem, is whether or not every Hilbert space operator T admits a half-space which is almost invariant both for T and T^*.

A7.28. The spectral theorem for compact normal operators shows that every such operator can be diagonalised. As such, it mimics the finite-dimensional result. For general normal operators on an infinite dimensional Hilbert space, this fails miserably. For instance, if M_x is the multiplication operator acting on $L^2([0,1], dx)$, where dx represents Lebesgue measure, then we have seen that M_x is normal, but has no eigenvalues. It follows immediately from this observation that M_x can not be diagonalisable. A wonderful result known as the Weyl-von Neumann-Berg/Sikonia Theorem [**6**] shows that once again, the result is true *up to a small compact perturbation.* More precisely, we have:

Theorem. **[The Weyl-von Neumann-Berg/Sikonia Theorem.]** *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal. Then, given $\varepsilon > 0$, there exists $U \in \mathcal{B}(\mathcal{H})$ unitary, $K \in \mathcal{K}(\mathcal{H})$ satisfying $\|K\| < \varepsilon$ and $D \in \mathcal{B}(\mathcal{H})$ diagonal such that*

$$N = U^* D U + K.$$

Moreover, D can be chosen to have the same spectrum and essential spectrum *(see Appendix A) as N.*

A7.29. Using this, we are now in a position to give a very simple proof of Halmos' result on the density of the irreducible operators. This proof is due to H. Radjavi and P. Rosenthal [**43**]. Let us agree to say that an operator $D \in \mathcal{B}(\mathcal{H})$ is *diagonalisable* if there exists a unitary operator U such that the matrix of $U^* D U$ with respect to the standard orthonormal basis is diagonal.

Theorem. **[Radjavi and Rosenthal.]** *Let $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Then there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ is irreducible.*

Proof. By the Weyl-von Neumann-Berg/Sikonia Theorem, there exists a self-adjoint operator D whose matrix is diagonal with respect to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ such that

$$\|D - (T^* - T)/2\| < \frac{\varepsilon}{4}.$$

Then there is a self-adjoint operator D_1 diagonal with respect to $\{e_n\}_{n=1}^{\infty}$ such that all of the eigenvalues of D_1 are distinct and $\|D - D_1\| < \frac{\varepsilon}{4}$. Now let D_2 be any self-adjoint compact operator within $\varepsilon/2$ of $(T - T^*)/2i$ whose matrix with respect to $\{e_n\}_{n=1}^{\infty}$ has all entries different from 0 (such operators exist in profusion - why?). Then the operator $D_1 + iD_2$ is within ε of T. Also, $D_1 + iD_2$ is irreducible, since the invariant subspaces of D_1 are the subspaces spanned by subcollections of $\{e_n\}_{n=1}^{\infty}$, and none of these are invariant under D_2 except $\{0\}$ and $\mathcal{H}$.

$\square$

Exercises for Chapter 7

Exercise 7.1. Compact operators as limits of finite-rank operators

Suppose that $\mathfrak{X}$ is a Banach space and $\mathcal{H}$ is a Hilbert space. Prove that if $K : \mathfrak{X} \to \mathcal{H}$ is compact, then K is a limit of finite-rank operators from $\mathfrak{X}$ into $\mathcal{H}$.

Exercise 7.2. Adjoints of operator matrices

Suppose that $\mathcal{H}_k$ is a complex Hilbert space, $1 \leq k \leq n$, and that $T \in \mathcal{B}(\oplus_{k=1}^n \mathcal{H}_k)$ admits an operator-matrix form

$$T = [T_{i,j}],$$

where $T_{i,j} = P_{\mathcal{H}_i} T|_{\mathcal{H}_j}$, $1 \leq i, j \leq n$. Then

$$T^* = [T_{j,i}^*].$$

(This was used in Proposition 7.29.)

Exercise 7.3. Spectrum of operator matrices

Suppose $T = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ as an operator on $\mathcal{H} \oplus \mathcal{H}$.

(i) Is $\sigma(A) \subseteq \sigma(T)$?
(ii) Is $\sigma(D) \subseteq \sigma(T)$?
(iii) What can be said about the sets of eigenvalues of A and D with respect to those of T?
(iv) Is $\sigma(T) \subseteq \sigma(A) \cup \sigma(D)$?

Exercise 7.4. Injective with dense range

Find an operator $T \in \mathcal{B}(\mathcal{H})$ such that T is injective, the range of T is dense, but T is not invertible.

Exercise 7.5. Distance to the set of invertible elements

Let S be the unilateral shift opeator acting on a Hilbert space $\mathcal{H}$. Show that the distance from S to the set of invertible operators on $\mathcal{H}$ is exactly 1.

Exercise 7.6. Polynomially compact operators

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **polynomially compact** if there exists a non-zero polynomial $p \in \mathbb{C}[x]$ such that $p(T) \in \mathcal{K}(\mathcal{H})$.

Describe the spectrum of a polynomially compact operator.

Exercise 7.7. Distance to the compact operators

What is the distance from the unilateral shift $S \in \mathcal{B}(\mathcal{H})$ to the set $\mathcal{K}(\mathcal{H})$ of compact operators?

Exercise 7.8. Weighted shift operators

Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $\{w_n\}_{n=1}^{\infty} \in \ell^{\infty}(\mathbb{N})$ and define

$$\begin{array}{rccc} W: & \mathcal{H} & \to & \mathcal{H} \\ & e_n & \mapsto & w_n e_{n+1} \end{array}$$

for each $n \geq 1$. Extend W by linearity and continuity to all of $\mathcal{H}$. Such an operator is called a *unilateral forward weighted shift with weight sequence* $\{w_n\}_{n=1}^{\infty}$.

(i) Calculate $\|W\|$.
(ii) Calculate $\sigma(W)$ in the case where $w_n = 1$ for all n. This particular operator is called *the* unilateral shift.
(iii) Calculate $\operatorname{spr}(W)$ in general.
(iv) When is W nilpotent?
(v) When is W compact?
(vi) If W is compact, compute $\sigma(W)$.
(vii) When is W quasinilpotent? (Recall that $T \in \mathcal{B}(\mathcal{H})$ is *quasinilpotent* if $\sigma(W) = \{0\}$.)
(viii) Find a unilateral weighted shift W of norm 1 such that W is quasinilpotent but not nilpotent. Is it possible to find one that is nilpotent but not quasinilpotent?

Exercise 7.9. Similarity and unitary equivalence

Two operators $A, B \in \mathcal{B}(\mathcal{H})$ are said to be **unitarily equivalent** if there exists $U \in \mathcal{B}(\mathcal{H})$ unitary such that $A = U^*BU$. We say that A and B are **similar** if there exists an invertible operator $R \in \mathcal{B}(\mathcal{H})$ such that $A = R^{-1}BR$.

(a) Prove that unitary equivalence and similarity define equivalence relations on $\mathcal{B}(\mathcal{H})$.
(b) Give necessary and sufficient conditions for two diagonal operators to be unitarily equivalent.
(c) Give necessary and sufficient conditions for two diagonal operators to be similar.
(d) Prove that two injective unilateral weighted shifts W with weight sequence $(w_n)_n$ and V with weight sequence $(v_n)_n$ are unitarily equivalent if and only if $|w_n| = |v_n|$ for all $n \geq 1$.

Is the same true for injective bilateral shifts?

Exercise 7.10. Weighted shift operators II

Which weighted shifts (bilateral or unilateral) are:

(i) normal?
(ii) self-adjoint?
(iii) unitary?
(iv) essentially unitary?
(v) essentially normal?
(vi) essentially self-adjoint?

Exercise 7.11. Hilbert-Schmidt operators

Suppose $\mathcal{H}$ is a Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that the matrix of T with respect to this basis is $[t_{ij}]$ Finally, suppose that

$$\|T\|_2 := \left(\sum_{i,j} |t_{ij}|^2 \right)^{\frac{1}{2}} < \infty.$$

(i) Show that $\|T\| < \|T\|_2$.
(ii) Show that $T \in \mathcal{K}(\mathcal{H})$.
(iii) The set of all operators T for which $\|T\|_2$ is finite is called the *Hilbert-Schmidt class* and is sometimes denoted by $\mathcal{C}_2(\mathcal{H})$. Show that $\mathcal{C}_2(\mathcal{H})$ is a *proper* subset of $\mathcal{K}(\mathcal{H})$.

CHAPTER 8

Representations of Banach algebras

He was happily married – but his wife wasn't.

Victor Borge

Definitions and Examples

8.1. The notion of representations is ubiquitous in mathematics, and the theory of Banach algebras is no exception. The guiding philosophy is that we have many techniques to study operators acting on a Banach space $\mathfrak{X}$ that are not available to us in general Banach algebras. For example, we have seen that an operator $T \in \mathcal{B}(\mathfrak{X})$ is invertible if and only if it is bounded below and has dense range. If $T \in \mathcal{A}$ for some unital Banach subalgebra of $\mathcal{B}(\mathfrak{X})$, this offers us a necessary (but not sufficent) condition for verifying that $T \in \mathcal{A}^{-1}$.

8.2. Definition. *Let $\mathcal{A}$ be a Banach algebra and $\mathfrak{X}$ be a Banach space. A* ***representation*** *of $\mathcal{A}$ on $\mathfrak{X}$ is a non-zero homomorphism $\varphi : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$.*

Unsurprisingly, our main focus will be on *continuous* representations.

8.3. Definition. *Let $\mathcal{A}$ be a Banach algebra and $\mathfrak{X}$ be a Banach space. We say that $\mathfrak{X}$ is a* ***left Banach $\mathcal{A}$-module*** *if $\mathfrak{X}$ is a left $\mathcal{A}$-module and there exists a constant $\kappa > 0$ such that for all $a \in \mathcal{A}$, $x \in \mathfrak{X}$ we have:*

$$\|a \bullet x\|_{\mathfrak{X}} \leq \kappa \, \|a\|_{\mathcal{A}} \, \|x\|_{\mathfrak{X}}.$$

8.4. Remark. Note that if $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ is a continuous representation of $\mathcal{A}$, then $\mathfrak{X}$ is imbued with a left Banach $\mathcal{A}$-module structure over $\mathcal{A}$, namely: given $x \in \mathfrak{X}$ and $a \in \mathcal{A}$, we may define $a \bullet x := \varrho(a)x \in \mathfrak{X}$. That this is a left $\mathcal{A}$-module structure is routine, and given $a \in \mathcal{A}$, $x \in \mathfrak{X}$ we see that

$$\|a \bullet x\|_{\mathfrak{X}} = \|\varrho(a)x\|_{\mathfrak{X}} \leq \|\varrho\| \, \|a\|_{\mathcal{A}} \, \|x\|_{\mathfrak{X}}.$$

Setting $\kappa := \|\varrho\|$ (or $\kappa = 1$ if $\varrho = 0$) shows that this is indeed a left Banach $\mathcal{A}$-module structure.

Conversely, suppose that $\mathfrak{X}$ is a left Banach $\mathcal{A}$-module, with constant $\kappa > 0$ as above. Given $a \in \mathcal{A}$, define a map $\varrho(a) : \mathfrak{X} \to \mathfrak{X}$ via $\varrho(a)x = a \bullet x$. It is not hard to see that $\varrho(a)$ is linear, and the inequality

$$\|\varrho(a)x\|_{\mathfrak{X}} = \|a \bullet x\|_{\mathfrak{X}} \leq \kappa \|a\|_{\mathcal{A}} \, \|x\|_{\mathfrak{X}}$$

implies that $\|\varrho(a)\| \leq \kappa \|a\|$.

It is equally straightforward to verify that the map $\varrho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defined by $a \mapsto \varrho(a)$ is an algebra homomorphism, and the inequality above shows that $\|\varrho\| \leq \kappa < \infty$, so that ϱ is a continuous representation of $\mathcal{A}$ on $\mathfrak{X}$.

We have just argued that continuous representations of $\mathcal{A}$ on Banach spaces and left Banach $\mathcal{A}$-modules are two sides of the same coin: that is, they are two ways of viewing the same object.

8.5. Example. Let $\mathcal{A}$ be a Banach algebra. With $\mathfrak{X} := \mathcal{A}$, we may define a representation

$$\begin{array}{rccc} \lambda : & \mathcal{A} & \to & \mathcal{B}(\mathcal{A}) \\ & a & \mapsto & L_a, \end{array}$$

where $L_a x = ax$ for all $x \in \mathcal{A}$. Clearly $\|L_a\| \leq \|a\|$, so that λ is a contractive representation of $\mathcal{A}$.

8.6. Example. In Example 7.14 we saw that if μ is a finite, positive, regular Borel measure on a non-empty set X, then for each $f \in L^\infty(X, \mu)$ we may define the multiplication operator $M_f \in \mathcal{B}(L^2(X, \mu))$ via $M_f g = fg$ for all $g \in L^2(X, \mu)$.

We leave it to the reader to verify that the map

$$\begin{array}{rccc} \varrho : & L^\infty(X, \mu) & \to & \mathcal{B}(L^2(X, \mu)) \\ & f & \mapsto & M_f \end{array}$$

defines an isometric representation of $L^\infty(X, mu)$ on $L^2(X, \mu)$. In fact, this representation has an extra property, namely: $\varrho(f^*) = \varrho(\overline{f}) = M_{\overline{f}} = M_f^*$ for all $f \in L^\infty(X, \mu)$. We normally say that ϱ is a *-representation. Of course, most Banach algebras do not admit an involution. We shall have more to say about this later when we examine C^*-algebras.

8.7. Suppose that $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ is a continuous representation of a Banach algebra $\mathcal{A}$ on a Banach space $\mathfrak{X}$. Let

$$\mathfrak{Y} := \overline{\text{span}} \ \varrho(\mathcal{A})\mathfrak{X} = \overline{\text{span}}\{\varrho(a)x : a \in \mathcal{A}, x \in \mathfrak{X}\}.$$

It is not hard to see that $\mathfrak{Y}$ is a closed subspace of $\mathfrak{X}$, and that $\varrho(a)\mathfrak{Y} \subseteq \mathfrak{Y}$ for all $a \in \mathcal{A}$. That is, $\mathfrak{Y}$ is invariant for $\varrho(\mathcal{A}) \subseteq \mathcal{B}(\mathfrak{X})$.

In trying to study any mathematical object, it is worthwhile to break it up into "simpler components". The next definition tells us what we mean by this in the context of representations of Banach algebras.

8.8. Definition. *Let $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ be a continuous representation of a Banach algebra $\mathcal{A}$ on a Banach space $\mathfrak{X}$. We say that ϱ is **algebraically irreducible** if $0 \neq x \in \mathfrak{X}$ implies that*

$$\varrho(\mathcal{A})x := \{\varrho(a)x : a \in \mathcal{A}\} = \mathfrak{X}.$$

Equivalently, ϱ is algebraically irreducible if the only $\mathcal{A}$-invariant linear manifolds in $\mathfrak{X}$ are $\{0\}$ and $\mathfrak{X}$.

*We say that ϱ is **topologically irreducible** if $0 \neq x \in \mathfrak{X}$ implies that*

$$\varrho(\mathcal{A})x := \{\varrho(a)x : a \in \mathcal{A}\}$$

*is dense in $\mathfrak{X}$. Equivalently, ϱ is topologically irreducible if the only **closed** $\mathcal{A}$-invariant linear subspaces in $\mathfrak{X}$ are $\{0\}$ and $\mathfrak{X}$.*

8.9. The Cohen-Hewitt Factorisation Theorem below will give us some greater insight into the relationship between algebraically irreducible and topologically irreducible representations of certain Banach algebras. But first we turn our attention to a theorem of Jacobson.

Jacobson's Density Theorem

8.10. Suppose that $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ is an algebraically irreducible representation of a Banach algebra $\mathcal{A}$ on a Banach space $\mathfrak{X}$. Then $\varrho(\mathcal{A})$ is (by definition) **transitive**. That is, for all $0 \neq x \in \mathfrak{X}$ and all $y \in \mathfrak{X}$ there exists $a \in \mathcal{A}$ such that $\varrho(a)x = y$.

Our goal is to extend this result by showing that for each $n \geq 1$, $\varrho(\mathcal{A})$ is in fact n**-transitive**; that is, given linearly independent vectors $x_1, x_2, \ldots, x_n \in \mathfrak{X}$ and arbitrary vectors $y_1, y_2, \ldots, y_n \in \mathfrak{X}$, we can find $a \in \mathcal{A}$ such that $\varrho(a)x_k = y_k$, $1 \leq k \leq n$.

8.11. Lemma. *Let $\mathcal{A}$ be a Banach algebra, $\mathfrak{X}$ be a Banach space, and $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ be a continuous, algebraically irreducible representation of $\mathcal{A}$. Suppose that $T : \mathfrak{X} \to \mathfrak{X}$ is a non-zero linear map commuting with ϱ; that is,*

$$T\varrho(a)x = \varrho(a)Tx \text{ for all } x \in \mathfrak{X}, a \in \mathcal{A}.$$

Then $T = \lambda I$ for some $\lambda \in \mathbb{C}$.

Proof. If $Tx_0 = 0$ for some $0 \neq x_0 \in \mathfrak{X}$, then

$$\{0\} = \varrho(\mathcal{A})(Tx_0) = T(\varrho(\mathcal{A})x_0) = T\mathfrak{X},$$

contradicting the fact that T is non-zero. Thus T is injective.

Since $T \neq 0$, there exists $x_1 \in \mathfrak{X}$ such that $Tx_1 \neq 0$. In particular, $x_1 \neq 0$. Then

$$T\mathfrak{X} = T(\varrho(\mathcal{A})x_1) = \varrho(\mathcal{A})(Tx_1) = \mathfrak{X},$$

implying that T is surjective.

Combining these two facts, we deduce that T is bijective, hence invertible (as a linear but not necessarily continuous linear map).

Let $\mathfrak{D} := \{D : \mathfrak{X} \to \mathfrak{X} : D \text{ is linear and } D \text{ commutes with } \varrho\}$. It is easy to see that $\mathfrak{D}$ is an algebra over $\mathbb{C}$ which contains all scalar operators, and the argument above shows that every non-zero element of D is invertible. Since the only division algebra over $\mathbb{C}$ is (isomorphic to) $\mathbb{C}$ itself, we conclude that $\mathfrak{D} := \{\alpha I : \alpha \in \mathbb{C}\}$.

□

8.12. Proposition. *Let $\mathcal{A}$ be a Banach algebra, $\mathfrak{X}$ be a Banach space, and $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ be a continuous representation of $\mathcal{A}$. Let $0 \neq x \in \mathfrak{X}$. If ϱ is algebraically irreducible, then*

$$\mathcal{M} := \{a \in \mathcal{A} : \varrho(a)x = 0\}$$

is a maximal left ideal of $\mathcal{A}$.

Proof. That $\mathcal{M}$ is a left ideal of $\mathcal{A}$ is routine and is left as an exercise for the reader.

Suppose that $\mathcal{K} \subseteq \mathcal{A}$ is a left ideal of $\mathcal{A}$ which contains $\mathcal{M}$ properly. Thus there exists $k \in \mathcal{K} \setminus \mathcal{M}$, whence $\varrho(k)x \neq 0$. Since ϱ is algebraically irreducible, $\varrho(\mathcal{A})(\varrho(k)x) = \mathfrak{X}$, and thus there exists an element $a \in \mathcal{A}$ such that $\varrho(a)\varrho(k)x = x$. Set $e := ak \in \mathcal{K}$.

Given $b \in \mathcal{A}$, observe that $\varrho(be - b)x = \varrho(b)\varrho(e)x - \varrho(b)x = 0$, and so $be - b \in \mathcal{M} \subseteq \mathcal{K}$. But $e \in \mathcal{K}$ implies that $be \in \mathcal{K}$, and thus $b = be - (be - b) \in \mathcal{K}$. Since $b \in \mathcal{A}$ was arbitrary, $\mathcal{A} \subseteq \mathcal{K}$, and thus $\mathcal{M}$ is maximal.

□

8.13. Lemma. *Let $\mathcal{A}$ be a Banach algebra and suppose that $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ is a continuous, algebraically irreducible representation of $\mathcal{A}$ on the Banach space $\mathfrak{X}$. Let $x_1, x_2 \in \mathfrak{X}$ be two linearly independent vectors. Then there exists $a_0 \in \mathcal{A}$ such that $\varrho(a_0)x_1 = 0 \neq \varrho(a_0)x_2$.*

Proof. We argue by contradiction. Suppose to the contrary that $\varrho(a)x_1 = 0$ implies that $\varrho(a)x_2 = 0$. Let $\mathcal{M}_k := \{a \in \mathcal{A} : \varrho(a)x_k = 0\}$, $k = 1, 2$. Then $\mathcal{M}_1 \subseteq \mathcal{M}_2$. By Proposition 8.12 above, $\mathcal{M}_1$ and $\mathcal{M}_2$ are maximal left ideals of $\mathcal{A}$, and thus $\mathcal{M}_1 = \mathcal{M}_2$.

The fact that neither of x_1, x_2 is zero implies that $\varrho(\mathcal{A})x_1 = \mathfrak{X} = \varrho(\mathcal{A})x_2$. Consider the map

$$\begin{array}{cccc} T: & \mathfrak{X} & \to & \mathfrak{X} \\ & \varrho(a)x_1 & \mapsto & \varrho(a)x_2. \end{array}$$

We claim that T is a well-defined linear map that commutes with ϱ.

Indeed, if $\varrho(a_1)x_1 = \varrho(a_2)x_1$, then $\varrho(a_1 - a_2)x_1 = 0$, so $a_1 - a_2 \in \mathcal{M}_1 = \mathcal{M}_2$, implying that

$$T\varrho(a_1)x_1 = \varrho(a_1)x_2 = \varrho(a_2)x_2 = T\varrho(a_2)x_1.$$

That is, T is well-defined.

Also, for all $\alpha \in \mathbb{C}$,

$$\begin{aligned} T(\alpha\varrho(a_1)x_1 + \varrho(a_2)x_1) &= T(\varrho(\alpha a_1 + a_2)x_1) \\ &= \varrho(\alpha a_1 + a_2)x_2 \\ &= \alpha\varrho(a_1)x_2 + \varrho(a_2)x_2 \\ &= \alpha T\varrho(a_1)x_1 + T\varrho(a_2)x_1, \end{aligned}$$

proving that T is linear.

Next, for all $b \in \mathcal{A}$,

$$\begin{aligned}\varrho(b)T\varrho(a)x_1 &= \varrho(b)\varrho(a)x_2\\ &= \varrho(ba)x_2\\ &= T\varrho(ba)x_1\\ &= T\varrho(b)\varrho(a)x_1,\end{aligned}$$

so that $\varrho(b)T = T\varrho(b)$; i.e. T commutes with ϱ.

By Lemma 8.11 above, $T = \lambda I$ for some $\lambda \in \mathbb{C}$. But then

$$\lambda\varrho(a)x_1 = T\varrho(a)x_1 = \varrho(a)x_2 \text{ for all } a \in \mathcal{A},$$

and thus

$$\varrho(a)(\lambda x_1 - x_2) = 0 \text{ for all } a \in \mathcal{A}.$$

Since ϱ is algebraically irreducible, we conclude that $x_2 = \lambda x_1$, contradicting our hypothesis that x_1, x_2 are linearly independent vectors. This concludes the proof.

□

The next Lemma is adapted from [**8**].

8.14. Lemma. *Let $\mathcal{A}$ be a Banach algebra and suppose that $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ is a continuous, algebraically irreducible representation of $\mathcal{A}$ on the Banach space $\mathfrak{X}$. Let $2 \le n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \mathfrak{X}$ be linearly independent. Then there exists $b \in \mathcal{A}$ such that $\varrho(b)x_n \ne 0 = \varrho(b)x_k$, $1 \le k < n$.*

Proof. We shall argue by induction. The case $n = 2$ is handled by Lemma 8.13 above.

Suppose that $3 \le N \in \mathbb{N}$ and that the result holds whenever $n < N$. We prove that the result holds for $n = N$.

Now $x_2, x_3, \ldots, x_N$ are $N-1$ linearly independent vectors in $\mathfrak{X}$, so by our induction hypothesis, there exists $a_1 \in \mathcal{A}$ such that $\varrho(a_1)x_k = 0$, $2 \le k \le N-1$ and $\varrho(a_1)x_N \ne 0$.

- If $\varrho(a_1)x_1 = 0$, then we set $b = a_1$ and we are done.
- If $\varrho(a_1)x_1$ and $\varrho(a_1)x_N$ are linearly independent, then by Lemma 8.13, we can find $a_2 \in \mathcal{A}$ such that $\varrho(a_2)\varrho(a_1)x_1 = 0 \ne \varrho(a_2)\varrho(a_1)x_N$. Setting $b = a_2a_1$, we find that $\varrho(b)x_k = 0$, $1 \le k \le N-1$, while $\varrho(b)x_N \ne 0$, and we are done.

We have reduced the problem to the case where there exists $\alpha \in \mathbb{C}$ such that $\alpha\varrho(a_1)x_1 = \varrho(a_1)x_N$.

Observe that the set $\{\alpha x_1 - x_N, x_2, x_3, \ldots, x_{N-1}\}$ is linearly independent. By the induction hypothesis, we can find $a_3 \in \mathcal{A}$ such that $\varrho(\alpha x_1 - x_N) \ne 0 = \varrho(a_3)x_k$, $2 \le k \le N-1$.

- If $\varrho(a_3)x_1 = 0$, then we also have that $\varrho(a_3)x_N \ne 0$. If we set $b = a_3$, then we are done.

- If $\varrho(a_3)x_1 \neq 0$ and $\{\varrho(a_3)x_1, \varrho(a_3)x_N\}$ is linearly independent, then we can find $a_4 \in \mathcal{A}$ such that $\varrho(a_4a_3)x_1 = 0 \neq \varrho(a_4a_3)x_N$. We may then set $b = a_4a_3$ and we are done.

We have reduced the problem to the case where $\varrho(a_3)x_1 \neq 0$ and $\{\varrho(a_3)x_1, \varrho(a_3)x_N\}$ is linearly independent.

- If $\varrho(a_3)x_N = 0$, then we set $b = a_3$. We then have $\varrho(b)x_1 \neq 0 = \varrho(b)x_k$, $2 \leq k \leq N$. By interchanging the roles of x_1 and x_N, we are done.

We have reduced the problem to the case where there exists some $\mu \in \mathbb{C}$ such that $\mu\varrho(a_3)x_1 = \varrho(a_3)x_N$, so that

$$\varrho(a_3)(\mu x_1 - x_N) = 0 \neq \varrho(a_3)(\lambda x_1 - x_N).$$

From this it follows that $\mu \neq \lambda$. But then $\varrho(a_3)x_1 \neq 0$ implies that there exists $a_5 \in \mathcal{A}$ such that

$$\varrho(a_5)\varrho(a_3)x_1 = \varrho(a_1)x_1.$$

Let $b = a_1 - a_5a_3$. An easy calculation shows that $\varrho(b)x_k = 0$, $1 \leq k \leq N-1$, while $\varrho(b)x_N \neq 0$.

□

8.15. Theorem. [The Jacobson Density Theorem.] *Let $\mathcal{A}$ be a Banach algebra and suppose that $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ is a continuous, algebraically irreducible representation of $\mathcal{A}$ on the Banach space $\mathfrak{X}$. If $N \in \mathbb{N}$, $x_1, x_2, \ldots, x_N \in \mathfrak{X}$ are linearly independent and $y_1, y_2, \ldots, y_N \in \mathfrak{X}$, then there exists $d \in \mathcal{A}$ such that $\varrho(d)x_n = y_n$, $1 \leq n \leq N$.*

In other words, $\varrho(\mathcal{A})$ is N-transitive.

Proof. For each $1 \leq k \leq N$, choose $a_k \in \mathcal{A}$ such that $\varrho(a_k)x_k \neq 0$ while $\varrho(a_k)x_n = 0$, $1 \leq n \neq k \leq N$. Since ϱ is algebraically irreducible, we can find $b_n \in \mathcal{A}$, $1 \leq n \leq N$ such that $\varrho(b_na_n)x_n = y_n$.

Let $d = \sum_{k=1}^N b_ka_k$. Then, for each $1 \leq n \leq N$,

$$\varrho(d)x_n = \sum_{k=1}^N \varrho(b_k)\varrho(a_k)x_n = \varrho(b_n)\varrho(a_n)x_n = y_n.$$

□

The Cohen-Hewitt factorisation theorem

8.16. We have seen that every non-unital Banach algebra $\mathcal{A}$ embeds isometrically into its unitisation $\tilde{\mathcal{A}}$. Without this embedding, there is in general no element – or collection of elements – that can serve to replace the notion of the missing identity element. For example, if $\mathcal{A} = \left\{\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} : z \in \mathbb{C}\right\}$, then $ab = 0$ for all $a, b \in \mathcal{A}$, which is about as far from being unital as one can hope for. Many other such examples can be produced. We now examine Banach algebras which might fail to be unital, but which nevertheless admit "the next best thing" to an identity element. This will have many important consequences for the algebra. We shall only consider the one major theorem mentioned above.

8.17. Definition. *Let $\mathcal{A}$ be a Banach algebra. A net $(e_\lambda)_\lambda$ in $\mathcal{A}$ is said to be a* ***bounded left approximate identity*** *for $\mathcal{A}$ if there exists a real number $\mu > 0$ such that $\|e_\lambda\| \le \mu$ for all $\lambda \in \Lambda$ and for each $a \in \mathcal{A}$,*

$$\lim_\lambda \|e_\lambda a - a\| = 0.$$

When this is the case, we shall refer to $(e_\lambda)_\lambda$ as a μ-bounded left approximate identity.

8.18. Remark. Not every Banach algebra admits a bounded left approximate identity, let alone a **left approximate identity** (i.e. a net $(e_\lambda)_\lambda$ for which $\lim_\lambda \|e_\lambda a - a\| = 0$ for all $a \in \mathcal{A}$, regardless of whether or not $(e_\lambda)_\lambda$ is bounded).

Suppose, however that $\mathcal{A}$ is a Banach algebra that *does* admit a bounded left approximate identity. For every representation $\varrho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, we then find that

$$\lim_\lambda \|\varrho(e_\lambda)\varrho(a)x - \varrho(a)x\| = 0$$

for all $x \in \mathfrak{X}$, and thus $\lim_\lambda \|\varrho(e_\lambda)y - y\| = 0$ for all $y \in \overline{\operatorname{span}}\varrho(\mathcal{A})\mathfrak{X}$. In particular, if ϱ is topologically irreducible, then

$$\lim_\lambda \|\varrho(e_\lambda)x - x\| = 0.$$

8.19. Suppose that $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ is a continuous representation of a Banach algebra $\mathcal{A}$ on a Banach space $\mathfrak{X}$. Recall that the unitisation of $\mathcal{A}$ is the space $\tilde{\mathcal{A}} = \mathbb{C} \oplus \mathcal{A}$ equipped with the norm $\|(\alpha, a)\|_{\tilde{\mathcal{A}}} := |\alpha| + \|a\|_{\mathcal{A}}$.

We leave it to the reader to verify that the map

$$\begin{array}{cccc} \varrho_u : & \tilde{\mathcal{A}} & \to & \mathcal{B}(\mathfrak{X}) \\ & (\alpha, a) & \mapsto & \alpha I + \varrho(a) \end{array}$$

is a continuous representation of $\tilde{\mathcal{A}}$ on $\mathfrak{X}$. This will come in very handy below.

8.20. Our proof is based upon that of Kisyński [**33**]. There is another (similar) proof by Mortini [**36**] that is also worth looking at, though the form of Cohen's Theorem he proves is slightly different from the one presented below. Both owe much to the original proofs of Cohen [**12**] and of Hewitt [**29**].

8.21. Lemma. *Let $\mathcal{A}$ be a Banach algebra and $(e_\lambda)_{\lambda\in\Lambda}$ be a μ-bounded left approximate identity for $\mathcal{A}$. Suppose that $\mathfrak{X}$ is a Banach space and that $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ is a continuous representation of $\mathcal{A}$.*

Suppose furthermore that

- *$y \in \overline{\operatorname{span}}\varrho(\mathcal{A})\mathfrak{X}$;*
- *$b = \beta 1 + b_0 \in \tilde{\mathcal{A}}$ is invertible (here $\beta \in \mathbb{C}$ and $b_0 \in \mathcal{A}$);*
- *$0 < \eta < \frac{1}{\mu+1}$; and*
- *$\delta > 0$.*

Then there exist $\lambda \in \Lambda$ *such that if*

$$g = (\beta - \eta\beta)1 + (b_0 + \eta\beta e_\lambda) \in \tilde{\mathcal{A}},$$

then g *is invertible and*

$$\|\varrho(g^{-1})y - \varrho(b^{-1})y\| < \delta.$$

Proof. For each $\lambda \in \Lambda$, define

$$g_\lambda = (\beta - \eta\beta)1 + (b_0 + \eta\beta e_\lambda) = b + h_\lambda,$$

where $h_\lambda = \eta\beta(e_\lambda - 1) \in \tilde{\mathcal{A}}$.

Clearly it suffices to prove that

(I) there exists $\lambda_0 \in \Lambda$ such that $g_\lambda \in (\tilde{\mathcal{A}})^{-1}$ for all $\lambda \geq \lambda_0$, and
(II) $\lim_\lambda \|\varrho_u(g_\lambda^{-1})y - \varrho_u(b^{-1})y\|_{\mathfrak{X}} = 0$.

Write $b^{-1} = \beta^{-1}1 + r$ for some $r \in \mathcal{A}$, and note that $g_\lambda = (1 + h_\lambda b^{-1})b$. Now

$$\begin{aligned} h_\lambda b^{-1} &= \eta\beta(e_\lambda - 1)(\beta^{-1}1 + r) \\ &= \eta(e_\lambda - 1)1 + \eta\beta(e_\lambda - 1)r, \end{aligned}$$

whence

$$\|h_\lambda b^{-1}\| \leq \eta \ \|e_\lambda - 1\|_{\tilde{\mathcal{A}}} + |\eta\beta| \|e_\lambda r - r\|_{\mathcal{A}}.$$

Now $\eta < \frac{1}{\mu+1}$. Choose $0 < \theta$ such that $\eta\|e_\lambda - 1\|_{\tilde{\mathcal{A}}} < \theta < 1$. Since $\lim_\lambda e_\lambda r = r$, we see that there exists $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies

$$\|h_\lambda b^{-1}\| < \theta.$$

It then follows that $1 + h_\lambda b^{-1}$ is invertible in $\tilde{\mathcal{A}}$, and that

$$\|(1 + h_\lambda b^{-1})^{-1}\| \leq (1 - \theta)^{-1} \text{ for all } \lambda \geq \lambda_0.$$

From this we deduce that for $\lambda \geq \lambda_0$,

$$\|g_\lambda^{-1}\| = \|b^{-1}(1 + h_\lambda b^{-1})^{-1}\| \leq \|b^{-1}\|_{\tilde{\mathcal{A}}}(1 - \theta)^{-1}.$$

This proves item (I) above.

As for (II), if $\lambda \geq \lambda_0$, then

$$\begin{aligned} g_\lambda^{-1} - b^{-1} &= g_\lambda^{-1}(b - g_\lambda)b^{-1} \\ &= -g_\lambda^{-1} h_\lambda b^{-1} \\ &= -\eta\beta g_\lambda^{-1}(e_\lambda - 1)b^{-1}. \end{aligned}$$

This implies that for all $\lambda \geq \lambda_0$,

$$\begin{aligned} \|\varrho_u(g_\lambda^{-1})y - \varrho_u(b^{-1})y\|_{\mathfrak{X}} &= \|\varrho_u(-\eta\beta g_\lambda^{-1}(e_\lambda - 1)b^{-1})y\|_{\mathfrak{X}} \\ &= |\eta\beta| \ \|\varrho_u\| \ \|g_\lambda^{-1}\| \ \|\varrho(e_\lambda)\varrho_u(b^{-1})y - \varrho_u(b^{-1})y\|_{\mathfrak{X}} \\ &\leq (1 - \theta)^{-1}|\eta\beta| \ \|\varrho_u\| \ \|b^{-1}\|_{\tilde{\mathcal{A}}} \ \|\varrho(e_\lambda)\varrho_u(b^{-1})y - \varrho_u(b^{-1})y\|_{\mathfrak{X}}. \end{aligned}$$

From this (II) certainly follows, which completes the proof of the lemma.

□

8.22. Theorem. [The Cohen-Hewitt Factorisation Theorem.] *Let $\mathcal{A}$ be a Banach algebra and suppose that $(e_\lambda)_\lambda$ is a μ-bounded left approximate identity for $\mathcal{A}$. Let $\mathfrak{X}$ be a Banach space and $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ be a continuous representation of $\mathcal{A}$. Then*

$$\varrho(\mathcal{A}) \bullet \mathfrak{X} := \{\varrho(a)x : a \in \mathcal{A}, x \in \mathfrak{X}\}$$

is a closed subspace of $\mathfrak{X}$.

Moreover, given $y \in \overline{\operatorname{span}}\, \varrho(\mathcal{A})\mathfrak{X}$ and $\varepsilon > 0$, there exist $x \in \overline{\varrho(\mathcal{A})y}$ satisfying $\|x - y\| < \varepsilon$ and $b \in \mathcal{A}$ such that $\varrho(b)x = y$.

Proof. First let us choose a positive, strictly decreasing sequence $(t_n)_{n=0}^\infty$ such that

- $t_0 = 1$;
- $\lim_n t_n = 0$; and
- $0 < \eta_n := 1 - \dfrac{t_n}{t_{n-1}} < \dfrac{1}{\mu + 1}$ for all $n \ge 1$.

For each $n \ge 1$, set $p_n := t_{n-1} - t_n > 0$. Now $t_n = (1 - \eta_n)t_{n-1}$, so $p_n = \eta_n t_{n-1}$ and $\sum_n p_n = 1$. In particular, $\lim_n p_n = 0$.

(As noted by Kisyński, the exact form of the sequence is not essential; Cohen choose a constant $0 < \eta < \frac{1}{\mu+1}$ and set $t_n = (1 - \eta)^n$, $n \ge 1$. Thus $\eta_n = \eta$ for all $n \ge 1$.)

Let $y \in \overline{\operatorname{span}} \varrho(\mathcal{A})\mathfrak{X}$. By Lemma 8.21, given $\varepsilon > 0$ there exists a sequence $(g_n)_{n=0}^\infty$ of invertible elements in $\tilde{\mathcal{A}}$ such that $g_0 = 1$ and for all $n \ge 0$,

(a) $g_n = t_n 1 + d_n$ for some $d_n \in \mathcal{A}$;
(b) $\|d_n - d_{n-1}\|_{\mathcal{A}} \le \mu p_n$; and
(c)
$$\|\varrho_u(g_n^{-1})y - \varrho_u(g_{n-1}^{-1})y\| < \varepsilon\, p_n.$$

It follows from item (b) above that if $m > n$ are positive integers, then

$$\|d_m - d_n\| \le \sum_{k=n+1}^{m} \|d_k - d_{k-1}\| \le \sum_{k=n+1}^{m} \mu p_k.$$

Combining this with the fact that each $p_n > 0$ and $\sum_n p_n = 1$, we see that $(d_n)_n$ is convergent in $\mathcal{A}$, and since $\lim_n t_n = 0$, we see that there exists $b \in \mathcal{A}$ such that

$$\lim_n \|g_n - b\|_{\tilde{\mathcal{A}}} = 0.$$

Moreover,

$$\|b\|_{\mathcal{A}} = \|\sum_n d_n - d_{n-1}\|_{\mathcal{A}} \le \mu \sum_n p_n = \mu.$$

Suppose that $x_n := \varrho_u(g_n^{-1})y$, $n \ge 0$. Then $\lim_n \|x_n - x\|_{\mathfrak{X}} = 0$ for some $x \in \mathfrak{X}$ such that

$$\|x - y\| = \|x - x_0\|_{\mathfrak{X}} \le \sum_{n \ge 1} \|x_n - x_{n-1}\|_{\mathfrak{X}} \le \varepsilon \sum_{n \ge 1} p_n = \varepsilon.$$

But then $y = \varrho_u(g_n)x_n$, $n \ge 0$ and thus by continuity,

$$y = \varrho_u(b)x = \varrho(b)x.$$

By Remark 8.18, $y = \lim_\lambda \varrho(e_\lambda)y \in \overline{\varrho(\mathcal{A})y}$, and thus – writing $g_n^{-1} = t_n^{-1}1 + h_n$ for all $n \ge 0$ – we have

$$x_n = \varrho_u(g_n^{-1})y = t_n^{-1}y + \varrho(h_n)y \in \overline{\varrho(\mathcal{A})y}.$$

Hence

$$x = \lim_n x_n \in \overline{\varrho(\mathcal{A})y}.$$

□

8.23. Remark. In fact, it can be shown that $b = \sum_n p_n e_{\lambda_n}$ for an appropriate choice of $\lambda_n \in \Lambda$.

8.24. Corollary. *Let $\mathcal{A}$ be a Banach algebra and suppose that $\mathcal{A}$ admits a bounded left approximate unit. Then*

$$\mathcal{A}^2 := \{ab : a, b \in \mathcal{A}\} = \mathcal{A}.$$

Proof. Let $\mathfrak{X} = \mathcal{A}$ and consider the left regular representation $\varrho(a)x = ax$ for all $a \in \mathcal{A}$, $x \in \mathfrak{X} = \mathcal{A}$. By the Cohen-Hewitt Factorisation Theorem (with $(e_\lambda)_\lambda$ as our bounded left approximate identity for $\mathcal{A}$),

$$\varrho(\mathcal{A}) \bullet \mathfrak{X} = \{\varrho(a)x : a \in \mathcal{A}, x \in \mathcal{X}\}$$

is a closed subspace of $\mathfrak{X}$.

But for all $x \in \mathfrak{X} = \mathcal{A}$, we have that $x = \lim_\lambda e_\lambda x = \lim_\lambda \varrho(e_\lambda)x \in \overline{\varrho(\mathcal{A}) \bullet \mathfrak{X}}$, so that $x = \varrho(a)b$ for some $b \in \mathcal{A}$; i.e. $x = ab$.

□

Supplementary Examples

S8.1. Example. If G is a locally compact abelian group with Haar measure μ, it's known that $L^1(G,\mu)$ is a Banach algebra under convolution, and if G is not discrete, then $L^1(G,\mu)$ has a bounded approximate identity (though it doesn't have an identity element).

S8.2. Example. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and let

$$\Lambda := \{F \subseteq \mathcal{H} : F \text{ is a finite-dimensional subspace}\}.$$

We may partially order Λ by containment: $F_1 \le F_2$ if $F_1 \subseteq F_2$.

For each $F \in \Lambda$, let P_F denote the orthogonal projection of $\mathcal{H}$ onto F. Then $(P_F)_{F\in\Lambda}$ is a bounded approximate identity for $\mathcal{K}(\mathcal{H})$, the set of compact operators on $\mathcal{H}$.

When $\mathcal{H}$ is separable, we can do better. Let $\{e_n\}_n$ be an ONB for $\mathcal{H}$ and let P_n denote the orthogonal projection of $\mathcal{H}$ onto $\operatorname{span}\{e_k : 1 \le k \le n\}$, $n \ge 1$. Then $(P_n)_n$ is a countable, bounded approximate unit for $\mathcal{K}(\mathcal{H})$.

S8.3. Example. The space $(\ell^1(\mathbb{Z}), \|\cdot\|_1)$ is also a Banach algebra using *point-wise* multiplication. That is, given $x = (x_n)_n$ and $y = (y_n)_n$, we may define $x \bullet y := (x_n y_n)_n$. Then $\ell^1(\mathbb{Z})$ is clearly a Banach space, an algebra, and

$$\|x \bullet y\|_1 = \|(x_n y_n)_n\|_1 \le (\sup_n |x_n|)\ \|y\|_1 \le \|x\|_1\ \|y\|_1.$$

Thus it is a Banach algebra. It is a worthwhile exercise for the reader to prove that $\ell^1(\mathbb{Z})$ does not admit a bounded approximate identity using this multiplication.

Appendix

A8.1. In the next Chapter, we shall begin the study of C^*-algebras. As we shall eventually prove, any such algebra $\mathcal{A}$ is (up to isometric *-isomorphism - the canonical isomorphism in this category) a self-adjoint Banach subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We shall also demonstrate that every C^*-algebra admits a bounded approximate identity. Indeed, we shall see that the elements $(X_\lambda)_\lambda \subseteq \mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ of the bounded approximate identity may be chosen to be positive of norm at most one, and that the net is *increasing* in the sense that $\lambda_1 \le \lambda_2$ implies that $X_{\lambda_2} - X_{\lambda_1}$ is a positive operator.

A8.2. The class of C^*-algebras will also be shown to enjoy the property that if the identity map ϱ on a C^*-algebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is topologically irreducible, then it is in fact algebraically irreducible. This is a result known as Kadison's Transitivity Theorem. Thus the Jacobson Density Theorem may be applied to $\mathcal{A}$ to show that $\mathcal{A}$ is (so-called) N-transitive for all N.

Exercises for Chapter 8

Exercise 8.1. SINCLAIR'S THEOREM.

Let $\mathcal{A}$ be a Banach algebra and suppose that $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ is a continuous, algebraically irreducible representation of $\mathcal{A}$ on the Banach space $\mathfrak{X}$. Prove that if $N \in \mathbb{N}$, $x_1, x_2, \dots, x_N \in \mathfrak{X}$ are linearly independent and $y_1, y_2, \dots, y_N \in \mathfrak{X}$ are also linearly independent, then there exists $a \in \mathcal{A}^{-1}$ such that $\varrho(a)x_n = y_n$, $1 \le n \le N$.

Hint. Let $\mathcal{N} = \operatorname{span}\{x_1, x_2, \dots, x_N, y_1, y_2, \dots, y_N\}$. It follows from basic linear algebra that we can find $T \in \mathcal{B}(\mathcal{N})$ invertible such that $Tx_n = y_n$, $1 \le n \le N$. Moreover, $T = \exp(R)$ for some $R \in \mathcal{B}(\mathcal{N})$ (why?).

Choose a basis $\mathfrak{B} := \{x_1, x_2, \dots, x_N, z_1, \dots, z_K\}$ for $\mathcal{N}$ and choose $a \in \mathcal{A}$ such that $\varrho(a)w = Rw$ for all $w \in \mathfrak{B}$. Prove that $\varrho(a)^k$ and R^k coincide on $\mathcal{N}$ for all $k \ge 1$.

CHAPTER 9

C^*-algebras: An introduction

I don't have to tell you folks about scuba diving. So, that'll save some time.

Emo Philips

Definitions and examples.

9.1. In this chapter we turn our attention to an important class of Banach algebras known as *C^*-algebras*. After introducing the notion of an involution on a Banach algebra, we shall see that the C^*-equation (defined below) imposes an incredible amount of structure upon an involutive Banach algebra in which it holds. Our main goal will be to prove that every C^*-algebra admits an isometric *-representation, a result known as the *Gelfand-Naimark Theorem*. Before getting there, however, we shall consider the basic structure of these algebras, and we shall obtain an extension of the holomorphic functional calculus for normal elements of C^*-algebras (to be defined below).

9.2. Definition. *Let $\mathcal{A}$ be an algebra. Then an **involution** on $\mathcal{A}$ is a map*

$$\begin{array}{rccc} *: & \mathcal{A} & \to & \mathcal{A} \\ & a & \mapsto & a^* \end{array}$$

satisfying

(i) $(a^*)^* = a$ *for all* $a \in \mathcal{A}$*;*
(ii) $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$ *for all* $a, b \in \mathcal{A}$*,* $\alpha, \beta \in \mathbb{C}$*;*
(iii) $(ab)^* = b^* a^*$ *for all* $a, b \in \mathcal{A}$*.*

*If $\mathcal{A}$ carries an involution, we say that $\mathcal{A}$ is an **involutive algebra**, or a ***-algebra**. A subset $\mathcal{F}$ of $\mathcal{A}$ is said to be **self-adjoint** if $x \in \mathcal{F}$ implies $x^* \in \mathcal{F}$.*

*A homomorphism $\tau : \mathcal{A} \to \mathcal{B}$ between two involutive algebras is said to be a ***-homomorphism** if τ respects the involution. That is, $\tau(a^*) = (\tau(a))^*$ for all $a \in \mathcal{A}$.*

*Finally, a **Banach *-algebra** is an involutive Banach algebra $\mathcal{A}$ whose involution satisfies $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$.*

9.3. Remarks.

- If $\mathcal{A}$ is an involutive algebra, we say that an element $h \in \mathcal{A}$ is **self-adjoint** if $\{h\}$ is self-adjoint, namely $h = h^*$.
- Observe that if $\mathcal{A}$ is a unital involutive algebra with unit $e_{\mathcal{A}}$, then for all $a \in \mathcal{A}$ we have $(e_{\mathcal{A}}^* a) = (a^* e_{\mathcal{A}})^* = (a^*)^* = a = (e_{\mathcal{A}} a^*)^* = (a e_{\mathcal{A}}^*)$. Thus $e_{\mathcal{A}} = e_{\mathcal{A}}^*$, since the unit must be unique.
- The condition that a homomorphism τ from an involutive Banach algebra $\mathcal{A}$ to an involutive Banach algebra $\mathcal{B}$ be a *-homomorphism is equivalent to the condition $\tau(h) = \tau(h)^*$ whenever $h = h^*$. To see this, note that if this condition is met, then given $a \in \mathcal{A}$, we may write $a = h + i\,k$, where $h = (a + a^*)/2$ and $k = (a - a^*)/2i$. Then $h = h^*$, $k = k^*$, and $\tau(a^*) = \tau(h - i\,k) = (\tau(h) + i\tau(k))^* = \tau(a^*)$, implying that τ is a *-homomorphism. The other direction is clear.

9.4. Example. Let $\mathcal{A} = (\mathbb{C}, |\cdot|)$. Then $* : \lambda \mapsto \overline{\lambda}$ defines an involution on $\mathbb{C}$.

9.5. Example. Consider the disk algebra $\mathcal{A}(\mathbb{D})$. For each $f \in \mathcal{A}(\mathbb{D})$, define $f^*(z) = \overline{f(\overline{z})}$ for each $z \in \mathbb{D}$. Then the map $* : f \mapsto f^*$ defines an involution on $\mathcal{A}(\mathbb{D})$, under which it becomes a Banach *-algebra.

9.6. Example. Recall from Remark 7.6 that the map that if $\mathcal{H}$ is a Hilbert space, then the map that sends a continuous linear operator T to its Hilbert space adjoint T^* is an isometric involution. Thus $\mathcal{B}(\mathcal{H})$ is a Banach *-algebra.

Suppose $\dim \mathcal{H} = 2$, and identify $\mathcal{B}(\mathcal{H})$ with $\mathbb{M}_2$. Let $S \in \mathcal{B}(\mathcal{H})$ be the invertible operator $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Consider the map

$$\begin{array}{llll} \mathrm{Ad}_S : & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & T & \mapsto & S^{-1} T S \end{array}.$$

Then Ad_S is a multiplicative homomorphism of $\mathcal{B}(\mathcal{H})$, but it is not a *-homomorphism.

For example,

$$\mathrm{Ad}_S \begin{bmatrix} 1-i & 2-i \\ 3-i & 4-i \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 3-i & 7-2i \end{bmatrix},$$

while

$$\left(\mathrm{Ad}_S \begin{bmatrix} 1+i & 3+i \\ 2+i & 4+i \end{bmatrix}\right)^* = \begin{bmatrix} -1 & 2-i \\ -2 & 6-2i \end{bmatrix}.$$

On the other hand, if $U \in \mathcal{B}(\mathcal{H})$ is unitary, then it is not hard to verify that Ad_U *does* define a *-automorphism.

9.7. Example. Let $\mathcal{T}_n$ denote the algebra of $n \times n$ upper triangular matrices, viewed as a Banach subalgebra of $\mathcal{B}(\mathbb{C}^n)$ equipped with the operator norm. We can define an involution on $\mathcal{T}_n$ via the map: $[t_{ij}]^* = [\overline{t_{(n+1)-j\,(n+1)-i}}]$.

We leave it as an exercise for the reader to prove that this involution is isometric, and thus $\mathcal{T}_n$ becomes a Banach *-algebra with this involution.

9.8. Definition. *A C^*-**algebra** $\mathbb{A}$ is an involutive Banach algebra which satisfies the* C*-equation*:*

$$\|a^* a\| = \|a\|^2 \text{ for all } a \in \mathbb{A}.$$

*A norm on an involutive Banach algebra which satisfies this equation will be called a **C*-norm**.*

9.9. Remark. First observe that if $\mathbb{B}$ is an involutive Banach algebra and $\|b^* b\| \geq \|b\|^2$ for all $b \in \mathbb{B}$, then $\|b\|^2 \leq \|b^*\| \, \|b\|$, which implies that $\|b\| \leq \|b^*\|$. But then $\|b^*\| \leq \|(b^*)^*\| = \|b\|$, so that $\|b\| = \|b^*\|$. In particular, $\mathbb{B}$ is a Banach *-algebra.

Moreover, $\|b\|^2 \leq \|b^* b\| \leq \|b^*\| \, \|b\| = \|b\|^2$, showing that the norm on $\mathbb{B}$ is a C*-norm.

Finally, if $\mathbb{B}$ is a non-zero unital C^*-algebra with unit $e_{\mathbb{B}}$, then

$$\|e_{\mathbb{B}}\| = \|e_{\mathbb{B}}^2\| = \|e_{\mathbb{B}}^* \, e_{\mathbb{B}}\| = \|e_{\mathbb{B}}\|^2,$$

and hence $\|e_{\mathbb{B}}\| = 1$.

9.10. Example. It is always useful to have counterexamples as well as examples. To that end, consider the following:

(i) The disk algebra $\mathbb{A}(\mathbb{D})$ is not a C^*-algebra with the involution $f^*(z) = \overline{f(\overline{z})}$. Indeed, if $f(z) = i\,z + z^2$, then $f^*(z) = -i\,z + z^2$. Thus

$$\|f^* \, f\| = \sup_{|z|=1} |z^4 + z^2| = 2,$$

while

$$\|f\|^2 \geq |f(i)|^2 = 4.$$

(ii) $\mathcal{T}_n$ is not a C^*-algebra with the involution defined in Example 9.7. To see this, note that if E_{1n} denotes the standard $(1, n)$ matrix unit, then $E_{1n}^* = E_{1n}$, so that $\|E_{1n}^* \, E_{1n}\| = \|E_{1n}^2\| = \|0\| = 0$, while $\|E_{1n}\| = 1$, as is readily verified.

(iii) Recall that $\ell^1(\mathbb{Z})$ is a Banach algebra, where for $f, g \in \ell^1(\mathbb{Z})$, we defined the product via convolution:

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(n-k) \, g(k)$$

and

$$\|f\|_1 = \sum_{k \in \mathbb{Z}} |f(k)|.$$

Consider the involution $f^*(n) = \overline{f(-n)}$. Let $g \in \ell^1(\mathbb{Z})$ be the element defined by: $g(n) = 0$ if $n \notin \{0, 1, 2\}$; $g(0) = -i = g(2)$, and $g(1) = 1$. We leave it to the reader to verify that $\|g^* \, g\|_1 = 5$, while $\|g\|_1 = 3$. Again, this is not a C^*-norm.

9.11. Example. Let X be a locally compact, Hausdorff space. Consider $(\mathcal{C}_0(X), \|\cdot\|_\infty)$. For $f \in \mathcal{C}_0(X)$, define $f^*(x) = \overline{f(x)}$ for each $x \in X$. Then $\mathcal{C}(X)$ is a C^*-algebra. The details are left to the reader.

This C^*-algebra is unital precisely when X is compact.

9.12. Example. Let $\mathcal{H}$ be a Hilbert space. As we have just recalled, $\mathcal{B}(\mathcal{H})$ is an involutive Banach algebra using the Hilbert space adjoint as our involution. We now check that equipped with this involution, $\mathcal{B}(\mathcal{H})$ verifies the C*-equation.

Let $T \in \mathcal{B}(\mathcal{H})$. Then $\|T^*T\| \le \|T\|^2$ from above. For the reverse inequality, observe that

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\|=1} \|Tx\|^2 \\ &= \sup_{\|x\|=1} \langle Tx, Tx\rangle \\ &= \sup_{\|x\|=1} \langle T^*Tx, x\rangle \\ &\le \sup_{\|x\|=1} \|T^*T\|\,\|x\|^2 \\ &= \|T^*T\|. \end{aligned}$$

Thus $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. By considering the case where $\mathcal{H} = \mathbb{C}^n$ is finite dimensional, we find that $\mathbb{M}_n$ equipped with the operator norm and Hilbert space adjoint is a C^*-algebra.

9.13. Remark. Suppose that $\mathbb{A}$ is a C^*-algebra and that $\mathbb{B}$ is a norm-closed, self-adjoint subalgebra of $\mathcal{A}$. Then the C*-equation is trivially satisfied for all $b \in \mathbb{B}$, because it is already satisfied in $\mathcal{A}$, and the norm is inherited from $\mathbb{A}$. It follows that $\mathbb{B}$ is also a C^*-algebra.

In particular, if $a \in \mathcal{A}$ and $\mathbb{A}$ is unital, then we denote by $C^*(a)$ the **unital C^*–algebra generated by** a. It is the smallest unital, norm-closed, self-adjoint subalgebra of $\mathbb{A}$ containing a, that is, it is the intersection of all C^*-subalgebras of $\mathbb{A}$ containing a, and it is easily seen to coincide with the closure of

$$\{p(a, a^*) : p \in \mathbb{C}[x, y]\},$$

where $\mathbb{C}[x, y]$ denotes the set of all polynomials in two non-commuting variables x and y with complex coefficients.

We shall also define the set $C_\circ(a)$ to be the **non-unital C^*-algebra generated by** a.

It coincides with the closure of

$$\{p(a, a^*) : p \in \mathbb{C}_\circ[x, y]\},$$

where $\mathbb{C}_\circ[x, y]$ denotes the set of all polynomials in two non-commuting variables x and y with complex coefficients but without constant term; equivalently, $p(0, 0) = 0$.

9.14. Example. More generally, if $\mathbb{A}$ is any C^*-algebra, and if $\mathbb{X} \subseteq \mathbb{A}$, we denote by $C^*(\mathbb{X})$ the unital C^*-subalgebra of $\mathbb{A}$ generated by $\mathbb{X}$. This of course only makes sense if $\mathbb{A}$ is unital. As before, it is the intersection of all C^*-algebras of $\mathbb{A}$ containing $\mathbb{X}$. We write $C_\circ(\mathbb{X})$ to denote the smallest non-unital C^*-subalgebra of $\mathbb{A}$ that contains $\mathbb{X}$. This makes sense even if $\mathbb{A}$ is non-unital.

9.15. Example. Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{K}(\mathcal{H})$ is a closed, self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, and thus $\mathcal{K}(\mathcal{H})$ is a C^*-algebra. $\mathcal{K}(\mathcal{H})$ is not unital unless $\mathcal{H}$ is finite-dimensional.

9.16. Example. Let $\{\mathbb{A}_\alpha\}_{\alpha\in\Lambda}$ be a family of C^*-algebras indexed by a set Λ. It is elementary to verify that

$$\mathbb{A} = \{(a_\alpha)_{\alpha\in\Lambda} : a_\alpha \in \mathcal{A}_\alpha, \alpha \in \Lambda, \sup_\alpha \|a_\alpha\| < \infty\}$$

is a C^*-algebra, where the involution is given by $(a_\alpha)^* = (a_\alpha^*)$, and the norm is given by $\|(a_\alpha)\| = \sup_\alpha \|a_\alpha\|$.

Let $\mathbb{K} = \{(a_\alpha) \in \mathcal{A} :$ for all $\varepsilon > 0$, $\{\alpha \in \Lambda : \|a_\alpha\| \geq \varepsilon\}$ is finite$\}$. Then $\mathbb{K}$ is a C^*-algebra; in fact, $\mathbb{K}$ is a closed, self-adjoint ideal of $\mathbb{A}$.

In particular, if $\{k_n\}_{n=1}^\infty \subseteq \mathbb{N}$, then $\mathbb{A} = \oplus_{n=1}^\infty \mathbb{M}_{k_n}$ is a C^*-algebra under this norm. Setting $k_n = 1$ for all $n \geq 1$ shows that ℓ^∞ is a C^*-algebra, and that c_0 is a closed, self-adjoint ideal in ℓ^∞.

9.17. Example. More generally, let μ be a finite regular Borel measure on the measure space X. Then $L^\infty(X,\mu)$ is a C^*-algebra with the standard norm. As in the case of $\mathcal{C}(X)$, the involution here is $f^*(x) = \overline{f(x)}$ for all $x \in X$.

In fact, we can think of $L^\infty(X,\mu)$ as a commutative C^*-subalgebra of $\mathcal{B}(L^2(X,\mu))$ as follows. Recall from Example 6.7 for each $\varphi \in L^\infty(X,\mu)$, we define the multiplication operator

$$\begin{array}{rccc} M_\varphi : & L^2(X,\mu) & \to & L^2(X,\mu) \\ & f & \mapsto & \varphi f. \end{array}$$

In Example 8.6, we showed that the map

$$\begin{array}{rccc} \varrho : & L^\infty(X,\mu) & \to & \mathcal{B}(L^2(X,\mu)) \\ & f & \mapsto & M_f \end{array}$$

is an isometric *-representation. That is, an isometric *-homomorphism of $L^\infty(X,\mu)$ into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (which happens to be $L^2(X,\mu)$ in this case). We then identify $L^\infty(X,\mu)$ with its image under this map ϱ, and use the same notation for both algebras. Since ϱ preserves products, the image algebra is clearly also abelian.

We shall write $\mathfrak{M}^\infty(X,\mu)$ to denote the range of ϱ, so that

$$\mathfrak{M}^\infty(X,\mu) = \{M_f : f \in \text{Ł}^\infty(X,\mu)\} \subseteq \mathcal{B}(L^2(X,\mu)).$$

Thus $L^\infty(X,\mu)$ is isometrically *-isomorphic to $\mathfrak{M}^\infty(X,\mu)$. Since isometric *-isomorphisms are the isomorphisms in the category of C^*-algebras, the above identification of $L^\infty(X,\mu)$ with $\mathfrak{M}^\infty(X,\mu)$ is natural.

9.18. Example. Let $\mathcal{H}$ be a Hilbert space and $\mathbb{F} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint family of operators on $\mathcal{H}$. Consider the *commutant* $\mathbb{F}'$ of $\mathbb{F}$ defined as:

$$\mathbb{F}' = \{T \in \mathcal{B}(\mathcal{H}) : TF = FT \text{ for all } F \in \mathbb{F}\}.$$

We claim that $\mathbb{F}'$ is a C^*-algebra.

That it is an algebra is an easy exercise. If $(T_n)_n \subseteq \mathcal{F}'$ and if $\lim_{n\to\infty} T_n = T \in \mathcal{B}(\mathcal{H})$, then for any $F \in \mathbb{F}$, we have $T\,F = \lim_{n\to\infty} T_n\,F = \lim_{n\to\infty} F\,T_n = F\,T$. Hence $T \in \mathbb{F}'$, and so $\mathbb{F}'$ is closed in $\mathcal{B}(\mathcal{H})$. Finally, if $T \in \mathbb{F}'$ and $F \in \mathbb{F}$, then $F^* \in \mathcal{F}$ by assumption. Thus $T\,F^* = F^*\,T$. Taking adjoints, we obtain $T^*\,F = F\,T^*$, and therefore $T^* \in \mathbb{F}'$, proving that $\mathcal{F}'$ is a closed, self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. By Remark 9.13, it is a C^*-algebra.

9.19. It is difficult to overstate the importance of the C^*-equation. Hold on, that's a bit hyperbolic. What we mean by that is that this equation is *really* important. That is, they are really important to the study of operator theory and operator algebras, and to those disciplines (I'm talking to you, Physics) that rely heavily upon operator theory and operator algebras. If you are alone in a cage with a starving lion, a new hair-do and your wits, you will most likely be excused for not having the C^*-equation be one the first things that spring to your mind, and your estimation of their general usefulness will undoubtedly not be of the impossible to overstate variety. But if you are invited to talk about your hobbies at a pyjama party at Angela Merkel's house, well, the sky's the limit. (The C^*-equation is not unlike eigenvalues and spectrum in this regard, but that's another matter.)

The C^*-equation allows us to relate analytic information to algebraic information. For example, consider the following Lemma, which relates the norm of an element of a C^*-algebra to its spectral radius, and its consequence, Theorem 9.21.

9.20. Lemma. *Let $\mathbb{A}$ be a C^*-algebra, and suppose $h = h^* \in \mathbb{A}$. Then*

$$\|h\| = \mathrm{spr}(h).$$

More generally, if $a \in \mathbb{A}$, then $\|a\| = (\mathrm{spr}(a^*a))^{1/2}$.

Proof. Now $\|h\|^2 = \|h^*h\| = \|h^2\|$. By induction, we find that $\|h\|^{2^n} = \|h^{2^n}\|$ for all $n \ge 1$. Using Beurling's Spectral Radius Formula,

$$\mathrm{spr}(h) = \lim_{n\to\infty} \|h^{2^n}\|^{1/2^n} = \lim_{n\to\infty} (\|h\|^{2^n})^{1/2^n} = \|h\|.$$

In general, a^*a is self-adjoint, and hence $\|a\|^2 = \|a^*a\| = \mathrm{spr}(a^*a)$.

□

9.21. Theorem. *Let $\alpha : \mathbb{A} \to \mathbb{B}$ be a *-isomorphism from a C^*-algebra $\mathbb{A}$ to a C^*-algebra $\mathbb{B}$. Then α is isometric. In particular, each C^*-algebra possesses a unique C^*-norm.*

Proof. First note that since α is a *-isomorphism, $\sigma_{\mathbb{A}}(a) = \sigma_{\mathbb{B}}(\alpha(a))$ for all $a \in \mathcal{A}$. As such,

$$\begin{aligned} \|a\|_{\mathbb{A}} &= [\mathrm{spr}_{\mathbb{A}}(a^* a)]^{1/2} \\ &= [\mathrm{spr}_{\mathbb{B}}(\alpha(a)^* \alpha(a))]^{1/2} \\ &= \|\alpha(a)\|_{\mathbb{B}}. \end{aligned}$$

Thus α is isometric.

If $\mathbb{A}$ has two C*-norms $\|\cdot\|_1$ and $\|\cdot\|_2$, then the identity map $id(a) = a$ is clearly a *-isomorphism of $\mathbb{A}$ onto itself, and thus is isometric from above, implying that the two norms coincide.

□

Recall that we stated that isometric *-isomorphisms are the isomorphisms in the category of C^*-algebras. By Theorem 9.21, we see that the word "isometric" is extraneous.

Unitisations of C^*-algebras.

9.22. Given a non-unital Banach algebra $\mathcal{A}$, we defined its unitisation to be the algebra

$$\mathcal{A}_u = \mathbb{C} \oplus \mathcal{A},$$

equipped with the multiplication $(\alpha, a) \cdot (\beta, b) = (\alpha\beta, \alpha b + \beta a + ab)$ and the norm $\|(\alpha, a)\| := |\alpha| + \|a\|_{\mathcal{A}}$.

When $\mathcal{A}$ is in fact a C^*-algebra, the above norm on its Banach algebra-unitisation fails to be a C^*-norm. Our present goal is to define a separate unitisation of $\mathcal{A}$ which is a C^*-algebra. This will require the notion of an *essential ideal*, which we now define.

9.23. Definition. *Let $\mathbb{K}$ be an ideal of a C^*-algebra $\mathbb{A}$. The* ***annihilator*** *of $\mathbb{K}$ in $\mathcal{A}$ is the set*

$$\mathbb{K}^\perp = \{a \in \mathbb{A} : a\,k = 0 \textit{ for all } k \in \mathbb{K}\}.$$

$\mathbb{K}$ is said to be ***essential*** *in $\mathbb{A}$ if its annihilator $\mathbb{K}^\perp = \{0\}$.*

9.24. The apparent asymmetry (why do we multiply by a on the left?) of this definition is illusory. Suppose $\mathbb{A}$ and $\mathbb{K}$ are as above. Let $a \in \mathbb{K}^\perp$. Given $k \in \mathbb{K}$, $a^* k \in \mathbb{K}$ and hence $a\,a^* k = 0$. But then $\|a^* k\|^2 = \|k^* a\,a^* k\| = 0$, forcing $a^* \in \mathbb{K}^\perp$. This in turn implies that $k\,a = (a^* k^*)^* = 0$ for all $k \in \mathcal{K}$. As such, $\mathbb{K}^\perp = \{a \in \mathbb{A} : k\,a = 0 \text{ for all } k \in \mathbb{K}\}$.

It is routine to verify that $\mathbb{K}^\perp$ is a closed subalgebra of $\mathbb{A}$, and from above, we see that $\mathbb{K}^\perp$ is self-adjoint, implying that $\mathbb{K}^\perp$ is a C^*-subalgebra of $\mathbb{A}$.

9.25. Example. Let $\mathcal{H}$ be a complex, infinite dimensional Hilbert space.

Given $x, y \in \mathcal{H}$, let us denote by $x \otimes y^*$ the rank-one operator $(x \otimes y^*)(z) = \langle z, y \rangle x$, $z \in \mathcal{H}$. It is not hard to see that $\|x \otimes y^*\| = \|x\| \, \|y\|$. In particular, if $\|x\| = 1$, then $x \otimes x^*$ represents the orthogonal projection of $\mathcal{H}$ onto $\mathbb{C}x$.

We claim that $\mathcal{K}(\mathcal{H})$ is essential in $\mathcal{B}(\mathcal{H})$. Indeed, if $0 \ne T \in \mathcal{B}(\mathcal{H})$, choose a non-zero vector $x \in \mathcal{H}$ such that $y = Tx \ne 0$. Then $0 \ne T(x \otimes x^*)$, and hence $T \notin \mathcal{K}(\mathcal{H})^\perp$.

9.26. Example. Recall that if X is a compact, Hausdorff space, then there is a bijective correspondence between the closed subsets Y of X and the closed ideals $\mathbb{K}$ of $\mathcal{C}(X)$. Given $Y \subseteq X$ closed, the associated ideal $\mathbb{K}_Y = \{f \in \mathcal{C}(X) : f(x) = 0 \text{ for all } x \in Y\}$, while given an ideal $\mathbb{K}$ in $\mathcal{C}(X)$, the corresponding closed subset of X is $Y_{\mathbb{K}} = \{x \in X : f(x) = 0 \text{ for all } f \in \mathbb{K}\}$.

Let $Y \subseteq X$ be closed. We claim that $\mathbb{K}_Y$ is an essential ideal of $\mathcal{C}(X)$ if and only if Y is nowhere dense in X.

Suppose first that Y is nowhere dense. Let $f \in \mathbb{K}^\perp$. If $x \in X \backslash \overline{Y} = X \backslash Y$, then by Urysohn's Lemma we can find $g_x \in \mathbb{K}_Y$ such that $g_x(x) \ne 0$. Since $f g_x = 0$, we have $f(x) = 0$. But $X \backslash Y$ is dense in X and f is continuous, and so $f = 0$ and $\mathbb{K}_Y$ is essential.

To prove the converse, suppose Y is *not* nowhere dense. Then we can find an open set $G \subseteq \overline{Y} = Y$. Choose $y_0 \in G$. Again, by Urysohn's Lemma, we can find $f \in \mathcal{C}(X)$ such that $f(y_0) = 1$ and $f(x) = 0$ for all $x \in X \backslash G$. It is routine to verify that $f \in \mathbb{K}_Y^\perp$, and hence $\mathbb{K}_Y$ is not essential.

9.27. Example. Let X be a locally compact, Hausdorff space. Then $\mathcal{C}_\circ(X)$ is an essential ideal in $\mathcal{C}_b(X)$, the space of bounded continuous functions on X with the supremum norm.

9.28. Definition. *Let $\mathbb{A}$ be a C^*-algebra. A C^*-algebra $\mathbb{B}$ is said to be a* ***unitisation*** *of $\mathbb{A}$ if $\mathbb{B}$ is unital and $\mathbb{A}$ is *-isomorphic to an essential ideal in $\mathbb{B}$.*

Unless one specifically indicates otherwise, when referring to the *unitisation* of a C^*-algebra, it will be assumed that one is always referring to this notion, as opposed to the unitisation in the sense of Banach algebras previously defined.

9.29. Example. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $\mathbb{B} \subseteq \mathcal{B}(\mathcal{H})$ be any unital C^*-algebra containing $\mathcal{K}(\mathcal{H})$. Then $\mathbb{B}$ is a unitisation of $\mathcal{K}(\mathcal{H})$.

9.30. Example. Let $\mathbb{A}$ be a unital C^*-algebra and suppose $\mathbb{B}$ is a unitization of $\mathbb{A}$. Let $\varrho : \mathbb{A} \to \mathbb{B}$ be the *-monomorphic embedding of $\mathbb{A}$ into $\mathbb{B}$ as an essential ideal. Then for each $a \in \mathbb{A}$,

$$(e_{\mathbb{B}} - \varrho(e_{\mathbb{A}}))\,(\varrho(a)) = 0,$$

and hence $e_{\mathbb{B}} = \varrho(e_{\mathbb{A}})$. But $\varrho(\mathbb{A})$ is an ideal in $\mathbb{B}$, and hence $\varrho(\mathbb{A}) = \mathbb{B}$. Thus any unitisation of $\mathbb{A}$ is *-isomorphic to $\mathbb{A}$ itself.

9.31. Theorem. *Every C^*-algebra $\mathbb{A}$ possesses a unitisation $\mathbb{A}_e$.*

Proof. If $\mathbb{A}$ is unital, then it serves as its own unitisation. Suppose, therefore, that $\mathbb{A}$ is not unital. Consider the map:

$$\begin{array}{rccc} \kappa : & \mathbb{A} & \to & \mathcal{B}(\mathbb{A}) \\ & a & \mapsto & L_a \end{array}$$

where $L_a(x) = a\,x$ for all $x \in \mathbb{A}$. Then κ is clearly a homomorphism. Denote by $\mathbb{A}_e$ the subalgebra of $\mathcal{B}(\mathbb{A})$ generated by $\kappa(\mathbb{A})$ and I, the identity operator. While there is no obvious candidate for an involution on $\mathcal{B}(\mathbb{A})$, nevertheless we may define one on $\mathbb{A}_e$ via $(L_a + \lambda I)^* = L_{a^*} + \overline{\lambda} I$.

Now $\|L_a\| = \sup_{\|x\|=1} \|L_a x\| = \sup_{\|x\|=1} \|a\,x\| \le \|a\|$, so that κ is continuous. In fact, $\|L_a\| \ge \|L_a\left(\dfrac{a^*}{\|a\|}\right)\| = \|a\|$, so that κ is an isometric *-monomorphism. In particular, therefore, $\kappa(\mathbb{A})$ is closed in $\mathcal{B}(\mathbb{A})$. Since $\mathbb{A}_e$ is a finite dimensional extension of $\kappa(\mathbb{A})$, $\mathbb{A}_e$ is closed as well.

Next,

$$\begin{aligned} \|(L_a + \lambda I)^*(L_a + \lambda I)\| &= \sup_{\|x\|=1} \|(a^* + \overline{\lambda})(a + \lambda)x\| \\ &\ge \sup_{\|x\|=1} \|x^*(a^* + \overline{\lambda})(a + \lambda)x\| \\ &= \sup_{\|x\|=1} \|ax + \lambda x\|^2 \\ &= \|L_a + \lambda\|^2. \end{aligned}$$

By Remark 9.9, $\mathbb{A}_e$ is a C^*-algebra.

That $\kappa(\mathbb{A})$ is an ideal in $\mathbb{A}_e$ is easily checked. Suppose $(L_a + \lambda I)\, L_b = 0$ for all $b \in \mathbb{A}$. Then for all $b, x \in \mathbb{A}$, we have $(a\,b + \lambda\,b)x = 0$. Letting $x = (a\,b + \lambda\,b)^*$, we find that $a\,b = -\lambda\,b$. Since b is arbitrary, this implies that $-\lambda^{-1} a$ is a unit for $\mathbb{A}$, a contradiction. This implies that $\kappa(\mathcal{A})$ is essential in $\mathbb{A}_e$, completing the proof.

□

9.32. Two observations are in order. First, it will be useful to keep in mind that for any $x \in \mathbb{A}_e$, $\|x\|_{\mathbb{A}_e} = \sup\{\|x\,a\|_{\mathbb{A}} : \|a\|_{\mathbb{A}} = 1\}$. Second, the unitisation of $\mathbb{A}$ above is unique in the following sense:

If $\mathbb{B}$ is any unital C^*-algebra containing $\mathbb{A}$, then $\mathbb{B}$ contains an *-isomorphic copy of $\mathbb{A}_e$. Indeed, if $\mathbb{B}_0$ is the algebra generated by $\mathbb{A}$ and $e_{\mathbb{B}}$, then either $e_{\mathbb{B}} \in \mathbb{A}$, in which case $\mathbb{A} = \mathbb{A}_e \subseteq \mathbb{B}$, or $\mathbb{B}_0$ is a 1-dimensional extension of $\mathbb{A}$, and hence is closed in $\mathbb{B}$. Since $\mathbb{B}_0$ is clearly self-adjoint, it is a C^*-algebra. The map:

$$\begin{array}{rccc} \Phi : & \mathbb{A}_e & \to & \mathbb{B}_0 \\ & L_a + \lambda I & \mapsto & a + \lambda\, e_{\mathbb{B}} \end{array}$$

is easily seen to be a *-isomorphism, and thus is isometric, by Theorem 9.21. $\mathbb{B}_0$ is our desired copy of $\mathbb{A}_e$.

9.33. Remark. Let $\mathbb{A}$ be a non-unital C^*-algebra and $a \in \mathbb{A}$. We now have two possibilities for defining the spectrum of a its "unitisation". On the one hand, $\mathbb{A}$ is a Banach algebra and as such it admits a Banach algebra unitisation $\mathbb{A}_u := \{(\alpha, a) : \alpha \in \mathbb{C}, a \in \mathbb{A})\}$ with norm $\|(\alpha, a)\|_{\mathbb{A}_u} := |\alpha| + \|a\|_{\mathcal{A}}$. Previously we have defined the spectrum of a to be $\sigma_{\mathbb{A}_u}((0, a))$.

Now that we know that $\mathbb{A}$ admits a C^*-unitisation (i.e. a unitisation $\mathbb{A}_e$ which is a C^*-algebra), it seems more natural to define the spectrum of $a \in \mathbb{A}$ to be $\sigma_{\tilde{(\mathcal{A})}}(L_a)$.

Fortunately we shall not have to decide between the two, since both unitisations yield the same spectrum.

(a) Set $\mathbb{A}_u := \mathbb{C} \oplus \mathbb{A}$, equipped with the multiplication $(\alpha, a)(\beta, b) = (\alpha\beta, \alpha b + \beta a + ab)$ and norm $\|(\alpha, a)\| = |\alpha| + \|a\|_{\mathcal{A}}$. This is the Banach algebra unitisation of $\mathbb{A}$, and it is *not* a C^*-algebra in general.

(b) Set $\mathbb{A}_e$ to be the C^*-algebra unitisation of $\mathbb{A}$ as defined in Theorem 9.31. That is, we have an isometric embedding $\kappa : \mathbb{A} \to \mathcal{B}(\mathbb{A})$ defined by $\kappa(a) = L_a$, where $L_a(x) = ax$ for all $x \in \mathbb{A}$. We define an involution * on $\kappa(\mathbb{A}) + \mathbb{C}I \subseteq \mathcal{B}(\mathbb{A})$ via $(L_a + \alpha I)^* := L_{a^*} + \overline{\alpha} I$. We proved in that Theorem that $\mathbb{A}_e := \kappa(\mathbb{A}) + \mathbb{C}I$ is a unital C^*-algebra.

Consider the map

$$\begin{array}{cccc} \Theta: & \mathbb{A}_u & \to & \mathbb{A}_e \\ & (\alpha, a) & \mapsto & L_a + \alpha I. \end{array}$$

(The next part of the argument just shows that Θ is an isomorphism.) Then

$$\Theta(\lambda(\alpha, a) + (\beta, b)) = \Theta(\lambda\alpha + \beta, \lambda a + b) = L_{\lambda a + b} + (\lambda\alpha + \beta)I.$$

But for all $x \in \mathbb{A}$,

$$(L_{\lambda a+b} + (\lambda\alpha+\beta)I)x = (\lambda a+b)x + (\lambda\alpha+\beta)x = \lambda(ax+\alpha x) + (bx+\beta x) = \lambda L_{(\alpha,a)}x + L_{(\beta,b)}x,$$

and therefore

$$L_{\lambda\alpha+\beta, \lambda a+b} = \lambda L_{(\alpha,a)} + L_{(\beta,b)}.$$

In other words, Θ is linear.

Similarly,

$$\Theta((\alpha, a)(\beta, b)) = \Theta((\alpha\beta, \alpha b + \beta a + ab)) = L_{\alpha b + \beta a + ab} + (\alpha\beta)I,$$

while

$$\Theta((\alpha, a))\ \Theta((\beta, b)) = (L_a + \alpha I)(L_b + \beta I) = L_a L_b + \alpha L_b + \beta L_a + (\alpha\beta)I.$$

A routine calculation shows that

$$\Theta((\alpha, a)(\beta, b)) = \Theta((\alpha, a))\ \Theta((\beta, b)).$$

In other words, Θ is multiplicative, and thus Θ is a homomorphism.

By definition of $\mathbb{A}_e$, Θ is clearly onto.

Let us check that Θ is injective. If $\Theta(\alpha, a) = \Theta(\beta, b)$, then $L_a + \alpha I - (L_b + \beta I) = 0$, which implies that

$$L_{a-b} = L_a - L_b = (\beta - \alpha)I.$$

If $\beta \neq \alpha$, then $(\beta - \alpha)^{-1} L_{a-b} = L_{(\beta-\alpha)^{-1}(b-a)} = I$, which implies that $e := (\beta - \alpha)^{-1}(b - a) \in \mathcal{A}$ satisfies $ex = x$ for all $x \in \mathcal{A}$. Thus e is a left multiplicative identity for $\mathbb{A}$.

But then $x \in \mathbb{A}$ implies that $x^* \in \mathcal{A}$, and $xe^* = (ex^*)^* = (x^*)^* = x$. That is, e^* is a right multiplicative identity for $\mathbb{A}$.

Finally, we see that this forces $e^* = ee^* = e$, so $e = e^*$ is a multiplicative identity for $\mathcal{A}$, contradicting our assumption that $\mathbb{A}$ was non-unital. Hence $\alpha = \beta$.

But then

$$L_{a-b} = 0I,$$

so that

$$0 = L_{a-b}(a - b)^* = (a - b)(a - b)^*,$$

which in a C^*-algebra implies that $a - b = 0$ (by the C^*-equation!!!); i.e. $a = b$.

Thus $\Theta(\alpha, a) = \Theta(\beta, b)$ implies that $(\alpha, a) = (\beta, b)$, and Θ is injective.

Next, we prove that Θ is a homeomorphism. By the Inverse Mapping Theorem (see Corollary 9.4 of my online Functional Analysis notes, for example), it suffices to prove that Θ is continuous (because it is a bijection).

Note that

$$\|\Theta(\alpha, a)\|_{\mathbb{A}_e} = \|L_a + \alpha I\| \le \|L_a\| + \|\alpha I\| = \|a\| + |\alpha| = \|(\alpha, a)\|_{\mathbb{A}_u}.$$

Thus $\|\Theta\| \le 1$; i.e. Θ is contractive (and thus continuous).

We have just proven that $\mathbb{A}_u$ is *Banach algebra*-isomorphic to $\mathbb{A}_e$. This in turn implies that

$$\sigma_{\mathbb{A}_u}((0, a)) = \sigma_{\Theta(\mathbb{A}_u)}(\Theta(0, a)) = \sigma_{\mathbb{A}_e}(L_a).$$

In other words, the C^*-algebraic and Banach algebraic spectra of $a \in \mathbb{A}$ coincide.

9.34. Example. Let X be a locally compact, Hausdorff space, and denote by X_0 the one point compactification of X. Then $\mathcal{C}(X_0)$ is the minimal unitization of $\mathcal{C}_0(X)$.

9.35. We mention that there is also a notion of a *largest* unitization for a C^*-algebra $\mathbb{A}$, called the **multiplier algebra** of $\mathbb{A}$. It plays an analogous rôle for abstract C^*-algebras that $\mathcal{B}(\mathcal{H})$ plays for $\mathcal{K}(\mathcal{H})$.

Positivity, normality and the abstract spectral theorem.

9.36. The most important C^*-algebra is the C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. When studying Hilbert space operators, it was useful to define the notions of positive operators, self-adjoint operators, normal and unitary operators. Fortunately, each of these notions may be generalised to the C^*-algebra setting, and they are extremely useful there as well.

9.37. Let $\mathbb{A}$ be a C^*-algebra and let $x \in \mathbb{A}$. We write $\sigma(x)$ to denote the spectrum of x in the unitisation $\mathbb{A}_e$ of $\mathbb{A}$. (If $\mathbb{A}$ is unital, recall that $\mathbb{A}_e = \mathbb{A}$, and so no confusions arises.)

It is easy to verify that $\sigma(x^*) = \sigma(x)^* = \{\overline{\lambda} : \lambda \in \sigma(x)\}$. Moreover, x is invertible if and only if both $x^* x$ and $x\,x^*$ are invertible. Indeed, if $x \in \mathbb{A}^{-1}$, then so is x^*. Thus $x^*\,x$ and $x\,x^*$ lie in $\mathbb{A}^{-1}$, since this latter is a group. Conversely, if $x^*\,x$ is invertible with inverse z, then $z\,x^*\,x = e_{\mathbb{A}}$ and so x is left invertible. But $(x\,x^*)r = e_{\mathbb{A}}$ for some $r \in \mathbb{A}$, and so x is right invertible.

Finally, we remark that the invertibility of *both* $x\,x^*$ and of $x^*\,x$ is required. Indeed, if S denotes the unilateral backward shift operator on $\mathcal{B}(\ell^2)$, then $S\,S^* = I$, but S is not invertible, as we have seen.

9.38. Definition. *For each a in a C^*-algebra $\mathbb{A}$, we define the* ***real part***

$$\operatorname{Re} a = (a + a^*)/2$$

and the ***imaginary part***

$$\operatorname{Im} a = (a - a^*)/2i$$

of a.

The terminology is of course inspired from the C^*-algebra $\mathbb{C}$.

9.39. Definition. *An element x of a C^*-algebra $\mathbb{A}$ is called*

- ***hermitian*** *if $x = x^*$;*
- ***normal*** *if $x\,x^* = x^* x$;*
- ***positive*** *if $x = x^*$ and $\sigma(x) \subseteq [0, \infty)$;*
- ***unitary*** *if $x^* = x^{-1}$;*
- ***idempotent*** *if $x = x^2$;*
- *a* ***projection*** *if $x = x^* = x^2$*
- *a* ***partial isometry*** *if $x\,x^*$ and $x^*\,x$ are projections (called the* range projection *and the* initial projection *of x, respectively).*

9.40. Example. Consider the C^*-algebra c_0. A sequence $\mathbf{x} = (x_n)_n \in c_0$ is

- hermitian if and only if $x_n \in \mathbb{R}$ for all $n \ge 1$;
- always normal;
- positive if and only if $x_n \ge 0$ for all $n \ge 1$;
- unitary if and only if $|x_n| = 1$ for all $n \ge 1$;
- idempotent (or a projection, or a partial isometry) if and only if $x_n \in \{0, 1\}$ for all $n \ge 1$, and only finitely many x_n's are non-zero.

We should think of hermitian elements of C^*-algebras as generalisations of real numbers (or real-valued functions), and unitary elements of C^*-algebras generalisations of complex numbers (or complexed-valued functions) of modulus 1. The next result shows us one reason for doing this.

9.41. Proposition. *Let $\mathbb{A}$ be a C^*-algebra.*

(i) *If $u \in \mathbb{A}$ is unitary, then $\sigma(u) \subseteq \mathbb{T}$.*
(ii) *If $h \in \mathbb{A}$ is hermitian, then $\sigma(h) \subseteq \mathbb{R}$.*

Proof.

(i) First observe that $1 = \|e_{\mathbb{A}}\| = \|u^*u\| = \|u\|^2$. Thus $\mathrm{spr}(u) \le \|u\| = 1$ implies $\sigma(u) \subseteq \overline{\mathbb{D}}$. But $\|u^{-1}\| \ge 1/\mathrm{dist}(0, \sigma(u))$ implies that $\mathrm{dist}(0, \sigma(u)) \ge \|u^*\| = 1$, and so $\sigma(u) \subseteq \mathbb{T}$.
(ii) Suppose $h = h^* \in \mathbb{A}$. Consider $u = \exp(ih)$. Using the uniform convergence of the power series expansion of $\exp(ih)$ we see that $u^* = \exp(-ih^*) = \exp(-ih)$. Since (ih) and $(-ih)$ obviously commute, we obtain:

$$\begin{aligned} u^*u &= \exp(-ih)\, exp\,(ih) \\ &= \exp(-ih + ih) \\ &= \exp(0) \\ &= 1 \\ &= uu^*. \end{aligned}$$

Thus u is unitary. By (i) and the holomorphic functional calculus, $\sigma(u) = \exp(i\sigma(h)) \subseteq \mathbb{T}$, from which we conclude that $\sigma(h) \subseteq \mathbb{R}$.

□

9.42. Suppose $\mathbb{S}$ is a unital, self-adjoint linear manifold in a C^*-algebra $\mathbb{A}$. If $h = h^* \in \mathbb{S}$, then $\mathrm{spr}(h) \le \|h\|$, and hence $\sigma(h) \subseteq [-\|h\|, \|h\|]$. Letting $p_1 = h + \|h\|e_{\mathbb{A}}$ and $p_2 = \|h\|e_{\mathbb{A}}$, we find that both p_1 and p_2 are positive and $h = p_1 - p_2$. Thus for any $s \in \mathbb{S}$, we may apply this to the real and imaginary parts of s to see that s is a linear combination of four positive elements. This linear combination is *far* from unique. (Another such linear combination is obtained by simply letting $q_1 = p_1 + e_{\mathbb{A}}$, $q_2 = p_2 + e_{\mathbb{A}}$.)

Such linear manifolds $\mathbb{S}$ as above are referred to as **operator systems**. For example, $\{\alpha_{-1}\overline{z} + \alpha_0 + \alpha_1 z : \alpha_{-1}, \alpha_0, \alpha_1 \in \mathbb{C}\}$ is an operator system in $\mathcal{C}(\mathbb{T})$. Many results stated for C^*-algebras carry over to operator systems. We refer the reader to [**39**] for an excellent treatment of this vast topic.

9.43. Theorem. *Suppose that $\mathbb{A} \subseteq \mathbb{B}$ are C^*-algebras and $x \in \mathbb{A}$. Then $\sigma_{\mathbb{A}}(x) = \sigma_{\mathbb{B}}(x)$.*

Proof. By considering $\mathbb{A}_e$ instead of $\mathbb{A}$, we may assume that $\mathbb{A}$ is unital. Clearly it suffices to prove that $\sigma_{\mathbb{A}}(x) \subseteq \sigma_{\mathbb{B}}(x)$.

First consider the case where $h = h^* \in \mathbb{A}$. Then $\sigma_{\mathbb{A}}(h) \subseteq \mathbb{R}$, and as such $\sigma_{\mathbb{A}}(h) = \partial\sigma_{\mathbb{A}}(h)$. By Proposition 4.7, $\partial\sigma_{\mathbb{A}}(h) \subseteq \sigma_{\mathbb{B}}(h)$ for any C^*-algebra $\mathbb{B}$ containing $\mathbb{A}$.

In general, if $x \in \mathbb{A}$ is not invertible in $\mathbb{A}$, then either $h_1 = x^*x$ or $h_2 = xx^*$ is not invertible. As h_1 and h_2 are self-adjoint, from above we have either $0 \in \sigma_{\mathbb{B}}(h_1)$ or $0 \in \sigma_{\mathbb{B}}(h_2)$. Either way, it follows that x is not invertible in $\mathbb{B}$.

□

9.44. Theorem. [The Gelfand-Naimark Theorem.] *Let $\mathbb{A}$ be an abelian C^*-algebra. Then the Gelfand Transform $\Gamma : \mathbb{A} \to \mathcal{C}_0(\Sigma_{\mathbb{A}})$ is an (necessarily isometric) *-isomorphism.*

Proof. We have seen that the Gelfand Transform is a norm decreasing homomorphism from $\mathbb{A}$ into $\mathcal{C}_0(\Sigma_{\mathbb{A}})$. By Theorem 5.17, $\sigma(a) \cup \{0\} = \operatorname{ran}\Gamma(a) \cup \{0\}$ in both the unital and non-unital cases. In particular, if $h = h^* \in \mathbb{A}$, then $\operatorname{ran}\Gamma(a) \subseteq \mathbb{R}$, and so $\Gamma(h) = \Gamma(h)^*$. Thus Γ is a *-homomorphism.

Also, $\|\Gamma(a)\|^2 = \|\Gamma(a^*a)\| = \operatorname{spr}(\Gamma(a^*a)) = \operatorname{spr}(a^*a) = \|a\|^2$, and so Γ is isometric. Finally, $\Gamma(\mathbb{A})$ is a closed, self-adjoint subalgebra of $\mathcal{C}_0(\Sigma_{\mathbb{A}})$ which (by Theorem 5.16) separates the points of $\Sigma_{\mathbb{A}}$. By the Stone-Weierstraß Theorem, $\Gamma(\mathbb{A}) = \mathcal{C}_0(\Sigma_{\mathbb{A}})$.

□

9.45. Theorem. [The Abstract Spectral Theorem.] *Let $\mathbb{A}$ be a unital C^*-algebra and $n \in \mathbb{A}$ be normal. Then $\Sigma_{C^*(n)}$ is homeomorphic to $\sigma(n)$. As such, $C^*(n)$ is isometrically *-isomorphic to $(\mathcal{C}(\sigma(n)), \|\cdot\|)$.*

Proof. We claim that $\Gamma(n)$ implements the homeomorphism between $\Sigma_{C^*(n)}$ and $\sigma(n)$. Since $\Sigma_{C^*(n)}$ is compact, $\sigma(n)$ is Hausdorff, and $\Gamma(n)$ is continuous, it suffices to show that $\Gamma(n)$ is a bijection. By Theorem 5.17, $\operatorname{ran}\Gamma(n) = \sigma(n)$, and so $\Gamma(n)$ is onto. Suppose $\phi_1, \phi_2 \in \Sigma_{C^*(n)}$ and $\phi_1(n) = \Gamma(n)(\phi_1) = \Gamma(n)(\phi_2) = \phi_2(n)$. Since Γ is a *-homomorphism, $\phi_1(n^*) = \Gamma(n^*)(\phi_1) = \overline{\Gamma(n)(\phi_1)} = \overline{\Gamma(n)(\phi_2)} = \Gamma(n^*)(\phi_2) = \phi_2(n^*)$. Then $\phi_1(p(n, n^*)) = \phi_2(n, n^*))$ for all polynomials p in two non-commuting variables, as both ϕ_1 and ϕ_2 are multiplicative. By the continuity of ϕ_1 and ϕ_2 and the density of $\{p(n, n^*) : p \text{ a polynomial in two non-commuting variables }\}$ in $C^*(n)$, we find that $\phi_1 = \phi_2$ and $\Gamma(n)$ is injective. By the Gelfand-Naimark Theorem 9.44, $C^*(n) \simeq^* \mathcal{C}(\Sigma_{C^*(n)})$. It follows immediately that $C^*(n) \simeq^* \mathcal{C}(\sigma(n))$.

□

9.46. Remark. It is worth drawing attention to the fact that if $\Gamma : C^*(n) \to \mathcal{C}(\sum_{C^*(n)})$ is the Gelfand Transform and for $x \in C^*(n)$ we set $\Gamma'(x) = \Gamma(x) \circ (\Gamma(n))^{-1}$, then Γ' implements the *-isomorphism between $C^*(n)$ and $\mathcal{C}(\sigma(n))$. Furthermore $\Gamma'(n)(z) = z$ for all $z \in \sigma(n)$; that is, $\Gamma'(n) = q$, where $q(z) = z$. In practice, we usually identify $\mathcal{C}(\Sigma_{C^*(n)})$ and $\mathcal{C}(\sigma(n))$, and still refer to the induced map Γ' as the Gelfand Transform, relabelling it as Γ.

When $\mathbb{A}$ is non-unital, we have $\mathbb{A} \subseteq \mathbb{A}_e$ and $C^*(1_{\mathbb{A}_e}, n) \simeq^* \mathcal{C}(\sigma(n))$. But then $C_0^*(n) \subseteq \mathbb{A}_e$ corresponds to the functions in $\mathcal{C}(\sigma(n))$ which vanish at 0, namely $\mathcal{C}_0(\sigma(n)\backslash\{0\})$.

As an immediate Corollary to the above theorem, we are able to extend the holomorphic functional Calculus developed in Chapter Two to a broader class of functions.

9.47. Theorem. [The Continuous Functional Calculus.] *Let $\mathbb{A}$ be a unital C^*-algebra and $n \in \mathbb{A}$ be normal. Then $\Gamma^{-1} : \mathcal{C}(\sigma(n)) \to C^*(n)$ is an isometric *-isomorphism and for all $f, g \in \mathcal{C}(\sigma(n))$, $\lambda \in \mathbb{C}$, we have*

(i) $(\lambda f + g)(n) = \lambda f(n) + g(n)$;
(ii) $(f\,g)(n) = f(n)\,g(n)$;
(iii) *the Spectral Mapping Theorem:* $\sigma(f(n)) = f(\sigma(n))$;
(iv) $\|f(n)\| = \mathrm{spr}(f(n)) = \mathrm{spr}(f) = \|f\|$.

In particular, if $q(z) = z$, $z \in \sigma(n)$, then $n = \Gamma^{-1}(q)$.

Remark. When $\mathbb{A}$ is non-unital, the Gelfand Transform induces a functional calculus for continuous functions vanishing at 0.

9.48. Corollary. *Let $\mathbb{A}$ be a unital C^*-algebra and $n \in \mathbb{A}$ be normal. Then*

(i) $n = n^*$ *if and only if* $\sigma(n) \subseteq \mathbb{R}$;
(ii) $n \ge 0$ *if and only if* $\sigma(n) \subseteq [0, \infty)$;
(iii) $n^* = n^{-1}$ *if and only if* $\sigma(n) \subseteq \mathbb{T}$;
(iv) $n = n^* = n^2$ *if and only if* $\sigma(n) \subseteq \{0, 1\}$.

Proof. This is an immediate consequence of identifying $C^*(n)$ with $\mathcal{C}(\sigma(n))$.

□

It is worth observing that all of the above notions are C^*-notions; that is, if $\varphi : \mathbb{A} \to \mathbb{B}$ is a *-isomorphism of C^*-algebras, then each of the above notions is preserved by φ.

9.49. Proposition. *Let $\mathbb{A}$ be a C^*-algebra and $0 \le r \in \mathbb{A}$. Then there exists a unique element $q \in \mathbb{A}$ such that $0 \le q$ and $q^2 = r$. Moreover, if $a \in \mathbb{A}$ and $a\,r = r\,a$, then a commutes with q.*

Proof. Consider the function $f(z) = z^{\frac{1}{2}} \in \mathcal{C}(\sigma(r))$, and note that $f(0) = 0$. Thus $q := f(r) \in C_0^*(r)$ and thus is a normal element of $\mathbb{A}$. In fact, $\sigma(q) = f(\sigma(r)) \subseteq [0, \infty)$, and so $q \ge 0$. Next, $q^2 = (f(r))^2 = f^2(r) = j(r) = r$, where $j(z) = z$, $z \in \sigma(r)$.

Suppose $0 \le s \in \mathbb{A}$ and $s^2 = r$. Then $s\,r = s(s^2) = (s^2)s = r\,s$, so that $C_0^*(r, s)$ is abelian. The Gelfand Map $\Gamma_1 : C_0^*(r, s) \to \mathcal{C}_0(\Sigma_{C_0^*(r,s)})$ is an isometric *-isomorphism and $\Gamma_1(q)$, $\Gamma_1(s)$ are two positive functions whose square is $\Gamma_1(r)$. Thus $\Gamma_1(q) = \Gamma_1(s)$. Since Γ_1 is injective, $q = s$. This shows that q is unique.

Finally, if $a\,r = r\,a$, then a commutes with every polynomial in r. Since $q = f(r)$ is a limit of polynomials in r, and since multiplication is jointly continuous, $a\,q = q\,a$.

□

For obvious reasons, we write $q = r^{\frac{1}{2}}$ and refer to q as the (positive) **square root** of r.

Let us momentarily pause to address a natural question which arises. For $\mathcal{H}$ a Hilbert space and $R \in \mathcal{B}(\mathcal{H})$, we currently have two apparently distinct notions of positivity. That is, we have the operator notion (1): $R = R^*$ and $\langle Rx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and the C^*-algebra notion (2): R is normal and $\sigma(R) \subseteq [0, \infty)$. The following proposition reconciles these two notions.

9.50. Proposition. *Let $\mathcal{H}$ be a complex Hilbert space and $R \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(i) *$R = R^*$ and $\langle Rx, x >\geq 0$ for all $x \in \mathcal{H}$;*
(ii) *R is normal and $\sigma(R) \subseteq [0, \infty)$.*

Proof.

(i) $\Rightarrow$ (ii) Clearly $R = R^*$ implies R is normal, and $\sigma(R) \subseteq \mathbb{R}$. Let $\lambda \in \mathbb{R}$ with $\lambda < 0$. Then

$$\begin{aligned}\|(R-\lambda I)x\|^2 &= \langle (R-\lambda I)x, (R-\lambda I)x\rangle \\ &= \langle Rx, Rx\rangle - 2\lambda\langle Rx, x\rangle + \lambda^2\langle x, x\rangle \\ &\geq \lambda^2\langle x, x\rangle.\end{aligned}$$

Thus $(R - \lambda I)$ is bounded below. Since R is normal, $\sigma(R) = \sigma_a(R)$ by Proposition 7.17, and therefore $\lambda \notin \sigma(R)$. Hence $\sigma(R) \in [0, \infty)$.

(ii) $\Rightarrow$ (i) Suppose R is normal and $\sigma(R) \subseteq [0, \infty)$. Then by Proposition 9.49, the operator $Q = R^{\frac{1}{2}}$ is positive. Let $x \in \mathcal{H}$. Then

$$\begin{aligned}\langle Rx, x\rangle &= \langle Q^2 x, x\rangle \\ &= \langle Qx, Qx\rangle \\ &= \|Qx\|^2 \geq 0.\end{aligned}$$

□

9.51. Remark. Of course, the above Proposition fails spectacularly when R is not normal. For example, if V is the classical Volterra operator from Example 7.23, then $\sigma(V) = \{0\} \subseteq [0, \infty)$. But V is not positive, or even normal, for the only normal quasinilpotent operator is 0.

9.52. Definition. *Let $\mathbb{A}$ be a C^*-algebra and $h = h^* \in \mathbb{A}$. Consider the function $f_+ : \mathbb{R} \to \mathbb{R}$, $f_+(x) = \max\{x, 0\}$. We define the **positive** part h_+ of h to be $h_+ = f_+(h)$, and the **negative** part h_- of h to be $h_- = h_+ - h$. It follows easily from the continuous functional calculus that $h_- = f_-(h)$, where $f_-(x) = -\min\{x, 0\}$ for all $x \in \mathbb{R}$. Both $h_+, h_- \geq 0$, as h_+, h_- are normal and $\sigma(h_+) = \sigma(f_+(h)) = f_+(\sigma(h)) \subseteq [0, \infty)$ (with a parallel proof holding for h_-). We therefore have $h = h_+ - h_-$.* Clearly, given $x \in \mathbb{A}$, we can write x in terms of its real and imaginary parts, $x = y + iz$, and $y = y_+ - y_-$, $z = z_+ - z_-$. Thus every element of $\mathbb{A}$ is a linear combination of (at most 4) positive elements.

A useful result that follows from the functional calculus is:

9.53. Proposition. *Let $\mathbb{A}$ be a C^*-algebra and $h = h^* \in \mathbb{A}$. Then*

$$\|h\| = \max(\|h_+\|, \|h_-\|).$$

Proof. Consider $\|h_+\| = \operatorname{spr}(h_+) = \operatorname{spr}(f_+(h)) = \|f_+|_{\sigma(h)}\| = \max(\{0\}, \{\lambda : \lambda \in \sigma(h), \lambda \geq 0\})$, while $\|h_-\| = \operatorname{spr}(h_-) = \operatorname{spr}(f_-(h)) = \|f_-|_{\sigma(h)}\| = \max(\{0\}, \{-\lambda : \lambda \in \sigma(h), \lambda \leq 0\})$. A moment's reflection shows that $\max(\|h_+\|, \|h_-\|) = \operatorname{spr}(h) = \|h\|$.

□

9.54. Lemma. *Let $\mathbb{A}$ be a unital C^*-algebra and $h = h^* \in \mathbb{A}$. The following are equivalent:*

(i) $h \geq 0$;
(ii) $\|t1 - h\| \leq t$ *for some* $t \geq \|h\|$;
(iii) $\|t1 - h\| \leq t$ *for all* $t \geq \|h\|$.

Proof. First let us identify $C^*(h)$ with $\mathcal{C}(\sigma(h))$ via the Gelfand Transform Γ. Let $\hat{h} = \Gamma(h)$ so that $\hat{h}(z) = z$ for all $z \in \sigma(h)$. The equivalence of the above three conditions is a result of their equivalence in $\mathcal{C}(\sigma(h))$, combined with the fact that positivity is a C^*-notion, as noted in the comments following Corollary 9.48. Thus we have

(i) $\Rightarrow$ (iii)

$$\begin{aligned} \|t1 - h\| &= \|\Gamma(t1 - h)\| \\ &= \|t\mathbf{1} - \hat{h}\| \\ &\leq t \qquad \text{for all } t \geq \|\hat{h}\| = \|h\|. \end{aligned}$$

(iii) $\Rightarrow$ (ii) Obvious.
(ii) $\Rightarrow$ (i) If $\|t1 - h\| = \|t\mathbf{1} - \hat{h}\| \leq t$, then $\hat{h} \geq 0$, and so $h \geq 0$.

□

9.55. Definition. *Let $\mathbb{A}$ be a Banach space. A* ***real cone*** *in $\mathbb{A}$ is a subset $\mathbb{V}$ of $\mathbb{A}$ satisfying:*

(i) $0 \in \mathbb{V}$;
(ii) *if* $x, y \in \mathbb{V}$ *and* $\lambda \geq 0$ *in* $\mathbb{R}$, *then* $\lambda x + y \in \mathbb{V}$;
(iii) $\mathbb{V} \cap \{-x : x \in \mathbb{V}\} = \{0\}$.

For the sake of convenience, we shall write $-\mathbb{V}$ for $\{-x : x \in \mathbb{V}\}$.

9.56. Example. Let $\mathbb{A} = \mathbb{C}$, the complex numbers viewed as a 1-dimensional Banach space over itself. The set $\mathbb{V} = \{z \in \mathbb{C} : \operatorname{Re}(z) \in [0, \infty),\ \operatorname{Im}(z) \in [0, \infty)\}$ is a real cone in $\mathbb{A}$. More generally, any of the four "quadrants" in $\mathbb{C}$ determined by two lines passing through the origin forms a real cone.

9.57. Example. Let $\mathbb{A} = \mathcal{C}(X)$, X a compact Hausdorff space. The set $\mathbb{V} = \{f \in \mathcal{C}(X) : f \geq 0\}$ is a real cone in $\mathbb{A}$.

9.58. Proposition. *Let $\mathbb{A}$ be a C^*-algebra. Then $\mathbb{A}_+ = \{p \in \mathbb{A} : p \geq 0\}$ is a norm-closed, real cone in $\mathbb{A}$, called* ***the positive cone*** *of $\mathbb{A}$.*

Proof. We may assume without loss of generality that $1 \in \mathbb{A}$. Clearly $0 \in \mathbb{A}_+$, and if $p \in \mathbb{A}_+$ and $0 \leq \lambda \in \mathbb{R}$, then $(\lambda p)^* = \overline{\lambda} p^* = \lambda p$ and $\sigma(\lambda p) = \lambda \sigma(p) \subseteq [0, \infty)$, so that $\lambda p \in \mathbb{A}_+$.

Next suppose that $x, y \in \mathbb{A}_+$. By Lemma 9.54, we obtain:

$$\begin{aligned} \| \, (\|x\| + \|y\|)1 - (x+y) \, \| &\leq \| \, \|x\|1 - x \, \| + \| \, \|y\|1 - y \, \| \\ &\leq \|x\| + \|y\|, \end{aligned}$$

imply by the same Lemma that $x + y \geq 0$. Suppose $x \in \mathcal{F} \cap (-\mathcal{F})$. Then $x = x^*$ and $\sigma(x) \subseteq [0, \infty) \cap (-\infty, 0] = \{0\}$. Since $\|x\| = \text{spr}(x) = 0$, we have $x = 0$. So far we have shown that $\mathbb{A}_+$ is a real cone.

Finally, suppose that we have $\{x_n\}_{n=1}^\infty \subseteq \mathbb{A}_+$ and $x = \lim_{n\to\infty} x_n$. Then $x^* = \lim_{n\to\infty} x_n^* = \lim_{n\to\infty} x_n = x$, so that x is self-adjoint. By dropping to a subsequence if necessary, we may assume that $\|x\| \geq \|x_n\|/2$ for all $n \geq 1$. Then

$$\begin{aligned} \| \, (2\|x\|)1 - x \, \| &= \lim_{n\to\infty} \| \, (2\|x\|)1 - x_n \, \| \\ &\leq \lim_{n\to\infty} 2\|x\| \\ &= 2\|x\|. \end{aligned}$$

By Lemma 9.54 yet again, $x \geq 0$ and so $\mathbb{A}_+$ is norm-closed, completing the proof.

□

9.59. Let $\mathcal{H}$ be a Hilbert space and $Z \in \mathcal{B}(\mathcal{H})$. If $R = Z^*Z$, then for any $x \in \mathcal{H}$,

$$\langle Rx, x \rangle = \langle Z^*Zx, x \rangle = \|Zx\|^2 \geq 0,$$

and so $R \geq 0$. Our next goal is to show that in any C^*-algebra, $r \in \mathcal{A}$ is positive precisely if r factors as z^*z for some $z \in \mathcal{A}$. The proof is rather more delicate than in the $\mathcal{B}(\mathcal{H})$ setting. The next Lemma comes in handy.

9.60. Lemma. *Let $\mathcal{A}$ be a Banach algebra and $a, b \in \mathcal{A}$. Then*

$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}.$$

Proof. Clearly it suffices to consider the case where $\mathcal{A}$ is unital. The proof, while completely unmotivated, is a simple algebraic calculation.

Suppose $0 \neq \lambda \in \rho(ab)$. Let $c = \lambda^{-1}(\lambda - ab)^{-1}$, and verify that

$$\begin{aligned} (\lambda - ba)^{-1} &= (\lambda^{-1} + bca) \\ &= \lambda^{-1} + \lambda^{-1} b(\lambda - ab)^{-1} a \\ &= \lambda^{-1}(1 + b(\lambda - ab)^{-1} a). \end{aligned}$$

□

9.61. Theorem. *Let $\mathbb{A}$ be a C^*-algebra and $r \in \mathbb{A}$. Then $r \geq 0$ if and only if $r = z^* z$ for some $z \in \mathbb{A}$.*

Proof. First suppose that $r \geq 0$. By Proposition 9.49, there exists a unique $z \geq 0$ so that $r = z^2 = z^* z$.

Next, suppose $r = z^* z$ for some $z \in \mathbb{A}$. Clearly $r = r^*$. Let us write r as the difference of its positive and negative parts, namely $r = r_+ - r_-$. Our goal is to show that $r_- = 0$.

Now $r_- \geq 0$ and so r_- has a positive square root. Consider $y = z r_-^{\frac{1}{2}}$. Then $y^* y$ is self-adjoint and

$$\begin{aligned} y^* y &= (z r_-^{\frac{1}{2}})^* (z r_-^{\frac{1}{2}}) \\ &= r_-^{\frac{1}{2}} z^* z r_-^{\frac{1}{2}} \\ &= r_-(r_+ - r_-) r_-^{\frac{1}{2}} \\ &= r_-^{\frac{1}{2}} r_+ r_-^{\frac{1}{2}} - r_-^2 \\ &= -r_-^2 \\ &\leq 0. \end{aligned}$$

(Note that the last equality follows from the fact that $f_-^{\frac{1}{2}} f_+ = 0$.) Thus $\sigma(y^* y) \subseteq (-\infty, 0]$. Writing $y = h + ik$, where $h = \mathrm{Re}y$, $k = \mathrm{Im}y$, we have

$$\begin{aligned} yy^* &= h^2 + i\,k\,h - i\,h\,k + k^2 \\ y^* y &= h^2 - i\,h\,h + i\,h\,k + k^2 \end{aligned}$$

so that

$$yy^* = (yy^* + y^* y) - (y^* y) = 2(h^2 + k^2) - (y^* y).$$

Since $h^2 + k^2 \geq 0$ and $y^* y \leq 0$ from above, the fact that $\mathbb{A}_+$ is a positive cone implies that $yy^* \geq 0$. Thus $\sigma(yy^*) \subseteq [0, \infty)$.

By the previous Lemma, $\sigma(yy^*) \cup \{0\} = \sigma(y^* y) \cup \{0\}$, from which we deduce that $\sigma(yy^*) = \{0\} = \sigma(y^* y)$. But then $\|y\|^2 = \|y^* y\| = \mathrm{spr}(y^* y) = 0$, and so $y = 0$. That is, $\| - r_-^2\| = \|y^* y\| = 0$, so that $r_- = 0$ and $r = r_+ \geq 0$, as claimed.

□

9.62. Remark.

(a) Given $a \in \mathbb{A}$, a C^*-algebra, we can now define $|a| = (a^* a)^{\frac{1}{2}}$, and we call this the **absolute value** of a.

(b) The above theorem has a partial extension to involutive unital Banach algebras. Suppose $\mathcal{B}$ is such an algebra with unit e, and $x \in \mathcal{B}$ satisfies $\|x - e\| < 1$. Then $x = y^2$ for some $y \in \mathcal{B}$. Indeed, when $\|e - x\| < 1$, we have $\sigma(x) \subseteq \{z \in \mathbb{C} : |z - 1| < 1\}$. As such the function $f(z) = z^{\frac{1}{2}}$ is analytic on $\sigma(x)$, and so we set $y = f(x)$.

9.63. A partial order on $\mathbb{A}_{sa}$. Given a C^*-algebra $\mathbb{A}$, we denote by $\mathbb{A}_{sa}$ the set of self-adjoint elements of $\mathbb{A}$, i.e.

$$\mathbb{A}_{sa} := \{h \in \mathbb{A} : h = h^*\}.$$

This is a *real* subspace of $\mathbb{A}$. If $a \in \mathbb{A}$, we define

$$\mathrm{Re}(a) := \frac{a + a^*}{2} \text{ and } \mathrm{Im}(a) = \frac{a - a^*}{2i}.$$

It is clear that $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ are self-adjoint and $a = \mathrm{Re}(a) + i\mathrm{Im}(a)$. This way of writing a is referred to as the **Cartesian decomposition** of a.

Given two self-adjoint elements $x, y \in \mathbb{A}$, a C^*-algebra, we set $x \le y$ if $y - x \ge 0$. It is easy to check that this defines a partial order. Certain, but not all properties of the order on $\mathbb{R}$ carry over to this setting. Consider the following:

9.64. Proposition. *Let $\mathbb{A}$ be a C^*-algebra.*

(i) *If $a, b \in \mathbb{A}_{sa}$ and $c \in \mathbb{A}$, then $a \le b$ implies $c^*ac \le c^*bc$.*
(ii) *If $0 \le a \le b$, then $\|a\| \le \|b\|$.*
(iii) *If $1 \in \mathbb{A}$, $a, b \in \mathbb{A}_+$ are invertible and $a \le b$, then $b^{-1} \le a^{-1}$.*

Proof.

(i) Since $a \le b$, $b - a$ is positive, and so we can find $z \in \mathbb{A}$ so that $b - a = z^*z$. Then $c^*z^*zc = (zc)^*(zc) \ge 0$ by Theorem 9.61. That is, $c^*bc - c^*ac \ge 0$, which is equivalent to our claim.

(ii) It suffices to consider the case where $1 \in \mathbb{A}$. Then the unital C^*-algebra generated by b, namely $C^*(b) \simeq^* \mathcal{C}(\sigma(b))$. Then $\Gamma(b) \le \|\Gamma(b)\|\mathbf{1} = \|b\|\mathbf{1}$, and since positivity is a C^*-property, $b \le \|b\|1$.

But then $a \le b$ and $b \le \|b\|1$ implies $a \le \|b\|1$. Again, by the Gelfand-Naimark Theorem, $\Gamma(a) \le \|b\|\mathbf{1}$, and so $\|a\| = \|\Gamma(a)\| \le \|b\|$.

(iii) First suppose $c \ge 1$. Then $\Gamma(c) \ge \mathbf{1}$, and so $\Gamma(c)$ is invertible and $\Gamma(c)^{-1} \le \mathbf{1}$. This in turn implies that c is invertible and that $c^{-1} \le 1$.

More generally, given $a \le b$, $1 = a^{-\frac{1}{2}}aa^{-\frac{1}{2}} \le a^{-\frac{1}{2}}ba^{-\frac{1}{2}}$, and so the above argument implies that $1 \ge (a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{-1} = a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}$.

Finally, $a^{-1} = a^{-\frac{1}{2}}1a^{-\frac{1}{2}} \ge b^{-1}$, by (i) above.

$\square$

Supplementary Examples

S9.1. Example. Our work on the Jacobson radical in the Appendix to Chapter 5 allows us to obtain a very useful result without much more effort.

Theorem. *Every C^*-algebra $\mathbb{A}$ is **semisimple**; that is, the Jacobson radical $J(\mathbb{A})$ of $\mathbb{A}$ is equal to $\{0\}$.*

Proof. Recall that $J(\mathbb{A})$ is a left quasi-regular ideal in $\mathbb{A}$. Suppose that $J(\mathbb{A}) \neq \{0\}$. Let $0 \neq q \in J(\mathbb{A})$ and set $q_0 := -\frac{1}{\|q\|^2} q^* q$. Then $q_0 \in J(\mathbb{A})$ since the latter is an ideal. Since $0 \leq q^* q$, it follows that $q_0 \leq 0$ and $\|q_0\| = 1$. That is, $q_0 = q_0^*$, $\sigma(q_0) \subseteq [-1, 0]$, and $= \mathrm{spr}(q_0) = \|q_0\| = 1$, implying that $-1 \in \sigma(q_0) := \sigma_{\mathbb{A}_e}(q_0)$. In other words, $1 + q_0$ is not invertible in $\mathbb{A}_e$.

Since $J(\mathbb{A})$ is left quasi-regular, there exists $w \in \mathbb{A}$ such that $w \diamond q = 0$. By identifying $\mathbb{A}$ with its image in $\mathbb{A}_e$, we may suppose that $w, q \in \mathbb{A}_e$. By Proposition A5.7, it then follows that $1 + q_0$ is left invertible in $\mathbb{A}_e$. Let $b \in \mathbb{A}_e$ denote the left-inverse of $1 + q_0$. That is, $b(1 + q_0) = 1$ in $\mathbb{A}_e$. But then

$$1 = 1^* = (b(1 + q_0))^* = (1 + q_0)^* b^* = (1 + q_0) b^*,$$

implying that $1 + q_0$ is invertible in $\mathbb{A}_e$, and contradiction.

Thus $J(\mathbb{A}) = \{0\}$, as claimed.

□

S9.2. Example. If follows that if $\mathbb{A}$ is a finite-dimensional C^*-algebra, then – by the Wedderburn-Artin Theorem [**5, 54**] – $\mathbb{A}$ is isomorphic to a direct sum of full matrix algebras. That is, one can find a sequence $(k_1, k_2, \ldots, k_m)$ in $\mathbb{N}^m$ and a bijective algebra homomorphism $\varrho : \mathcal{A} \to \oplus_{i=1}^m \mathbb{M}_{k_i}(\mathbb{C})$.

Of course, the canonical isomorphisms in the category of C^*-algebras are the *-isomorphisms, and it is natural to ask whether or not $\mathbb{A}$ is *-isomorphic to $\oplus_{i=1}^m \mathbb{M}_{k_i}(\mathbb{C})$. The answer is "yes", and the proof is left as an exercise for the reader.

It is also worth pointing out that even if $\mathbb{A} \subseteq \mathbb{M}_n(\mathbb{C})$ for some $n \geq 1$, this still leaves open the question of **multiplicity**. For example, the algebras

$$\mathbb{A} = \{\mathrm{diag}(\alpha, \alpha, \alpha, A) : \alpha \in \mathbb{C}, A \in \mathbb{M}_2(\mathbb{C})\}$$

and

$$\mathbb{B} := \{\mathrm{diag}(\alpha, A, A) : \alpha \in \mathbb{C}, A \in \mathbb{M}_2(\mathbb{C})\}$$

are both subalgebras of $\mathbb{M}_5(\mathbb{C})$ which are isomorphic to $\mathbb{C} \oplus \mathbb{M}_2(\mathbb{C})$, but they are inherently different as subalgebras of $\mathbb{M}_5(\mathbb{C})$. One way of identifying this difference is to consider their **commutants**, that is, by comparing the operators which commute with each algebra.

For example, an easy computation shows that the commutants $\mathbb{A}'$ of $\mathbb{A}$ and $\mathbb{B}'$ of $\mathbb{B}$ are the algebras

$$\mathbb{A}' = \{\mathrm{diag}(B, \beta I_2) : B \in \mathbb{M}_3(\mathbb{C}), \beta \in \mathbb{C}\},$$

while

$$\mathbb{B}' = \left\{ \begin{bmatrix} \beta & 0 & 0 \\ 0 & \gamma_1 I_2 & \gamma_2 I_2 \\ 0 & \gamma_3 I_2 & \gamma_4 I_2 \end{bmatrix} : \beta, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{C} \right\}$$

so that $\dim(\mathbb{A}') = 10$, while $\dim(\mathbb{B}') = 5$.

The classification of C^*-algebras acting on a finite-dimensional Hilbert space then says that if $\mathbb{A} \subseteq \mathbb{M}_n(\mathbb{C})$ is a C^*-algebra, then there exist two sequences

$$(k_1, k_2, \ldots, k_m), (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{N}^m$$

and a unitary matrix U such that

$$U^* \mathbb{A} U = \mathbb{M}_{k_1}^{(\mu_1)}(\mathbb{C}) \oplus \mathbb{M}_{k_2}^{(\mu_2)}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_{k_m}^{(\mu_m)}(\mathbb{C}) \oplus 0_r,$$

where $r = n - (\sum_{i=1}^m k_i \mu_i)$.

S9.3. Example. There are many interesting and important classes of C^*-algebras; more than one can shake a stick at, even if it were a very light stick, one had burly arms, and one were the stick-shaking champion of one's university. Unfortunately, even the definitions of many of these algebras (for example AF-algebras, Bunce-Deddens algebras, crossed-product C^*-algebras, full/reduced group C^*-algebras) require more background than we currently have.

Let us at least give an alternate definition of a **uniformly hyperfinite** (UHF-) C^*-algebra which avoids a description of inductive limits.

Let $\mathcal{H} = \ell^2$ with ONB $(e_n)_{n \ge 1}$. Given $n \ge 1$, we let

$$\mathbb{A}_n := \mathbb{M}_n(\mathbb{C}) \otimes I := \{T^{(\infty)} := T \oplus T \oplus T \oplus T \cdots : T \in \mathbb{M}_n(\mathbb{C})\}.$$

It is not difficult to verify that if $n \mid m$, then $\mathbb{A}_n \subseteq \mathbb{A}_m$.

Let $(k_n)_{n \ge 1}$ be a sequence of positive integers such that

- $k_n \mid k_{n+1}$ for all $n \ge 1$; and
- $\lim_n k_n = \infty$.

We leave it to the reader to verify that

$$\mathbb{B}_0 := \cup_{n \ge 1} \mathbb{A}_{k_n}$$

is a self-adjoint, unital subalgebra of $\mathcal{B}(\mathcal{H})$. It is not, however, a C^*-algebra because it is not norm-closed. The norm-closure $\mathbb{B} := \overline{\mathbb{B}_0}$ of $\mathbb{B}_0$ *is* a C^*-algebra, called a UHF C^*-algebra.

It can be shown that UHF C^*-algebras are simple, unital and admit a unique **tracial state**. That is, there exists a unique linear function $\tau : \mathbb{B} \to \mathbb{C}$ such that $\|\tau\| = 1 = \tau(1)$, and $\tau(xy) = \tau(yx)$ for each $x, y \in \mathbb{B}$.

Although each element of $\mathbb{B}$ is *almost a matrix* (in the sense that it can be approximated arbitrarily well by an operator of the form $T^{(\infty)}$ for some $T \in \mathbb{M}_{k_n}(\mathbb{C})$, nevertheless, $\mathbb{B} \cap \mathcal{K}(\mathcal{H}) = \{0\}$, and so elements of $\mathbb{B}$ are in this sense *very far* from being matrices.

Appendix

A9.1. The question of determining whether or not two C^*-algebras $\mathbb{A}$ and $\mathbb{B}$ are *-isomorphic is a deep and often extremely difficult problem. Attempts to answer this problem for various classes of C^*-algebras have resulted in entirely new fields of mathematics.

One very successful approach has been to use tools from (algebraic) K-theory to bring to bear upon this question. The idea is to associate to each C^*-algebra $\mathbb{A}$ two groups $K_0(\mathbb{A})$ and $K_1(\mathbb{A})$ which are related to projections and unitaries in (matrix algebras over) $\mathbb{A}$. More precisely, the map $\mathbb{A} \mapsto (K_0(\mathbb{A}), K_1(\mathbb{A}))$ is a functor from the category of C^*-algebras to the category of abelian groups. Given two C^*-algebras $\mathbb{A}$ and $\mathbb{B}$, if $(K_0(\mathbb{A}), K_1(\mathbb{B}))$ is not isomorphic to $(K_0(\mathbb{B}), K_1(\mathbb{B}))$, then $\mathbb{A}$ and $\mathbb{B}$ are not isomorphic. (The converse is false.) The basic idea behind this approach is that the pairs of abelian groups should be easier to study than the original C^*-algebras themselves.

This approach is not always successful in distinguishing between non-isomorphic C^*-algebras, and other techniques have also been brought to bear upon this problem, including other types of K-theory (such as KK-theory and KL-theory).

We refer the reader to [**48**] and to [**55**] for more information on K-theory. Be aware that this field is vast, which is both exciting an daunting.

A9.2. C^*-algebra theory also connects with many other areas of math, including graph theory. Given a graph $\mathcal{G}$, it is possible to associate to $\mathcal{G}$ a C^*-algebra known as the **graph algebra** of $\mathcal{G}$. *Very roughly*, vertices of the graph correspond to projections in the $C{\ast}$-algebras and edges in the graph correspond to **partial isometries** in the C^*-algebra. A description of these is beyond the scope of these notes, and we refer the interested reader to the wonderful book by Raeburn [**44**] for more information on these.

A9.3. The author would be remiss in failing to mention the excellent monograph by K.R. Davidson [**19**], which also provides a cornucopia of C^*-algebras and highlights the basic results which everyone studying C^*-algebras should know.

Exercises for Chapter 9

Exercise 9.1. $\mathcal{T}_n$ IS A BANACH *-ALGEBRA

Prove that the involution on $\mathcal{T}_n$ defined in Example 9.7 is isometric.

Exercise 9.2. THE POSITIVE CONE

Let $\mathbb{A}$ be a C^*-algebra, and let a, b lie in the positive cone of $\mathbb{A}$.

(i) Show that $a \leq b$ implies $a^{1/2} \leq b^{1/2}$.
(ii) Show that $a \leq b$ does NOT imply that $a^2 \leq b^2$.

Exercise 9.3. COMPACT NORMAL OPERATORS

If M and N are compact normal operators with the same spectrum, then $C^*(M)$ is isometrically isomorphic to $C^*(N)$. Do M and N have to be unitarily equivalent?

Exercise 9.4. FINITE-DIMENSIONAL C^*-ALGEBRAS

Suppose that $\mathbb{A}$ is a finite-dimensional C^*-algebra and that $\varrho : \mathbb{A} \to \mathbb{M}_n(\mathbb{C})$ is an algebra homomorphism. Prove that there exists $S \in \mathbb{M}_n(\mathbb{C})$ invertible such that the map $\pi : \mathbb{A} \to \mathbb{M}_n(\mathbb{C})$ defined by

$$\pi(a) = S^{-1}\varrho(a)S, a \in \mathbb{A}$$

is a *-homomorphism.

Exercise 9.5. FINITE-DIMENSIONAL C^*-ALGEBRAS, PART II

Prove that if $\mathbb{A} \subseteq \mathbb{M}_n(\mathbb{C})$ is a C^*-algebra, then there exist two sequences

$$(k_1, k_2, \ldots, k_m), (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{N}^m$$

and a unitary matrix U such that

$$U^*\mathbb{A}U = \mathbb{M}_{k_1}^{(\mu_1)}(\mathbb{C}) \oplus \mathbb{M}_{k_2}^{(\mu_2)}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_{k_m}^{(\mu_m)}(\mathbb{C}) \oplus 0_r,$$

where $r = n - (\sum_{i=1}^m k_i\mu_i)$.

CHAPTER 10

C^*-algebras: approximate identities and ideals

A thesaurus is great. There's no other word for it.

Ross Smith

Approximate identities.

10.1. In Chapter 9, we briefly discussed ideals of C^*-algebras in connection with unitisations. Now we return for a more detailed and structured look at ideals and their elements. As with Banach algebras in general, at times it is not desirable to adjoin a unit to a C^*-algebra. As we are about to see, however, there is a distinct advantage of C^*-algebras over Banach algebras when it comes to approximate identities. For one thing, they always exist. Secondly, one can always choose them to consist of positive elements, bounded above in norm by 1. Thirdly, one can choose the net to be increasing. The C^*-equation is a wondrous thing.

10.2. Definition. *Let $\mathbb{A}$ be a C^*-algebra and suppose $\mathbb{K}$ is a linear manifold in $\mathbb{A}$. Then a **right approximate identity** for $\mathbb{K}$ is an increasing net (u_λ) of positive elements in $\mathbb{K}$ such that $\|u_\lambda\| \le 1$ for all λ, and such that*

$$\lim_\lambda \|k - k\, u_\lambda\| = 0$$

for all $k \in \overline{\mathbb{K}}$.

Analogously, one can define a **left approximate identity** for a linear manifold $\mathbb{K}$ of $\mathbb{A}$.

By an **algebraic** (**left**, **right**, or **two-sided**) **ideal** of a C*-algebra $\mathbb{A}$, we shall simply mean a linear manifold $\mathbb{K}$ which is invariant under multiplication (on the left, right, or both sides) by elements of $\mathbb{A}$. The notion of a (left, right or two-sided) ideal differs only in that ideals are assumed to be norm-closed. Unless otherwise specified, *algebraic ideals and ideals are assumed to be two-sided.*

10.3. Example. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then $\mathcal{F}(\mathcal{H}) = \{F \in \mathcal{B}(\mathcal{H}) : \operatorname{rank} F < \infty\}$ is an algebraic ideal whose closure is the set $\mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$ (by Theorem 7.19).

10.4. Example. Let $\mathcal{C}_{00}(\mathbb{R}) = \{f \in \mathcal{C}(\mathbb{R}) : \text{ supp}(f) \text{ is compact } \}$. Then $\mathcal{C}_{00}(\mathbb{R})$ is an algebraic ideal of $\mathcal{C}(\mathbb{R})$ whose norm closure is $\mathcal{C}_0(\mathbb{R})$, the set of continuous functions which vanish at infinity.

10.5. Proposition. *Let $\mathbb{A}$ be a C^*-algebra and suppose $\mathbb{K}$ is an **algebraic** left ideal in $\mathbb{A}$. Then $\mathbb{K}$ has a right approximate identity.*

Proof. We may assume without loss of generality that $\mathbb{A}$ is unital. Given a finite subset $F = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{K}$, we define

$$h_F = \sum_{i=1}^{n} a_i^* a_i$$

and

$$v_F = h_F \Big(h_F + \frac{1}{n} 1\Big)^{-1} = \Big(h_F + \frac{1}{n} 1\Big)^{-1} h_F.$$

Note that $v_F \in \mathbb{K}$, since $h_F \in \mathbb{K}$.

Let $\mathcal{F} = \{F : F \subseteq \mathbb{K},\ F \text{ finite } \}$ be directed by inclusion.

Claim: the set $(v_F : F \in \mathcal{F}, \supseteq)$ is a right approximate identity for $\mathbb{K}$.

To see this, note first that $h_F \geq 0$ and that $0 \leq t(t + \frac{1}{n})^{-1} \leq 1$ for all $t \in \mathbb{R}^+$. Thus $0 \leq v_F \leq 1$ by the functional calculus. Suppose $F, G \in \mathcal{F}$ and $F \supseteq G$. We may assume that $F = \{a_1, a_2, \ldots, a_m\}$ and that $G = \{a_1, a_2, \ldots, a_n\}$, where $n \leq m$.

Now $h_F \geq h_G$ since $h_F - h_G = \sum_{i=n+1}^{m} a_i^* a_i \geq 0$. Thus $h_F + \frac{1}{n} 1 \geq h_G + \frac{1}{n} 1$ and hence

$$(h_F + \frac{1}{n} 1)^{-1} \leq (h_G + \frac{1}{n} 1)^{-1}.$$

From this, and since $\frac{1}{m}(t + \frac{1}{m})^{-1} \leq \frac{1}{n}(t + \frac{1}{n})^{-1}$ for all $t \in \mathbb{R}^+$, we have

$$\frac{1}{m}(h_F + \frac{1}{m} 1)^{-1} \leq \frac{1}{n}(h_F + \frac{1}{n} 1)^{-1} \leq \frac{1}{n}(h_G + \frac{1}{n} 1)^{-1}.$$

But

$$\frac{1}{m}(h_F + \frac{1}{m} 1)^{-1} = 1 - v_F$$

and

$$\frac{1}{n}(h_G + \frac{1}{n} 1)^{-1} = 1 - v_G,$$

and so $1 - v_F \leq 1 - v_G$, implying that $v_G \leq v_F$ when $F \supseteq G$.

Suppose $k \in \mathbb{K}$. Given $n \in \mathbb{N}$, choose $F_0 \in \mathcal{F}$ such that F_0 has n elements and $k \in F_0$. If $F \in \mathcal{F}$ and $F_0 \subseteq F$, then F has $m\,(\geq n)$ elements, including k. Thus $k^* k \leq h_F$, and

$$\begin{aligned} (k - k v_F)^* (k - k v_F) &= (1 - v_F) k^* k (1 - v_F) \\ &\leq (1 - v_F) h_F (1 - v_F) \\ &= \tfrac{1}{m^2} (h_F + \tfrac{1}{m} 1)^{-2} h_F. \end{aligned}$$

Since $\frac{1}{m^2}(t+\frac{1}{m}1)^{-2}t \le \frac{1}{4m}$ for all $t \in \mathbb{R}^+$, we have

$$\begin{aligned}\|k - kv_F\|^2 &= \|(k-kv_F)^*(k-kv_F)\| \\ &\le \|\tfrac{1}{m^2}(h_F + \tfrac{1}{m}1)^{-2}h_F\| \\ &\le \tfrac{1}{4m} \\ &\le \tfrac{1}{4n}.\end{aligned}$$

Thus $\|k - kv_F\| \le \frac{1}{2\sqrt{n}}$ for all $F \supseteq F_0$. By definition, $\lim_{F\in\mathcal{F}} \|k - kv_F\| = 0$.

Finally, suppose that $k \in \overline{\mathbb{K}}$, and let $\varepsilon > 0$. Choose $k_0 \in \mathbb{K}$ such that $\|k - k_0\| < \frac{\varepsilon}{3}$. By the previous paragraph, there exists $F_0 \in \mathcal{F}$ such that $F_0 \subseteq F \in \mathcal{F}$ implies that $\|k_0 - k_0 v_F\| < \frac{\varepsilon}{3}$. Thus $F_0 \subseteq F \in \mathcal{F}$ implies that

$$\begin{aligned}\|k - kv_F\| &\le \|k - k_0\| + \|k_0 - k_0 v_F\| + \|k_0 v_F - kv_F\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|k - k_0\|\,\|v_F\| \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3}\cdot 1 \\ &= \varepsilon.\end{aligned}$$

Thus $k = \lim_{F\in\mathcal{F}} kv_F$. This concludes the proof.

□

If $\mathbb{K}$ is an algebraic right ideal of $\mathbb{A}$, then $\mathbb{K}^*$ is an algebraic left ideal of $\mathbb{A}$. By applying the above Proposition to $\mathbb{K}^*$ and interpreting it in terms of $\mathbb{K}$ itself we obtain:

10.6. Corollary. *Let $\mathbb{A}$ be a C^*-algebra and suppose $\mathbb{K}$ is an* ***algebraic*** *right ideal in $\mathbb{A}$. Then $\mathbb{K}$ has a left approximate identity.*

10.7. Remark. We have shown that an algebraic two sided ideal of a C*-algebra $\mathbb{A}$ possesses both a left and a right approximate identity. We now wish to show that these two identities can be chosen to coincide. First we require a lemma.

10.8. Lemma. *Let $\mathbb{A}$ be a C^*-algebra and suppose that $\mathbb{K}$ is an algebraic, self-adjoint ideal in $\mathbb{A}$. Then any left approximate identity for $\mathbb{K}$ is also a right approximate identity for $\mathbb{K}$, and vice-versa.*

Proof. Suppose (u_λ) is a right approximate unit for $\mathbb{K}$. Then $\lim_\lambda \|k - ku_\lambda\| = 0$ for all $k \in \overline{\mathbb{K}}$. But then $\lim_\lambda \|k^* - k^* u_\lambda\| = 0 = \lim_\lambda \|k - u_\lambda k\|$, so that (u_λ) is also a left approximate unit for $\mathbb{K}$.

□

10.9. Theorem. *Every C^*-algebra has an approximate identity. If the C^*-algebra is separable, then a countable approximate identity may be chosen*

Proof. Let $\mathbb{A}$ be the C*-algebra. It is clearly a self-adjoint left ideal in itself, and therefore has a right approximate identity by Proposition 10.5, which is an approximate identity by Lemma 10.8.

Next, suppose $\mathbb{A}$ is separable, and let $\{a_n\}_{n=1}^\infty$ be a countable dense subset of $\mathbb{A}$. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for $\mathbb{A}$. Choose $\lambda_0 \in \Lambda$ arbitrarily. For each $k \geq 1$, we can find $\lambda_k \in \Lambda$ such that $\lambda_k \geq \lambda_{k-1}$ and $\max(\|u_{\lambda_k} a_n - a_n\|, \|a_n u_{\lambda_k} - a_n\|) < \varepsilon$ for each $1 \leq n \leq k$. A relatively routine approximation argument then implies that $(u_{\lambda_k})_{k=1}^\infty$ is the desired countable approximate identity.

□

10.10. Corollary. *Every closed ideal $\mathbb{K}$ in a C^*-algebra $\mathbb{A}$ is self-adjoint.*

Proof. Let $k \in \mathbb{K}$, and let $(u_\lambda)_\lambda$ denote the approximate identity for $\mathbb{K}$. Then $k^* = \lim_\lambda k^* u_\lambda$, but $u_\lambda \in \mathbb{K}$ for all λ, implying that each $k^* u_\lambda$ and therefore k^* lies in $\overline{\mathbb{K}} = \mathbb{K}$.

□

The above result is, in general, false if the ideal is not closed. For example, if $\mathbb{A} = \mathcal{C}(\overline{\mathbb{D}})$, the continuous functions on the closed unit disk, and if $\mathbb{K} = q\,\mathbb{A}$, where $q \in \mathbb{A}$ is the identity function $q(z) = z$, then $\mathbb{K}$ is an algebraic ideal in $\mathbb{A}$, but q^* does not lie in $\mathbb{K}$.

10.11. Corollary. *Every algebraic ideal $\mathbb{K}$ in a C^*-algebra $\mathbb{A}$ has an approximate identity.*

Proof. Since $\overline{\mathbb{K}}$ is a *closed* ideal in $\mathbb{A}$, it must be self-adjoint, by the previous Corollary. The left approximate identity (u_λ) for $\mathbb{K}$ is again a left approximate identity for $\overline{\mathbb{K}}$. By Lemma 10.8, (u_λ) is a right approximate identity for $\overline{\mathbb{K}}$, and since it already lies in $\mathbb{K}$, it is therefore an approximate unit for $\mathbb{K}$.

□

10.12. Example. Let $\mathbb{A}$ be a unital C*-algebra. For each $n \geq 1$, set $u_n = e_{\mathbb{A}}$. Then $\{u_n\}_{n=1}^\infty$ is an approximate unit for $\mathbb{A}$.

10.13. Example. Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. For each $k \geq 1$, let P_k denote the orthogonal projection of $\mathcal{H}$ onto the span of $\{e_1, e_2, \ldots, e_k\}$. Then from the arguments of Theorem 7.19, $\{P_k\}_{k=1}^\infty$ is an approximate identity for $\mathcal{K}(\mathcal{H})$.

10.14. Example. Consider the ideal $\mathcal{C}_0(\mathbb{R})$ of $\mathcal{C}(\mathbb{R})$. For each $n \geq 1$, let

$$u_n(x) = \begin{cases} 1 & \text{if } |x| \leq n, \\ (n+1) - |x| & \text{if } |x| \in (n, n+1), \\ 0 & \text{if } |x| \geq n. \end{cases}$$

Then $\{u_n\}_{n=1}^{\infty}$ is an approximate identity for $\mathcal{C}_0(\mathbb{R})$.

10.15. Proposition. *Suppose $\mathbb{A}$ is a C*-algebra and $\mathbb{L}$ is an ideal in $\mathbb{A}$. If $\mathbb{K}$ is an ideal in $\mathbb{L}$, then $\mathbb{K}$ is also an ideal in $\mathbb{A}$.*

Proof. Since $\mathbb{K}$ is the linear span of its positive elements, it suffices to prove that ak and ka lie in $\mathbb{K}$ for all $a \in \mathbb{A}$ and $0 \leq k \in \mathbb{K}$. Since $ak = (a\,k^{\frac{1}{2}})k^{\frac{1}{2}}$, and since $k^{\frac{1}{2}} \in \mathbb{K}$, we have $ak \in \mathbb{K}\,\mathbb{L} \subseteq \mathbb{K}$.

□

10.16. Definition. *Let $\mathbb{B}$ be a C*-algebra. A C*-subalgebra $\mathbb{A}$ of $\mathbb{B}$ is said to be **hereditary** if $b \in \mathbb{B}_+,\ a \in \mathbb{A}_+$ with $0 \leq b \leq a$ implies $b \in \mathbb{A}$.*

10.17. Example. Let $\mathbb{B} = \mathcal{C}([0,1])$ and $\mathbb{A} = \{f \in \mathcal{C}([0,1]) : f(x) = 0 \text{ for all } x \in [\frac{1}{4}, \frac{3}{4}]\}$. If $g \in \mathbb{B}_+,\ f \in \mathbb{A}_+$ and $0 \leq g \leq f$, then $0 \leq g(x) \leq f(x) = 0$ for all $x \in [\frac{1}{4}, \frac{3}{4}]$, and hence $g \in \mathbb{A}$. Thus $\mathbb{A}$ is a hereditary C*-subalgebra of $\mathbb{B}$.

10.18. Proposition. *Let $\mathbb{B}$ be a C*-algebra and $0 \neq p \neq 1$ be a projection in $\mathbb{B}$. Then $\mathbb{A} = p\mathbb{B}p$ is hereditary.*

Proof. That $\mathbb{A} = p\mathbb{B}p$ is a C*-subalgebra of $\mathbb{B}$ is routine. Suppose $0 \leq b \leq a$ for some $a \in \mathbb{A},\ b \in \mathbb{B}$. Then by Proposition 9.64, $0 \leq (1-p)b(1-p) \leq (1-p)a(1-p) = 0$, and so $(1-p)b(1-p) = 0$.

Next, $\|b^{\frac{1}{2}}(1-p)\|^2 = \|(1-p)b(1-p)\| = 0$, so that $b(1-p) = b^{\frac{1}{2}}(b^{\frac{1}{2}}(1-p)) = 0$. Finally, since $b = b^*$, $(1-p)b = (b(1-p))^* = 0$, and so $b = pbp \in \mathbb{A}$, as required.

□

10.19. Lemma. *Let $\mathbb{A}$ be a C*-algebra and $a, b \in \mathbb{A}$. Suppose $0 \leq b,\ \|b\| \leq 1$ and $aa^* \leq b^4$. Then there exists $c \in \mathbb{A}$, $\|c\| \leq 1$ such that $a = b\,c$.*

Proof. Let $\mathbb{A}_e$ denote the minimal unitisation of $\mathbb{A}$, and denote by $\mathbf{1}$ the identity in $\mathbb{A}_e$. For $0 < \lambda < 1$, let $c_\lambda = (b + \lambda 1)^{-1}a$, which lies in $\mathbb{A}$ because $\mathbb{A}$ is an ideal of $\mathbb{A}_e$. Our goal is to prove that $c = \lim_{\lambda \to 0} c_\lambda$ exists, and that this is the element we want. Now

$$\begin{aligned} c_\lambda c_\lambda^* &= (b+\lambda 1)^{-1} aa^* (b+\lambda 1)^{-1} \\ &\leq (b+\lambda 1)^{-1} b^4 (b+\lambda 1)^{-1} \\ &\leq b^2, \end{aligned}$$

and hence $\|c_\lambda\|^2 = \|c_\lambda^*\|^2 \le \|b^2\| \le 1$. Next we prove that $\{c_\lambda\}_{\lambda\in(0,1)}$ is Cauchy. If $\lambda, \beta \in (0,1)$, then

$$\begin{aligned}
\|c_\lambda - c_\beta\|^2 &= \|(c_\lambda - c_\beta)^*\|^2 \\
&= \|(c_\lambda - c_\beta)(c_\lambda - c_\beta)^*\| \\
&= \|((b+\lambda 1)^{-1} - (b+\beta 1)^{-1})aa^*(b+\lambda 1)^{-1} - (b+\beta 1)^{-1})\| \\
&= |\lambda - \beta|^2 \|(b+\lambda 1)^{-1}(b+\beta 1)^{-1}aa^*(b+\lambda 1)^{-1}(b+\beta 1)^{-1}\| \\
&\le |\lambda - \beta|^2 \|(b+\lambda 1)^{-1}(b+\beta 1)^{-1}b^4(b+\lambda 1)^{-1}(b+\beta 1)^{-1}\| \\
&\le |\lambda - \beta|^2,
\end{aligned}$$

as $b^4(b+\lambda 1)^{-2}(b+\beta 1)^{-2} \le 1$. Let $c = \lim_{\lambda\to 0} c_\lambda$. Then $bc = \lim_{\lambda\to 0} bc_\lambda = a$.

□

10.20. Proposition. *Let $\mathbb{A}$ be a C^*-algebra. Then every ideal in $\mathbb{A}$ is hereditary.*

Proof. Suppose that $0 \ne \mathbb{K}$ is an ideal in $\mathbb{A}$, $a \in \mathbb{A}, k \in \mathbb{K}$ and $0 \le a \le k$. Then we can write $a = zz^*$ for some $z \in \mathbb{A}$ and $k = (k^{\frac{1}{4}})^4$, where $k^{\frac{1}{4}} \in \mathbb{K}$ by the continuous functional calculus. Then

$$0 \le zz^* \le (k^{\frac{1}{4}})^4.$$

By Lemma 10.19, $z = (k^{\frac{1}{4}})c$ for some $c \in \mathbb{A}$. In particular, $z \in \mathbb{K}$ and hence $a = zz^* \in \mathbb{K}$, as required.

□

10.21. Theorem. *Let $\mathbb{A}$ be a C^*-algebra and $\mathbb{K}$ be an ideal in $\mathbb{A}$. Let (u_λ) be any approximate unit for $\mathbb{K}$. Then $\mathbb{A}/\mathbb{K}$ is a C^*-algebra, and for $a \in \mathbb{A}$, we have*

$$\|\pi_{\mathbb{K}}(a)\| = \lim_\lambda \|a - au_\lambda\|.$$

Proof. Fix $a \in \mathbb{A}$. Clearly

$$\begin{aligned}
\|\pi_{\mathbb{K}}(a)\| &= \inf\{\|a+k\| : k \in \mathbb{K}\} \\
&\le \inf\{\|a - au_\lambda\| : \lambda \in \Lambda\},
\end{aligned}$$

as each u_λ and hence au_λ lies in $\mathbb{K}$.

Now, given $\epsilon > 0$, choose $k \in \mathbb{K}$ so that $\|\pi_{\mathbb{K}}(a)\| + \epsilon > \|a+k\|$. Then

$$\begin{aligned}
\|a - au_\lambda\| &= \|(a+k) - k - au_\lambda\| \\
&= \|(a+k) - (k - ku_\lambda) - (a+k)u_\lambda\| \\
&\le \|(a+k) - (a+k)u_\lambda\| + \|k - ku_\lambda\|.
\end{aligned}$$

We shall work in the unitisation $\mathbb{A}_e$ of $\mathbb{A}$ in order to obtain our desired norm estimates.

$$\begin{aligned} \|a - au_\lambda\| &\leq \|(a+k)(1-u_\lambda)\| + \|k - ku_\lambda\| \\ &\leq \|a+k\| \; \|1-u_\lambda\| + \|k - ku_\lambda\|. \end{aligned}$$

Hence $\lim_\lambda \|a - au_\lambda\| \leq (\|\pi_{\mathbb{K}}(a)\| + \epsilon)1 + 0$. Since $\epsilon > 0$ was arbitrary, $\|\pi_{\mathbb{K}}(a)\| = \lim_\lambda \|a - au_\lambda\|$.

We saw in Proposition 2.17 that $\mathbb{A}/\mathbb{K}$ is a Banach algebra. Since $\mathbb{K}$ is self-adjoint, we can set $\pi_{\mathbb{K}}(a)^* = \pi_{\mathbb{K}}(a^*)$, and this is a well-defined involution on $\mathbb{A}/\mathbb{K}$. There remains only to verify the C^*-equation.

Given $a \in \mathbb{A}$,

$$\begin{aligned} \|\pi_{\mathbb{K}}(a)^* \, \pi_{\mathbb{K}}(a)\| &= \|\pi_{\mathbb{K}}(a^* a)\| \\ &= \inf_\lambda \|a^* a - a^* a u_\lambda\| \\ &\geq \inf_\lambda \|(1-u_\lambda)(a^* a)(1-u_\lambda)\| \\ &= \inf_\lambda \|a(1-u_\lambda)\|^2 \\ &= \|\pi_{\mathbb{K}}(a)\|^2. \end{aligned}$$

By Remark 9.9, we see that the quotient norm is a C^*-norm, and thus $\mathbb{A}/\mathbb{K}$ is a C^*-algebra.

□

A subspace of a Banach space is said to be **proximinal** if the distance from an arbitrary vector to that subspace is always attained. Although we shall not prove it here, it can be shown that ideals of C^*-algebras are proximinal - that is, the quotient norm is attained.

10.22. Theorem. *Let $\tau : \mathbb{A} \to \mathbb{B}$ be a *-homomorphism between C^*-algebras $\mathbb{A}$ and $\mathbb{B}$. Then $\|\tau\| \leq 1$, and τ is isometric if and only if τ is injective.*

Proof. First suppose $0 \leq r \in \mathbb{A}$. Then $r = z^* z$ for some $z \in \mathbb{A}$, and hence $\tau(r) = \tau(z)^* \tau(z) \geq 0$. In particular, $\tau(r)$ is normal and so $C_0^*(\tau(r))$ is abelian. Let $\varphi \in \sum_{C_0^*(\tau(r))}$. Then $\varphi \circ \tau \in \sum_{C_0^*(r)}$ and hence $\|\varphi \circ \tau\| \leq 1$. But

$$\begin{aligned} \|\tau(r)\| &= \sup\{\varphi(\tau(r)) : \varphi \in \textstyle\sum_{C_0^*(r)}\} \\ &\leq \|r\|. \end{aligned}$$

More generally, if $a \in \mathbb{A}$, then $a^* a \geq 0$, and hence from above,

$$\|\tau(a)\|^2 = \|\tau(a^* a)\| \leq \|a^* a\| = \|a\|^2.$$

Thus we have shown that τ is continuous, with $\|\tau\| \leq 1$.

Clearly if τ is isometric, it must be injective.

Next, suppose that τ is not isometric, and choose $a \in \mathbb{A}$ such that $\|a\| = 1$, but $\|\tau(a)\| < 1$. Let $r = a^* a$. Then $\|r\| = \|a\|^2 = 1$, but $\|\tau(r)\| = \|\tau(a)\|^2 = 1 - \delta < 1$ for some $\delta > 0$. We shall work with $r \geq 0$ instead of a. Choose $f \in \mathcal{C}([0,1])$ such that $f(x) = 0$ for all $x \in [0, 1-\delta]$, but $f(1) = 1$. By the Stone-Weierstraß Theorem, f is a limit of polynomials p_n in one variable with $p_n(0) = 0$ for each $n \geq 1$. For any such polynomial,

$$\tau(p_n(r)) = p_n(\tau(r)),$$

since τ is a *-homomorphism. Since τ is continuous from above,

$$\begin{aligned} \tau(f(r)) &= \tau(\lim_{n\to\infty} p_n(r)) \\ &= \lim_{n\to\infty} \tau(p_n(r)) \\ &= \lim_{n\to\infty} p_n(\tau(r)) \\ &= f(\tau(r)). \end{aligned}$$

Now $\mathrm{spr}(r) = \|r\| = 1$, and since $0 \leq r$, we conclude that $1 \in \sigma(r)$. Thus $1 = f(1) \in f(\sigma(r)) = \sigma(f(r))$, so that $f(r) \neq 0$. Finally, $\tau(r) \geq 0$ and $\mathrm{spr}(\tau(r)) \leq \|\tau(r)\| \leq 1-\delta$. Since $f|_{[0,1-\delta]} = 0$, we have $f(\tau(r)) = 0 = \tau(f(r))$, implying that τ is not injective.

□

10.23. Corollary. *Let $\tau : \mathbb{A} \to \mathbb{B}$ be a *-homomorphism between C^*-algebras $\mathbb{A}$ and $\mathbb{B}$. Then τ can be factored as $\tau = \overline{\tau} \circ \pi$, where $\pi : \mathbb{A} \to \mathbb{A}/\ker\tau$ is the canonical map, and $\overline{\tau} : \mathbb{A}/\ker\tau \to \mathrm{ran}\,\tau$ is an isometric *-isomorphism. In particular, $\tau(\mathbb{A})$ is a C^*-algebra.*

Proof. Since τ is continuous, $\ker\tau$ is a norm-closed ideal of $\mathbb{A}$, and hence is self-adjoint by Corollary 10.10. By Theorem 10.21, $\mathbb{A}/\ker\tau$ is a C^*-algebra and from elementary algebra arguments, τ factors as $\tau = \overline{\tau} \circ \pi$, where $\pi : \mathbb{A} \to \mathbb{A}/\ker\tau$ is the canonical map and $\overline{\tau}$ is the *-homomorphism

$$\begin{array}{rccc} \overline{\tau}: & \mathbb{A}/\ker\tau & \to & \mathrm{ran}\,\tau \\ & a + \ker\tau & \mapsto & \tau(a). \end{array}$$

Since $\ker\overline{\tau} = 0$, $\overline{\tau}$ is an isometric map onto its range, and thus $\mathrm{ran}\,\overline{\tau} = \mathrm{ran}\,\tau$ is a C^*-subalgebra of $\mathbb{B}$.

□

10.24. Proposition. *Let $\mathbb{A}$ be a C^*-subalgebra of a C^*-algebra $\mathbb{B}$, and let $\mathbb{K}$ be an ideal of $\mathbb{B}$. Then $\mathbb{A} \cap \mathbb{K}$ is an ideal in $\mathbb{A}$, and*

$$\frac{\mathbb{A}+\mathbb{K}}{\mathbb{K}} \simeq \frac{\mathbb{A}}{\mathbb{A}\cap\mathbb{K}}.$$

In particular, $\mathbb{A} + \mathbb{K}$ is a C^*-subalgebra of $\mathbb{B}$.

Proof. The first statement is a routine exercise. Consider the map

$$\begin{array}{rccc} \beta: & \mathbb{A} & \to & \mathbb{B}/\mathbb{K} \\ & a & \mapsto & a + \mathbb{K}. \end{array}$$

It is readily seen to be a *-homomorphism. Moreover, $\ker\beta = \mathbb{A} \cap \mathbb{K}$. By Corollary 10.23, $\mathrm{ran}\,\beta = \mathbb{A} + \mathbb{K}/\mathbb{K}$ is isometrically *-isomorphic to $\mathbb{A}/(\mathbb{A}\cap\mathbb{K})$, and so $\mathbb{A}+\mathbb{K}/\mathbb{K}$ is a C^*-algebra. Thus it is complete. Since $\mathbb{K}$ is also complete, we conclude that $\mathbb{A} + \mathbb{K}$ is complete as well. Hence $\mathbb{A} + \mathbb{K}$ is a closed, self-adjoint subalgebra of $\mathbb{B}$, as was required to prove.

□

10.25. Theorem. *Let $\mathbb{A}$ be a hereditary C^*-subalgebra of a C^*-algebra $\mathbb{B}$. Suppose $\mathbb{K}$ is an ideal in $\mathbb{A}$. Then $\mathbb{K} = \mathbb{A} \cap \mathbb{L}$ for some ideal $\mathbb{L}$ of $\mathbb{B}$.*
Proof.

□

Supplementary Examples

S10.1. Example. In progress.

Appendix

A10.1. In progress.

Exercises for Chapter 10

Exercise 10.1. In progress.

CHAPTER 11

C^*-algebras: the GNS construction

My opinions may have changed, but not the fact that I'm right.

Ashleigh Brilliant

The state space.

11.1. Let us now turn our attention to the dual space of a C^*-algebra. As we shall see, the linear functionals generalise the notion of measures on spaces of continuous functions, and are crucial to the representation theory of C^*-algebras.

11.2. Definition. *Let $\mathbb{A}$ and $\mathbb{B}$ be C^*-algebras, and let φ denote a linear map from $\mathbb{A}$ to $\mathbb{B}$. We define the **adjoint** of φ as $\varphi^* : \mathbb{A} \to \mathbb{B}$ via $\varphi^*(a) = (\varphi(a^*))^*$ for all $a \in \mathbb{A}$. Then φ is said to be **self-adjoint** if $\varphi = \varphi^*$.*

*The map φ is said to be **positive** if $\varphi(x^*x) \geq 0$ for all $x \in \mathbb{A}$. (Equivalently, if $\varphi(p) \geq 0$ whenever $p \geq 0$.) We write $\varphi \geq 0$ when this is the case.*

*An element $\varphi \in \mathbb{A}^*$ is called a **state** if φ is a positive linear functional of norm one. We denote by $\mathcal{S}(\mathbb{A})$ the set of all states on $\mathbb{A}$, and refer to this as the **state space** of $\mathbb{A}$.*

11.3. Remarks. A few comments are in order.

- By definition, a linear map φ between C^*-algebras is self-adjoint if and only if $\varphi(x^*) = (\varphi(x))^*$ for all $x \in \mathbb{A}$. It is routine to verify that this is equivalent to asking that φ send hermitian elements of $\mathbb{A}$ to hermitian elements of $\mathbb{B}$.
- If $\varphi \geq 0$, then φ preserves order. That is, if $x \leq y$ in $\mathbb{A}$, then $y - x \geq 0$, and hence $\varphi(y - x) = \varphi(y) - \varphi(x) \geq 0$ in $\mathbb{B}$.
- Finally, it is easy to see that every positive linear map φ is automatically self-adjoint. Indeed, given $h = h^* \in \mathbb{A}$, write $h = h_+ - h_-$, and observe that $\varphi(h) = \varphi(h_+) - \varphi(h_-)$ is self-adjoint.

11.4. Example. Let $\varphi : \mathbb{A} \to \mathbb{B}$ be any *-homomorphism between C^*-algebras $\mathbb{A}$ and $\mathbb{B}$. Then

$$\varphi(x^*x) = \varphi(x)^* \ \varphi(x) \geq 0$$

for all $x \in \mathbb{A}$, and hence $\varphi \geq 0$.

11.5. Example. Let X be a compact, Hausdorff space. The Riesz-Markov Theorem [**49**] asserts that $\mathcal{C}(X)^* \simeq \mathcal{M}(X)$, the space of complex-valued regular Borel measures on X. The action of a measure μ on $f \in \mathcal{C}(X)$ is through integration, that is: $\mu(f) := \int_X f \, \mathrm{d}\mu$.

When $X = [0,1]$, we can identify $\mathcal{M}(X)$ with the space $BV[0,1]$ of functions of bounded variation on $[0,1]$. Now, given $F \in BV[0,1]$, we define $\mu_F \in \mathcal{M}(X)$ via

$$\mu_F(f) = \int_X f \, \mathrm{d}F,$$

the quantity on the right being a Riemann-Stieltjes integral. For example, the evaluation functional $\delta_x(f) = f(x)$ for some $x \in X$ corresponds to the point mass at x.

Observe that μ is a self-adjoint (resp. positive) linear functional precisely when the measure $\mathrm{d}\mu$ is real-valued (resp. positive).

11.6. Example. Let $n, m \geq 1$ be integers, and consider the C^*-algebra $\mathbb{A} = \mathbb{M}_n \oplus \mathbb{M}_m \subseteq \mathcal{B}(\mathbb{C}^{n+m})$. For each $k \geq 1$, let $\mathrm{tr} : \mathbb{M}_k \to \mathbb{C}$ denote the **normalized trace functional**

$$\mathrm{tr}\left([a_{ij}]\right) = \frac{1}{k} \sum_{i=1}^{k} a_{ii}.$$

For $a = (a_1, a_2) \in \mathbb{A}$ and $\lambda \in [0,1]$, we can define $\varphi_\lambda(a) = \lambda \mathrm{tr}_n(a_1) + (1-\lambda)\mathrm{tr}_m(a_2)$. Then $\{\varphi_\lambda\}_{\lambda \in [0,1]}$ is a family of states on $\mathbb{A}$.

11.7. Example. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and let P be a non-trivial projection on $\mathcal{H}$. The map

$$\begin{array}{rccc} \varphi : & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & T & \mapsto & PTP \end{array}$$

is a positive linear map.

Indeed, if $T \geq 0$, the $\varphi(T^*T) = PT^*TP = (TP)^*(TP) \geq 0$. Observe that φ is *not* a *-homomorphism!

11.8. Remark. We have shown that every element of a C^*-algebra is, in a natural way, a linear combination of four positive elements. Of course, this is a generalisation of the corresponding fact for complex numbers.

In a similar vein, every complex measure possesses a **Jordan decomposition** [**49**] as a linear combination of four positive measures. Because of the association between linear functionals on commutative C^*-algebras and measures as outlined above, we shall think of linear functionals on C^*-algebras as abstract measures, and obtain a corresponding Jordan decomposition for these as well. This will imply that the state space of a C^*-algebra $\mathbb{A}$ is in some sense "large", a fact which we shall exploit in the proof of the Gelfand-Naimark Construction below.

We have seen that every multiplicative linear functional on an abelian Banach algebra is automatically continuous of norm one. In the case of C^*-algebras, the same applies to positive linear functionals.

11.9. Theorem. *Let $\mathbb{A}$ be a C^*-algebra and $\varphi : \mathbb{A} \to \mathbb{C}$ be a positive linear map. Then φ is continuous.*

Proof. First observe that φ is bounded if and only if there exists $K > 0$ so that $0 \le r \in \mathbb{A}_+$ with $\|r\| \le 1$ implies $\varphi(r) \le K$. Indeed, if φ is bounded, we can trivially choose $K = \|\varphi\|$.

Conversely, if $0 \le r \in \mathbb{A}_+$ with $\|r\| \le 1$ implies $\varphi(r) \le K$, then given any $x \in \mathbb{A}$, we can write $x = y + iz$, where $y = \mathrm{Re}x$, $z = \mathrm{Im}x$. Then we set $y = y_+ - y_-$ and $z = z_+ - z_-$, and recall that $\max(\|y_+\|, \|y_-\|, \|z_+\|, \|z_-\|) \le \|x\|$. From this we obtain

$$\begin{aligned} |\varphi(x)| &= |\varphi(y_+) - \varphi(y_-) + i\varphi(z_+) - i\varphi(z_-)| \\ &\le \varphi(y_+) + \varphi(y_-) + \varphi(z_+) + \varphi(z_-) \\ &\le 4K\|x\|, \end{aligned}$$

and so $\|\varphi\| \le 4K < \infty$.

Now we argue by contradiction. Suppose, to the contrary, that for every $n \ge 1$ we can find $0 \le r_n$ in $\mathbb{A}$ so that $\|r_n\| \le \frac{1}{2^n}$ and $\varphi(r_n) \ge 1$. Then for each $k \ge 1$, $s_k = \sum_{n=1}^k r_n \in \mathbb{A}_+$, and $s_k \le s = \sum_{n=1}^\infty r_n \in \mathbb{A}_+$. From Remark 11.3, we see that $k \le \varphi(s_k) \le \varphi(s)$ for all $k \ge 1$, which is absurd. It follows that φ must be bounded on $\mathbb{A}_+$, and hence on $\mathbb{A}$.

□

11.10. Given a positive linear functional φ on a C^*-algebra $\mathbb{A}$, we can construct a pseudo-inner product on $\mathbb{A}$ by setting

$$[a, b] := \varphi(b^* a)$$

for $a, b \in \mathbb{A}$. Then we have

(i) $[a, b]$ is clearly a sesquilinear function, linear in a and conjugate linear in b;
(ii) $[a, a] \ge 0$ for all $a \in \mathbb{A}$, as $\varphi \ge 0$ and $a^* a \ge 0$;
(iii) Since φ is self-adjoint, $[a, b] = \varphi(b^* a) = \overline{\varphi(a^* b)} = \overline{[b, a]}$;
(iv) If $x \in \mathbb{A}$, then $[xa, b] = \varphi(b^*(xa)) = \varphi((x^* b)^* a) = [a, x^* b]$.

The following will also prove useful in the GNS construction.

11.11. Lemma. *Let $[\cdot, \cdot]$ be a positive sesquilinear function on a C^*-algebra $\mathbb{A}$. Then $[\cdot, \cdot]$ satisfies the Cauchy-Schwarz Inequality:*

$$|[a, b]|^2 \le [a, a]\ [b, b].$$

Proof.

(a) If $[a, b] = 0$, there is nothing to prove.

(b) If $[a,a] = 0$, then we claim that $[a,b] = 0$ for all $b \in \mathbb{A}$. To see this, note that for all $\beta \in \mathbb{C}$,

$$\begin{aligned} 0 &\leq [a+\beta b, a+\beta b] \\ &= [a,a] + |\beta|^2[b,b] + 2\mathrm{Re}(\beta[a,b]). \end{aligned}$$

Suppose there exists $b \in \mathbb{A}$ such that $[a,b] \neq 0$. We may then scale b so that $[a,b] = -1$. Now choose $\beta > 0$. The above equation then becomes:

$$\begin{aligned} 0 &\leq [a,a] - 2\beta + \beta^2[b,b] \\ &= -2\beta + \beta^2[b,b], \end{aligned}$$

which implies $0 \leq \beta[b,b] - 2$. This yields a contradiction when β is chosen sufficiently small and positive. Thus $[a,a] = 0$ implies $|[a,b]|^2 = 0 \leq [a,a]\,[b,b]$, which is clearly true.

(c) If $[a,b] \neq 0$ and $[a,a] \neq 0$, we may choose $\beta = -[a,a]/[a,b]$. Then, as above,

$$\begin{aligned} 0 &\leq [a,a] + |\beta|^2[b,b] + 2\mathrm{Re}(\beta[a,b]) \\ &= [a,a] - 2[a,a] + \frac{[a,a]^2[b,b]}{|[a,b]|^2}, \end{aligned}$$

which implies

$$|[a,b]|^2 \leq [a,a]\,[b,b],$$

as claimed.

□

11.12. Lemma. *Let* $\mathbb{A}$ *be a* C^**-algebra, and* $0 \leq \varphi \in \mathbb{A}^*$. *Then*

(i) $|\varphi(b^*a)| \leq \varphi(a^*a)^{\frac{1}{2}}\,\varphi(b^*b)^{\frac{1}{2}}$;

(ii) $|\varphi(a)|^2 \leq \|\varphi\|\varphi(a^*a)$.

Proof.

(i) This is just a reformulation of the Cauchy-Schwarz Inequality which we deduced for the pseudo-inner product associated to φ in the previous Lemma.

(ii) Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\mathbb{A}$. Then

$$\begin{aligned} |\varphi(a)|^2 &= \lim_\lambda |\varphi(au_\lambda)|^2 \\ &\leq \sup_\lambda \varphi(a^*a)\,\varphi(u_\lambda^*u_\lambda) \\ &\leq \sup_\lambda \varphi(a^*a)\|\varphi\|\,\|u_\lambda\|^2 \\ &\leq \varphi(a^*a)\|\varphi\|. \end{aligned}$$

□

11.13. Theorem. *Let $\mathbb{A}$ be a C^*-algebra and $\varphi \in \mathbb{A}^*$. The following are then equivalent:*

(i) $0 \leq \varphi$;

(ii) $\|\varphi\| = \lim_\lambda \varphi(u_\lambda)$ *for some approximate unit* $(u_\lambda)_{\lambda\in\Lambda}$ *of* $\mathbb{A}$;

(iii) $\|\varphi\| = \lim_\lambda \varphi(u_\lambda)$ *for every approximate unit* $(u_\lambda)_{\lambda\in\Lambda}$ *of* $\mathbb{A}$.

Proof.

(i) implies (iii) Consider $\{\varphi(u_\lambda)\}_{\lambda\in\Lambda}$, which is an increasing net of positive real numbers, bounded above by $\|\varphi\|$. Then $\lim_\lambda \varphi(u_\lambda)$ exists. Clearly $\lim_\lambda \varphi(u_\lambda) \leq \sup_\lambda \|\varphi\| \, \|u_\lambda\| \leq \|\varphi\|$.

For the other inequality, first observe that if $0 \leq r$ and $\|r\| \leq 1$, then $0 \leq r^2 \leq r$. This follows from the Gelfand-Naimark Theorem by indentifying r with the identity function $q(z) = z$ on $\sigma(r) \subseteq [0,1]$. Then, given $a \in \mathbb{A}$,

$$\begin{aligned}
|\varphi(a)| &= \lim_\lambda |\varphi(u_\lambda a)| \\
&\leq \lim_\lambda \varphi(u_\lambda^* u_\lambda)^{\frac{1}{2}} \, \varphi(a^* a)^{\frac{1}{2}} \\
&\leq \lim_\lambda \varphi(u_\lambda)^{\frac{1}{2}} \, \varphi(a^* a)^{\frac{1}{2}} \\
&\leq \lim_\lambda \varphi(u_\lambda)^{\frac{1}{2}} \|\varphi\|^{\frac{1}{2}} \, \|a^* a\|^{\frac{1}{2}} \\
&\leq (\lim_\lambda \varphi(u_\lambda)^{\frac{1}{2}}) \|\varphi\|^{\frac{1}{2}} \, \|a\|.
\end{aligned}$$

By taking the supremum over $a \in \mathbb{A}$, $\|a\| = 1$, we find that $\|\varphi\|^{\frac{1}{2}} \leq \lim_\lambda \varphi(u_\lambda)^{\frac{1}{2}}$, and hence $\|\varphi\| = \lim_\lambda \varphi(u_\lambda)$.

(iii) implies (ii) Obvious.

(ii) implies (i) Let us scale φ so that $\|\varphi\| = 1$. Consider $h = h^* \in \mathbb{A}$ with $\|h\| = 1$. Let $\varphi(h) = s + it \in \mathbb{C}$ where $s, t \in \mathbb{R}$. Our first goal is to show that $t = 0$. By considering $-h$ instead of h, we may assume that $t \geq 0$. Fix an integer $n \geq 1$, and consider $x_{n,\lambda} = h + inu_\lambda$. Now

$$\begin{aligned}
\|x_{n,\lambda}\|^2 &= \|x_{n,\lambda}^* x_{n,\lambda}\| \\
&= \|h^2 + in(hu_\lambda - u_\lambda h) - n^2 u_\lambda\| \\
&\leq \|h\|^2 + n\|hu_\lambda - u_\lambda h\| + n^2 \\
&= 1 + n^2 + n\|hu_\lambda - u_\lambda h\|.
\end{aligned}$$

Now $\lim_\lambda \varphi(x_{n,\lambda}) = \lim_\lambda(\varphi(h) + in\varphi(u_\lambda)) = \varphi(h) + in = s + i(t+n)$. Furthermore, $|\varphi(x_{n,\lambda})|^2 \leq \|x_{n,\lambda}\|^2$, and so

$$s^2 + (t+n)^2 \leq \lim_\lambda (1 + n^2 + n\|hu_\lambda - u_\lambda h\|) = 1 + n^2.$$

Thus $s^2 + t^2 + 2tn + n^2 \leq 1 + n^2$. Unless $t = 0$, we obtain a contradiction by choosing n sufficiently large.

So far we have shown that φ is self-adjoint. We still want $0 \leq \varphi(r)$. Suppose $0 \leq r \leq 1$. Let $h_\lambda = r - u_\lambda$. Then $-1 \leq -u_\lambda \leq h_\lambda \leq r \leq 1$, and so $\|h_\lambda\| \le 1$. Now $\lim_\lambda \varphi(h_\lambda) = \varphi(r) - 1$, and since $|\varphi(h_\lambda)| \le 1$, we have $|\varphi(r) - 1| \le 1$, from which we conclude that $0 \le \varphi(r) \le 1$, which completes the proof.

□

11.14. Corollary. *Let $\mathbb{A}$ be an abelian C^*-algebra and $\varphi \in \Sigma_{\mathbb{A}}$ be a multiplicative linear functional on $\mathbb{A}$. Then $\varphi \ge 0$.*

Proof. Let $(u_\lambda)_\lambda$ be an approximate identity for $\mathbb{A}$. By Theorem 11.13, it suffices to prove that

$$\lim_\lambda \varphi(u_\lambda) = \|\varphi\|.$$

Now $\varphi \in \Sigma_{\mathbb{A}}$ implies that $\|\varphi\| \le 1$, by Proposition 5.11. Moreover, by definition, $\varphi \ne 0$. Choose $a \in \mathbb{A}$ such that $\varphi(a) \ne 0$. Then $a = \lim_\lambda u_\lambda a$, and thus

$$0 \ne \varphi(a) = \lim_\lambda \varphi(u_\lambda)\varphi(a).$$

From this it follows that $\lim_\lambda \varphi(u_\lambda) = 1$, and thus

$$\|\varphi\| = 1 = \lim_\lambda \varphi(u_\lambda).$$

This completes the proof.

□

11.15. Corollary. *Suppose $\mathbb{A}$ is a C^*-algebra, and $\varphi, \alpha, \beta \in \mathbb{A}^*$.*

(i) *If $\alpha, \beta \ge 0$, then $\|\alpha + \beta\| = \|\alpha\| + \|\beta\|$.*
(ii) *Suppose $\mathbb{A}$ is unital. Then $\varphi \ge 0$ if and only if $\|\varphi\| = \varphi(e_{\mathbb{A}})$. In particular, φ is a state on $\mathbb{A}$ if and only if $\varphi(e_{\mathbb{A}}) = 1 = \|\varphi\|$.*

Proof.

(i) Since $\alpha, \beta \ge 0$, so is $\alpha + \beta$. But then if $(u_\lambda)_{\lambda\in\Lambda}$ is any approximate unit for $\mathbb{A}$,

$$\begin{aligned} \|\alpha + \beta\| &= \lim_\lambda (\alpha + \beta)(u_\lambda) \\ &= \lim_\lambda \alpha(u_\lambda) + \lim_\lambda \beta(u_\lambda) \\ &= \|\alpha\| + \|\beta\|. \end{aligned}$$

(ii) This is an immediate consequence of Theorem 11.13, after observing that $u_\lambda = e_{\mathbb{A}}$ is an approximate identity for $\mathbb{A}$.

□

11.16. Proposition. *Let $\mathbb{A}$ be a C^*-algebra and $0 \le \varphi \in \mathbb{A}^*$. Then for all $a, b \in \mathbb{A}$,*

$$\varphi(b^* a^* a b) \le \|a^* a\| \varphi(b^* b).$$

Proof. We claim that $b^* a^* a b \le \|a^* a\| b^* b$, from which the above equation clearly follows. For the sake of convenience, we shall work in $\mathbb{A}_e$.

We know that $a^* a \le \|a^* a\| e_{\mathbb{A}}$ in $\mathbb{A}_e$, and thus

$$b^* a^* a b \le b^* (\|a^* a\| e_{\mathbb{A}}) b = \|a^* a\| b^* b.$$

Since φ is positive, it preserves order, and we are done.

□

11.17. Theorem. *Let $\mathbb{A}$ be a unital C^*-algebra. Then the state space $\mathcal{S}(\mathbb{A})$ is a weak*-compact, convex subset of the unit ball $\mathbb{A}_1^*$ of $\mathbb{A}^*$.*

Proof. Clearly $\mathcal{S}(\mathbb{A}) \subseteq \mathbb{A}_1^*$. Since $\mathbb{A}_1^*$ is weak*-compact by the Banach-Alaoglu Theorem, it suffices to show that $\mathcal{S}(\mathbb{A})$ is weak*-closed.

Suppose $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is a net in $\mathcal{S}(\mathbb{A})$ converging in the weak*-topology to $\varphi \in \mathbb{A}^*$. Again, the weak*-compactness of $\mathbb{A}_1^*$ implies that $\|\varphi\| \le 1$. Moreover,

$$\varphi(1) = \lim_\lambda \varphi_\lambda(1) = 1,$$

and so by Corollary 11.15, $\varphi \in \mathcal{S}(\mathbb{A})$. Thus $\mathcal{S}(\mathbb{A})$ is weak*-closed, as required.

If $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{A})$ and $0 < t < 1$, then clearly $\varphi = t\varphi_1 + (1-t)\varphi_2$ is positive, and $\varphi(1) = 1$. Since $\|\varphi\| \le t\|\varphi_1\| + (1-t)\|\varphi_2\| = 1$, $\varphi \in \mathcal{S}(\mathbb{A})$, which is therefore convex.

□

11.18. Our next goal is to prove that if $\mathbb{A}$ and $\mathbb{B}$ are C^*-algebras with $\mathbb{A} \subseteq \mathbb{B}$, then every state on $\mathbb{A}$ can be extended to a state on $\mathbb{B}$. Before doing that, let us observe that the restriction of a state on $\mathbb{B}$ is not necessarily a state on $\mathbb{A}$, although it is still clearly a positive linear functional.

For example, let c denote the C^*-subalgebra of $\ell^\infty(\mathbb{N})$ consisting of *convergent* sequences. Then $c_0 = \{(a_n)_{n=1}^\infty \in c : \lim_{n\to\infty} a_n = 0\}$ is a non-unital C^*-subalgebra of c. Consider the states β_1 and β_2 on c, where

$$\beta_1(a_n) = \lim_{n\to\infty} a_n \quad \text{and} \quad \beta_2(a_n) = a_1.$$

Then $\beta = \frac{1}{2}(\beta_1 + \beta_2)$ is again a state on c, by Theorem 11.17. The restriction of β to c_0 is $\frac{1}{2}\beta_2$, which is *not* a state on c_0.

11.19. Theorem. *Let $\mathbb{A}$ and $\mathbb{B}$ be C^*-algebras with $\mathbb{A} \subseteq \mathbb{B}$. Suppose $\varphi \in \mathcal{S}(\mathbb{A})$. Then there exists $\beta \in \mathcal{S}(\mathbb{B})$ whose restriction to $\mathbb{A}$ coincides with φ.*

Proof. Consider first the case where $\mathbb{B} = \mathbb{A}_e$, the unitization of $\mathbb{A}$.

Here we have no choice as to the definition of β since $\beta \in \mathcal{S}(\mathbb{B})$ implies $\beta(e_{\mathbb{B}}) = 1$. In other words, we must have $\beta(a + \alpha e_{\mathbb{B}}) = \varphi(a) + \alpha$. It remains only to verify that this β is in fact a state, which reduces to verifying that $\|\beta\| = 1$. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\mathbb{A}$. Now

$$\begin{aligned} |\beta(a + \alpha e_{\mathbb{B}})| &= |\varphi(a) + \alpha| \\ &= \lim_\lambda |\varphi(au_\lambda) + \alpha\varphi(u_\lambda)| \\ &= \lim_\lambda |\varphi(au_\lambda + \alpha u_\lambda)| \\ &\leq \liminf_\lambda \|\varphi\| \; \|a + \alpha e_{\mathbb{B}}\| \; \|u_\lambda\| \\ &\leq \|a + \alpha e_{\mathbb{B}}\|. \end{aligned}$$

It follows that $\|\beta\| \leq 1$. Since β is an extension of φ, it has norm at least 1, i.e. $\beta \in \mathcal{S}(\mathbb{B})$.

Consider next the case where $\mathbb{B}$ is any unital C^*-algebra containing $\mathbb{A}$. Then we can assume, using the above paragraph, that $\mathbb{A}$ is unital as well. If $\varphi \in \mathcal{S}(\mathbb{A})$ and β is any extension of φ to $\mathbb{B}$ given us by the Hahn-Banach Theorem (with $\|\beta\| = \|\varphi\| = 1$), then $\|\beta\| = 1 = \varphi(e_{\mathbb{B}}) = \beta(e_{\mathbb{B}})$, and so $\beta \in \mathcal{S}(\mathbb{B})$.

Finally, suppose $\mathbb{B}$ is not unital. First we extend φ to a state $\tilde{\varphi}$ on $\mathbb{A}_e$ by the first paragraph. From the second paragraph, $\tilde{\varphi}$ extends to a state $\tilde{\beta}$ on $\tilde{\mathbb{B}}$. Let β be the restriction of $\tilde{\beta}$ to $\mathbb{B}$. Clearly β is positive, and $1 = \|\tilde{\beta}\| \geq \|\beta\| \geq \|\varphi\| = 1$, since β is an extension of φ. Thus $\beta \in \mathcal{S}(\mathbb{B})$.

□

11.20. Corollary. *Suppose that $\mathbb{A}$ and $\mathbb{B}$ are C^*-algebras and that $\mathbb{A} \subseteq \mathbb{B}$. Then every positive linear functional on $\mathbb{A}$ extends to a positive linear functional on $\mathbb{B}$ with the same norm.*

Proof. If $0 < \varphi$ is a positive linear functional on $\mathbb{A}$, then $\alpha = \varphi/\|\varphi\|$ is a state on $\mathbb{A}$, which extends to a state β on $\mathbb{B}$ by Theorem 11.19 above. Hence $\|\varphi\|\ \beta$ extends φ.

□

11.21. Proposition. *If $\mathbb{A}$ is an ideal of a C^*-algebra $\mathbb{B}$, then any positive linear functional φ on $\mathbb{A}$ extends in a* unique *way to a positive linear functional β on $\mathbb{B}$ with $\|\beta\| = \|\varphi\|$.*

Proof. Suppose $\mathbb{A} \subseteq \mathbb{B}$ is an ideal. From Corollary 11.20, given $0 \leq \varphi \in \mathbb{A}^*$, we can find $0 \leq \gamma_1 \in \mathbb{B}^*$ so that $\|\gamma_1\| = \|\varphi\|$ and $\gamma_1|_{\mathbb{A}} = \varphi$. Let γ_2 be any positive extension of φ to $\mathbb{B}$ with $\|\gamma_2\| = \|\varphi\|$. Let $(u_\lambda)_\lambda$ be an approximate identity for $\mathbb{A}$.

Then $\lim_\lambda \gamma_2(1 - u_\lambda) = 0$. Moreover, $(1 - u_\lambda)^2 \leq (1 - u_\lambda)$, and so

$$\lim_\lambda \gamma_2((1 - u_\lambda)^2) = 0.$$

For all $b \in \mathbb{B}$,

$$\begin{aligned}|\gamma_2(b) - \gamma_2(u_\lambda b)|^2 &= |\gamma_2((1-u_\lambda)b)|^2 \\ &\le \gamma_2((1-u_\lambda)^2)\ \gamma_2(b^*b)\end{aligned}$$

by the Cauchy-Schwarz inequality. It follows that $\lim_\lambda |\gamma_2(b) - \gamma_2(u_\lambda b)| = 0$, so that $\gamma_2(b) = \lim_\lambda \gamma_2(u_\lambda b)$.

Since $\mathbb{A}$ is an ideal, we have $u_\lambda b \in \mathbb{A}$, and hence $\gamma_2(b) = \lim_\lambda \varphi(u_\lambda b)$. In particular, the values of γ on $\mathbb{B}$ are completely determined by the values of φ on $\mathbb{A}$, and so γ_1 is unique.

□

11.22. Proposition. *Let $\mathbb{A}$ be a C^*-algebra and $0 \ne n \in \mathbb{A}$ be normal.*

(a) *If $\tau \in \mathcal{S}(\mathbb{A})$, then $\tau(n) \in \overline{\mathrm{co}}(\sigma(n))$, the closed convex hull of the spectrum of n.*
(b) *There exists $\tau \in \mathcal{S}(\mathbb{A})$ such that $|\tau(n)| = \|n\|$.*

Proof.

(a) First recall that the closed convex hull of a compact subset subset $\Omega \subseteq \mathbb{C}$ is the intersection of all closed disks which contain Ω.

Suppose that $\tau \in \mathcal{S}(\mathbb{A})$ and that $\tau(n) \notin \overline{\mathrm{co}}(\sigma(n))$. Then there exists $z_0 \in \mathbb{C}$ and $r > 0$ so that $\sigma(n) \in \overline{D}(z_0, r) := \{\lambda \in \mathbb{C} : |z_0 - \lambda| \le r\}$, but $|\tau(n) - z_0| > r$. Let $\tilde{\tau}$ denote the positive extension of τ to $\mathbb{A}_e$, with $\|\tilde{\tau}\| = \|\tau\| = 1$. Let e denote the identity in $\mathbb{A}_e$. Now $n - z_0 e$ is normal and $\sigma(n - z_0 e) = \sigma(n) - z_0 \subseteq \overline{D}(0, r)$, so that

$$\|n - z_0 e\| = \mathrm{spr}(n - z_0 e) \le r,$$

while $|\tilde{\tau}(n - z_0 e)| = |\tilde{\tau}(n) - z_0| > r \le \|n - z_0 e\|$, implying that the extension $\tilde{\tau}$ has norm greater than one, a contradiction since $\|\tilde{\tau}\| = \|\tau\| = 1$.

(b) We may assume that $n \ne 0$. Now $C_0^*(n) \simeq^* \mathcal{C}_0(\sigma(n)\backslash\{0\})$. Let $\lambda \in \sigma(n)\backslash\{0\}$ such that $|\lambda| = \mathrm{spr}(n) = \|n\|$. Let $\tau \in \Sigma_{C_0^*(n)}$ be the corresponding multiplicative linear functional, so that

$$\tau(m) = [\Gamma(m)](\lambda), \quad m \in C_0^*(n).$$

Then $\tau \in \mathcal{S}(\mathbb{A})$ and

$$|\tau(n)| = |[\Gamma(n)](\lambda)| = \|n\|.$$

□

The GNS Construction.

11.23. We have defined a C^*-algebra as an involutive Banach algebra $\mathbb{A}$ which satisfies the C^*-equation $\|a^*a\| = \|a\|^2$ for all $a \in \mathbb{A}$. We have seen that every norm-closed, self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. For a period of time, the former were referred to as "B^*-algebras", while the latter were referred to as "*concrete* B^*-algebras", or "C^*-algebras". In this section we prove that every C^*-algebra of operators is isometrically $*$-isomorphic to a C^*-algebra of operators on a Hilbert space, a result known as the **Gelfand-Naimark-Segal Construction**. This reconciled the two notions, and the name C^*-algebra won out.

11.24. Definition. *A* ****-representation*** *of a* C^**-algebra* $\mathbb{A}$ *is a pair* $(\mathcal{H}, \varrho)$ *where*

$$\varrho : \mathbb{A} \to \mathcal{B}(\mathcal{H})$$

is a **-homomorphism. The representation is said to be* ***faithful*** *if* ϱ *is injective.*

A ***cyclic*** *vector for the representation is a vector* $\nu \in \mathcal{H}$ *for which* $\varrho(\mathbb{A})\nu = \{\varrho(a)\nu : a \in \mathbb{A}\}$ *is dense in* $\mathcal{H}$*. The representation* $(\mathcal{H}, \varrho)$ *is said to be* ***cyclic*** *if it admits a cyclic vector* ν*, in which case we shall often write* $(\mathcal{H}, \varrho, \nu)$ *to emphasize the fact that* ν *is cyclic for* $(\mathcal{H}, \varrho)$*.*

We note that it is common to refer to ϱ as the representation, and to apply adjectives such as "faithful" or "cyclic" to ϱ.

11.25. Example.

(a) Let $\mathbb{A} = \mathcal{C}([0,1])$, and $\mathcal{H} = L^2([0,1], dx)$, where dx denotes Lebesgue measure on the interval $[0,1]$. Then $(\mathcal{H}, \varrho)$ is a representation of $\mathbb{A}$, where

$$\varrho(f) = M_f, \quad f \in \mathcal{C}([0,1])$$

and $M_f g = fg$, $g \in \mathcal{H}$. Since $\|M_f\| = \|f\|_\infty$ by Example 6.7, ϱ is injective, and hence $(\mathcal{H}, \varrho)$ is faithful.

Consider the constant function $\nu(x) = 1, x \in [0,1]$ as an element of $\mathcal{H}$. (Strictly speaking, of course, ν is an equivalence class of this function in $L^2([0,1], dx)$.) For any $a \in \mathbb{A}$, $\varrho(a)\nu = a$, and so $\varrho(\mathbb{A})\nu = \mathcal{C}([0,1])$, which is dense in $L^2([0,1], dx)$. Thus ν is cyclic for $(\mathcal{H}, \varrho)$.

(b) Let $\mathcal{H}$ be a separable complex Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Let $\mathbb{A} = \mathcal{K}(\mathcal{H})$, and consider the representation

$$\begin{array}{cccc} \varrho : & \mathbb{A} & \to & \mathcal{B}(\mathcal{H}^{(2)}) \\ & K & \mapsto & K \oplus K. \end{array}$$

Let $\nu = e_1 \oplus e_2 \in \mathcal{H}^{(2)}$. For each $y, z \in \mathcal{H}$, $y \otimes e_1^*$ and $z \otimes e_2^* \in \mathcal{K}(\mathcal{H})$, being rank one operators. Then

$$\begin{array}{rcl} \varrho(y \otimes e_1^*)(\nu) & = & y \oplus 0 \\ \varrho(z \otimes e_2^*)(\nu) & = & 0 \oplus z \end{array}$$

and so $\mathcal{H}^{(2)} = \varrho(\mathbb{A})\nu$, i.e. ν is cyclic for $(\mathcal{H}^{(2)}, \varrho)$.

(c) Let $\mathbb{A} = \mathcal{C}([0,1])$ once again and let $\mathcal{H} = \mathbb{C}$. Then $(\mathbb{C}, \varrho)$ is a representation, where $\varrho(f) = f(1)$, $f \in \mathcal{C}([0,1])$. Note that $(\mathbb{C}, \varrho)$ is not faithful, since, for example, if $g(x) = 1 - x$, $x \in [0,1]$, then $g \neq 0$, but $\varrho(g) = g(1) = 0$.

(d) With $\mathbb{A}$ as above, consider $\mathcal{H} = \mathbb{C}^3$ along with the representation

$$\begin{array}{rccc} \varrho: & \mathcal{C}([0,1]) & \mapsto & \mathcal{B}(\mathbb{C}^3) \\ & f & \mapsto & f(0) \oplus f(0) \oplus f(1). \end{array}$$

We leave it as an exercise for the reader to verify that $(\mathcal{H}, \varrho)$ is not cyclic.

11.26. Let $\mathbb{A}$ be a C^*-algebra. We now describe a process that allows us to identify a certain quotient of $\mathbb{A}$ by a closed left ideal with a pre-Hilbert space.

Suppose $0 \le \varphi \in \mathbb{A}^*$. Recall from paragraph 11.10 that we obtain a pseudo-inner product on $\mathbb{A}$ via

$$[a, b] := \varphi(b^* a).$$

Let $\mathbb{L} = \{m \in \mathbb{A} : [m, m] = 0\}$. It follows from the Cauchy-Schwarz Inequality (Lemma 11.11) that $m \in \mathbb{L}$ if and only if $[m, b] = 0$ for all $b \in \mathbb{A}$. In particular, if $m_1, m_2 \in \mathbb{L}$ and $\lambda \in \mathbb{C}$, $b \in \mathbb{A}$, then $[\lambda m_1 + m_2, b] = \lambda[m_1, b] + [m_2, b] = 0 + 0$, so that $\mathbb{L}$ is easily seen to be a subspace of $\mathbb{A}$. Moreover, by Paragraph 11.10 (iv), if $m \in \mathbb{L}$ and $a \in \mathbb{A}$, then $[am, am] = [m, a^* am] = 0$ from above, and so $am \in \mathbb{L}$. Thus $\mathbb{L}$ is in fact a left ideal of $\mathbb{A}$.

It is routine to verify that $\mathbb{A}/\mathbb{L}$ is a pre-Hilbert space when equipped with the inner product $\langle a + \mathbb{L}, b + \mathbb{L}\rangle := [a, b] := \varphi(b^* a)$. Furthermore, we can define a left module action of $\mathbb{A}$ upon $\mathbb{A}/\mathbb{L}$ via

$$a \circ (x + \mathbb{L}) = ax + \mathbb{L}, \qquad a \in \mathbb{A}, x + \mathbb{L} \in \mathbb{A}/\mathbb{L}.$$

This map is well-defined because if $x + \mathbb{L} = y + \mathbb{L}$, then $x - y \in \mathbb{L}$. Since this latter is a left ideal of $\mathbb{A}$, $ax - ay \in \mathbb{L}$, and so $ax + \mathbb{L} = ay + \mathbb{L}$.

11.27. Theorem. [The Gelfand-Naimark-Segal Construction.] *Let $\mathbb{A}$ be a C^*-algebra and $0 \le \varphi \in \mathbb{A}^*$. Then there exists a cyclic representation $(\mathcal{H}, \varrho, \nu)$ of $\mathbb{A}$ where ν is a cyclic vector satisfying $\|\nu\| = \|\varphi\|^{\frac{1}{2}}$ and*

$$\langle \varrho(a)\nu, \nu\rangle = \varphi(a), \qquad a \in \mathbb{A}.$$

Proof. Using the notation above, let $\mathcal{H}$ denote the completion of the pre-Hilbert space $\mathbb{A}/\mathbb{L}$, where $\mathbb{L} = \{m \in \mathbb{A} : \varphi(m^* m) = 0\}$. For $a, x \in \mathbb{A}$,

$$\begin{aligned} \|a \circ (x + \mathbb{L})\|^2 &= \|ax + \mathbb{L}\|^2 \\ &= \langle ax + \mathbb{L}, ax + \mathbb{L}\rangle \\ &= [ax, ax] \\ &= \varphi(x^* a^* a x) \\ &\le \|a^* a\| \ \varphi(x^* x) \\ &= \|a\|^2 \ [x, x] \\ &= \|a\|^2 \ \|x + \mathbb{L}\|^2, \end{aligned}$$

and so if we define

$$\begin{array}{rccc} \varrho_0(a): & \mathbb{A}/\mathbb{L} & \to & \mathbb{A}/\mathbb{L} \\ & x+\mathbb{L} & \mapsto & ax+\mathbb{L} \end{array}$$

then $\|\varrho_0(a)\| \le \|a\|$ and therefore $\varrho_0(a)$ extends to a bounded linear map $\varrho(a)$ on $\mathcal{H}$. It is now routine to verify that $a \mapsto \varrho(a)$ is a linear homomorphism of $\mathbb{A}$ into $\mathcal{B}(\mathcal{H})$.

Also,

$$\langle \varrho(a^*)x+\mathbb{L}, y+\mathbb{L}\rangle = [a^*x, y] = [x, ay] = \langle x+\mathbb{L}, \varrho(a)y+\mathbb{L}\rangle$$

for all $x, y, a \in \mathbb{A}$, and so by the density of $\mathbb{A}/\mathbb{L}$ in $\mathcal{H}$, we see that

$$\langle \varrho(a^*)\xi_1, \xi_2\rangle = \langle \xi_1, \varrho(a)\xi_2\rangle \quad \text{for all } \xi_1, \xi_2 \in \mathcal{H}.$$

Hence $\varrho(a^*) = \varrho(a)^*$ for all $a \in \mathbb{A}$, which implies that $(\mathcal{H}, \varrho)$ is a representation of $\mathbb{A}$.

Let $(u_\lambda)_\lambda$ be an approximate identity for $\mathbb{A}$. Then $(u_\lambda + \mathbb{L})_\lambda$ is a net of vectors in the unit ball of $\mathcal{H}$. Furthermore, since $(u_\lambda)_\lambda$ is increasing, so is $(\varphi(u_\lambda))_\lambda$ in $[0,1]$. Given $0 < \varepsilon < 1$, choose λ_0 so that $\lambda \ge \lambda_0$ implies that $0 \le \|\varphi\| - \varphi(u_\lambda) < \varepsilon/2$. If $\beta \ge \alpha \ge \lambda_0$, then

$$\begin{aligned} \|(u_\beta+\mathbb{L}) - (u_\alpha+\mathbb{L})\|^2 &= [(u_\beta - u_\alpha), (u_\beta - u_\alpha)] \\ &= \varphi((u_\beta - u_\alpha)^2) \\ &\le \varphi(u_\beta - u_\alpha) \\ &< |\|\varphi\| - \varphi(u_\beta)| + |\|\varphi\| - \varphi(u_\alpha)| \\ &< \varepsilon. \end{aligned}$$

Thus $(u_\lambda + \mathbb{L})_\lambda$ is Cauchy in the complete space $\mathcal{H}$, and therefore it converges to some vector ν in the unit ball of $\mathcal{H}$. Also,

$$\|\nu\|^2 = \lim_\lambda [u_\lambda + \mathbb{L}, u_\lambda + \mathbb{L}] = \lim_\lambda \varphi(u_\lambda^2) = \|\varphi\|,$$

since $(u_\lambda^2)_\lambda$ is also an approximate identity for $\mathbb{A}$.

For any $a \in \mathbb{A}$,

$$\begin{aligned} \varrho(a)\nu &= \lim_\lambda \varrho(a)(u_\lambda + \mathbb{L}) \\ &= \lim_\lambda au_\lambda + \mathbb{L} \\ &= a + \mathbb{L}. \end{aligned}$$

Thus $\overline{\varrho(\mathbb{A})\nu} = \overline{\mathbb{A}/\mathbb{L}} = \mathcal{H}$, and therefore ν is indeed a cyclic vector for $(\mathcal{H}, \varrho)$.

Finally,

$$\begin{aligned} \langle \varrho(a)\nu, \nu\rangle &= \langle a+\mathbb{L}, \nu\rangle \\ &= \lim_\lambda \langle a+\mathbb{L}, u_\lambda+\mathbb{L}\rangle \\ &= \lim_\lambda [a, u_\lambda] \\ &= \lim_\lambda \varphi(u_\lambda^* a) \\ &= \varphi(a) \end{aligned}$$

for all $a \in \mathbb{A}$, completing the proof.

$\square$

11.28. Let $(\mathcal{H}_\lambda, \varrho_\lambda)_\lambda$ be a family of representations of a fixed C^*-algebra $\mathbb{A}$. Let $\mathcal{H} = \oplus_\lambda \mathcal{H}_\lambda$ denote the Hilbert space direct sum of the family $(\mathcal{H}_\lambda)_\lambda$, and for $a \in \mathbb{A}$, define

$$\begin{array}{rccc} \varrho: & \mathbb{A} & \to & \mathcal{B}(\mathcal{H}) \\ & a & \mapsto & \oplus_\lambda \varrho_\lambda(a). \end{array}$$

Since each ϱ_λ is a representation, $\|\varrho_\lambda\| \le 1$, and thus $\|\varrho\| \le 1$. It is now routine to verify that $(\mathcal{H}, \varrho)$ is a representation of $\mathbb{A}$, call the **direct sum** of $(\mathcal{H}_\lambda, \varrho_\lambda)_\lambda$ and denoted by

$$(\mathcal{H}, \varrho) = \oplus_\lambda (\mathcal{H}_\lambda, \varrho_\lambda).$$

Clearly $\|\varrho(a)\| = \sup_\lambda \|\varrho_\lambda(a)\|$ for all $a \in \mathbb{A}$.

In particular, for each $\tau \in \mathcal{S}(\mathbb{A})$, the state space of $\mathbb{A}$, we have constructed a cyclic representation $(\mathcal{H}_\tau, \varrho_\tau, \nu_\tau)$ via the GNS Construction (Theorem 11.27).

11.29. Definition. *The **universal representation** of a C^*-algebra $\mathbb{A}$ is the direct sum representation*

$$(\mathcal{H}, \varrho) = \oplus\{(\mathcal{H}_\tau, \varrho_\tau, \nu_\tau) : \tau \in \mathcal{S}(\mathbb{A})\}.$$

11.30. Theorem. [Gelfand-Naimark.] *Let $\mathbb{A}$ be a C^*-algebra. The universal representation $(\mathcal{H}, \varrho)$ is a faithful representation of $\mathbb{A}$, and hence $\mathbb{A}$ is isometrically *-isomorphic to a C^*-algebra of operators on $\mathcal{H}$.*

Proof. Let $a \in \mathbb{A}$. Then $n = a^*a \ge 0$, and so by Proposition 11.21, there exists a state $\tau \in \mathcal{S}(\mathbb{A})$ with $|\tau(n)| = \|n\|$. Let $(\mathcal{H}_\tau, \varrho_\tau, \nu_\tau)$ be the corresponding cyclic representation and observe that $\|\nu\| = \|\tau\|^{\frac{1}{2}} = 1$.

Now $\|n\| = |\tau(n)| = |\langle \varrho_\tau(n)\nu_\tau, \nu_\tau\rangle| \le \|\varrho_\tau(n)\| \le \|n\|$, and so $\|n\| = \|\varrho_\tau(n)\|$.

It follows that

$$\begin{aligned} \|\varrho(a)\|^2 &= \|\varrho(a)^*\varrho(a)\| = \|\varrho(a^*a)\| \\ &= \|\varrho(n)\| \ge \|\varrho_\tau(n)\| \\ &= \|n\| = \|a^*a\| \\ &= \|a\|^2. \end{aligned}$$

Thus $\|a\| \le \|\varrho(a)\|$. Since $\|\varrho\| \le 1$, $\|\varrho(a)\| \le \|a\|$, and thus ϱ is isometric.

$\square$

Supplementary Examples

S11.1. Example. In progress.

Appendix

A11.1. In progress.

Exercises for Chapter 11

Question 1. In progress.

CHAPTER 12

von Neumann algebras

Health nuts are going to feel stupid someday, lying in hospitals, dying of nothing.

Redd Foxx

Basic theory.

12.1. In this Chapter, we study a class of concrete C^*-algebras which are closed in a second, weaker topology than the norm topology. These are the so-called *von Neumann algebras*. While various important and deep structure theorems for these algebras (based upon the projections which can be found in the algebra) exist, we shall restrict ourselves to that part of the theory necessary for us to prove the celebrated **Spectral Theorem for normal operators**.

Before doing so, we remind the reader about how one constructs locally convex space topologies (LCS topologies) on a vector space from a separating family of seminorms. (Full details, including all proofs, may be found in [**35**].) But first we recall that a function $p : \mathcal{V} \to \mathbb{R}$ is said to be a **seminorm** on the complex vector space $\mathcal{V}$ if

- $p(x) \ge 0$ for all $x \in \mathcal{V}$;
- $p(\lambda x) = |\lambda|\, p(x)$ for all $\lambda \in \mathbb{C}$, $x \in \mathcal{V}$; and
- $p(x + y) \le p(x) + p(y)$ for all $x, y \in \mathcal{V}$.

Thus the only thing distinguishing a norm from a seminorm is that if μ is a norm on $\mathcal{V}$, then $\mu(x) = 0$ implies that $x = 0$, which we do not necessarily have in the case of seminorms. Indeed, if we fix $x_0 \in [0, 1]$, then the map

$$\begin{array}{rccc} p_{x_0} : & \mathcal{C}([0,1], \mathbb{C}) & \to & \mathbb{R} \\ & f & \mapsto & |f(x_0)| \end{array}$$

is a prototypical example of a seminorm which is not a norm.

A family $\{p_\lambda : \lambda \in \Lambda\}$ of seminorms on $\mathcal{V}$ is said to be **separating** if for all $0 \ne x \in \mathcal{V}$, there exists $\lambda \in \Lambda$ such that $p_\lambda(x) \ne 0$.

We mention in passing that the definition of a seminorm can be easily modified to apply to real vector spaces.

12.2. Let Γ be a family of seminorms on a vector space $\mathcal{V}$. For $F \subseteq \Gamma$ finite, $x \in \mathcal{V}$ and $\varepsilon > 0$, set

$$N(x, F, \varepsilon) = \{y \in \mathcal{V} : p(x-y) < \varepsilon, p \in F\}.$$

Permitting ourselves a slight abuse of notation, we shall write $N(x, p, \varepsilon)$ in the case where $F = \{p\}$.

12.3. Theorem. *If Γ is a separating family of seminorms on a vector space $\mathcal{V}$, then*

$$\mathcal{B} = \{N(x, F, \varepsilon) : x \in \mathcal{V}, \varepsilon > 0, F \subseteq \Gamma \text{ finite}\}$$

is a base for a locally convex topology $\mathcal{T}$ on $\mathcal{V}$. Moreover, each $p \in \Gamma$ is $\mathcal{T}$-continuous.

12.4. The above result says that a separating family of seminorms on a vector space $\mathcal{V}$ gives rise to a locally convex topology on $\mathcal{V}$. The next result shows that all locally convex spaces arise in this manner.

12.5. Theorem. *Suppose that $(\mathcal{V}, \mathcal{T}_{\mathcal{V}})$ is a LCS. Then there exists a separating family Γ of seminorms on $\mathcal{V}$ which generate the topology $\mathcal{T}_{\mathcal{V}}$.*

12.6. Proposition. *Let $\mathcal{V}$ be a vector space and Γ be a separating family of seminorms on $\mathcal{V}$. Let $\mathcal{T}$ denote the locally convex topology on $\mathcal{V}$ generated by Γ.*

A net $(x_\lambda)_\lambda$ in $\mathcal{V}$ converges to a point $x \in \mathcal{V}$ if and only if

$$\lim_\lambda p(x - x_\lambda) = 0 \text{ for all } p \in \Gamma.$$

Proof.

- Suppose first that $(x_\lambda)_\lambda$ converges to x in the $\mathcal{T}$-topology. Given $p \in \Gamma$ and $\varepsilon > 0$, the set $N(x, p, \varepsilon) \subseteq \mathcal{T}$ and so there exists λ_0 so that $\lambda \ge \lambda_0$ implies that $x_\lambda \in N(x, p, \varepsilon)$. That is, $\lambda \ge \lambda_0$ implies that $p(x - x_\lambda) < \varepsilon$. Thus $\lim_\lambda p(x - x_\lambda) = 0$.

  Alternatively, one may argue as follows: suppose that $(x_\lambda)_\lambda$ converges to x in the $\mathcal{T}$-topology. Given $p \in \Gamma$, we know that p is continuous in the $\mathcal{T}$-topology by Theorem 12.3. Since $\lim_\lambda x - x_\lambda = 0$,

  $$\lim_\lambda p(x - x_\lambda) = p(\lim_\lambda (x - x_\lambda)) = p(0) = 0.$$

- Conversely, suppose that $\lim_\lambda p(x - x_\lambda) = 0$ for all $p \in \Gamma$. Let $U \in \mathcal{U}_x$ is the $\mathcal{T}$-topology. Then there exist $p_1, p_2, ..., p_m \in \Gamma$ and $\varepsilon > 0$ so that $N(x, \{p_1, p_2, ..., p_m\}, \varepsilon) \subseteq U$. For each $1 \le j \le m$, choose λ_j so that $\lambda \ge \lambda_j$ implies that $p_j(x_\lambda - x) < \varepsilon$. Choose $\lambda_0 \ge \lambda_1, \lambda_2, ..., \lambda_m$. If $\lambda \ge \lambda_0$, then $p_j(x_\lambda - x) < \varepsilon$ for all $1 \le j \le m$ so that $x_\lambda \in N(x, \{p_1, p_2, ..., p_m\}, \varepsilon) \subseteq U$. Hence $\lim_\lambda x_\lambda = x$ in $(\mathcal{V}, \mathcal{T})$.

□

12.7. Example. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and recall that $\mathcal{H}$ is a Hilbert space when equipped with the inner product $\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$.

Recall also that $\mathcal{B}(\mathcal{H})$ is a normed linear space with the operator norm $\|T\| := \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| \le 1\}$.

From above, we see that the norm topology on $\mathcal{B}(\mathcal{H})$ admits as a nbhd base at $T \in \mathcal{B}(\mathcal{H})$ the collection

$$\{N(T, \|\cdot\|, \varepsilon) : \varepsilon > 0\} = \{V_\varepsilon(T) : \varepsilon > 0\},$$

and that this is the locally convex topology generated by the separating family $\Gamma = \{\|\cdot\|\}$ of (semi)norms.

Convergence of a net of operators $(T_\lambda)_\lambda$ to $T \in \mathcal{B}(\mathcal{H})$ in the norm topology (i.e. $\lim_\lambda \|T_\lambda - T\| = 0$) should be thought of as uniform convergence on the closed unit ball of $\mathcal{H}$.

This is certainly not the only interesting topology one can impose upon $\mathcal{B}(\mathcal{H})$. Let us first consider the topology of "pointwise convergence".

THE STRONG OPERATOR TOPOLOGY (SOT).

For each $x \in \mathcal{H}$, consider

$$\begin{array}{rccc} p_x : & \mathcal{B}(\mathcal{H}) & \to & \mathbb{R} \\ & T & \mapsto & \|Tx\|. \end{array}$$

Then

(i) $p_x(T) \ge 0$ for all $T \in \mathcal{B}(\mathcal{H})$;
(ii) $p_x(\lambda T) = \|\lambda T x\| = |\lambda| \, \|Tx\| = |\lambda| \, p_x(T)$ for all $\lambda \in \mathbb{K}$;
(iii) $p_x(T_1 + T_2) = \|T_1 x + T_2 x\| \le \|T_1 x\| + \|T_2 x\| = p_x(T_1) + p_x(T_2)$,

so that p_x is a seminorm on $\mathcal{B}(\mathcal{H})$ for each $x \in \mathcal{H}$.

In general, p_x is not a norm because we can always find $T \in \mathcal{B}(\mathcal{H})$ so that $0 \ne T$ but $p_x(T) = 0$. Indeed, let $y \in \mathcal{H}$ with $0 \ne y$ and $y \perp x$. Define $T_y : \mathcal{H} \to \mathcal{H}$ via $T_y(z) = \langle z, y \rangle y$. Then $\|T_y(z)\| \le \|z\| \, \|y\|^2$ by the Cauchy-Schwarz Inequality and in particular $T_y(y) = \|y\|^2 y \ne 0$, but $T_y(x) = \langle x, y \rangle y = 0y = 0$. Thus $0 \ne T_y$ but $p_x(T_y) = 0$.

On the other hand, if $0 \ne T \in \mathcal{B}(\mathcal{H})$, then there exists $x \in \mathcal{H}$ so that $Tx \ne 0$. Thus $p_x(T) = \|Tx\| \ne 0$, proving that $\Gamma_{\mathrm{SOT}} := \{p_x : x \in \mathcal{H}\}$ separates the points of $\mathcal{B}(\mathcal{H})$.

The locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by Γ_{SOT} is called the **strong operator topology** and is denoted by SOT.

By Proposition 12.6 above, we see that a net $(T_\lambda)_\lambda \in \mathcal{B}(\mathcal{H})$ converges to $T \in \mathcal{B}(\mathcal{H})$ in the SOT if and only if

$$\lim_\lambda p_x(T_\lambda - T) = \lim_\lambda \|T_\lambda x - Tx\| = 0 \quad \text{ for all } x \in \mathcal{H}.$$

Thus the SOT is the topology of pointwise convergence. That is, it is the weakest topology that makes all of the evaluation maps $T \mapsto Tx$, $x \in \mathcal{H}$ continuous.

A nbhd base for the SOT at the point $T \in \mathcal{B}(\mathcal{H})$ is given by the collection

$$\{N(T, \{x_1, x_2, ..., x_m\}, \varepsilon) : m \ge 1, x_j \in \mathcal{H}, 1 \le j \le m, \varepsilon > 0\}$$

where, for $m \geq 1$, $F := \{x_j \in \mathcal{H} : 1 \leq j \leq m\}$ and $\varepsilon > 0$, we have

$$N(T, F, \varepsilon) = \{R \in \mathcal{B}(\mathcal{H}) : \|Rx_j - Tx_j\| < \varepsilon,\ 1 \leq j \leq m\}.$$

THE WEAK OPERATOR TOPOLOGY (WOT).

Next, for each pair $(x, y) \in \mathcal{H} \times \mathcal{H}$, consider the map

$$\begin{array}{cccc} q_{x,y} : & \mathcal{B}(\mathcal{H}) & \to & \mathbb{R} \\ & T & \mapsto & |\langle Tx, y\rangle|. \end{array}$$

Again, it is routine to verify that each $q_{x,y}$ is a seminorm but not a norm on $\mathcal{B}(\mathcal{H})$.

The locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by $\Gamma_{\text{WOT}} := \{q_{x,y} : (x, y) \in \mathcal{H} \times \mathcal{H}\}$ is called the **weak operator topology** on $\mathcal{B}(\mathcal{H})$ and is denoted by WOT.

A net $(T_\lambda)_\lambda \in \mathcal{B}(\mathcal{H})$ converges to $T \in \mathcal{B}(\mathcal{H})$ in the WOT if and only if

$$\lim_\lambda |\langle (T_\lambda - T)x, y\rangle| = \lim_\lambda |\langle T_\lambda x, y\rangle - \langle Tx, y\rangle| = 0$$

for all $x, y \in \mathcal{H}$. In other words, the WOT is the weakest topology that makes all of the functions $T \mapsto \langle Tx, y\rangle$, $x, y \in \mathcal{H}$ continuous.

A nbhd base for the WOT at the point $T \in \mathcal{B}(\mathcal{H})$ is given by the collection

$$\{N(T, \{x_1, x_2, ..., x_m, y_1, y_2, ..., y_m\}, \varepsilon) : m \geq 1, x_j, y_j \in \mathcal{H}, 1 \leq j \leq m, \varepsilon > 0\},$$

where, for $m \geq 1$, $F := \{(x_j, y_j) \in \mathcal{H} \times \mathcal{H} :\ 1 \leq j \leq m\}$ and $\varepsilon > 0$, we have

$$N(T, F, \varepsilon) = \{R \in \mathcal{B}(\mathcal{H}) : |\langle Rx_j - Tx_j, y_j\rangle| < \varepsilon,\ 1 \leq j \leq\ m\}.$$

12.8. Example. Let $\mathcal{H} = \mathbb{C}^n$ for some $n \geq 1$. Since a finite-dimensional (real or) complex vector space admits at most one topology making it into a topological vector space (see [**35**]), the WOT, SOT and norm topologies on $\mathcal{B}(\mathcal{H})$ all coincide.

12.9. Example. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Let P_n denote the orthogonal projection onto the span of $\{e_1, e_2, ..., e_n\}$, $n \geq 1$. Then the sequence $\{P_n\}_{n=1}^\infty$ converges to the identity in the SOT.

Indeed, if $x \in \mathcal{H}$, say $x = \sum_k x_k e_k$, then $\|x - P_n x\| = \|\sum_{k=n+1}^\infty x_k e_k\| = (\sum_{k=n+1}^\infty |x_k|^2)^{\frac{1}{2}}$ and this tends to 0 as n tends to infinity.

12.10. Remark. In infinite-dimensional Hilbert spaces, the SOT, WOT and norm topologies are all distinct. For example, if $\mathcal{H}$ is separable and infinite dimensional with orthonormal basis $\{e_n\}_{n=1}^\infty$, and if $F_n = e_1 \otimes e_n^*$, then it is easy to verify that SOT-$\lim_n F_n = 0$ but $\|F_n\| = 1$ for all $n \geq 1$, while if $G_n = e_n \otimes e_1^*$, then WOT-$\lim_n G_n = 0$, while $\|G_n e_1\| = 1$ for all $n \geq 1$, so that SOT-$\lim_n G_n \neq 0$.

These examples can easily be adapted to non-separable spaces.

12.11. Proposition. *Let $\mathcal{H}$ be a Hilbert space and let $A, B \in \mathcal{B}(\mathcal{H})$ be fixed. Then each of the functions*

(i) $\begin{array}{rccc} \sigma: & \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & (X, Y) & \mapsto & X + Y \end{array}$;

(ii) $\begin{array}{rccc} \mu: & \mathbb{C} \times \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & (z, X) & \mapsto & zX \end{array}$;

(iii) $\begin{array}{rccc} \lambda_A: & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & X & \mapsto & AX \end{array}$;

(iv) $\begin{array}{rccc} \rho_B: & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & X & \mapsto & XB \end{array}$;

(v) $\begin{array}{rccc} \alpha: & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & T & \mapsto & T^* \end{array}$.

is continuous in the WOT*. The first four are also* SOT*-continuous, while the adjoint operation α is not* SOT *continuous.*

Proof. That σ and μ are both WOT- and SOT-continuous is clear, since they are LCS, and thus topological vector space topologies. The remaining items are left as an exercise for the reader.

□

12.12. Definition. *Let $\mathcal{H}$ be a Hilbert space. Then a **von Neumann algebra** (also called a W^*-algebra) $\mathfrak{M}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the* WOT*.*

We remark that some authors require that the algebra $\mathfrak{M}$ contain the identity operator. As we shall see, every von Neumann algebra contains a maximal projection which serves as an identity for the algebra as a ring. By restricting our attention to the range of that projection, we can then assume that the identity operator lies in $\mathfrak{M}$.

12.13. Example. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.

12.14. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint subalgebra. Then $\overline{\mathcal{A}}^{\text{WOT}}$ is a von Neumann algebra. If $\mathcal{A}$ is abelian, then so is $\overline{\mathcal{A}}^{\text{WOT}}$.*

Proof. Suppose that $(A_\alpha)_{\alpha \in \Lambda}$ and $(B_\beta)_{\beta \in \Gamma}$ are nets in $\mathcal{A}$ with WOT-$\lim_\alpha A_\alpha = A$ and WOT-$\lim_\beta B_\beta = B$. Now $\Lambda \times \Gamma$ is a directed set with the lexicographic order, so that $(\alpha_1, \beta_1) \le (\alpha_2, \beta_2)$ if $\alpha_1 < \alpha_2$, or $\alpha_1 = \alpha_2$ and $\beta_1 \le \beta_2$. If we set $A_{\alpha,\beta} = A_\alpha$, $B_{\alpha,\beta} = B_\beta$ for all α, β, then $\lim_{\alpha,\beta} A_{\alpha,\beta} = A$ and $\lim_{\alpha,\beta} B_{\alpha,\beta} = B$. By Proposition 12.11, for all $z \in \mathbb{C}$, $zA + B = \text{WOT} - \lim_{\alpha,\beta} zA_{\alpha,\beta} + B_{\alpha,\beta} \in \overline{\mathcal{A}}^{\text{WOT}}$.

Next, for each $\beta \in \Gamma$, $AB_\beta = \text{WOT-}\lim_\alpha A_\alpha B_\beta \overline{\mathcal{A}}^{\text{WOT}}$, and thus

$$\text{WOT-}\lim_\beta AB_\beta = AB \in \overline{\mathcal{A}}^{\text{WOT}}.$$

Thus $\overline{\mathcal{A}}^{\text{WOT}}$ is an algebra. Since the adjoint operation is continuous in the WOT, and since $\mathcal{A}$ is self-adjoint, $A_\alpha \mapsto_{\text{WOT}} A$ implies $A_\alpha^* \mapsto_{\text{WOT}} A^*$, and so $A^* \in \overline{\mathcal{A}}^{\text{WOT}}$. Hence $\overline{\mathcal{A}}^{\text{WOT}}$ is a von Neumann algebra.

Suppose $\mathcal{A}$ is abelian. For all $\beta \in \Gamma$ and $x, y \in \mathcal{H}$,

$$\begin{aligned}
\langle AB_\beta x, y\rangle &= \text{WOT} - \lim_\alpha \langle A_\alpha B_\beta x, y\rangle \\
&= \text{WOT} - \lim_\alpha \langle B_\beta A_\alpha x, y\rangle \\
&= \text{WOT} - \lim_\alpha \langle A_\alpha x, B_\beta^* y\rangle \\
&= \langle Ax, B_\beta^* y\rangle = \langle B_\beta Ax, y\rangle.
\end{aligned}$$

Thus $AB_\beta = B_\beta A$ for all $\beta \in \Gamma$. The same argument then shows that $AB = BA$, and so $\overline{\mathcal{A}}^{\text{WOT}}$ is abelian.

□

12.15. Definition. *If $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ is any collection of operators, then*

$$\mathcal{C}' = \{T \in \mathcal{B}(\mathcal{H}) : TC = CT \text{ for all } C \in \mathcal{C}\}$$

is called the ***commutant*** *of $\mathcal{C}$.*

12.16. Proposition. *Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{K}(\mathcal{H})' = \mathbb{C}I$.*
Proof. Exercise.

□

12.17. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint collection of operators. Then the commutant $\mathcal{C}'$ of $\mathcal{C}$ is a von Neumann algebra.*
Proof. Suppose $A, B \in \mathcal{C}'$, $z \in \mathbb{C}$ and $C \in \mathcal{C}$. Then $(zA + B)C = zAC + BC = zCA + CB = C(zA + B)$ and $(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$, so that $\mathcal{C}'$ is an algebra. Also, $\mathcal{C}$ self-adjoint implies that $AC^* = C^*A$ and hence $CA^* = A^*C$ for all $C \in \mathcal{C}$. Thus $\mathcal{C}'$ is self-adjoint.

Finally, if $A_\alpha \in \mathcal{C}'$, $\alpha \in \Lambda$ and $\text{WOT} - \lim_\alpha A_\alpha = A$, then for all $x, y \in \mathcal{H}$,

$$\begin{aligned}
\langle ACx, y\rangle &= \lim_\alpha \langle CA_\alpha x, y\rangle \\
&= \lim_\alpha \langle A_\alpha x, C^* y\rangle \\
&= \langle Ax, C^* y\rangle \\
&= \langle CAx, y\rangle,
\end{aligned}$$

so that $A \in \mathcal{C}'$ and therefore $\mathcal{C}'$ is WOT-closed, which completes the proof.

□

12.18. Definition. *A **masa** $\mathbb{M}$ in a C^*-algebra $\mathbb{A}$ is a maximal abelian self-adjoint subalgebra. That is, $\mathbb{M}$ is a self-adjoint abelian subalgebra of $\mathbb{A}$, and is not properly contained in any abelian self-adjoint subalgebra of $\mathbb{A}$.*

12.19. Example. Let $\mathbb{A} = \mathbb{M}_n(\mathbb{C})$ for some $n \geq 1$. Then $\mathcal{D}_n = \{\mathrm{diag}(d_1, d_2, ..., d_n) : d_k \in \mathbb{C},\ 1 \leq k \leq n\}$ is a masa in $\mathbb{A}$. We leave the verification as an exercise, although this example will be covered by Proposition **??** below.

12.20. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators. The following are equivalent:*

(a) $\mathfrak{M} = \mathfrak{M}'$;
(b) $\mathfrak{M}$ *is a masa.*

In particular, every masa in $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.
Proof.

(a) implies (b): Since $\mathfrak{M} = \mathfrak{M}'$, $\mathfrak{M}$ is abelian. Suppose $\mathfrak{M} \subseteq \mathfrak{N}$, where $\mathfrak{N}$ is abelian and self-adjoint. Then $\mathfrak{N} \subseteq \mathfrak{M}'$, and so $\mathfrak{N} \subseteq \mathfrak{M}$. Thus $\mathfrak{M}$ is a masa.
(b) implies (a): Suppose that $\mathfrak{M}$ is a masa. Let $T \in \mathfrak{M}'$, $T = H + iK$, where $H = (T + T^*)/2$ and $K = (T - T^*)/2i$. If $M \in \mathfrak{M}$, then $M^* \in \mathfrak{M}$, so that $TM^* = M^*T$ and thus $T^*M = MT^*$ and $T^* \in \mathfrak{M}'$. But then $H, K \in \mathfrak{M}'$.

Now if $\mathfrak{N}$ is the WOT-closed algebra generated by $\mathfrak{M}$ and H, then $\mathfrak{N}$ is abelian and so $\mathfrak{N} = \mathfrak{M}$ by maximality. Thus $H \in \mathfrak{M}$. Similarly, $K \in \mathfrak{M}$ and therefore $T \in \mathfrak{M}$. That is, $\mathfrak{M}' \subseteq \mathfrak{M}$. Since $\mathfrak{M}$ is abelian, $\mathfrak{M} \subseteq \mathfrak{M}'$, from which equality follows.

□

Recall that a measure space (X, μ) is called a **probability space** if μ is a positive regular Borel measure on X for which $\mu(X) = 1$. Recall that the map $f \mapsto M_f$ is an isometric embedding of $L^\infty(X, \mu)$ into $\mathcal{B}(L^2(X, \mu))$. Let us use $\mathcal{M}^\infty(X, \mu)$ to denote the image of $L^\infty(X, \mu)$ under this embedding.

12.21. Proposition. *Let (X, μ) be a probability space. Then $\mathcal{M}^\infty(X, \mu)$ is a masa in $\mathcal{B}(L^2(X, \mu))$, and as such is a von Neumann algebra.*
Proof. Since $\mathcal{M}^\infty(X, \mu)$ is self-adjoint, by Proposition 12.20, it suffices to show that $\mathcal{M}^\infty(X, \mu) = \mathcal{M}^\infty(X, \mu)'$. Observe that $\mathcal{M}^\infty(X, \mu)$ is abelian, and so $\mathcal{M}^\infty(X, \mu) \subseteq \mathcal{M}^\infty(X, \mu)'$.

Suppose $T \in \mathcal{B}(\mathcal{H})$ satisfies $TM_f = M_fT$ for all $f \in L^\infty(X, \mu)$. Let $e \in L^2(X, \mu)$ denote the constant function $e(x) = 1 a.e.$, and set $g = Te$.

Then $Tf = TM_fe = M_fTe = fg$ for all $f \in L^\infty(X, \mu)$. If we can show that $g \in L^\infty(X, \mu)$, then it will follow from the continuity of T and the fact that $L^\infty(X, \mu)$ is dense in $L^2(X, \mu)$ that $T = M_g$.

Let $E = \{x \in X : |g(x)| \geq \|T\| + 1\}$, and let $f = \chi_E \in L^\infty(X, \mu)$. Then

$$\begin{aligned} \|Tf\|^2 &= \int_X |fg|^2 d\mu \\ &= \int_E |fg|^2 d\mu \\ &> \|T\|^2 \int_E |f|^2 d\mu \\ &= \|T\|^2 \ \|f\|_2^2, \end{aligned}$$

and so $\|f\|_2^2 = 0$, implying that $f = 0$ a.e.. Thus $|g(x)| \leq \|T\| + 1$ a.e., and hence $g \in L^\infty(X, \mu)$. From the argument above, $T = M_g \in \mathcal{M}^\infty(X, \mu)$, and hence $\mathcal{M}^\infty(X, \mu)' \subseteq \mathcal{M}^\infty(X, \mu)$.

□

12.22. Lemma. *Suppose $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra and $x \in \mathcal{H}$. Let P denote the orthogonal projection onto $[\mathcal{A}x]$, the closure of $\mathcal{A}x$ in $\mathcal{H}$. Then $P \in \mathcal{A}'$.*

Proof. We prove that $[\mathcal{A}x]$ is reducing for each element A of $\mathcal{A}$. Indeed, if $z \in [\mathcal{A}x]$, then $z = \lim_n A_n x$ for some sequence $\{A_n\}_n$ in $\mathcal{A}$. But then $Az = \lim_n AA_n x \in [\mathcal{A}x]$, and $A^* z = \lim_n A^* A_n x \in [\mathcal{A}x]$, so that $[\mathcal{A}x]$ is reducing for A by Proposition **??**.

Thus $AP = PAP$ and $A^*P = PA^*P$, from which $PA = AP$, and $P \in \mathcal{A}'$, as claimed.

□

12.23. Definition. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$. The **kernel** of $\mathcal{C}$ is the set*

$$\ker \mathcal{C} = \{x \in \mathcal{H} : Cx = 0 \textit{ for all } C \in \mathcal{C}\}.$$

12.24. Example.

(a) We leave it as an exercise for the reader to verify that $\ker \mathcal{K}(\mathcal{H}) = \{0\}$.
(b) If $T \in \mathcal{B}(\mathcal{H})$, and $\mathcal{C}$ is the algebra generated by T, then $\ker \mathcal{C} = \ker T$.

12.25. Lemma. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. Set $\mathcal{A}^{(n)} = \{A \oplus A \oplus \ldots \oplus A : A \in \mathcal{A}\} \subseteq \mathcal{B}(\mathcal{H}^{(n)})$. Then $(\mathcal{A}^{(n)})'' = \{B \oplus B \oplus \ldots \oplus B : B \in \mathcal{A}''\}$.*

Proof. Exercise.

□

12.26. Theorem. [The von Neumann Double Commutant Theorem.] *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators and suppose that $\ker \mathcal{A} = \{0\}$. Then $\overline{\mathcal{A}}^{\text{WOT}} = \overline{\mathcal{A}}^{\text{SOT}} = \mathcal{A}''$. In particular, if $\mathcal{A}$ is a von Neumann algebra, then $\mathcal{A} = \mathcal{A}''$.*

Remark: Before proving the result, let us pause to observe what a truly remarkable Theorem this is. Indeed, the conclusion of this Theorem allows us to identify a topological concept, namely the closure of a given algebra in a certain topology, with a purely algebraic concept, the second commutant of the algebra. It is difficult to overstate the usefulness of this Theorem.

Proof. Observe that $\mathcal{A} \subseteq \mathcal{A}''$ and that this latter is a von Neumann algebra by Proposition 12.17. Thus $\overline{\mathcal{A}}^{\text{WOT}} \subseteq \mathcal{A}''$. Since the strong operator topology is stronger than the weak operator topology,

$$\overline{\mathcal{A}}^{\text{SOT}} \subseteq \overline{\mathcal{A}}^{\text{WOT}} \subseteq \mathcal{A}''.$$

It therefore suffices to prove that if $B \in \mathcal{A}''$, then $B \in \overline{\mathcal{A}}^{\text{SOT}}$. This amounts to proving that if $\varepsilon > 0$ and $x_1, x_2, ..., x_n \in \mathcal{H}$, then there exists $A \in \mathcal{A}$ so that $\|(A - B)x_k\| < \varepsilon$, $1 \le k \le n$.

Let $\varepsilon > 0$.

(1) Case One: $n = 1$ Let $x \in \mathcal{H}$. Then by Lemma 12.22, if P is the orthogonal projection onto $[\mathcal{A}x]$, $P \in \mathcal{A}'$. Moreover, $x \in \operatorname{ran} P$, for if $C \in \mathcal{A}$, then $C(I-P)x = (I-P)Cx = 0$, and hence $(I-P)x \in \ker \mathcal{A} = \{0\}$. Since $P \in \mathcal{A}'$, we have $PB = BP$, and so $Bx = BPx = PBx \in \operatorname{ran} P$. That is, there exists $A \in \mathcal{A}$ so that $\|Bx - Ax\| < \varepsilon$.

(2) Case Two: $n > 1$ Let $x_1, x_2, ..., x_n \in \mathcal{H}$ and set $z = x_1 \oplus x_2 \oplus \cdots \oplus x_n \in \mathcal{H}^{(n)}$. By Lemma 12.25, $\mathcal{A}^{(n)}$ is a self-adjoint algebra of operators and it is routine to check that $\ker \mathcal{A}^{(n)} = \{0\}$. By Case One above, we can find $A_0 \in (\mathcal{A}^{(n)})''$ so that
$$\|(A_0 - B^{(n)})z\| < \varepsilon.$$
Since $(\mathcal{A}^{(n)})'' = (\mathcal{A}'')^{(n)}$, $A_0 = A^{(n)}$ for some $A \in \mathcal{A}$, and so we have $(\sum_{k=1}^n \|(A-B)x_k\|^2)^{\frac{1}{2}} < \varepsilon$, which in turn implies that $\|(A-B)x_k\| < \varepsilon$ for all $1 \le k \le n$.

□

12.27. Proposition. *Let $\mathcal{H}$ be a Hilbert space and suppose $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is a linear map. The following are equivalent:*

(a) *φ is* SOT*-continuous;*
(b) *φ is* WOT*-continuous;*
(c) *there exist $\{x_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n \in \mathcal{H}$ so that $\varphi(T) = \sum_{k=1}^n \langle Tx_k, y_k \rangle$ for all $T \in \mathcal{B}(\mathcal{H})$.*

Proof.

(c) implies (b): this is clear from the definition of the WOT.

(b) implies (a): this follows from the fact that the WOT is weaker than the SOT.

(a) implies (c): Let $\varepsilon > 0$. From the definition of a basic neighbourhood in the SOT, we can find vectors $x_1, x_2, ..., x_n \in \mathcal{H}$ such that $(\sum_{k=1}^n \|Tx_k - 0x_k\|^2)^{\frac{1}{2}} < \varepsilon$ implies $|\varphi(T) - \varphi(0)| < 1$. Consider $\begin{array}{rccc} \Psi: & \mathcal{B}(\mathcal{H}) & \to & \mathcal{H}^{(n)} \\ & T & \mapsto & (Tx_1, Tx_2, ..., Tx_n). \end{array}$

Then Ψ is linear and so $R = \operatorname{ran} T$ is a linear manifold. Consider

$$\begin{array}{rccc} \beta_R: & R & \to & \mathbb{C} \\ & (Tx_1, Tx_2, ..., Tx_n) & \mapsto & \varphi(T). \end{array}$$

Then from above it follows that β_R is well-defined, is continuous, and in fact $\|\beta_R\| \le 1/\varepsilon$. By the Hahn-Banach Theorem, β_R extends to a continuous linear functional $\beta \in (\mathcal{H}^{(n)})^* \simeq \mathcal{H}^{(n)}$. By the Riesz Representation Theorem, $\beta(Z) = \langle Zx, y\rangle$ for some $y \in \mathcal{H}^{(n)}$, say $y = (y_1, y_2, ..., y_n)$. In particular,

$$\begin{aligned} \varphi(T) &= \beta(Tx_1, Tx_2, ..., Tx_n) \\ &= \langle (Tx_1, Tx_2, ..., Tx_n), (y_1, y_2, ..., y_n)\rangle \\ &= \sum_{k=1}^n \langle Tx_k, y_k\rangle \end{aligned}$$

for all $T \in \mathcal{B}(\mathcal{H})$.

□

12.28. Remark. Suppose $\mathcal{H}$ is a separable, complex Hilbert space, $T \in \mathcal{B}(\mathcal{H})$ and $F \in \mathcal{F}(\mathcal{H})$ is a finite rank operator. Let $\{e_\alpha\}_\alpha$ be an orthonormal basis for $\mathcal{H}$. It can be shown that we can then define $\operatorname{tr}(TF) = \sum_i k_{\alpha\alpha}$, where $[TF] = [k_{\alpha,\beta}]$ with respect to the given basis. If $F = \sum_{i=1}^n y_{\alpha_i} \otimes x_{\alpha_i}^*$, then

$$\operatorname{tr}(TF) = \sum_{i=1}^n \langle Tx_{\alpha_i}, y_{\alpha_i}\rangle.$$

Thus the WOT-continuous (or SOT-continuous) linear functionals are those induced by φ_F, $F \in \mathcal{F}(\mathcal{H})$, where $\varphi_F(T) = \operatorname{tr}(TF)$.

12.29. Corollary. *The spaces* $(\mathcal{B}(\mathcal{H}), \text{WOT})$ *and* $(\mathcal{B}(\mathcal{H}), \text{WOT})$ *have the same closed, convex sets.*

Proof. By the Krein-Milman Theorem, the SOT-closed convex subsets are completely determined by the SOT-closed half-spaces which contain them. These in turn are determined by the SOT-continuous linear functionals on $\mathcal{B}(\mathcal{H})$. Since the SOT- and WOT-continuous linear functionals on $\mathcal{B}(\mathcal{H})$ coincide, every SOT-closed convex set is also WOT-closed.

Conversely, any WOT-closed set is automatically SOT-closed, and in particular, this applies to convex sets.

□

12.30. Proposition. *Let $\mathbb{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then the unit ball $\mathbb{M}_1$ of $\mathbb{M}$ is* WOT*-compact.*

Proof. First note that $\mathbb{M}_1$ is WOT-closed, since $(T_\alpha)_\alpha \subseteq \mathbb{M}_1$ and $T_\alpha \to T$ in the WOT implies that $T \in \overline{\mathbb{M}}^{\mathrm{WOT}}$ and

$$|\langle Tx, y\rangle| = \lim_\alpha |\langle T_\alpha x, y\rangle| \le \sup_\alpha \|T_\alpha\| \, \|x\| \, \|y\|$$

for all $x, y \in \mathcal{H}$. The remainder of the proof is similar to that of the Banach-Alaoglu Theorem.

For each $x, y \in \mathcal{H}$, consider $I_{x,y} = [-\|x\| \, \|y\|, \|x\| \, \|y\|]$. Let $B = \Pi_{x,y\in\mathcal{H}} I_{x,y}$, and suppose that B carries the product topology so that B is compact (since each $I_{x,y}$ clearly is). Now the map

$$\begin{array}{rccc} j: & \mathbb{M}_1 & \to & B \\ & T & \mapsto & \Pi_{x,y\in\mathcal{H}} \langle Tx, y\rangle \end{array}$$

is clearly an injective map from $\mathbb{M}_1$ into B. We claim that j is a homeomorphism of $(\mathbb{M}_1, \mathrm{WOT})$ with its range.

Indeed, $T_\alpha \to_{\mathrm{WOT}} T$ if and only if $\langle T_\alpha x, y\rangle \to \langle Tx, y\rangle$ for each $x, y \in \mathcal{H}$ if and only if $j(T_\alpha) \to j(T)$ in the product topology on B.

Moreover, $j(\mathbb{M}_1)$ is closed in B. To see this, suppose $(j(T_\alpha))_\alpha \subseteq j(\mathbb{M}_1)$. If $j(T_\alpha) \to (z_{x,y})_{x,y\in\mathcal{H}}$, then for each $y_0 \in \mathcal{H}$,

$$\phi_{y_0}(x) := z_{x,y_0}$$

defines a continuous linear functional on $\mathcal{H}$. By the Riesz Representation Theorem, there exists a vector $T^* y_0 \in \mathcal{H}$ so that $\phi_{y_0}(x) = \langle x, T^* y_0\rangle$. It is not difficult to verify that the function $y_0 \mapsto T^* y_0$ is linear. Moreover, $|z_{x,y}| \le \|x\| \, \|y\|$ for all $x, y \in \mathcal{H}$ and hence

$$\begin{aligned} \|T^* y_0\| &= \sup_{\|x\|=1} |\langle x, T^* y_0\rangle| \\ &= \sup_{\|x\|=1} |z_{x,y_0}| \\ &\le \|x\| \, \|y_0\| = \|y_0\|. \end{aligned}$$

Hence $\|T\| = \|T^*\| \le 1$.

Clearly $\langle T_\alpha x, y\rangle \mapsto z_{x,y} = \langle x, T^* y\rangle = \langle Tx, y\rangle$ for all $x, y \in \mathcal{H}$, and so $(z_{x,y})_{x,y\in\mathcal{H}} = \Pi_{x,y} = \Pi_{x,y} \langle Tx, y\rangle = j(T) \in \operatorname{ran} j$. Thus $\operatorname{ran} j$ is closed in the compact set B and hence $\operatorname{ran} j$ is compact. But then $(\mathbb{M}_1, \mathrm{WOT})$ is also compact, which is what we were trying to prove.

□

12.31. Proposition. *Let $(P_\beta)_{\beta\in\Gamma}$ be an increasing net of positive elements in the unit ball $\mathbb{M}_1$ of a unital von Neumann algebra $\mathbb{M}$. Then $P = \mathrm{SOT} - \lim_\beta P_\beta$ exists, $P \in \mathbb{M}_1$ and $0 \le P \le I$.*

Proof. Fix $x \in \mathcal{H}$. Then $\langle P_\beta x, x\rangle_\beta$ is an increasing net of positive real numbers in $[0,1]$ and hence $m_x := \lim_\beta \langle P_\beta x, x\rangle$ exists. Let $\varepsilon > 0$ and choose β_0 such that $\beta \ge \beta_0$ implies $|m_x - \langle P_\beta x, x\rangle| < \varepsilon$.

Since $(P_\beta)_\beta$ is increasing, if $\beta \ge \alpha$, then $P_\beta - P_\alpha \ge 0$, and so $(P_\beta - P_\alpha)^{\frac{1}{2}} \in \mathbb{M}$. Moreover, $0 \le P_\alpha \le P_\beta \le I$ implies $P_\beta - P_\alpha \le I - 0 = I$, and hence $(P_\beta - P_\alpha)^{\frac{1}{2}} \le I$. If $\beta \ge \alpha \ge \beta_0$, then

$$\begin{aligned} \|(P_\beta - P_\alpha)x\|^2 &\le \|(P_\beta - P_\alpha)^{\frac{1}{2}}\|^2 \ \|(P_\beta - P_\alpha)^{\frac{1}{2}}x\|^2 \\ &= \|P_\beta - P_\alpha\| \ \langle (P_\beta - P_\alpha)^{\frac{1}{2}}x, (P_\beta - P_\alpha)^{\frac{1}{2}}x\rangle \\ &\le \langle (P_\beta - P_\alpha)x, x\rangle < \varepsilon. \end{aligned}$$

Hence $(P_\beta x)_\beta$ is Cauchy. Since $\mathcal{H}$ is complete, $Px := \lim_\beta P_\beta x$ exists for all $x \in \mathcal{H}$. It is not hard to check that P is linear, and $\langle Px, x\rangle = \lim_\beta \langle P_\beta x, x\rangle \ge 0$, so that $P \ge 0$. Since $P_\beta \to P$ in the SOT, we also have $P_\beta \to P$ in the WOT. Since the unit ball $\mathbb{M}_1$ of $\mathbb{M}$ is WOT-compact from above, and since $P_\beta \in \mathbb{M}_1$ for all β, we get $P \in \mathbb{M}_1$.

□

Kaplansky's Density Theorem.

12.32. In order to prove our next approximation theorem, we shall require a few auxiliary results. The first of these addresses a special case of continuity of joint multiplication in $\mathcal{B}(\mathcal{H})$ relative to the SOT. The reader should compare the next Proposition with Proposition 12.11 above.

12.33. Proposition. *Let $\Omega \subseteq \mathcal{B}(\mathcal{H})$ be a non-empty, bounded set. The map*

$$\begin{array}{rccc} \mu: & \Omega \times \Omega & \to & \mathcal{B}(\mathcal{H}) \\ & (X, Y) & \mapsto & XY \end{array}$$

is SOT*-continuous.*

Proof. Let $\|\Omega\|_\infty := \sup\{\|T\| : T \in \Omega\}$. (We assume below that $\Omega \ne \{0\}$, otherwise the proof is obvious.)

Let $(X_\lambda, Y_\lambda)_{\lambda\in\Lambda}$ be a net in $\Omega \times \Omega$ converging to $(X, Y) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ in the product SOT-topology; i.e. $\mathrm{SOT} - \lim_\lambda X_\lambda = X$ and $\mathrm{SOT} - \lim_\lambda Y_\lambda = Y$. Let $z \in \mathcal{H}$ and $\varepsilon > 0$. Then

$$\begin{aligned} \|XYz - X_\lambda Y_\lambda z\| &\le \|XYz - X_\lambda Yz\| + \|X_\lambda Yz - X_\lambda Y_\lambda z\| \\ &\le \|XYz - X_\lambda Yz\| + \|X_\lambda\| \ \|Yz - Y_\lambda z\|. \end{aligned}$$

Choose $\lambda_0 \in \Lambda$ such that $\lambda \ge \lambda_0$ implies that

- $\|X(Yz) - X_\lambda(Yz)\| < \frac{\varepsilon}{2}$; and
- $\|Yz - Y_\lambda z\| < \dfrac{\varepsilon}{2\|\Omega\|_\infty}$.

From this it readily follows that $\lambda \ge \lambda_0$ implies that

$$\|XYz - X_\lambda Y_\lambda z\| < \frac{\varepsilon}{2} + \|\Omega\|_\infty \, \frac{\varepsilon}{2\|\Omega\|_\infty} = \varepsilon.$$

By definition, $\mathrm{SOT} - \lim_\lambda X_\lambda Y_\lambda = XY$.

□

12.34. Remark. The corresponding statement for the WOT is false. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, and let $X_n := e_1 \otimes e_n^*$, $Y_n = e_n \otimes e_1^*$ be rank-one partial isometries in $\mathcal{B}(\mathcal{H})$. Notice, in particular, that $\|X_n\| = 1 = \|Y_n\|$ for all $n \ge 1$.

We leave it to the reader to verify that $\mathrm{WOT} - \lim_n X_n = 0 = \mathrm{WOT} - \lim_n Y_n$. On the other hand,

$$\mathrm{WOT} - \lim_n X_n Y_n = \mathrm{WOT} - \lim_n e_1 \otimes e_1^* = e_1 \otimes e_1^* \ne 0.$$

Recall that the involution map $\iota : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $\iota(T) = T^*$ is not SOT-continuous. Our next result says tht if we restrict its domain to the set of normal operators in $\mathcal{B}(\mathcal{H})$, then the restricted map *is* continuous. Let us write $\mathrm{NOR}(\mathcal{H})$ to denote the set of normal operators in $\mathcal{B}(\mathcal{H})$. Recall that if $N \in \mathrm{NOR}(\mathcal{H})$ and $x \in \mathcal{H}$, then $\|Nx\| = \|N^*x\|$. Also, it is clear that if $(T_\lambda)_\lambda$ is any net converging strongly to $T \in \mathcal{B}(\mathcal{H})$, then $\lim_\lambda \|T_\lambda x\| = \|Tx\|$.

12.35. Lemma. *The involution map $\iota^\circ : \mathrm{NOR}(\mathcal{H}) \to \mathrm{NOR}(\mathcal{H})$ defined by $\iota^\circ(N) = N^*$ is* SOT*-continuous.*

Proof. Let $(N_\lambda)_\lambda$ be a net of normal operators converging in the SOT to a normal operator N, and let $x \in \mathcal{H}$. Then

$$\begin{aligned}
\|N_\lambda^* x - N^* x\|^2 &= \langle N_\lambda^* x - N_\lambda x, N_\lambda^* x - N_\lambda x\rangle \\
&= \|N_\lambda^* x\|^2 + \|N^* x\|^2 - 2\mathrm{Re}\langle N_\lambda^* x, N^* x\rangle \\
&= \|N_\lambda^* x\|^2 + \|N^* x\|^2 - 2\mathrm{Re}\langle N^* x, N^* x\rangle - 2\mathrm{Re}\langle (N_\lambda^* - N^*)x, N^* x\rangle \\
&= \|N_\lambda^* x\|^2 - \|N^* x\|^2 - 2\mathrm{Re}\langle x, (N_\lambda - N)N^* x\rangle \\
&\le \|N_\lambda x\|^2 - \|N x\|^2 - 2\|x\| \, \|(N_\lambda - N)(N^* x)\|
\end{aligned}$$

Since $\lim_\lambda \|N_\lambda x\| = \|Nx\|$, and since $\lim_\lambda \|(N_\lambda - N)(N^*x)\| = 0$, we see that

$$\lim_\lambda \|N_\lambda^* x - N^* x\| = 0,$$

implying that $(N_\lambda^*)_\lambda$ converges in the SOT-topology to N^*, and therefore that ι° is SOT-continuous.

□

12.36. Proposition. *Let $\Omega \subseteq \textsc{nor}(\mathcal{H})$ be a non-empty bounded set, and let $p(x, y)$ be a polynomial in two commuting variables x and y. The map*

$$\begin{array}{rccc} p^\circ : & \Omega & \to & \mathcal{B}(\mathcal{H}) \\ & T & \mapsto & p(T, T^*) \end{array}$$

is SOT*-continuous on Ω.*

Proof. We leave this as an Assignment Exercise.

□

Our next Proposition asserts that every continuous function on $f : \mathbb{C} \to \mathbb{C}$ is SOT-continuous as a function from a bounded set Ω of normal operators in $\mathcal{B}(\mathcal{H})$ into $\textsc{nor}(\mathcal{H})$.

12.37. Proposition. *Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function, and let $\Omega \subseteq \textsc{nor}(\mathcal{H})$ be a bounded set of normal operators. The map*

$$\begin{array}{rccc} f : & \Omega & \mapsto & \textsc{nor}(\mathcal{H}) \\ & N & \mapsto & f(N) \end{array}$$

is SOT*-continuous.*

Proof. Once again, we assume that $\Omega \neq \{0\}$, for otherwise the result is trivial. Let us define $\|\Omega\|_\infty := \sup\{\|M\| : M \in \Omega\}$. Fix $M_0 \in \Omega$; we shall prove SOT-continuity of f at M_0.

Let $x \in \mathcal{H}$ and $\varepsilon > 0$. Since the disc of radius $\|\Omega\|_\infty$ in $\mathbb{C}$ centred at the origin is compact, we may apply the Stone-Weierstraß Theorem to produce a polynomial $p(z, \overline{z})$ which approximates f to within $\delta := \frac{\varepsilon}{3\|x\|+1}$ on that disc; that is,

$$\sup\{|f(z) - p(z, \overline{z})| : |z| \leq \|\Omega\|_\infty\} \leq \delta.$$

Observe that if $M \in \Omega$, then the fact that the Gelfand Transform is isometric, coupled with the fact that $\sigma(M) \subseteq B(0, \|\Omega\|) := \{z \in \mathbb{C} : |z| \leq \|\Omega\|_\infty\}$ implies that

$$\|f(M) - p(M, M^*)\| = \sup\{|f(z) - p(z, \overline{z})| : z \in \sigma(M)\} \leq \delta.$$

Let $(M_\lambda)_{\lambda \in \Lambda}$ be a net in Ω converging in the SOT to M. Then

$$\begin{aligned} \|f(M_\lambda)x - f(M_0)x\| &\leq \|f(M_\lambda)x - p(M_\lambda, M_\lambda^*)x\| \\ &\qquad + \|p(M_\lambda, M_\lambda^*)x - p(M_0, M_0^*)x\| \\ &\qquad\qquad + \|p(M_0, M_0^*)x - f(M_0)x\| \\ &\leq \|f(M_\lambda) - p(M_\lambda, M_\lambda^*)\| \ \|x\| \\ &\qquad + \|p(M_\lambda, M_\lambda^*)x - p(M_0, M_0^*)x\| \\ &\qquad\qquad + \|p(M_0, M_0^*) - f(M_0)\| \ \|x\| \\ &\leq \delta \ \|x\| + \|p(M_\lambda, M_\lambda^*)x - p(M_0, M_0^*)x\| + \delta \ \|x\| \\ &\leq \frac{\varepsilon}{3} + \|p(M_\lambda, M_\lambda^*)x - p(M_0, M_0^*)x\| + \frac{\varepsilon}{3}. \end{aligned}$$

By Proposition 12.36, there exists $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies that

$$\|p(M_\lambda, M_\lambda^*)x - p(M_0, M_0^*)x\| < \frac{\varepsilon}{3}.$$

Then $\lambda \geq \lambda_0$ clearly implies that

$$\|f(M_\lambda)x - f(M_0)x\| < \varepsilon,$$

proving that $f(M_\lambda)$ converges in the SOT to $f(M_0)$. That is, f is SOT-continuous on Ω.

□

Our immediate goal is to prove Kaplansky's Density Theorem, which states that if $I \in \mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra, then every self-adjoint operator of norm one in the SOT-closure of $\mathcal{A}$ is a limit of a net of self-adjoint operators in the *unit ball* of $\mathcal{A}$. This is far from obvious. If $T \in \overline{A}^{\text{SOT}}$, then any SOT-nbhd of T contains an infinite-dimensional subspace of $\mathcal{A}$, and as a result, it contains operators of arbitrarily large norm. The above proposition and its proof required the set Ω above to be bounded. We need a way to get around this. Our proof of Kaplansky's Density Theorem below is modelled after that appearing in [**31**].

12.38. Definition. *Let $H \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. The **Cayley Transform** of H is the operator*

$$U_H := (H - iI)(H + iI)^{-1} = (H + iI)^{-1}(H - iI).$$

12.39. Remarks.

(a) First note that $H = H^*$ implies that $\sigma(H) \subseteq \mathbb{R}$, and thus $H + iI$ is indeed invertible. Also, a routine calculation shows that U_H is unitary. Indeed,

$$U_H^* = ((H + iI)^{-1})^*(H - iI)^* = (H - iI)^{-1}(H + iI) = U_H^{-1}.$$

(b) Consider the continuous function $\gamma : \mathbb{R} \to \mathbb{T} \setminus \{1\}$ defined by $\gamma(x) = \dfrac{x - i}{x + i}$. It is not hard to verify that γ is bijective with continuous inverse

$$\gamma^{-1}(z) = \frac{-i(z + 1)}{z - 1}.$$

By the continuous functional calculus (i.e. the Gelfand Transform for normal operators), we see that $U_H = \gamma(H)$, and thus

$$H = \text{id}(H) = \gamma^{-1} \circ \gamma(H) = \gamma^{-1}(U_H).$$

A moment's thought should convince the reader that if $V \in \mathcal{B}(\mathcal{H})$ is any unitary operator with $1 \notin \sigma(V)$, then $K := \gamma^{-1}(V)$ is a self-adjoint operator for which $\gamma(K) = U_K = V$.

(c) If $x \in \mathbb{R}$, then $|x+i| = \sqrt{x^2+1} \geq 1$, and thus $|x+i|^{-1} \leq 1$. Combining this with the functional calculus and the Spectral Mapping Theorem, we see that if $H = H^* \in \mathcal{B}(\mathcal{H})$, then $(H+iI)^{-1}$ is a normal operator with spectral radius at most 1, and thus $\|(H+iI)^{-1}\| \leq 1$. This will be used in the Lemma below.

12.40. Lemma. *Let $\mathcal{U}(\mathcal{H}) := \{U \in \mathcal{B}(\mathcal{H}) : U$ is unitary$\}$. The Cayley transform map*

$$\begin{array}{rccc} \gamma: & \mathcal{B}(\mathcal{H})^{sa} & \to & \mathcal{U}(\mathcal{H}) \\ & H & \mapsto & U_H \end{array}$$

is continuous in the SOT*-topology.*

Proof. Let $H_0 = H_0^* \in \mathcal{B}(\mathcal{H})$, and let $(H_\lambda)_{\lambda\in\Lambda}$ be a net of hermitian operators tending in the SOT-topology to H_0. Let $x \in \mathcal{H}$, and let $\varepsilon > 0$. A routine calculation reveals that

$$(H_\lambda + iI)(U_{H_\lambda} - U_{H_0})(H_0 + iI) = 2i(H_\lambda - H_0),$$

and so (keeping in mind that $\|(H_\lambda + iI)^{-1}\| \leq 1$),

$$\begin{aligned} \|\gamma(H_\lambda)x - \gamma(H_0)x\| &= \|U_{H_\lambda}x - U_{H_0}x\| \\ &= 2\|(H_\lambda + iI)^{-1}(H_\lambda - H_0)(H_0 + iI)^{-1}x\| \\ &\leq 2\|(H_\lambda - H_0)((H_0 + iI)^{-1}x)\|. \end{aligned}$$

To prove the result, it suffices to note that we can choose $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies that

$$\|(H_\lambda - H_0)((H_0 + iI)^{-1}x)\| < \frac{\varepsilon}{2}.$$

□

12.41. Proposition. *Let $f \in \mathcal{C}_0(\mathbb{R}, \mathbb{C})$. Then $f : \mathcal{B}(\mathcal{H})^{sa} \to \mathcal{B}(\mathcal{H})$ is continuous in the* SOT*-topology.*

Proof. Recall that the map $\gamma : \mathbb{R} \to \mathbb{T}\setminus 1$ defined by $\gamma(x) = \dfrac{x-i}{x+i}$ is a homeomorphism with inverse $\gamma^{-1}(z) = \dfrac{-i(z+1)}{z-1}$. Define the function

$$\begin{array}{rccc} \varrho: & \mathbb{T} & \to & \mathbb{C} \\ & z & \mapsto & \begin{cases} 0 & \text{if } z = 1 \\ f \circ \gamma^{-1}(z) & \text{if } z \neq 1. \end{cases} \end{array}$$

Since f tends to zero as $|x|$ tends to infinity, we find that ϱ is continuous on $\mathbb{T}$. It follows from Proposition 12.37 that ϱ is continuous in the SOT-topology on the set of all normal operators of norm at most 1, and more specifically on the set $\mathcal{U}(\mathcal{H})$ of all unitary operators.

Finally, since ϱ and γ are both SOT-continuous, it follows that their composition

$$f = \varrho \circ \gamma$$

is again SOT-continuous.

□

12.42. Theorem. [Kaplansky's Density Theorem.] *Let $\mathcal{H}$ be a Hilbert space and let $I \in \mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators. Then*

$$(\overline{\mathbb{A}}^{\mathrm{SOT}})_1^{sa} \subseteq \overline{(\mathbb{A}_1^{sa})}^{\mathrm{SOT}}.$$

Proof. Let $H = H^* \in \overline{\mathbb{A}}^{\mathrm{SOT}}$ be an operator of norm at most one, and choose a net $(A_\lambda)_\lambda$ in $\mathbb{A}$ with $\mathrm{SOT} - \lim_\lambda A_\lambda = H$. Since the WOT is weaker than the SOT, it follows that

$$H = \mathrm{WOT} - \lim_\lambda A_\lambda.$$

But then

$$H = H^* = \mathrm{WOT} - \lim_\lambda A_\lambda^*,$$

since the involution map $\iota : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $\iota(T) = T^*$ is WOT-continuous. From this we conclude that

$$H = \mathrm{WOT} - \lim_\lambda \mathrm{Re}\, A_\lambda,$$

and thus $H \in \overline{(\mathbb{A}^{sa})}^{\mathrm{WOT}}$. Now, as we saw in Corollary 12.29, $(\mathcal{B}(\mathcal{H}), \mathrm{SOT})$ and $(\mathcal{B}(\mathcal{H}), \mathrm{WOT})$ have the same closed, convex sets, and since $\mathbb{A}^{sa}$ is convex, it follows that there exists a net $(H_\alpha)_\alpha$ in $\mathbb{A}^{sa}$ which converges in the SOT to H.

Next, we wish to replace $(H_\alpha)_\alpha$ by a net of hermitian operators in the *unit ball* of $\mathbb{A}^{sa}$.

To that end, let $f \in \mathcal{C}_0(\mathbb{R}, \mathbb{C})$ be any *real-valued* function, bounded in the uniform norm by 1, which satisfies $f(x) = x$, $x \in [-1, 1]$. The standard example appearing in a number of sources is

$$f(x) = \begin{cases} x & x \in [-1,1] \\ x^{-1} & |x| \geq 1. \end{cases}$$

Set $K_\alpha := f(H_\alpha)$ for each α. By Proposition 12.41, the net $(K_\alpha)_\alpha$ converges in the SOT to $f(H)$. But $\|H\| \leq 1$ implies that $\sigma(H) \subseteq [-1, 1]$, and $f(x) = x$ for all $x \in [-1, 1]$, implying that $f(H) = H$. Meanwhile, $K_\alpha = f(H_\alpha)$ is a self-adjoint operator (f is real-valued!) with $\sigma(K_\alpha) = f(\sigma(H_\alpha)) \subseteq \mathrm{ran}\, f \subseteq [-1, 1]$, from which we deduce that $\|K_\alpha\| \leq 1$. Since f is continuous and $f(0) = 0$, we also see that $K_\alpha = f(H_\alpha)$ lies in the (non-unital) C^*-algebra generated by H_α, and thus in $\overline{\mathbb{A}}^{\|\cdot\|}$.

Of course, every hermitian element of norm one in $\overline{\mathbb{A}}^{\|\cdot\|}$ is a *norm*-limit of hermitian elements of norm one in $\mathbb{A}$ (check!), and thus $K_\alpha \in \overline{(\mathbb{A}_1^{sa})}^{\mathrm{SOT}}$ for each α. Since

$$H = f(H) = \mathrm{SOT} - \lim f(H_\alpha) = \mathrm{SOT} - \lim K_\alpha,$$

we have that $H \in \overline{(\mathbb{A}_1^{sa})}^{\mathrm{SOT}}$, which concludes the proof.

□

The next Corollary is also referred to as (part of) Kaplansky's Density Theorem.

12.43. Corollary. *Let $\mathcal{H}$ be a Hilbert space and let $I \in \mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators. Then*

$$(\overline{\mathbb{A}}^{\mathrm{SOT}})_1 \subseteq \overline{(\mathbb{A}_1)}^{\mathrm{SOT}}.$$

Proof. The proof of this Corollary is left as an Assignment exercise.

□

Polar Decomposition

12.44. Given a complex number z, we can write z as a product of a positive number (its modulus) and a complex number of magnitude one. We wish to generalize this to operators on a Hilbert space. Our reason for waiting until this section to prove the result will be made clear from Proposition 12.52.

12.45. Definition. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $V \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. We say that V is a* ***partial isometry*** *if $\|Vx\| = \|x\|$ for all $x \in (\ker V)^\perp$. If $\ker V = \{0\}$, we say that V is an* ***isometry****.*

The space $(\ker V)^\perp$ is called the initial space *of V, while $\mathrm{ran} V$ is called the* final space *of V. Observe that $\mathrm{ran} V$ is automatically closed in $\mathcal{H}_2$.*

12.46. Example. Fix $n \in \mathbb{N}$ and let $\mathcal{H} = \mathbb{C}^n$. Then V is an isometry if and only if V is unitary.

12.47. Example. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Consider the unilateral forward shift $Se_n = e_{n+1}$, $n \ge 1$. Then S is an isometry, and S^* is a partial isometry with initial space $\{e_1\}^\perp$.

12.48. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_1, \mathcal{H}_2$ be closed subspaces of $\mathcal{H}$ with $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$. Then there exists a partial isometry $V \in \mathcal{B}(\mathcal{H})$ with initial space $\mathcal{H}_1$ and final space $\mathcal{H}_2$.*

Proof. The fact that $\dim\, \mathcal{H}_1 = \dim\, \mathcal{H}_2$ implies that there exists an index set Λ and orthonormal bases $\mathfrak{B}_1 = \{b_\lambda\}_{\lambda\in\Lambda}$ and $\mathfrak{B}_2 = \{c_\lambda\}_{\lambda\in\Lambda}$ for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Extend $\mathfrak{B}_1$ to an orthonormal basis $\mathfrak{B}_0 := \{b_\lambda\}_{\lambda\in\Lambda} \cup \{e_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$, and define $V \in \mathcal{B}(\mathcal{H})$ via:

$$Vb_\lambda := c_\lambda, \quad \lambda \in \Lambda$$
$$Ve_\omega := 0, \quad \omega \in \Omega.$$

We leave it to the reader to verify that this map extends by linearity and continuity to a continuous map on all of $\mathcal{H}$, and that the resulting map is indeed a partial isometry with initial space $\mathcal{H}_1$ and final space $\mathcal{H}_2$.

□

The next result says that an operator is determined entirely by all of its diagonal values (relative to an arbitrary orthonormal basis).

12.49. Lemma. *Let A and $B \in \mathcal{B}(\mathcal{H})$ and suppose that $\langle Ax, x\rangle = \langle Bx, x\rangle$ for all $x \in \mathcal{H}$. Then $A = B$.*

Proof. The proof of this is left as an Assignment Exercise.

□

12.50. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $V \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(a) *V is a partial isometry.*
(b) *V^* is a partial isometry.*
(c) *VV^* is a projection - in which case it is the orthogonal projection onto the range of V.*
(d) *V^*V is a projection, in which case it is the orthogonal projection onto the range of V^*.*

Proof.

(a) implies (d). Let us decompose $\mathcal{H} = \operatorname{ran} V^* \oplus (\operatorname{ran} V^*)^\perp = \operatorname{ran} V^* \oplus \ker V$. Let Q denote the orthogonal projection onto the range of V^*. Given $x \in \mathcal{H}$, we may write $x = y + z$ relative to this decomposition. Note that $Vx = Vy$ and that $Q^*Q = Q$. For each $x \in \mathcal{H}$,

$$\langle V^*Vx, x\rangle = \langle Vx, Vx\rangle = \langle Vy, Vy\rangle = \langle y, y\rangle = \langle Qx, Qx\rangle = \langle Qx, x\rangle.$$

By Lemma 12.49, $V^*V = Q$.

(d) implies (c). Suppose that V^*V is the orthogonal projection Q onto range V^*. Clearly VV^* is self-adjoint. Furthermore,

$$(VV^*)^2 = V(V^*V)V^* = VQV^* = VV^*,$$

implying that $P := VV^*$ is a (self-adjoint) idempotent, hence a projection. Clearly $\operatorname{ran} P \subseteq \operatorname{ran} V$, and if $y \in \operatorname{ran} V$ – say $y = Vx$ for some $x \in \mathcal{H}$, then

$$\begin{aligned}\langle Py, y\rangle &= \langle VV^*(Vx), (Vx)\rangle = \langle VQx, Vx\rangle \\ &= \langle Q^2x, x\rangle = \langle Qx, x\rangle \\ &= \langle Vx, Vx\rangle = \langle y, y\rangle,\end{aligned}$$

so that $\operatorname{ran} P \supseteq \operatorname{ran} V$; i.e. P is the orthogonal projection onto $\operatorname{ran} V$.

(c) implies (b). Let $y \in (\ker V^*)^\perp = \operatorname{ran} V$. Then

$$\begin{aligned}\|V^*y\|^2 &= \langle V^*y, V^*y\rangle = \langle VV^*y, y\rangle \\ &= \langle y, y\rangle = \|y\|^2.\end{aligned}$$

Thus $\|V^*y\| = \|y\|$, and V^* is a partial isometry.

(b) implies (a). Combining the above implications, we now know that (a) implies (b). That is, if V is a partial isometry, then so is V^*. Applying this with V replaced by V^* shows that if V^* is a partial isometry, then so is $(V^*)^* = V$.

□

12.51. Theorem. [***Polar Decomposition.***] *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. There there exists a partial isometry V and a positive operator P such that $T = VP$. If we further require that* $\ker V = \ker P$*, then V and P are unique.*

Proof. Let $P := |T| = (T^*T)^{\frac{1}{2}}$. Observe first that for all $x \in \mathcal{H}$,

$$\|Px\|^2 = \langle Px, Px\rangle = \langle P^2 x, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2.$$

Define $V_0 : \operatorname{ran} P \to \operatorname{ran} T$ via $V_0 Px = Tx$. If $Px_1 = Px_2$, then $x_1 - x_2 \in \ker P = \ker T$, which proves that $Tx_1 = Tx_2$ and thus V_0 is well-defined and isometric. That V_0 is linear is routine, and thus V_0 extends uniquely (by continuity) to an isometric map $V_1 : \overline{\operatorname{ran} P} \to \operatorname{ran} T$. Let $\mathcal{H} = (\operatorname{ran} P)^\perp \oplus \overline{\operatorname{ran} P}$, and for $x = y + z \in \mathcal{H}$ (relative to this decomposition), define $Vx = V_1 z$. Then V is a partial isometry with initial space $\operatorname{ran} P$ and final space $\operatorname{ran} T$.

Suppose next that W is a partial isometry, $0 \le Q$ is a positive operator, $\ker W = \ker Q$ and $T = WQ$. Then $\operatorname{ran} W^* = (\ker W)^\perp = (\ker Q)^\perp = \operatorname{ran} Q^* = \operatorname{ran} Q$, so that W^*W is the orthogonal projection of $\mathcal{H}$ onto $\operatorname{ran} Q$. Thus

$$T^*T = QW^*WQ = Q^2,$$

and by the uniqueness of positive square roots, $Q = P = |T|$. Hence $T = W|T| = V|T|$. Thus $W|_{\operatorname{ran}|T|} = V|_{\operatorname{ran}|T|}$, and since $\ker W = \ker P = \ker |T| = \ker V = (\operatorname{ran}|T|)^\perp$, we find that $W = V$.

□

12.52. Proposition. *Let $T \in \mathcal{B}(\mathcal{H})$. The unique partial isometry V with* $\ker V = \ker T = \ker |T|$ *appearing in the polar decomposition of the operator $T = V|T|$ lies in the von Neumann algebra generated by T.*

Proof. Let $Z \in C^*(T)'$, so that $ZT = TZ$ and $ZT^* = T^*Z$. It then follows that $Z(T^*T) = (T^*T)Z$, and so Z commutes with $|T|$, since the latter is the norm-limit of polynomials in T^*T.

Thus

$$VZ|T| = V|T|Z = TZ = ZT = ZV|T|.$$

- If $x \in \operatorname{ran}|T|$, say $x = |T|w$, then $VZx = VZ|T|w = ZV|T|w = ZVx.$ By continuity of VZ and of ZV, we see that $VZx = ZVx$ for all $x \in \overline{\operatorname{ran}|T|}$.
- If $x \in (\operatorname{ran}|T|)^\perp = \ker T = \ker V$, then the equation $TZx = ZTx = Z0 = 0$ implies that $Zx \in \ker T = \ker V$, so

$$VZx = 0 = Z0 = Z(Vx) = ZVx.$$

It follows that $VZ = ZV$, so that $V \in C^*(T)'' = W^*(T)$.

□

12.53. Example. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$, and define the unilateral forward shift operator $We_n = \frac{1}{n} e_{n+1}$. Then W is compact, and so the non-unital C^*-algebra $C_0^*(W)$ generated by W satisfies $C_0^*(W) \subseteq \mathcal{K}(\mathcal{H})$.

If $Se_n = e_n + 1$, $n \geq 1$ denotes the unweighted shift and $De_n = \frac{1}{n} e_n$ is the diagonal operator operator whose weight sequence is that of W, then $W = SD$, where S is a (partial) isometry and $0 \leq D$. Since $\ker D = \{0\} = \ker S$, we see that this is the unique polar decomposition of W. While $D = |W| \in C_0^*(W)$, S does not lie in $C^*(W)$, (the unital C^*-algebra generated by W), since $\pi(S) \notin \mathbb{C}\pi(I)$, where $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the canonical quotient map.

We finish this section with a proposition that was proven in one of the Assignments, though at the time we did not give it its proper name.

12.54. Proposition. [The Wold Decomposition.] *Let $\mathcal{H}$ be a Hilbert space, and $W \in \mathcal{B}(\mathcal{H})$ be an isometry. Then there exist a unitary U and a cardinal number α so that $W \simeq U \oplus S^{(\alpha)}$, where S is the forward unilateral shift operator.*

Supplementary Examples

S12.1. Example. Note that when $\mathbb{A}$ is a von Neumann algebra, then any algebraic two-sided ideal is automatically self-adjoint. Indeed, if $K \in \mathbb{K}$, a left algebraic ideal, then we may write the polar decomposition for K, namely $K = U|K|$. Since $\mathbb{A}$ is a von Neumann algebra, both terms of the polar decomposition of $\mathbb{K}$ lie in $\mathbb{A}$.

As such, $U \in \mathbb{A}$, and so is U^*. But then $U^*KU^* = |K|U^* = K^* \in \mathbb{K}$.

Appendix

A12.1. In progress.

Exercises for Chapter 12

Question 1. In progress.

CHAPTER 13

The Spectral Theorem for normal operators

If any of you cry at my funeral I'll never speak to you again.

Stan Laurel

The spectral theorem for normal operators.

13.1. In this section we extend the functional calculus for normal operators on a separable Hilbert space beyond the continuous functional calculus we obtained in Chapter Four via the Gelfand transform. In the present setting, we show that if $\mathcal{H}$ is a separable Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ is normal, then the unital von Neumann algebra $W^*(N)$ generated by N is isometrically *-isomorphic to $L^\infty(\sigma(N), \mu)$, where μ is a finite, positive, regular Borel measure with support $\sigma(N)$. This identification leads us to an L^∞-functional calculus for normal operators.

13.2. Proposition. *Let (X, μ) be a measure space, where μ is a finite, positive, regular Borel measure on X. The map*

$$\begin{array}{rccc} \varrho: & L^\infty(X,\mu) & \to & \mathfrak{M}^\infty(X,\mu) \\ & f & \mapsto & M_f \end{array}$$

is a homeomorphism from $(L^\infty(X,\mu), weak^*)$ to $(\mathfrak{M}^\infty(X,\mu), \mathrm{WOT})$.

Proof. We are required to show that a net $(f_\alpha)_{\alpha\in\Lambda}$ in $L^\infty(X,\mu)$ converges in the weak*-topology to a function f if and only if $(M_{f_\alpha})_\alpha$ converges in the WOT to M_f.

- Suppose f_α converges in the weak*-topology to f. Then for all $g \in L^1(X,\mu)$,

$$\lim_\alpha \int_X f_\alpha g d\mu = \int_X f g d\mu.$$

If $h_1, h_2 \in L^2(X, \mu)$, then $h_1 \overline{h_2} \in L^1(X, \mu)$ by Hölder's Inequality and so

$$\begin{aligned} \lim_\alpha \langle M_{f_\alpha} h_1, h_2 \rangle &= \lim_\alpha \langle f_\alpha h_1, h_2 \rangle \\ &= \lim_\alpha \int_X f_\alpha h_1 \overline{h_2} d\mu \\ &= \int_X f h_1 \overline{h_2} d\mu \\ &= \langle f h_1, h_2 \rangle \\ &= \langle M_f h_1, h_2 \rangle. \end{aligned}$$

That is, (M_{f_α}) converges in the WOT to M_f.

- Conversely, if $(M_{f_\alpha})_\alpha$ is a net in $\mathfrak{M}^\infty(X, \mu)$ which converges in the WOT to M_f, then given $g \in L^1(X, \mu)$, we can find $h_1, h_2 \in L^2(X, \mu)$ so that $g = h_1 \overline{h_2}$. Then, as above,

$$\begin{aligned} \lim_\alpha \int_X f_\alpha h_1 \overline{h_2} d\mu &= \lim_\alpha \langle M_{f_\alpha} h_1, h_2 \rangle \\ &= \langle M_f h_1, h_2 \rangle \\ &= \int_X f h_1 \overline{h_2} d\mu. \end{aligned}$$

Thus $(f_\alpha)_\alpha$ converges in the weak*-topology to f.

□

13.3. Lemma. *Let X be a compact, Hausdorff space and suppose that μ is a positive, regular Borel measure on X with $\mu(X) = 1$. Let $\varepsilon > 0$ and $g : X \to \mathbb{C}$ be a μ-integrable function. Then there exists a compact subset $J \subseteq X$ such that*

(i) *$g|_J$ is bounded; and*

(ii)

$$\int_{X \setminus J} |g| d\mu < \varepsilon.$$

Proof. For each $1 \le k \in \mathbb{N}$, define $E_k = \{x \in X : k - 1 \le |g(x)| < k\}$. Each E_k is then μ-measurable, and clearly $\cup_{k=1}^\infty E_k = X$. Furthermore,

$$\sum_k \int_{E_k} |g| d\mu = \int_{\cup_k E_k} |g| d\mu = \int_X |g| d\mu < \infty,$$

and so there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^\infty \int_{E_k} |g| d\mu < \frac{\varepsilon}{2}.$$

Let $F := \cup_{k=1}^N E_k$, and note that $\sup_{x \in F} |g(x)| \le N$. Since μ is regular, we can find a compact subset $J \subseteq F$ such that $\mu(F \setminus J) < \frac{\varepsilon}{2N}$.

Clearly $|g(x)| \le N$ for all $x \in J$, and

$$\begin{aligned}\int_{X \smallsetminus J} |g| d\mu &= \int_{X \smallsetminus F} |g| d\mu + \int_{F \smallsetminus J} |g| d\mu \\ &< \frac{\varepsilon}{2} + N \mu(F \smallsetminus J) \\ &< \frac{\varepsilon}{2} + N \frac{\varepsilon}{2N} \\ &= \varepsilon.\end{aligned}$$

□

13.4. Theorem. *Let X be a compact, Hausdorff space and suppose that μ is a positive, regular Borel measure on X with $\mu(X) = 1$. Then*

(i) *The unit ball of $\mathcal{C}(X)$ is weak*-dense in the unit ball of $L^\infty(X, \mu)$, and thus*
(ii) *$\mathcal{C}(X)$ is weak*-dense in $L^\infty(X, \mu)$.*
(iii) *The set $\mathfrak{M}_{\mathcal{C}(X)} := \{M_f : f \in \mathcal{C}(X)\}$ is* WOT*-dense in $\mathfrak{M}^\infty(X, \mu)$.*

Proof. Our proof will deal with μ-integrable *functions*; the translations to elements of $L^1(X, \mu)$ and $L^\infty(X, \mu)$ (which are *equivalence classes of functions*) is hopefully clear.

(i) Let $g : X \to \mathbb{C}$ be a μ-integrable function, and let $\varepsilon > 0$. (In other words, the equivalence class $[g]$ of g lies in $L^1(X, \mu)$, the pre-dual of $L^\infty(X, \mu)$.)

By Lemma 13.3, we can find a compact subset $J \subseteq X$ such that $g|_J$ is bounded, say by $N \ge 1$, and

$$\int_{X \smallsetminus J} |g| d\mu < \frac{\varepsilon}{4}.$$

Let $f : X \to \mathbb{C}$ be a μ-measurable function with $\sup_{x \in X} |f(x)| \le 1$. (In other words, the equivalence class $[f]$ of f lies in the unit ball of $L^\infty(X, \mu)$.)

By Lusin's Theorem 1.30, there exists a measurable subset $E \subseteq X$ satisfying $\mu(X \smallsetminus E) < \frac{\varepsilon}{4N}$ and a continuous function $h : X \to \mathbb{C}$, such that $\|h\|_\infty = \sup_{x \in X} |h(x)| \le 1$ and $h(x) = f(x)$ for all $x \in E$.

Thus for $g : X \to \mathbb{C}$ as above, and keeping in mind that $\mu(J \smallsetminus E) \le \mu(X \smallsetminus E) < \frac{\varepsilon}{4N}$,

$$\begin{aligned}\int_X |f - h||g| d\mu &= \int_{X \smallsetminus J} |f - h||g| d\mu + \int_{J \smallsetminus E} |f - h||g| d\mu + \int_E |f - h||g| d\mu \\ &\le \int_{X \smallsetminus J} 2|g| d\mu + \int_{J \smallsetminus E} 2N d\mu + \int_E 0|g| d\mu \\ &< 2\frac{\varepsilon}{4} + 2N \frac{\varepsilon}{4N} + 0 \\ &= \varepsilon.\end{aligned}$$

This completes the proof.

(ii) This follows immediately from part (i).
(iii) This follows from part (ii), combined with Proposition 13.2.

□

13.5. Recall that two positive measures μ_1 and μ_2 on a sigma algebra $(X, \mathcal{S})$ are *mutually absolutely continuous* if for $E \in \mathcal{S}$, $\mu_1(E) = 0$ is equivalent to $\mu_2(E) = 0$. We write $\mu_1 \sim \mu_2$ in this case.

13.6. Theorem. *Let X be a compact, metric space and μ_1, μ_2 be finite, positive, regular Borel measures on X. Suppose that $\tau : L^\infty(X, \mu_1) \mapsto L^\infty(X, \mu_2)$ is an isometric *-isomorphism and $\tau(f) = f$ for all $f \in \mathcal{C}(X)$. Then $\mu_1 \sim \mu_2$, $L^\infty(X, \mu_1) = L^\infty(X, \mu_2)$, and $\tau(g) = g$ for all $g \in L^\infty(X, \mu_1)$.*

Proof. Suppose that $E \subseteq X$ is a Borel set. Then $\tau(\chi_E) \in L^\infty(X, \mu_2)$ is idempotent, and hence a characteristic function, say $\chi_F (= \chi_{F(E)})$. If we can show that $E = F$ *a.e.* $-\, \mu_2$, then

$$\begin{aligned} \mu_1(E) = 0 \quad &\text{iff} \quad \chi_E = 0 \text{ in } L^\infty(X, \mu_1) \\ &\text{iff} \quad \tau(\chi_E) = 0 \text{ in } L^\infty(X, \mu_2) \\ &\text{iff} \quad \chi_F = 0 \text{ in } L^\infty(X, \mu_2) \\ &\text{iff} \quad \mu_2(F) = 0 \\ &\text{iff} \quad \mu_2(E) = 0. \end{aligned}$$

From this it follows that $\mu_1 \sim \mu_2$ and therefore that $L^\infty(X, \mu_1) = L^\infty(X, \mu_2)$. Furthermore, since τ then fixes all characteristic functions, it fixes their spans, which are norm dense in $L^\infty(X, \mu_1)$. By continuity of τ, we see that τ fixes the entire algebra, so τ is the identity map.

Note that $\chi_{X \setminus E} = 1 - \chi_E$, and hence $\tau(\chi_{X \setminus E}) = 1 - \tau(\chi_E) = 1 - \chi_F = \chi_{X \setminus F}$. As such, if we can prove that $E \subseteq X$ implies that $\mu_2(F \setminus E) = 0$, then $X \setminus E \subseteq X$ implies $\mu_2(E \setminus F) = \mu_2((X \setminus F) \setminus (X \setminus E)) = 0$. Letting $\Delta = (E \setminus F) \cup (F \setminus E)$, we have $\mu_2(\Delta) = 0$, and hence $E = F$ *a.e.* $-\, \mu_2$.

Case One: E is compact: For each $n \geq 1$, define $f_n \in \mathcal{C}(X)$ as follows:

$$f_n(x) = \begin{cases} 1 - n \text{ dist}(x, E) & \text{if dist}(x, E) \leq 1/n \\ 0 & \text{otherwise} \end{cases}$$

Then $f_n \geq \chi_E$ for all $n \geq 1$, and $f_n(x) \to \chi_E(x)$ as $n \to \infty$ for all $x \in X$. Since τ is a *-homomorphism, it is positive, and as such, it preserves order. Thus $\tau(\chi_E) \leq \tau(f_n)$ for all $n \geq 1$. But $f_n \in \mathcal{C}(X)$ implies $\tau(f_n) = f_n$ so that $\chi_F = \tau(\chi_E) \leq f_n$ for all $n \geq 1$. Hence $\chi_F \leq \chi_E$ in $L^\infty(X, \mu_2)$. Thus $\mu_2(F \setminus E) = 0$, as required.

Case Two: $E \subseteq X$ is Borel: Since μ_1, μ_2 are regular, we can find an increasing sequence $(K_n)_n$ of compact subseteq of E so that $\mu_i(E \setminus K_n) \to 0$ as $n \to \infty$, $i = 1, 2$. (Indeed, choose K_1 so that $\mu_1(E \setminus K_1) < 1$, $K_2 \geq K_1$ so that $\mu_2(E \setminus K_2) < 1/2$, etc.).

Now τ preserves order, and therefore it also preserves suprema. That is, if $\sup g_n = g$ in $L^\infty(X, \mu_1)$, then $\sup \tau(g_n) = \tau(g)$ in $L^\infty(X, \mu_2)$. In our case,

$$\sup \chi_{K_n} = \chi_E \text{ in } L^\infty(X, \mu_1).$$

Thus $\sup \tau(\chi_{K_n}) = \tau(\chi_E) = \chi_F$ in $L^\infty(X,\mu_2)$. Since $\tau(\chi_{K_n}) \le \chi_{K_n}$ by Case One, we have

$$\chi_E = \sup \chi_{K_n} \ge \sup \tau(\chi_{K_n}) = \chi_F$$

in $L^\infty(X,\mu_2)$, and so again, $\mu_2(F\backslash E) = 0$, completing the proof.

□

13.7. Definition. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an algebra. A vector $x \in \mathcal{H}$ is said to be **cyclic** for $\mathcal{A}$ if $[\mathcal{A}x] = \mathcal{H}$. Also, x is said to be **separating** for $\mathcal{A}$ if $A \in \mathcal{A}$ and $Ax = 0$ imply that $A = 0$.*

13.8. Example. Let $X \subseteq \mathbb{C}$ be a compact set and μ be a positive regular Borel measure with $\operatorname{supp}\mu = X$.

Let $q(z) = z$, $z \in X$, and consider $M_q \in \mathcal{B}(L^2(X,\mu))$. Then $e(z) = 1, z \in X$ is cyclic for $C^*(M_q)$. Indeed, since $C^*(M_q) \simeq \mathcal{C}(X)$, we get $[C^*(M_1)e] = [\mathcal{C}(X)] = L^2(X,\mu)$.

Note that e is also separating for $C^*(M_q)$, since $T \in C^*(M_q)$ implies $T = M_f$ for some $f \in \mathcal{C}(X)$, and hence $Te = f = 0$ if and only if $T = 0$. This is not a coincidence.

13.9. Lemma. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an abelian algebra. If x is cyclic for $\mathcal{A}$, then x is separating for $\mathcal{A}$.*

Proof. Suppose $A \in \mathcal{A}$ and $Ax = 0$. Then for all $B \in \mathcal{A}$, $ABx = BAx = 0$. By continuity of A, $Ay = 0$ for all $y \in [\mathcal{A}x] = \mathcal{H}$. Thus $A = 0$ and x is separating for $\mathcal{A}$.

□

13.10. Theorem. **[The Spectral Theorem: Cyclic Case.]** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal. Suppose that $x \in \mathcal{H}$ is a cyclic vector for $C^*(N)$. Then there exists a finite, positive, regular Borel measure μ with $\operatorname{supp}\mu = \sigma(N)$ and a unitary $U : \mathcal{H} \to L^2(\sigma(N),\mu)$ so that*

$$\begin{array}{rccc} \Gamma^* : & W^*(N) & \mapsto & \mathcal{B}(L^2(\sigma(N),\mu)) \\ & T & \mapsto & UTU^* \end{array}$$

is an isometric *-isomorphism onto $\mathfrak{M}^\infty(\sigma(N),\mu)$. Furthermore, up to the isomorphism between $\mathfrak{M}^\infty(\sigma(N),\mu)$ and $L^\infty(\sigma(N),\mu)$, $\Gamma^*|_{C^*(N)} = \Gamma$, the Gelfand transform.

Proof. First we observe that since $C^*(N)$ is separable and $x \in \mathcal{H}$ is cyclic for $C^*(N)$, it follows that $\mathcal{H}$ is separable as well. Without loss of generality, we may assume that $\|x\| = 1$.

Consider

$$\begin{array}{rccc} \varphi : & C^*(N) & \to & \mathbb{C} \\ & T & \mapsto & \langle Tx, x\rangle. \end{array}$$

Then φ is a positive linear functional. Also, $\Gamma : C^*(N) \to \mathcal{C}(\sigma(N))$ is an isometric *-isomorphism, so

$$\varphi \circ \Gamma^{-1} : \mathcal{C}(\sigma(N)) \to \mathbb{C}$$

is a positive linear functional on $\mathcal{C}(\sigma(N))$. By the Riesz-Markov Theorem 1.31, there exists a finite, positive, regular Borel measure μ on $\sigma(N)$ such that

$$\varphi(f(N)) = \varphi \circ \Gamma^{-1}(f) = \int_{\sigma(N)} f d\mu.$$

We claim that $\operatorname{supp}\mu = \sigma(N)$. For otherwise, there exists $G \subseteq \sigma(N)$ open so that $\mu(G) = 0$. Choose a non-zero positive continuous function f with $f \leq \chi_G$. Then $0 \neq f(N)$ and hence

$$\begin{aligned} \varphi(f(N)) &= \varphi((f(N)^{1/2})^2) \\ &= \|f(N)^{1/2}x\|^2 \\ &\neq 0 \end{aligned}$$

since $f(N)^{1/2} \in C^*(N)$ and x is cyclic, hence separating for $C^*(N)$. But then $0 \neq \varphi(f(N)) = \int_{\sigma(N)} f d\mu \leq \int_G 1 d\mu = \mu(G) = 0$, a contradiction. Thus $\operatorname{supp}\mu = \sigma(N)$.

Consider

$$\begin{aligned} U_0 : \quad C^*(N) &\to \mathcal{C}(\sigma(N)) \\ g(N)x &\mapsto g. \end{aligned}$$

Then

$$\begin{aligned} \|g\|_2^2 &= \int_{\sigma(N)} |g|^2 d\mu \\ &= \varphi \circ (|g|^2(N)) \\ &= \langle |g|^2(N)x, x\rangle \\ &= \langle g(N)^* g(N)x, x\rangle \\ &= \|g(N)x\|^2, \end{aligned}$$

so U_0 is isometric. We can and do extend U_0 to an isometry $U : \mathcal{H} = [C^*(N)x] \to [\mathcal{C}(\sigma(N))] = L^2(\sigma(N), \mu)$.

Now set

$$\begin{aligned} \Gamma^* : \quad W^*(N) &\to \mathcal{B}(L^2(\sigma(N), \mu)) \\ T &\mapsto UTU^*. \end{aligned}$$

Then Γ^* is an isometric *-preserving map. For $f, g \in \mathcal{C}(\sigma(N))$,

$$\Gamma^*(f(N))g = U f(N) (U^* g) = U f(N) g(N) x = fg,$$

so that $\Gamma^*(f(N)) = M_f$. In particular, $\operatorname{ran}\Gamma^* \supseteq \mathfrak{M}_{\mathcal{C}(X)}$.

Now Γ^* is WOT-WOT continuous. Indeed, suppose $f_\alpha(N) \to f(N)$ in the WOT. Then for all $g, h \in L^2(\sigma(N), \mu)$,

$$\langle U f_\alpha U^*(Ug), (Uh)\rangle = \langle f_\alpha g, h\rangle \to \langle fg, h\rangle = \langle U f U^*(Ug), (Uh)\rangle.$$

But $\mathfrak{M}_{\mathcal{C}(X)}$ is WOT-dense in $\mathfrak{M}^\infty(X, \mu)$ by Theorem 13.4 (iii), and thus it follows that $\operatorname{ran}\Gamma^* \supseteq \mathfrak{M}^\infty(\sigma(N), \mu)$.

□

13.11. We remark that the measure μ above is unique in the sense that if ν is a second finite, positive, regular Borel measure with support equal to $\sigma(N)$ and $\Gamma^*_\nu : W^*(N) \to \mathcal{B}(L^2(\sigma(N), \nu))$ extends the Gelfand map as Γ^* does, then $\mu \sim \nu$, $L^\infty(\sigma(N), \mu) = L^\infty(\sigma(N), \nu)$, and $\Gamma^*_\nu = \Gamma^*$.

Indeed, $\Gamma^*_\nu \circ (\Gamma^*)^{-1} : \mathfrak{M}^\infty(\sigma(N), \mu) \to \mathfrak{M}^\infty(\sigma, \nu)$ is an isometric *-isomorphism which, through Γ, induces an isometric *-isomorphism τ from $L^\infty(\sigma(N)), \mu$ to $L^\infty(\sigma(N), \nu)$ which fixes the continuous functions. By Theorem 13.6, τ is the identity map, so that $\Gamma^*_\nu = \Gamma^*$.

13.12. Proposition. *Suppose $\mathcal{H}$ is a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is an abelian C^*-algebra. Then there exists a masa $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$ so that $\mathcal{A} \subseteq \mathfrak{M}$.*

Proof. This is a straightforward application of Zorn's Lemma and the proof is left to the reader.

□

13.13. Theorem. *Let $\mathcal{H}$ be a separable Hilbert space and $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa. Then $\mathfrak{M}$ admits a cyclic vector x.*

Proof. The key to the first half of the proof is that if y and z are two non-zero vectors and z is orthogonal to $[\mathfrak{M}y]$, then $[\mathfrak{M}z]$ is orthogonal to $[\mathfrak{M}y]$. This follows from the fact that $[\mathfrak{M}y]$ is reducing for $\mathfrak{M}$.

Now consider the family

$$\mathfrak{F} = \{\{x_\alpha\}_\alpha \in \Lambda \subseteq \mathcal{H} : \|x_\alpha\| = 1 \text{ for all } \alpha \text{ and } [\mathfrak{M}x_{\alpha_1}] \perp [\mathfrak{M}x_{\alpha_2}] \text{ if } \alpha_1 \neq \alpha_2\},$$

partially ordered with respect to inclusion. If $\mathfrak{J} = \{(J_\beta)_\beta\}$ is a chain in $\mathfrak{F}$, it is routine to verify that $\cup_\beta J_\beta$ lies in $\mathfrak{F}$ and is an upper bound for $\mathfrak{J}$. By Zorn's Lemma, $\mathfrak{F}$ has a maximal element, say $\{x_\gamma\}_{\gamma \in \Xi}$. If $\mathcal{H}_0 = \vee[\mathfrak{M}x_\gamma] \neq \mathcal{H}$, then we can choose a unit vector $y \in \mathcal{H}_0$. From the comment in the first paragraph, we deduce that $\{x_\gamma\}_\gamma \cup \{y\} \in \mathfrak{F}$ and is greater than $\{x_\gamma\}_\gamma$, contradicting the maximality of $\{x_\gamma\}_\gamma$. Thus $\vee[\mathfrak{M}x_\gamma] = \mathcal{H}$.

Since $\mathfrak{M}$ is a masa, $I \in \mathfrak{M}$ and so $x_\gamma \in [\mathfrak{M}x_\gamma]$ for each γ and thus $\dim [\mathfrak{M}x_\gamma] \geq 1$. Since $\dim \mathcal{H} = \aleph_0 \geq \sum_\gamma \dim [\mathfrak{M}x_\gamma]$, it follows that the cardinality of Ξ is at most $\aleph_0$. Write $\Xi = \{n\}_{n=1}^m$, $m \leq \aleph_0$. Let $x = \sum_{n<m+1} x_n/n$. (The index set of the sum is merely a device to allow us to handle the cases where Ξ is infinite and where Ξ is finite simultaneously.) For each n, the orthogonal projection P_n onto $[\mathfrak{M}x_n]$ lies in $\mathfrak{M}' = \mathfrak{M}$, so that $[\mathfrak{M}x_n] = [\mathfrak{M}P_n x] \subseteq [\mathfrak{M}x]$ for all $n < m+1$. Thus $\mathcal{H} = \vee_{n<m+1}[\mathfrak{M}x_n] \subseteq [\mathfrak{M}x] \subseteq \mathcal{H}$, and x is a cyclic vector for $\mathfrak{M}$.

□

13.14. Corollary. *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be an abelian C^*-algebra. Then $\mathbb{A}$ has a separating vector.*

Proof. By Proposition 13.12, $\mathbb{A} \subseteq \mathfrak{M}$ for some masa $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$. By Theorem 13.13, $\mathfrak{M}$ has a cyclic vector x, and x is separating for $\mathfrak{M}$ by Lemma 13.9. Finally, if x is separating for $\mathfrak{M}$, then trivially x is also separating for $\mathbb{A}$.

□

13.15. Definition. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq \mathcal{H}$ be a closed subspace of $\mathcal{H}$. Let $P_{\mathcal{M}}$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. Given $T \in \mathcal{B}(\mathcal{H})$, we define the* ***compression*** *of T onto $\mathcal{M}$ to be the map*

$$P_{\mathcal{M}} T|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M}).$$

It is clear that $\|P_{\mathcal{M}} T|_{\mathcal{M}}\| \le \|T\|$.

Let $\mathcal{H}$ be a separable Hilbert space, $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be a C^*-algebra, and $x \in \mathcal{H}$. Denote by $\mathcal{H}_x$ the space $\overline{\mathbb{A}x}$, and for $Z \in \mathcal{B}(\mathcal{H})$, denote by Z_x the compression of Z to $\mathcal{H}_x$. Note that $\mathcal{H}_x$ is reducing for $\mathbb{A}$.

13.16. Proposition. *Let $\mathcal{H}$ be a separable Hilbert space, $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be a C^*-algebra, and $x \in \mathcal{H}$ be a separating vector for $\mathbb{A}$. The map*

$$\begin{array}{rccc} \Phi: & \mathbb{A} & \to & \mathcal{B}(\mathcal{H}_x) \\ & T & \mapsto & T_x \end{array}$$

is an isometric *-isomorphism of $\mathbb{A}$ onto $\operatorname{ran}\Phi$. *Moreover, $\sigma(T) = \sigma(T_x)$ for all $T \in \mathbb{A}$.*

Proof. Recall from Lemma 12.22 that the orthogonal projection P_x onto $\mathcal{H}_x$ lies in $\mathbb{A}'$, and $x \in \operatorname{ran} P_x$. From this the fact that Φ is a *-homomorphism easily follows.

Suppose $0 \ne T \in \mathbb{A}$. Then $T_x(x) = TP_x(x) = Tx \ne 0$, as x is separating for $\mathbb{A}$. Thus $\ker \Phi = 0$, and so Φ is an isometric map as well. It follows that $\Phi(\mathbb{A})$ is complete, and thus a C^*-algebra. In particular,

$$\sigma(T) = \sigma_{\mathbb{A}}(T) = \sigma_{\Phi(\mathbb{A})}(\Phi(T)) = \sigma_{\mathcal{B}(\mathcal{H}_x)}(T_x) = \sigma(T_x),$$

completing the proof.

□

13.17. Theorem. [The Spectral Theorem for normal operators.] *Let $\mathcal{H}$ be a separable Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal. Then there exists a finite, positive, regular Borel measure μ with support equal to $\sigma(N)$ and an isometric *-isomorphism*

$$\Gamma^* : W^*(N) \to \mathfrak{M}^\infty(\sigma(N), \mu)$$

which extends the Gelfand map $\Gamma_m : C^*(N) \to \mathfrak{M}_{\mathcal{C}(\sigma(N))}$ defined by $\Gamma_m(f(N)) = M_f$.

Moreover, the measure μ is unique up to mutual absolute continuity, while Γ_m^* and $M^\infty(\sigma(N), \mu)$ are unique.

Proof. By Corollary 13.14, $W^*(N)$ an abelian C^*-algebra implies that $W^*(N)$ has a separating vector x, which we may assume has norm one. Let $\mathcal{H}_x = [W^*(N)x]$, and consider (using the same notation as before)

$$\begin{array}{rccc} \Phi: & W^*(N) & \to & \mathcal{B}(\mathcal{H}_x) \\ & T & \mapsto & T_x. \end{array}$$

By Proposition **??**, Φ is an isometric *-isomorphism, and $\sigma(T_x) = \sigma(T)$ for all $T \in W^*(N)$ - in particular, $\sigma(N_x) = \sigma(N)$. By identifying $W^*(N)$ with its range $\Phi(W^*(N))$, we may assume that $W^*(N)$ already has a cyclic vector. But $\overline{C^*(N)}^{\mathrm{WOT}} = W^*(N)$, and so if $T \in W^*(N)$, then there exists a net $(T_\alpha)_\alpha \in C^*(N)$ so that $Tx = \lim_\alpha T_\alpha x \in [C^*(N)x]$. It follows that $\mathcal{H}_x = [C^*(N)x]$, so that x is also a cyclic vector for $C^*(N)$.

By the Cyclic Version of the Spectral Theorem for normal operators, Theorem 13.10, we obtain a finite, positive, regular Borel measure μ with support $\sigma(N)$ so that $\Gamma_m^* : W^*(N) \to \mathcal{B}(\mathcal{H}_x)$ is an isometric *-isomorphism. Also, $\operatorname{ran}\Gamma_m^* = \mathfrak{M}^\infty(\sigma(N),\mu)$. From the proof of that Theorem, we saw that Γ_m^* is WOT-WOT continuous, and so $\Gamma^* = \Gamma_m^* \circ \Phi$ is WOT-WOT continuous as well. Also, Γ^* extends the Gelfand map because Γ_m^* does.

Finally, Φ surjective implies that $\operatorname{ran}\Gamma^* = \mathfrak{M}^\infty(\sigma(N),\mu)$. Uniqueness follows as before.

□

13.18. Remarks.

(i) Let $\mathcal{H}$ be a separable Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal. The map Γ^* from Theorem **??** now defines L^∞-functional calculus for N. That is, for each $f \in L^\infty(X,\mu)$, we can define

$$f(N) := (\Gamma^*)^{-1}(M_f) \in W^*(N).$$

If f is continuous, then this agrees with the previously defined $f(N) \in C^*(N)$.

Of great importance is the fact that if $\iota(z) = z$ for all $z \in \sigma(N)$ is the identity function, then $\iota(N) = N$; in other words, the map Γ^* sends N to the operator M_ι, which corresponds to the L^∞-function ι.

(ii) From basic measure theory, any element of $L^\infty(X,\mu)$ is a limit of simple functions. Let $g = \sum_{k=1}^n \alpha_k \chi_{E_k}$, where E_k are (without loss of generality, mutually disjoint) measurable subsets of X. Then

$$g(N) = \sum_{k=1}^n \alpha_k \chi_{E_k}(N).$$

But $\chi_{E_k} = \chi_{E_k}^* = \chi_{E_k}^2$ is a self-adjoint idempotent in $L^\infty(X,\mu)$, and thus $Q_k := \chi_{E_k}(N)$ is a self-adjoint projection in $W^*(N)$. Let $f \in L^\infty(X,\mu)$, $\varepsilon > 0$, and choose a simple function g_ε such that

$$\|f - g_\varepsilon\|_\infty < \varepsilon.$$

Then

$$\|f(N) - g_\varepsilon(N)\| < \varepsilon.$$

In particular, this shows that every element of $W^*(N)$ can be approximated by linear combinations of projections, each of which lies in $W^*(N)$. The moral of the story is that von Neumann algebras have lots and lots of projections.

(iii) Another formulation of the Spectral Theorem for normal operators deals with **projection-valued spectral measures**. The rough idea is to find a function

$$\begin{array}{rccc} E: & \Omega & \to & \mathcal{B}(\mathcal{H}) \\ & Y & \mapsto & E_Y \end{array}$$

defined on the set Ω of Borel subsets of $\sigma(N)$ such that for each $Y \in \Omega$, $E_Y \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection in $\mathcal{B}(\mathcal{H})$ and for each $x, y \in \mathcal{H}$, the map $\mu_{x,y} : Y \mapsto \langle E_Y x, y \rangle$ defines a complex-valued Borel measure on $\sigma(N)$. It is then possible to define $\int_{\sigma(N)} f dE$ for each $f \in L^\infty(\sigma(N), \mu)$, and $N = \int_{\sigma(N)} \iota dE$, where $\iota(z) = z$ is the identity function on $\sigma(N)$. We refer the interested reader to Murphy's treatment [**38**] of this.

Kadison's Transitivity Theorem

13.19. In Chapter 8, we proved Jacobson's Density Theorem, which shows that if $\mathcal{A}$ is a Banach algebra, $\mathfrak{X}$ is a Banach space and $\varrho : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ is a continuous, algebraically irreducible representation of $\mathcal{A}$ on $\mathfrak{X}$, then for all choices of $N \geq 1$ vectors $y_1, y_2, \dots, y_N \in \mathfrak{X}$ and any choice of N linearly independent vectors $x_1, x_2, \dots, x_N \in \mathfrak{X}$, there exists $a \in \mathcal{A}$ such that $\varrho(a)x_n = y_n$, $1 \leq n \leq N$. We also say that $\varrho(\mathcal{A})$ is N-transitive (for each N).

Our present goal is to exhibit of theorem of R.V. Kadison [**30**] from 1957, who showed that if $\mathbb{A}$ is a C^*-algebra, and if $\varrho : \mathbb{A} \to \mathcal{B}(\mathcal{H})$ is a continuous, *topologically* irreducible representation of $\mathbb{A}$ onto a Hilbert space $\mathcal{H}$, then ϱ is *algebraically* irreducible, and as a consequence of Jacobson's Density Theorem, it is N-transitive for all $N \geq 1$.

13.20. Lemma. *Let $\mathcal{H}$ be a Hilbert space and $\mathbb{F} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint set. The following are equivalent.*

(i) $\mathbb{F}$ *acts **topologically irreducibly** on $\mathcal{H}$; that is, the only invariant (closed) subspaces of $\mathcal{H}$ which are invariant under $\mathbb{F}$ are $\{0\}$ and $\mathcal{H}$ itself.*

(ii) $\mathbb{F}' = \mathbb{C}I$.

Proof.

(i) implies (ii). Suppose that $\mathbb{F}$ acts topologically transitively. Since $\mathbb{F}$ is self-adjoint, $\mathbb{F}'$ is a von Neumann algebra, by Proposition 12.17. From this it is easy to see that it suffices to prove that every positive element of $\mathbb{F}'$ is scalar.

Let $0 \leq H \in \mathbb{F}'$. If $\sigma(H)$ is disconnected, say $\sigma(H) = \sigma_0 \cup \sigma_1$ for two non-empty, relatively closed sets σ_0 and σ_1 of $\sigma(H)$, then the characteristic functions χ_{σ_k} are non-sclar continuous functions on $\sigma(H)$, $k = 1, 2$. It follows that $P_0 := \chi_{\sigma_0}$ is a non-trivial projection in $C^*(H) \subseteq \mathbb{F}'$, and thus

$\operatorname{ran} P_0$ is a non-trivial invariant subspace for $\mathbb{F}$, a contradiction. Thus $\sigma(H)$ must be connected.

If $\sigma(H)$ is a singleton set, say $\sigma(H) = \{\alpha\}$, then $H = \alpha I$ is scalar operator and we are done.

Thus we may assume that $\sigma(H)$ is the interval $[\delta, \|H\|]$ for some appropriate $0 \le \delta < \|H\|$. By the Spectral Theorem for normal operators (Theorem **??**), we can find a finite, positive, regular Borel measure μ *with support equal to* $\sigma(H)$ and an isometric *-isomorphism

$$\Gamma^* : W^*(H) \to \mathfrak{M}^\infty(\sigma(H), \mu).$$

Choose $\eta \in (\delta, \|H\|)$, and let $Q := \chi_{[\eta, \|H\|]}(H)$. Then Q is a non-trivial projection in $W^*(H) \subseteq \mathbb{F}'$. As before, this implies that $\operatorname{ran} Q$ is a non-trivial invariant subspace for $\mathbb{F}$, a contradiction.

We conclude that every positive operator in $\mathbb{F}'$ is scalar, whence every hermitian and therefore every operator in $\mathbb{F}'$ is scalar as well.

(ii) implies (i). Suppose that $\mathbb{F}' = \mathbb{C}I$. Let $\{0\} \ne \mathcal{M} \subseteq \mathcal{H}$ be a closed, invariant subspace for $\mathbb{F}$. Since $\mathbb{F} = \mathbb{F}^*$, $\mathcal{M}$ is reducible for $\mathbb{F}$, and thus the projection P of $\mathcal{H}$ onto $\mathcal{M}$ lies in $\mathbb{F}'$. This implies that $P = I$, and thus $\mathcal{M} = \mathcal{H}$. In other words, $\mathbb{F}$ acts topologically irreducibly upon $\mathcal{H}$.

□

13.21. The above result, combined with Kaplansky's Density Theorem, is the key to Kadison's Transitivity Theorem below.

If $\mathbb{A}$ is a C^*-algebra which acts topologically irreducibly on a Hilbert space $\mathcal{H}$, then by Lemma **??**, $\mathbb{A}' = \mathbb{C}I$. It therefore follows that $\mathbb{A}'' = \mathcal{B}(\mathcal{H})$. By Kaplansky's Density Theorem, *any* operator $T \in \mathcal{B}(\mathcal{H})$ can be approximated in the SOT by an element of $\mathbb{A}$ *of the same norm.* In particular, if x and y are two non-zero vectors in $\mathcal{H}$ with $\|x\| = 1$, then $T = y \otimes x^*$ is an operator of norm $\|T\| = \|y\|$ and as such, given $\varepsilon > 0$, by Kaplansky's Density Theorem there exists $A \in \mathbb{A}$ such that $\|A\| \le \|y\|$ and

$$\|Ax - y\| = \|Ax - Tx\| < \varepsilon.$$

Let us refer to this as the "approximation technique" in the proof below.

13.22. Theorem. [Kadison's Transitivity Theorem.] *If the C^*-algebra $\mathbb{A}$ acts topologically irreducibly on a Hilbert space $\mathcal{H}$, then $\mathbb{A}$ acts algebraically irreducibly on $\mathcal{H}$.*

Proof. Let $x, y \in \mathcal{H}$ be two non-zero vectors. We must find an operator $B \in \mathbb{A}$ such that $Bx = y$. We may suppose (without loss of generality) that $\|x\| = 1$.

Let $\varepsilon > 0$. By the approximation technique from Paragraph **??**, there exists $A_0 \in \mathbb{A}$ with $\|A_0\| = \|y\|$ such that $\|A_0 x - y\| < \frac{\varepsilon}{2}$.

Set $y_1 = A_0 x - y$, and note that $\|y_1\| < \frac{\varepsilon}{2^1}$. By a second application of the approximation technique, there exists $A_1 \in \mathbb{A}$ with $\|A_1\| \le \frac{\varepsilon}{2}$ such that $\|A_1 x - y_1\| < \frac{\varepsilon}{4}$. Let $y_2 = y_1 - A_1 x = y - (A_0 x + A_1)x$ and note that $\|y_2\| < \frac{\varepsilon}{2^2}$.

In general, given $A_0, A_1, \ldots, A_{N-1} \in \mathbb{A}$ and $y_1, y_2, \ldots, y_N$ as above with $\|y_N\| < \frac{\varepsilon}{2^N}$, by the approximation technique we can find an element $A_N \in \mathbb{A}$ such that $\|A_N\| \leq \|y_N\| < \frac{\varepsilon}{2^N}$ and

$$\|y_N - A_N x\| < \frac{\varepsilon}{2^{N+1}}.$$

We set $y_{N+1} = y_N - A_N x = y - (A_0 + A_1 + \cdots A_N)x$.

Then

$$\sum_{n=0}^{\infty} \|A_n\| \leq \|A_0\| + \sum_{n=1}^{\infty} \|A_n\| \leq \|A_0\| + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} < \|y\| + \varepsilon < \infty,$$

and thus $B := \sum_{n=0}^{\infty} A_n \in \mathbb{A}$. Moreover,

$$Bx = \sum_{n=0}^{\infty} A_n x = \lim_{N \to \infty} (A_0 + A_1 + \cdots + A_N)x = \lim_{N \to \infty} y - y_{N+1} = y.$$

□

It is sometimes the next result which is explicitly referred to as Kadison's Transitivity Theorem.

13.23. Corollary. *If the C^*-algebra $\mathbb{A}$ acts topologically irreducibly on a Hilbert space $\mathcal{H}$, then $\mathbb{A}$ is N-transitive for all $N \geq 1$.*

Proof. This is an now an immediate consequence of Theorem 13.23 and the Jacobson Density Theorem (Theorem 8.15), using the inclusion representation

$$\begin{array}{rccc} \varrho : & \mathbb{A} & \to & \mathcal{B}(\mathcal{H}) \\ & A & \mapsto & A. \end{array}$$

□

Supplementary Examples

S13.1. In progress.

Appendix

A13.1. For those who prefer to avoid Lusin's Theorem, , we provide second, more elementary proof of the fact that the continuous functions on a compact, Hausdorff measure space (X, μ) are weak*-dense in the space $L^\infty(X, \mu)$.

Lemma. *Suppose X is a compact, Hausdorff space and that μ is a positive, regular Borel measure on X with $\mu(X) = 1$. If X can be written as the disjoint union of measurable sets $\{E_j\}_{j=1}^n$, $g \in L^1(X, \mu)$ and $\|g\|_1 = 1$, then for all $\varepsilon > 0$ there exist compact sets $K_1, K_2, ..., K_n$ such that $K_j \subseteq E_j$ and with $K = \cup_{j=1}^n K_j$,*

$$\int_{X \setminus K} |g| d\mu < \varepsilon.$$

Proof. For each $1 \le j \le n$, let $E_j(m) = \{x \in E_j : m - 1 \le |g(x)| < m\}, m \ge 1$. Then $E_j(m)$ is measurable for all m, j and

$$1 = \|g\|_1 = \sum_{j=1}^n \sum_m \int_{E_j(m)} |g| d\mu.$$

Let $\varepsilon > 0$. Then there exists $N > 0$ so that for each $1 \le j \le n$,

$$\sum_{m=N+1}^\infty \int_{E_j(m)} |g| d\mu < \varepsilon/2n.$$

For each $1 \le j \le n$, $1 \le m \le N$, the regularity of μ allows us to find a compact set $K_j(m) \subseteq E_j(m)$ so that $\mu(E_j(m) \setminus K_j(m)) < \varepsilon/2N^2 n$.

Let $K_j = \cup_{m=1}^N K_j(m)$. Since each $K_m(j)$ is compact, so is K_j. It follows that if $K = \cup_{j=1}^n K_j$, then

$$\begin{aligned}
\int_{X \setminus K} |g| d\mu &= \sum_{j=1}^n \sum_{m=N+1}^\infty \int_{E_j(m)} |g| d\mu + \sum_{j=1}^n \sum_{m=1}^N \int_{E_j(m) \setminus K_j(m)} |g| d\mu \\
&\le \sum_{j=1}^n \varepsilon/2n + \sum_{j=1}^n \sum_{m=1}^N (\varepsilon/2nN^2) \\
&< \varepsilon/2 + \sum_{j=1}^n \sum_{m=1}^N (\varepsilon/2nN^2) N \\
&= \varepsilon/2 + \sum_{j=1}^n \varepsilon/2n = \varepsilon.
\end{aligned}$$

□

Remark: If L is compact and $K \subseteq L$, then $\int_{X \setminus L} |g| d\mu \le \int_{X \setminus K} |g| d\mu < \varepsilon$.

A13.2. Proposition. *Let X be a compact, Hausdorff set and μ be a positive, regular Borel measure on X with $\mu(X) = 1$. Then the unit ball $(\mathcal{C}(X))_1$ of $\mathcal{C}(X)$ is weak*-dense in $(L^\infty(X,\mu))_1$, and as such, $\mathcal{C}(X)$ is weak*-dense in $L^\infty(X,\mu)$.*

Proof. First observe that the simple functions in $(L^\infty(X,\mu))_1$ are norm dense in $(L^\infty(X,\mu))_1$, and hence they are weak*-dense. As such, it suffices to prove that each simple function can be approximated in the weak*-topology on $L^\infty(X,\mu)$ by continuous functions.

Consider $\varphi(x) = \sum_{j=1}^n a_j \chi_{E_j}$, where E_j is measurable, $1 \le j \le n$ and $\cup_{j=1}^n E_j = X$. (We can suppose without loss of generality that the E_j's are also disjoint. Suppose furthermore that $\|\varphi\|_\infty \le 1$. Let $K_j \subseteq E_j$ be a compact set for all $1 \le j \le n$. Then $K = \cup_{j=1}^n K_j$ is compact, and so by Tietze's Extension Theorem we can find a function $f_K \in \mathcal{C}(X)$ so that $f_K(x) = a_j$ if $x \in K_j$ and $0 \le f_K \le 1$.

Let $\Lambda = \{K : K = \cup_{j=1}^n K_j, K_j \subseteq E_j \text{compact}\}$, and partially order Λ by inclusion, so that $K_1 \le K_2$ if $K_1 \subseteq K_2$. Then Λ is a directed set and $(f_K)_{K\in\Lambda}$ is a net in $\mathcal{C}(X)$. Let $\varepsilon > 0$. For $g \in L^1(X,\mu)$, by Lemma 13.21 and the remark which follows it, we can find $K_0 \in \Lambda$ so that $K \ge K_0$ implies $\int_{X\setminus K} |g| d\mu < \varepsilon/2$. But then $K \ge K_0$ implies

$$\begin{aligned} \left|\int_X (f_K - \varphi) g d\mu\right| &\le \sum_{j=1}^n \int_{E_j\setminus K_j} |f_K - \varphi|\,|g| d\mu \\ &\le 2\sum_{j=1}^n \int_{E_j\setminus K_j} |g| d\mu \\ &= 2\int_{X\setminus K} |g| d\mu < \varepsilon, \end{aligned}$$

and so weak*-$\lim_K f_K = \varphi$.

Thus $(\mathcal{C}(X))_1$ is weak*-dense in $(L^\infty(X,\mu))_1$. The second statement is straightforward.

□

Exercises for Chapter 13

Question 1. In progress.

Appendix A: The essential spectrum

A.1. Definition. *Given an operator $T \in \mathcal{B}(\mathcal{H})$, we define the* ***essential spectrum*** *of T to be the spectrum of the image $\pi(T)$ in the Calkin algebra $\mathcal{A}(\mathcal{H})$.*

In this note, we wish to prove a result due to Putnam and Schechter , namely:

A.2. Theorem. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\lambda \in \partial\sigma(T)$. Then either λ is isolated in $\sigma(T)$, or $\lambda \in \sigma_e(T)$.*

The proof below uses a description of the *singular points* of the semi-Fredholm domain of T, due to C. Apostol [**1**].

A.3. Definition. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then the* ***semi-Fredholm domain*** *$\rho_{\mathrm{sF}}(T)$ of T is the set of all complex numbers λ such that $\lambda 1 - \pi(T)$ is either left or right invertible in the Calkin algebra.*

*If $\mu \in \mathbb{C}$, then μ is called a $(T)-$**singular point** if the function*

$$\lambda \mapsto P_{\ker(T-\lambda)}$$

*is discontinuous at μ. Otherwise, μ is said to be $(T)-$**regular** .*

If $\mu \in \rho_{\mathrm{sF}}(T)$ and μ is singular (resp. μ is regular), then we write $\mu \in \rho^{\mathrm{s}}_{\mathrm{sF}}(T)$ (resp. $\rho^{\mathrm{r}}_{\mathrm{sF}}(T)$).

A.4. Lemma. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that μ is a regular point of the semi-Fredholm domain of T. Then*

$$\ker(T-\mu)^* \subseteq (\ \textit{span}\ \{\ker(T-\lambda) : \lambda \in \mathbb{C}\})^{\perp}.$$

Proof. First note that $\operatorname{ran}(T-\mu) \supseteq \ker(T-\lambda)$ for all $\lambda \neq \mu$. For if $x \in \ker(T-\lambda)$ and $\lambda \neq \mu$, then $(T-\mu)x = (\lambda-\mu)x$ and so $x \in \operatorname{ran}(T-\mu)$.

Also $\overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \in \mathbb{C}\} = \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \neq \mu\}$. This follows from the regularity of μ. Basically, we must show that $\ker(T-\mu) \subseteq \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \neq \mu\}$. But if $x \in (T-\mu)$ and $\|x\| = 1$, then again by the regularity of μ, for any $\epsilon > 0$ we can find $\lambda_n \to \mu$ such that $\|P_{\ker(T-\lambda_n)} - P_{\ker(T-\mu)}\| < \epsilon$.

But then

$$\begin{aligned} \epsilon &> \|P_{\ker(T-\lambda_n)}x - P_{\ker(T-\mu)}x\| \\ &= \|P_{\ker(T-\lambda_n)}x - x\|, \end{aligned}$$

and so $x \in \ker(T-\mu) \subseteq \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \neq \mu\}$.

Combining these two arguments,

$$\begin{aligned} \ker(T-\mu)^* &= (\operatorname{ran}(T-\mu))^\perp \\ &\subseteq \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \in \mathbb{C}\}. \end{aligned}$$

$\square$

A.5. Theorem. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

(i) *$\rho^{\mathrm{r}}_{\mathrm{sF}}(T)$ is open;*
(ii) *$\rho^{\mathrm{r}}_{\mathrm{sF}}(T) = \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*)^* := \{\overline{\lambda} : \lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T)\}$;*
(iii) *$\rho^{\mathrm{r}}_{\mathrm{sF}}(T)$ has no accumulation points in $\rho^{\mathrm{s}}_{\mathrm{sF}}(T)$.*

Proof.

(i) Let $\mu \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T)$ and put $Y = \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \in \mathbb{C}\}$. We claim that $TY \subseteq Y$.

Consider $y \in \operatorname{span}\{\ker(T-\lambda) : y \in \mathbb{C}\}$, say $y = \sum_{n=1}^m y_n$ with each $y_n \in \ker(T-\lambda_n)$. Then $Ty = \sum_{n=1}^m Ty_n = \sum_{n=1}^m \lambda_n y_n$ which lies in $\operatorname{span}\{\ker(T-\lambda_n) : 1 \le n \le m\}$. By the continuity of T, we have $TY \subseteq Y$. Let $T_Y = T|_Y$.

Since $\ker(T-\mu) \subseteq Y$, $\operatorname{ran}(T_Y-\mu)$ is closed. To see this, suppose that $\{x_n\}$ is a sequence in $\operatorname{ran}(T_Y-\mu)$ such that $\{x_n\}$ converges to $x \in Y$. Then there exists a sequence $\{y_n\} \subseteq Y$ such that $(T_Y-\mu)y_n = x_n$.

In fact, since $\ker(T-\mu) \subseteq Y$, we can let

$$z_n = P_{Y \ominus \ker(T-\mu)} y_n$$

and then

$$(T-\mu)z_n = (T_Y-\mu)z_n = (T_Y-\mu)y_n = x_n$$

for all $n \ge 1$.

Since $\operatorname{ran}(T-\mu)$ is closed, (i.e. $\mu \in \rho_{\mathrm{sF}}$), there exists $z \in \mathcal{H}$ such that $(T-\mu)z = x$. But $(T-\mu)$ is bounded below on $(\ker(T-\mu))^\perp$, and therefore $(T_Y-\mu)$ is bounded below on $Y \ominus \ker(T-\lambda)$. From this we get a $\delta > 0$ such that

$$\begin{aligned} \|x_n - x\| &= \|(T-\mu)z_n - (T-\mu)z\| \\ &\ge \delta\|z_n - z\| \end{aligned}$$

for all $n \ge 1$.

But then $z = \lim_{n\to\infty} z_n$, and so $z \in Y$. This gives us $(T_Y-\mu)z = (T-\mu)z = x$, and so $x \in \operatorname{ran}(T_Y-\mu)$, i.e. $\operatorname{ran}(T_Y-\mu)$ is closed.

We next claim that

$$\begin{aligned}(T_Y - \mu)Y &\supseteq \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \neq \mu\} \\ &= \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \in \mathbb{C}\} \\ &= Y.\end{aligned}$$

The first equality we saw in the previous Lemma, while the second *is* the definition of Y. As for the containment, let $y \in \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \neq \mu\}$, say $y = \sum_{n=1}^{m} y_n$. Then

$$\begin{aligned}(T_Y - \mu)y &= \sum_{n=1}^{m}(T_Y - \mu)y_n \\ &= \sum_{n=1}^{m}(\lambda_n - \mu)y_n\end{aligned}$$

where $y_n \in \ker(T - \lambda_n)$. Thus if $z = \sum_{n=1}^{m}(\lambda_n - \mu)^{-1}y_n$, we have $z \in Y$ and $(T_Y - \mu)z = y$. Since $\operatorname{ran}(T_Y - \mu)$ is closed, the desired conclusion follows.

Since $(T_Y - \mu)$ is onto, we have $\mu \in \rho_{\mathrm{r}}(T_Y)$, the right resolvent set of T_Y. For $A \in \mathcal{B}(\mathcal{H})$, define the *right resolvent* as

$$R_{\mathrm{r}}(\lambda; A) = (\lambda - A)^*[(\lambda - A)(\lambda - A)^*]^{-1}$$

so that

$$P_{\ker(A-\lambda)} = I - R_{\mathrm{r}}(\lambda; A)(\lambda - A)$$

for all $\lambda \in \rho_{\mathrm{r}}(A)$.

Since $\ker(T_Y - \mu) = \ker(T - \mu)$ for all $\lambda \in \mathbb{C}$, we infer that the map $\lambda \mapsto P_{\ker(T-\mu)}$ is continuous in an open neighbourhood G_μ of μ, as $\rho_{\mathrm{r}}(T)$ is open. Thus μ is an interior point of $\rho^{\mathrm{r}}_{\mathrm{sF}}(T)$, and so $\rho^{\mathrm{r}}_{\mathrm{sF}}(T)$ is open.

(ii) $\rho^{\mathrm{r}}_{\mathrm{sF}}(T) = \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*)^*$

Let $Z = (\overline{\operatorname{span}}\{\ker(T-\lambda)^* : \lambda \in \mathbb{C}\})^\perp$. Then the proof of (i) shows that $\rho^{\mathrm{r}}_{\mathrm{sF}}(T) \subseteq \rho_{\mathrm{r}}(T_Y)$ and $\rho^{\mathrm{r}}_{\mathrm{sF}}(T^*) \subseteq \rho_{\mathrm{r}}(T^*_{Z^\perp})$.

We now claim that $\rho^{\mathrm{r}}_{\mathrm{sF}}(T) \subseteq \rho_{\mathrm{l}}(T_{Y^\perp}) = \rho_{\mathrm{r}}(T^*_{Y^\perp})^*$. For suppose that $\lambda \in \rho_{\mathrm{r}}(T_Y)$. If $w \in \ker(T_{Y^\perp} - \lambda)$, then

$$(T-\lambda)\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} T_y - \lambda & T_Z \\ 0 & T_{Y^\perp} - \lambda \end{bmatrix}\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} T_Z w \\ 0 \end{bmatrix}.$$

Since $T_Y - \lambda$ is right invertible, $(T_Y - \lambda)R = I$ for some $R \in \mathcal{B}(Y)$ and so $(T_Y - \lambda)R(-T_Z w) = (-T_Z w)$. Letting $v = -RT_Z w$, we have

$$(T-\lambda)\begin{bmatrix} v \\ w \end{bmatrix} \text{ and so } \begin{bmatrix} v \\ w \end{bmatrix} \in \ker(T-\lambda) \subseteq Y.$$

Thus $w = 0$. But then $T_{Y^\perp}$ is injective.

If $\lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T)$, then $\operatorname{ran}(T_{Y^\perp} - \lambda) = P_{Y^\perp}(\operatorname{ran}(T-\lambda))$ and since $\operatorname{ran}(T-\lambda)$ is closed, so is $\operatorname{ran}(T_{Y^\perp} - \lambda)$. Thus $\lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T)$ implies that $\ker(T_{Y^\perp} - \lambda) = \{0\}$ and $\operatorname{ran}(T_{Y^\perp} - \lambda)$ is closed, so that $\lambda \in \rho_{\mathrm{l}}(T_{Y^\perp}) = \rho_{\mathrm{r}}(T^*_{Y^\perp})^*$.

Similarly, $\rho^{\mathrm{r}}_{\mathrm{sF}}(T^*) \subseteq \rho_{\mathrm{r}}(T_Z)^*$.

Since the maps

$$\begin{aligned} \lambda &\mapsto P_{\ker(T^*_{Y^\perp} - \overline{\lambda})} \lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T) \\ \lambda &\mapsto P_{\ker(T^*_{Z} - \overline{\lambda})} \lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*) \end{aligned}$$

are continuous, and by the first Lemma we have

$$\begin{aligned} \ker(T-\lambda)^* &= \ker(T^*_{Y^\perp} - \overline{\lambda}) \text{ for all } \lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T) \\ \ker(T-\overline{\lambda}) &= \ker(T_Z - \overline{\lambda}) \text{ for all } \lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*), \end{aligned}$$

we infer that

$$\begin{aligned} \rho^{\mathrm{r}}_{\mathrm{sF}}(T)^* &\subseteq \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*) \\ \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*) &\subseteq \rho^{\mathrm{r}}_{\mathrm{sF}}(T)^* \end{aligned}$$

so that $\rho^{\mathrm{r}}_{\mathrm{sF}}(T) = \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*)^*$.

(iii) $\rho^{\mathrm{s}}_{\mathrm{sF}}(T)$ has no accumulation points in $\rho_{\mathrm{sF}}(T)$.

Let $z \in \rho^{\mathrm{s}}_{\mathrm{sF}}(T)$. Then we may assume that $z \in \rho_{\mathrm{le}}(T)$, for otherwise, by (2), we may consider $\overline{z}$ and T^*. Put $Y_0 = \overline{\mathrm{span}}\{\ker(T-\lambda) : \lambda \neq z\}$. As $T_{Y_0} - z$ has dense range (the proof follows as from (1)), and since $\mathrm{ran}\,(T_{Y_0})$ is closed (i.e. $z \in \rho_{\mathrm{le}}(T)$), we get $z \in \rho_{\mathrm{r}}(T_{Y_0})$.

Now for $\lambda \neq z$, we have $\ker(T-\lambda) = \ker(T_{Y_0} - \lambda)$. Since the map

$$\lambda \mapsto R_{\mathrm{r}}(\lambda; T_{Y_0}) \qquad \lambda \in \rho_{\mathrm{r}}(T_{Y_0})$$

is continuous, we have that

$$\lambda \mapsto P_{\ker(T_{Y_0} - \lambda)} = I - R_{\mathrm{r}}(\lambda; T_{Y_0})(\lambda - T_{Y_0}) \qquad \lambda \in \rho_{\mathrm{r}}(T_{Y_0})$$

is continous, and so

$$\lambda \mapsto P_{\ker(T-\lambda)}$$

is continous in some punctured neighbourhood of z. Since $\rho_{\mathrm{sF}}(T)$ is open, we have that z is an isolated point in $\rho^{\mathrm{s}}_{\mathrm{sF}}(T)$.

Finally, suppose $\rho^{\mathrm{s}}_{\mathrm{sF}}(T)$ has an accumulation point $\mu \in \rho_{\mathrm{sF}}(T)$. Then by (1), $\mu \in \rho^{\mathrm{s}}_{\mathrm{sF}}(T)$ and μ is isolated, a contradiction. This concludes the proof.

□

A.6. Proposition. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\sigma(T) = \sigma_{\mathrm{e}}(T) \cup \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{p}}(T^*)^*,$$

where $\sigma_{\mathrm{p}}(T^*)^* = \{\overline{\lambda} : \lambda \in \sigma_{\mathrm{p}}(T^*)\}$.

Proof. Suppose $\lambda \notin \cup \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{p}}(T^*)^*$. Then $\mathrm{nul}(T-\lambda) = \mathrm{nul}(T-\lambda)^* = 0$.

Thus $(T-\lambda)$ is injective and has dense range. If $\lambda \notin \sigma_{\mathrm{e}}$, then $(T-\lambda)$ is Fredholm and thus $\mathrm{ran}\,(T-\lambda)$ is closed. But then $(T-\lambda)$ is bijective and hence $\lambda \notin \sigma(T)$. Thus $\sigma(T) \subseteq \sigma_{\mathrm{e}}(T) \cup \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{p}}(T^*)^*$. The other inclusion is obvious.

$\square$

A.7. Theorem. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose $\lambda \in \partial\sigma(T)$. Then either λ is isolated or $\lambda \in \sigma_e(T)$.*

Proof. Suppose $\lambda \notin \sigma_e(T)$. Then by the above Proposition, we may assume that $\lambda \in \sigma_p(T)$ (otherwise consider $\overline{\lambda}$ and T^*. Since $\lambda \in \partial\sigma(T)$, we can find a sequence $\{\lambda\}_n \subseteq \rho(T)$ such that $\lambda = \lim_{n\to\infty} \lambda_n$.

Since $\ker(T - \lambda_n) = \{0\}$ for all $n \ge 1$ while $\ker(T - \lambda) \ne \{0\}$, we conclude that $\lambda \in \rho_{\mathrm{sF}}^{\mathrm{s}}(T)$. Since $\rho_{\mathrm{sF}}^{\mathrm{s}}(T)$ has no accumulation points in $\rho_{\mathrm{sF}}(T)$, and since $\lambda \notin \sigma_e(T)$, we conclude that λ is isolated in $\sigma(T)$.

$\square$

A.8. Corollary. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma(T) = \sigma_e(T) \cup \Omega$, where Ω consists of some bounded components of the Fredholm domain of T and a sequence of isolated points in the Fredholm domain which converge to $\sigma_e(T)$.*

Appendix B. von Neumann algebras as dual spaces

Let $\mathcal{H}$ be a separable Hilbert space. In this note we show the von Neumann algebras are precisely the class of C*-algebras of $\mathcal{B}(\mathcal{H})$ which can be identified with the dual space of some Banach space $\mathfrak{X}$. Much of the material in the second half of this note is borrowed from the book of Pedersen [**40**] .

Let us first recall how $\mathcal{B}(\mathcal{H})$ is itself a dual space. By $\mathcal{K}(\mathcal{H})$ we denote the set of compact operators on $\mathcal{H}$.

Given an operator $K \in \mathcal{K}(\mathcal{H})$, we may consider $|K| = (K^*K)^{\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$. Then $|K| \geq 0$, and so by the Spectral Theorem for Compact Normal Operators, we know that $\sigma(|K|) = \{s_n(K)\}_{n=1}^{\infty}$, where $s_n(K) \geq 0$ for all $n \geq 1$ and $\lim_{n\to\infty} s_n(K) = 0$.

B.1. Definition. *We write $K \in \mathcal{C}_1(\mathcal{B}(\mathcal{H}))$ and say that K is a* ***trace class operator on*** *$\mathcal{H}$ if K is compact and $\sum_{n=1}^{\infty} s_n(K) < \infty$. The numbers $s_n = s_n(K)$ are called the* ***singular numbers*** *for K.*

More generally, we write $K \in \mathcal{C}_p(\mathcal{B}(\mathcal{H}))$ if $\sum_{n=1}^{\infty} {s_n}^p < \infty$.

We shall require the following two facts. Their proofs may be found in [**18**].

Facts:

- For each p, $1 \leq p < \infty$, $\mathcal{C}_p(\mathcal{B}(\mathcal{H}))$ is an ideal of $\mathcal{B}(\mathcal{H})$ called the *Schatten p-ideal.* Moreover, $\mathcal{C}_p(\mathcal{B}(\mathcal{H}))$ is closed in the $\mathcal{C}_p-$ *norm topology* which is the topology determined by the norm
$$\|K\|_p = (\sum_{n=1}^{\infty} {s_n}^p)^{1/p}.$$
- If $T \in \mathcal{B}(\mathcal{H})$, $K \in \mathcal{C}_1(\mathcal{B}(\mathcal{H}))$ and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$, then we can define $\mathrm{tr}(TK) = \sum_{n=1}^{\infty} a_{nn}$, where $TK = [a_{ij}]_{i,j\geq 1}$ with respect to the orthonormal basis $\{e_n\}_{n=1}^{\infty}$. One can then show that $\mathrm{tr}(TK)$ is well-defined; that is, it is independent of the orthonormal basis chosen.

From the above two facts, we see that given $T \in \mathcal{B}(\mathcal{H})$, we can define

$$\begin{array}{llll} \phi_T : & \mathcal{C}_1(\mathcal{B}(\mathcal{H})) & \to & \mathbb{C} \\ & K & \mapsto & \mathrm{tr}(TK) \end{array}.$$

The map that sends a trace class operator T to the functional ϕ_T proves to be an isometric isomorphism between $\mathcal{C}_1(\mathcal{B}(\mathcal{H}))^*$ and $\mathcal{B}(\mathcal{H})$, so that $\mathcal{B}(\mathcal{H})$ is a dual space and as such is endowed with the weak*-topology induced by its predual, $\mathcal{C}_1(\mathcal{B}(\mathcal{H}))$. This turns out to be precisely the ultraweak or σ-weak topology on $\mathcal{B}(\mathcal{H})$.

An alternate approach to this result is to realize $\mathcal{C}_1(\mathcal{B}(\mathcal{H}))$ as the closure of $\mathcal{H} \otimes \mathcal{H}$ in $\mathcal{B}(\mathcal{H})^*$.

In order to prove that every von Neumann algebra $\mathbb{A}$ is a dual space, we require some basic results form Linear Analysis.

B.2. Definition. *Let $\mathfrak{X}$ be a Banach space and $M \subset \mathfrak{X}$, $N \subseteq \mathfrak{X}^*$ be linear manifolds. Then*

$$\begin{array}{ll} M^\perp & = \{f \in \mathfrak{X}^* : f(m) = 0 \text{ for all } m \in M\} \\ {}^\perp N & = \{x \in \mathfrak{X} : g(x) = 0 \text{ for all } g \in N\}. \end{array}$$

B.3. Proposition. *Let $\mathfrak{X}$ be a Banach space and $M \subseteq \mathfrak{X}$, $N \subseteq \mathfrak{X}^*$ be linear manifolds. Then*

(1) *$M^\perp$ is a weak*-closed subspace of $\mathfrak{X}^*$.*
(2) *${}^\perp N$ is a norm closed subspace of $\mathfrak{X}$.*

Proof.

(1) Suppose $\{f_\alpha\}_{\alpha\in\Lambda}$ is a net in $M^\perp$ and f_α converges to f in the weak*-topology. Then for all x in $\mathfrak{X}$, $\lim_{\alpha\in\Lambda} f_\alpha(x) = f(x)$, and so in particular, $f(m) = \lim_\alpha f_\alpha(m) = 0$ for all $m \in M$, implying that $f \in M^\perp$. Thus M is weak*-closed.
(2) If $\{x_n\}_{n=1}^\infty \subseteq^\perp N$ and $x = \lim_{n\to\infty} x_n$, then $g(x) = \lim_{n\to\infty} g(x_n) = 0$ for all $g \in N$. Thus $x \in^\perp N$ and the latter is norm closed.

□

B.4. Theorem. *Let $\mathfrak{X}$ be a Banach space and let $M \subseteq \mathfrak{X}$ and $N \subseteq \mathfrak{X}^*$ be linear manifolds. Then $({}^\perp N)^\perp$ is the weak*-closure of N in $\mathfrak{X}^*$.*

Proof. Clearly, if $g \in N$, then $g(x) = 0$ for all $x \in {}^\perp N$, and so $g \in ({}^\perp N)^\perp$. But $({}^\perp N)^\perp$ is now a weak*-closed subspace of $\mathfrak{X}^*$ which contains the weak*-closure of N.

If f does not lie in the weak*-closure of N, then by the Hahn-Banach Theorem applied to $\mathfrak{X}^*$ with its weak*-topology (which separates points from convex sets), there exists $x \in {}^\perp N$ such that $f(x) \neq 0$. But then $f \notin ({}^\perp N)^\perp$, completing the proof.

□

B.5. Theorem. *Let $\mathfrak{X}$ be a Banach space and $M \subseteq \mathfrak{X}$ be a subspace of $\mathfrak{X}$. Let $\pi : \mathfrak{X} \to \mathfrak{X}/M$ denote the canonical quotient map. Then the map*

$$\begin{array}{rlcl} \tau : & (\mathfrak{X}/M)^* & \to & M^\perp \\ & f & \mapsto & f \circ \pi \end{array}$$

is an isometric isomorphism.

Proof. First we shall show that τ is injective.

If $\tau(f) = f \circ \pi = g \circ \pi = \tau(g)$, then

$$f(\pi(x)) = (f \circ \pi)(x) = (g \circ \pi)(x) = g(\pi(x)) \text{ for all } x \in \mathfrak{X},$$

and so $f = g$ as elements of $(\mathfrak{X}/M)^*$.

Next we show that τ is surjective.

Let $\phi \in M^\perp$ and define $g \in (\mathfrak{X}/M)^*$ by $g(\pi(x)) = \phi(x)$. To see that g is well-defined, note that if $\pi(x) = \pi(y)$, then

$$g(\pi(x)) - g(\pi(y)) = \phi(x) - \phi(y) = \phi(x - y).$$

But $\pi(x - y) = 0$ implies that $x - y \in M$, and so $\phi(x - y) = 0$. Thus g is well-defined, and since $\tau(g) = g \circ \pi = \phi$, τ is surjective.

Finally we show that τ is isometric. Let $\tau \in (\mathfrak{X}/M)^*$. Then

$$\begin{aligned} \|\tau(g)\| &= \|g \circ \pi\| \\ &= \sup_{\|x\|=1} \|g \circ \pi(x)\| \\ &= \sup_{\|\pi(x)\|=1} \|g(\pi(x))\| \\ &= \|g\|. \end{aligned}$$

□

B.6. Theorem. *Let $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Then $\mathbb{A}$ is isometrically isomorphic to the dual space of some Banach space.*

Proof. Let $\mathfrak{X} = \mathcal{C}_1(\mathcal{B}(\mathcal{H})) = (\mathcal{B}(\mathcal{H}))_*$, and let $M = {}^\perp\mathbb{A}$, so that M is closed in $\mathfrak{X}$. By Theorem B.5 above, we have

$$(\mathfrak{X}/M)^* \simeq M^\perp,$$

and this isomorphism is isometric. But then

$$(\mathcal{C}_1(\mathcal{B}(\mathcal{H}))/({}^\perp\mathbb{A}))^* \simeq ({}^\perp\mathbb{A})^\perp = \mathbb{A}^{-weak^*}.$$

But $\mathbb{A}$ is a von Neumann algebra and hence $\mathbb{A}$ is closed in the weak-operator topology, which is weaker than the weak*-topology on $\mathcal{B}(\mathcal{H})$. Thus $\mathbb{A}$ is weak*-closed as well, and so

$$(\mathcal{C}_1(\mathcal{B}(\mathcal{H}))/({}^\perp\mathbb{A}))^* \simeq \mathbb{A},$$

where the isomorphism is once again isometric.

□

To complete the analysis, one needs to show that if a C*-algebra $\mathbb{A}$ is isometrically isomorphic to the dual space of some Banach space $\mathfrak{X}$, then $\mathbb{A}$ is a von Neumann algebra. This is by far the more difficult implication.

We begin with the following Proposition, which may be found in [**50**].

B.7. Proposition. *Let $\mathbb{A}$ be a C*-algebra and let S denote its unit sphere. Then S has an extreme point if and only if $\mathbb{A}$ has an identity.*

Proof. Suppose first that $\mathbb{A}$ has an identity, say 1. We shall show that 1 is an extreme point in S. If $1 = (a+b)/2$ with $a, b \in S$, then put $c = (a+a^*)/2$ and $d = (b+b^*)/2$. Then $1 = (c+d)/2$ with $c, d \in S$. Since $d = 2-c$, d commutes with c and both c and d are self-adjoint.

Representing the C*-algebra generated by $1, c$, and d as continuous functions on some compact Hausdorff space, we can easily see that $c = d = 1$. Hence $a^* = 2-a$, so that a is normal. But then $a = a^* = 1$, again by norm considerations, so that $b = 1$ and thus 1 is an extreme point.

Conversely, suppose x is an extreme point in S. Let $C_0(\Omega)$ be the C*-subalgebra of $\mathbb{A}$ generated by x^*x. Then, since every C*-algebra has an approximate identity, we can take a sequence $\{y_n\}$ of positive elements in $C_0(\Omega)$ such that $\|y_n\| \le 1$ for all n, $\lim_{n\to\infty} \|(x^*x)y_n - (x^*x)\| = 0$, and $\lim_{n\to\infty} \|(x^*x)y_n^2 - (x^*x)\| = 0$. (This last step follows from the fact that if $\{y_n\}$ is a bounded approximate identity for $C_0(\Omega)$, then so is $\{y_n^2\}$.)

Suppose that at some point t of Ω, x^*x takes a non-zero value less than one. Then we can take a positive element c of $C_0(\Omega)$, non-zero at t, such that $\gamma_n = y_n + c$, $s_n = y_n - c$, $\|(x^*x)\gamma_n^2\| \le 1$, and $\|(x^*x)s_n^2\| \le 1$. Hence $x\gamma_n$ and xs_n are in S.

On the other hand,

$$\|(xy_n - x)^*(xy_n - x)\| = \|x^*xy_n^2 - x^*xy_n - x^*xy_n + x^*x\|,$$

and this tends to 0 as n tends to ∞. Hence $\lim_{n\to\infty} xy_n = x$, so that $x\gamma_n \to x + xc$ and $xs_n \to x - xc$.

Since $x + xc,\ x - xc \in S$ and $x = \dfrac{(x+xc)+(x-xc)}{2}$, $x = x + xc = x - xc$. Hence $xc = 0$ and so $\|cx^*xc\| = \|x^*xc^2\| = 0$. This is a contradiction, because $x^*x(t)c^2(t) \ne 0$.

Therefore, x^*x has no non-zero value less than one in Ω. In other words, x^*x is a projection.

Put $x^*x + xx^* = h$, and let $\mathbb{B}$ be a maximal commutative C*-algebra of $\mathbb{A}$ containing h. Suppose h is not invertible in $\mathbb{B}$. Then there exists a sequence $\{z_n\}$ of positive elements belonging to $\mathbb{B}$ which satisfies $\|z_n^2\| = 1$ for all n and $\lim_{n\to\infty} \|hz_n^2\| = 0$. Hence,

$$\|xz_n\| = \|z_nx^*\| = \|z_nx^*xz_n\|^{\frac{1}{2}} \le \|z_nhz_n\|^{\frac{1}{2}} \to 0 \quad (n \to \infty),$$

and analogously, $\|z_nx\| = \|x^*z_n\| \to 0$ $(n \to \infty)$. Therefore

$$\lim_{n\to\infty} \|z_n - xx^*z_n - z_nx^*x + xx^*z_nx^*x\| = 1.$$

Now we use the symbolic notation: $y(1-x) = y - yx,\ (1-x)y = y - xy$.

We shall show that $(1 - xx^*)\mathbb{A}(1 - x^*x) = 0$. Suppose

$$a \in (1 - xx^*)\mathbb{A}(1 - x^*x),$$

and $\|a\| \le 1$. Then

$$\|x \pm a\| = \|(x^* \pm a^*)(x \pm a)\|^{\frac{1}{2}} = \|x^*x \pm (x^*a + a^*x) + a^*a\|^{\frac{1}{2}}.$$

Since $a^*xx^*a = 0$, $x^*a = a^*x = 0$ and $x^*xa^*a = x^*x(1-x^*x)a^*a = 0$. Hence $\|x \pm a\| = \max(\|x^*x\|^{\frac{1}{2}}, \|a^*a\|^{\frac{1}{2}}) \le 1$, so that by the extremity of x, $a = 0$.

On the other hand,

$$z_n - xx^*z_n - z_nx^*x + xx^*z_nx^*x \in (1 - xx^*)\mathbb{A}(1 - x^*x);$$

hence it is zero, a contradiction.

Therefore h is invertible in $\mathbb{B}$, $h^{-1}h$ is the identity of $\mathbb{B}$, and so it is a projection in $\mathbb{A}$ and the identity of $h^{-1}h\mathbb{A}h^{-1}h$.

Suppose $\mathbb{A}(1-h^{-1}h) \ne 0$. Then there exists an element $a \ne 0$ in $\mathbb{A}(1-h^{-1}h)$. Since $a^*ah^{-1}h = 0$, a^*a commutes with $h^{-1}h\mathbb{A}h^{-1}h \supseteq \mathbb{B}$. But $a \notin \mathbb{B}$, since $a \ne 0$, $h^{-1}h = 1_{\mathbb{B}}$, and $ah^{-1}h = 0$. This contradicts the maximality of $\mathbb{B}$. Hence $h^{-1}h$ is the identity of $\mathbb{A}$, completing the proof.

□

Recall the following:

B.8. Theorem. [The Krein-Smulian Theorem] *A convex set in the dual space $\mathfrak{X}^*$ of a Banach space $\mathfrak{X}$ is weak*-closed if and only if its intersection with every positive multiple of the closed unit ball in $\mathfrak{X}^*$ is weak*-closed.*

We shall use the Krein-Smulian Theorem to prove the following Lemma.

B.9. Lemma. *Let $\mathbb{A}$ be a C*-algebra and $\mathfrak{X}$ be a Banach space such that $\mathbb{A}$ is isomorphic as a Banach space to $\mathfrak{X}^*$. Then $\mathbb{A}_h := \{a \in \mathbb{A} : a = a^*\}$is weak*-closed.*

Furthermore, the positive cone $\mathbb{A}_+$ of $\mathbb{A}$ is also weak-closed.*

Proof. By the Krein-Smulian Theorem above, it is sufficient to show that the unit ball $B_1(\mathbb{A}_h)$ is weak*-closed, for $\mathbb{A}_h$ is clearly convex. To that end, let $\{x_\alpha\}$ be a weak*-convergent net in $B_1(\mathbb{A}_h)$ and write the limit as $x + iy$, with $x, y \in \mathbb{A}_h$. Here, $x + iy \in B_1(\mathbb{A})$, which is weak*-closed by Alaoglu's Theorem. Then $\{x_\alpha + in\}$ is weak*-convergent to $x + i(y + n)$ for every n. Since $\|x_\alpha + in\| \le (1 + n^2)^{\frac{1}{2}}$ and the norm is weak*-lower semicontinuous, we have

$$(1 + n^2)^{\frac{1}{2}} \ge \|x + i(n + y)\| \ge \|n + y\|.$$

If $y \ne 0$, we may assume that $\sigma(y)$ contains a number $\lambda > 0$ (passing, if necessary, to $\{-x_\alpha\}$). But then

$$\lambda + n \le \|n + y\| \le (1 + n^2)^{\frac{1}{2}}$$

for all n, a contradiction. Thus $y = 0$. Again, since the norm is weak*-lower semi-continuous, we also have $\|x\| \leq 1$, that is, $x \in B_1(\mathbb{A}_h)$.

As for the positive cone, it again suffices to show that the unit ball $B_1(\mathbb{A}_+)$ of $\mathbb{A}$ is weak*-closed. But then simply note that $B_1(\mathbb{A}_+) = \frac{1}{2}(B_1(\mathbb{A}_h)+1)$, and translation and contraction do not affect weak*-closures.

□

B.10. Definition. *A C^*-algebra $\mathbb{A}$ is said to be* ***monotone complete*** *if each bounded increasing net in $\mathbb{A}_h$ has a least upper bound in $\mathbb{A}_h$.*

B.11. Example. *The most important example of a monotone complete C^*-algebra for our purposes is the space $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. To see that this is indeed monotone complete, it suffices (by translation) to show that increasing bounded nets of positive operators have a least upper bound. We do this by showing that such nets converge strongly.*

Let $\mathcal{H}$ be a Hilbert space and let $\{P_\alpha\}_{\alpha\in\Lambda}$ be a net of positive operators on $\mathcal{H}$ such that $0 \leq P_\alpha \leq P_\beta \leq I$ for $\alpha,\ \beta \in \Lambda$ with $\alpha \leq \beta$. Then there exits $P \in \mathcal{B}(\mathcal{H})$ such that $0 \leq P_\alpha \leq P \leq I$ for all α and the net $\{P_\alpha\}_{\alpha\in\Lambda}$ converges to P strongly.

Proof. *Indeed, if $Q \in \mathcal{B}(\mathcal{H})$ with $0 \leq Q \leq 1$, then $0 \leq Q \leq Q^2 \leq I$, since Q commutes with $(I-Q)^{\frac{1}{2}}$ by the functional calculus and*

$$\begin{aligned} < (Q-Q^2)x, x > &= < Q(I-Q)^{\frac{1}{2}}x, (I-Q)^{\frac{1}{2}}x > \\ &\geq 0 \end{aligned}$$

for all $x \in \mathcal{H}$.

Moreover, for all x, the net $\{< P_\alpha x, x >\}$ is nondecreasing and is bounded above by $\|x\|^2$, and thus is a Cauchy net. Now for $\alpha \leq \beta$, we have

$$\begin{aligned} \|(P_\beta - P_\alpha)x\|^2 &= < (P_\beta - P_\alpha)^2 x, x > \\ &\leq < (P_\beta - P_\alpha)x, x > \\ &= < P_\beta x, x > - < P_\alpha x, x > \end{aligned}$$

and so $\{P_\alpha x\}$ is a Cauchy net with respect to the Hilbert space norm.

For $x \in \mathcal{H}$, let $Px = \lim_\alpha P_\alpha x$. Then P is linear and $\|Px\| = \lim_\alpha \|P_\alpha x\| \leq \|x\|$, so that $\|P\| \leq 1$. Also,

$$0 \leq \lim_\alpha < P_\alpha x, x > = < Px, x >,$$

so that $P \geq 0$. This completes the proof.

□

B.12. Lemma. *Let $\mathbb{A}$ be a C^*-algebra and $\mathfrak{X}$ be a Banach space such that $\mathbb{A}$ is isomorphic as a Banach space to $\mathfrak{X}^*$. Then $\mathbb{A}$ is monotone complete.*

Proof. Let $\{x_i\}$ be a bounded increasing monotone net of self-adjoint elements of $\mathbb{A}$. Since $B_1(\mathbb{A}_h)$ is weak*-compact (being convex, norm bounded and weak*-closed), there is a subnet $\{x_j\}$ of $\{x_i\}$ which is weak*-convergent to an element $x \in \mathbb{A}_h$.

For each x_i we eventually have $x_j \geq x_i$ for $j \geq i$, and thus $x \geq x_i$ since $\mathbb{A}_+$ is weak*-closed. That is, consider the subnet $\{x_j - x_i\}_{j\geq i}$ which eventually lie in $\mathbb{A}_+$ and converges in the weak*-topology to $x - x_i$. In particular, x is an upper bound for $\{x_i\}$ in $\mathbb{A}_h$.

If $y \in \mathbb{A}_h$ and $y \geq x_i$ for all i then $y \geq x_j$ for all j, so that

$$y \geq \text{weak}^* - \lim x_j = x$$

as above. As such, x is the least upper bound for x_i and so $\mathbb{A}$ is monotone complete.

□

B.13. Definition. *Given a subset $\mathbb{M}$ of self-adjoint operators on some Hilbert space $\mathcal{H}$, we denote by $\mathbb{M}^m$ (resp. $\mathbb{M}_m$) the set of operators obtained by taking strong limits of increasing (resp. decreasing) nets in $\mathbb{M}$.*

Note that if $\mathbb{A}$ is a C*-algebra and $\mathbb{M} = \mathbb{A}_{sa}$, then $\mathbb{M}^m = \mathbb{M}_m$.

B.14. Lemma. *Let $\mathbb{A}$ be a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and let $\mathbb{M}$ denote the strong operator closure of $\mathbb{A}$. If P is a projection in $\mathbb{M}$ then given $x \in ranP$ and $y \in \mathrm{ran}P^\perp$ there is an element $B \in (\mathbb{M}_{sa})^m$ such that $Bx = x$ and $By = 0$.*

Proof. By Kaplansky's Density Theorem, we find find operators $A_n \in \mathbb{M}^1_+$ such that $\|A_n x - x\| < \frac{1}{2}$ and $\|A_n y\| < \frac{1}{n}2^{-n}$.

For $n < m$ define $B_{nm} = (1 + \sum_{k=n}^m kA_k)^{-1} \sum_{k=n}^m kA_k$. By spectral theory, $\|B_{nm}\| \leq 1$, $B_{nm} \in \mathbb{M}_+$, and $B_{nm} \leq \sum_{k=n}^m kA_k$.

Thus $< B_{nm}y, y >\leq< \sum_{k=n}^m kA_k y, y >\leq \sum_{k=n}^m 2_{-k} < 2^{-n+1}$.

Since $\sum_{k=n}^m kA_k \geq mA_m$, we have $B_{nm} \geq (1 + mA_m)^{-1} mA_m$ and so $1 - B_{nm} \leq (1 + mA_m)^{-1}$. But $A_m \in \mathbb{M}^1_+$ implies that $(1 + mA_m) \leq (1 + m)$, and hence $(1 + m)^{-1} \leq (1 + mA_m)^{-1}$. Then $(mA_m)^{\frac{1}{2}}(1 + m)^{-1}(mA_m)^{\frac{1}{2}} \leq (mA_m)^{\frac{1}{2}}(1 + mA_m)^{-1}(mA_m)^{\frac{1}{2}}$, and hence $(mA_m)(1 + m)^{-1} \leq (mA_m)(1 + mA_m)^{-1}$. It follows that $1 - (mA_m)(1 + mA_m)^{-1} \leq 1 - (mA_m)(1 + m)^{-1}$, i.e. $(1 + mA_m)^{-1} \leq (1 + m)^{-1}((1 + m) - mA_m)$, so that

$$1 - B_{nm} \leq ((1 + m) - mA_m).$$

Thus

$$\begin{aligned} < (1 - B_{nm})x, x > \quad &\leq< (1 + m)^{-1}(1 + m(1 - A_m))x, x > \\ &= (1 + m)^{-1}(< x, x > +m < (1 - A_m)x, x >) \\ &\leq (1 + m)^{-1}(1 + m(\tfrac{1}{m})) \\ &= 2(1 + m)^{-1}. \end{aligned}$$

For fixed n, the sequence $\{B_{nm}\}$ is monotone increasing, and since it is norm bounded, it is strongly convergent to an element $0 \le B_n \in (\mathbb{M}_{sa})^m$. Moreover, $\|B_n\| \le 1$.

Since $B_{n+1\,m} \le B_{nm}$ for each $m > (n+1)$, we see that $B_{n+1} \le B_n$, so that the sequence $\{B_n\}$ is monotone decreasing and bounded. Again, it is strongly convergent to an element $B \ge 0$, which lies in $(\mathbb{M}_{sa})^m$, again, as $(\mathbb{M}_{sa})^m = (\mathbb{M}_{sa})_m$.

Note that $\| < B_n y, y > \| = \| \lim_m < B_{nm} y, y > \| \le 2^{-n+1}$, and $\| < (1 - B_n)x, x > \| = \| \lim_m (1 - B_{nm})x, x > \| \le 0$. Since $0 \le B_n \le 1$, we deduce that $< (1 - B_n)x, x >= 0$, and hence that $B_n x = x$.

Finally, as $0 \le B \le 1$, $\| < By, y > \| = \| \lim_n < B_n y, y > \| = 0$, implying that $By = 0$. Similarly, $Bx = \lim_n B_n x = x$, completing the proof.

□

B.15. Theorem. *Let $\mathcal{H}$ be a Hilbert space. A unital C*-algebra $\mathbb{M}$ of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $(\mathbb{M}_{sa})^m = \mathbb{M}_{sa}$.*

Proof. Suppose that $\mathbb{M}$ is a von Neumann algebra. Let $T \in (\mathbb{M}_{sa})^m$. Then $T \in \mathbb{M}$ as the latter is closed in the strong operator topology. Since $T \in \mathcal{B}(\mathcal{H})_{sa}$ by definition, $T \in \mathbb{M}_{sa}$.

Conversely, to prove that $\mathbb{M}$ is a von Neumann algebra, it suffices to show that each projection P in the strong closure of $\mathbb{M}$ actually belongs to $\mathbb{M}$.

Suppose that $x \in P\mathcal{H}$ and $y \in (I - P)\mathcal{H}$. Then Lemma B.14 shows that there exists $R \in \mathbb{M}_+$ such that $Rx = x$ and $Ry = 0$. The range projection $P_{(x,y)}$ of R belongs to $\mathbb{M}$. Indeed, the sequence $(\frac{1}{n} + R)^{-1}$ is monotone increasing, and converges strongly to $P_{(x,y)}$. Thus $P_{(x,y)}x = x$, and $P_{(x,y)}y = 0$. The projections $P_{(x,y_1)} \wedge P_{(x,y_2)} \wedge \ldots \wedge P_{(x,y_n)}$ form a decreasing net in $\mathbb{M}_+$, when $\{y_1, y_2, \ldots, y_n\}$ runs through the finite subsets of $(I - P)\mathcal{H}$. Thus the limit projection $P_x \le P$, and lies in $\mathbb{M}_{sa}$. Clearly, P is the limit of the increasing net of projections $P_{x_1} \vee P_{x_2} \vee \ldots \vee P_{x_k}$ where $\{x_1, x_2, \ldots, x_k\}$ runs over the finite subsets of $P\mathcal{H}$. Thus $P \in (\mathbb{M}_{sa})^m \subseteq \mathbb{M}$, completing the proof.

□

B.16. Definition. *Let $\mathbb{A}$ be a von Neumann algebra. Then $\phi \in \mathbb{A}^*$ is said to be **normal** if for each bounded monotone increasing net $\{x_i\}$ in $\mathbb{A}_h$ with $\lim x_i = x$ we have $\{\phi(x_i)\}$ converging to $\phi(x)$.*

*More generally, if $\mathbb{A}$ and $\mathbb{B}$ are von Neumann algebras, then a positive linear map ρ of $\mathbb{A}$ into $\mathbb{B}$ is said to be **normal** if for each bounded monotone increasing net $\{x_i\}$ in $\mathbb{A}_h$, the net $\{\rho(x_i)\}$ increases to $\rho(x)$ in $\mathbb{B}_h$.*

B.17. Lemma. *If $\mathbb{A}$ is a unital monotone complete C*-algebra with a separating family of normal states, then there is a normal isomorphism of $\mathbb{A}$ onto a von Neumann algebra.*

Proof. Let $\mathcal{F}$ denote the separating family of normal states of $\mathbb{A}$ and consider the representation $\pi_{\mathcal{F}} = \oplus_{\phi\in\mathcal{F}}\pi_\phi$, acting on $\mathcal{H}_{\mathcal{F}} = \oplus_{\phi\in\mathcal{F}}\mathcal{H}_\phi$. Then $(\pi_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}})$ is faithful. Indeed, if $x \geq 0$ lies in the kernel of $\pi_{\mathcal{F}}$, then

$$\begin{aligned} \phi(x) &= < \pi_\phi(x)\xi_\phi, \xi_\phi > \\ &= 0 \end{aligned}$$

for each $\phi \in \mathcal{F}$, so that $x = 0$. Since $\ker\pi_{\mathcal{F}}$ is a C*-algebra, it is spanned by its positive elements, and therefore $\ker\pi_{\mathcal{F}} = \{0\}$.

Now if $\{x_\alpha\}$ is a bounded montone increasing net in $\mathbb{A}_{sa}$, then $\{x_\alpha\}$ has a least upper bound $x \in \mathbb{A}_{sa}$, as $\mathbb{A}$ is monotone complete. Also $\{\pi_{\mathcal{F}}(x_\alpha)\}_\alpha$ is a bounded monotone decreasing net in $\mathcal{B}(\mathcal{H}_{\mathcal{F}})$ as $\pi_{\mathcal{F}} \geq 0$, and thus $\{\pi_{\mathcal{F}}(x_\alpha)\}_\alpha$ has a least upper bound y in $\mathcal{B}(\mathcal{H}_{\mathcal{F}})$, as $\mathcal{B}(\mathcal{H}_{\mathcal{F}})$ is monotone complete. Since $x \geq x_\alpha$ for all α, $\pi_{\mathcal{F}}(x) \geq \pi_{\mathcal{F}}(x_\alpha)$ for all α, and hence $\pi_{\mathcal{F}}(x) \geq y$.

However, if $\phi \in \mathcal{F}$, and $(\pi_\phi, \mathcal{H}_\phi, z_\phi)$ is the cyclic representation associated with ϕ via the GNS construction, then for all unitaries u in $\mathbb{A}$,

$$\begin{aligned} < \pi_\phi(x)\pi_\phi(u)z_\phi, \pi_\phi(u)z_\phi > &= \phi(u^*xu) \\ &= \lim \phi(u^*x_\alpha u) \quad \text{as } \phi \text{ is normal} \\ &= \lim < \pi_\phi(x_\alpha)\pi_\phi(u)z_\phi, \pi_\phi(u)z_\phi > . \end{aligned}$$

Thus $(\pi_\phi(x) - y)\pi_\phi(u)z_\phi = 0$. But $\mathbb{A}$ is spanned by its unitaries, and hence

$$(\pi_\phi(x) - y)\left[\pi_\phi(\mathbb{A})z_\phi\right] = 0.$$

As $\mathcal{H}_{\mathcal{F}} = \oplus_{\phi\in\mathcal{F}}\mathcal{H}_\phi$, we conclude that $\pi_{\mathcal{F}}(x) = y$. Thus $\pi_{\mathcal{F}}(\mathbb{A})$ is monotone complete. By Theorem B.15, $\pi_{\mathcal{F}}(\mathbb{A})$ is a von Neumann algebra.

□

B.18. Theorem. *Let $\mathbb{A}$ be a C*-algebra and $\mathfrak{X}$ be a Banach space such that $\mathbb{A}$ is isomorphic as a Banach space to $\mathfrak{X}^*$. Then $\mathbb{A}$ has a faithful representation as a von Neumann algebra with $\mathbb{A}_* = \mathfrak{X}$.*

Proof. Consider the weak*-topology on $\mathbb{A}$ arising from $\mathfrak{X}$, and identify $\mathfrak{X}$ with the weak*-continuous elements of $\mathfrak{X}^*$. Since the unit ball $B_1(\mathbb{A})$ is weak*-compact, it has an extremal point, by the Krein-Milman Theorem. Hence $\mathbb{A}$ is unital, by Proposition B.7.

By Lemma B.9, $\mathbb{A}_h$ is weak*-closed, as well as the positive cone $\mathbb{A}_+$ of $\mathbb{A}$.

It now follows that the positive cone of $\mathfrak{X}$, namely $\mathfrak{X}_+$, is separating for $\mathbb{A}$. For if $a \in \mathbb{A}_h$ and $-a \notin \mathbb{A}_+$, then since $\mathbb{A}_+$ is a weak*-closed cone in $\mathbb{A}_h$, by the Hahn-Banach Theorem there exists an element $\phi \in \mathfrak{X}_h$ such that $\phi(\mathbb{A}_+) \geq 0$ and $\phi(a) > 0$. Namely, we can think of $\mathbb{A}_h$ as a real vector space and obtain a real linear functional on $\mathbb{A}_h$ satisfying these conditions. Then we complexify ϕ to $\mathbb{A}$.

By Lemma B.12, $\mathbb{A}$ is monotone complete.

Suppose $\phi \in \mathfrak{X}_+$. Then ϕ is normal, since if $\{x_i\}$ is a monotone increasing net in $\mathbb{A}$ with least upper bound x, then

$$\lim \phi(x_i) \leq \phi(x) = weak^* - \lim \phi(x_j) \leq \lim \phi(x_i).$$

Note that $\lim \phi(x_i)$ exists since it is a bounded monotone increasing net in $\mathbb{R}$. Thus $\mathbb{A}$ is a monotone complete C*-algebra with a separating family (namely $B_1(\mathfrak{X})_+$) of normal states.

By Lemma B.17, $\mathbb{A}$ has a faithful representation as a von Neumann algebra. Moreover, from the GNS construction, we has that $\mathfrak{X}_+ \subseteq \mathbb{A}_*$.

Also, if $x \in \mathbb{A}_h$ and $x \neq 0$, then $\phi(x) \neq 0$ for some $\phi \in \mathfrak{X}^+$. Thus the linear span of $\mathfrak{X}_+$ is norm dense in $\mathfrak{X}$, from which we conclude that $\mathfrak{X} \subseteq \mathbb{A}_*$.

Since the compact topology in $B_1(\mathbb{A})$ is unique, the weak*- and the σ-weak topologies coincide. Hence $\mathbb{A}_* = \mathfrak{X}$.

$\square$

Exercises

Hedgehogs. Why can't they just share the hedge?

Dan Antopolski

Question 1. A SPECTRAL ESTIMATE

Let $\mathcal{A}$ be a Banach algebra. For $a \in \mathcal{A}$ and $\lambda \in \rho(a)$, show that

$$\|(\lambda - a)^{-1}\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma(a))}.$$

Question 2. SIMILARITY ORBITS

Let $A \in \mathbb{M}_n$ for some $n \geq 1$. We define the **similarity orbit** of A to be the set

$$\mathcal{S}(A) = \{T \in \mathbb{M}_n : T = S^{-1}AS \text{ for some } S \in \mathbb{M}_n^{-1}\}.$$

Show that there exists $T \in \overline{\mathcal{S}(A)}$ for which $\|T\| = \operatorname{spr}(T) = \operatorname{spr}(A)$. Can this always be done if we ask T to lie in $\mathcal{S}(A)$ rather than in the norm closure of $\mathcal{S}(A)$?

Question 3. WEIGHTED SHIFTS I

Let $\mathcal{H}$ be an infinite dimensional, separable, *complex* Hilbert space. An operator $W \in \mathcal{B}(\mathcal{H})$ is said to be a **unilateral forward weighted shift** if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ and a sequence $(w_n)_n \in \ell^{\infty}$ so that $We_n = w_n e_{n+1}$ for all $n \geq 1$. Recall also from PMath 753 that a bijective operator $U \in \mathcal{B}(\mathcal{H})$ is said to be **unitary** if $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in \mathcal{H}$, and that unitaries are precisely the Hilbert space isomorphisms. Note that this relation is equivalent to the condition that $U^* = U^{-1}$, where U^* denotes the Hilbert space adjoint of U.

(a) Find $\|W\|$ in terms of the sequence $(w_n)_n$ of weights for W.
(b) Show that there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ so that U^*WU is a unilateral forward weighted shift with weight sequence $(|w_n|)_n$.

If $A, B \in \mathcal{B}(\mathcal{H})$, $V \in \mathcal{B}(\mathcal{H})$ is unitary and $A = V^* B V$, then we say that A and B are **unitarily equivalent**. Unitary equivalence of operators is easily seen to be an equivalence relation on $\mathcal{B}(\mathcal{H})$, and we write $A \simeq B$ to denote that A and B are unitarily equivalent. If there exists $S \in \mathcal{B}(\mathcal{H})$ so that S is invertible and $A = S^{-1} B S$, then we say that A and B are **similar**, and write $A \sim B$. From the paragraph above, we see that unitarily equivalent operators are similar.

(c) Show that if W is a unilateral forward weighted shift and $\lambda \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, then $W \simeq \lambda W$.

(d) Prove that $\sigma(W)$ has **circular symmetry**; that is, if $\alpha \in \sigma(W)$, then $\alpha\mathbb{T} := \{z \in \mathbb{C} : |z| = \alpha\} \subseteq \sigma(W)$.

Question 4. Weighted Shifts II

Let $\mathcal{H}$ be an infinite dimensional, separable complex Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Let W be the unilateral forward weighted shift operator satisfying $We_n = w_n e_{n+1}$ where $w_n = (\operatorname{GCD}(n, 2^n))^{-1}$.

(a) Prove that the spectral radius of W is greater than zero.

Hint. calculate W^{2^k}, $k \geq 1$.

(b) Prove that there exist *nilpotent* weighted shifts V_n (with respect to the same orthonormal basis $\{e_n\}_{n=1}^\infty$) so that $\lim_{n\to\infty} V_n = W$. (Recall that an element a of an algebra $\mathcal{A}$ is said to be *nilpotent of order* k if $a^{k-1} \neq 0 = a^k$. For this question, we are not imposing any restrictions on the order of nilpotence of V_n.)

(c) Conclude that the spectral radius function $\operatorname{spr} : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ is *not* continuous.

Question 5. The functional calculus

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ be two elements of $\mathbb{M}_3(\mathbb{C})$.

Let

$$T = \begin{bmatrix} 4 & 1 & 0 & -4 & 0 & 0 \\ 0 & 4 & -1 & 0 & -4 & 2 \\ 0 & 0 & 6 & 0 & 0 & -6 \\ 2 & 0 & 0 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 & -2 & 2 \\ 0 & 0 & 3 & 0 & 0 & -3 \end{bmatrix} \in \mathbb{M}_6(\mathbb{C}).$$

(a) Let $\exp(z) = e^z$, $z \in \mathbb{C}$. Find $\exp(T)$.

Hint. $S := \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ is invertible and

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -x+2y & 2x-2y \\ -x+y & 2x-y \end{bmatrix}.$$

(b) Let $g(z) = \begin{cases} 1 & \text{if } |z| < \frac{1}{2} \\ 0 & \text{if } |z-2| < \frac{1}{2} \\ 0 & \text{if } |z-3| < \frac{1}{2} \end{cases}$. Find $g(T)$.

Question 6. EXPONENTIALS

Let $\mathcal{A}$ be a unital Banach algebra, and let $a, b \in \mathcal{A}$, and $\exp(z) = e^z$, $z \in \mathbb{C}$.

(a) If $ab = ba$, prove that $\exp(a)\exp(b) = \exp(a+b)$.
(b) Does this necessarily hold if $ab \neq ba$?

Question 7. DIAGONAL OPERATORS

Let $\mathcal{H}$ be an infinite dimensional, separable, *complex* Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Recall that an operator $D \in \mathcal{B}(\mathcal{H})$ is said to be **diagonal** if there exists a sequence $(d_n)_n \in \ell^\infty$ so that $De_n = d_n e_n$ for all $n \geq 1$. (Of course, the reason for the terminology is that the matrix of D relative to this basis is the diagonal matrix $\operatorname{diag}(d_n)_n$.)

(a) Find $\|D\| := \sup\{\|Dx\| : \|x\| \leq 1\}$ in terms of the sequence $(d_n)_n$.
(b) Find $\sigma(D)$, the spectrum of D in $\mathcal{B}(\mathcal{H})$. Conclude that if $\Omega \subseteq \mathbb{C}$ is compact, then there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\sigma(T) = \Omega$.
(c) Determine the set of eigenvalues of D.
(d) State (and prove) necessary and sufficient conditions on the sequence $(d_n)_n$ for the operator D to be compact.

Let π denote the canonical quotient map from $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. If $T \in \mathcal{B}(\mathcal{H})$, the spectrum $\sigma(\pi(T))$ of $\pi(T)$ in $\mathcal{Q}(\mathcal{H})$ is known as the **essential spectrum** of T. It is also denoted by $\sigma_e(T)$.

(e) Find the essential spectrum $\sigma(\pi(D))$ of the diagonal operator D.

Question 8. WEIGHTED SHIFTS III

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Recall that an operator $N \in \mathcal{B}(\mathcal{H})$ is said to be **normal** if $N^*N = NN^*$, where N^* denotes the Hilbert space adjoint of N. We say that $M \in \mathcal{B}(\mathcal{H})$ is **essentially normal** if the image $\pi(M)$ of M in the Calkin algebra $\mathcal{Q}(\mathcal{H})$ under the canonical quotient map $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ is normal in the sense that $\pi(M)\pi(M^*) = \pi(M^*)\pi(M)$. [We may define an involution on $\mathcal{Q}(\mathcal{H})$ by setting $\pi(M)^* := \pi(M^*)$, so that M will be essentially normal if $\pi(M)\pi(M)^* = \pi(M)^*\pi(M)$, which looks like the usual notion of "normality".] It is clear that if $N \in \mathcal{B}(\mathcal{H})$ is normal and $K \in \mathcal{K}(\mathcal{H})$ is compact, then $M = N + K$ is essentially normal.

Suppose now that $\mathcal{H}$ is also separable. Recall from the first assignment that an operator $W \in \mathcal{B}(\mathcal{H})$ is said to be a **unilateral forward weighted shift** if there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ for $\mathcal{H}$ and a sequence $(w_n)_n \in \ell^\infty$ so that $We_n = w_n e_{n+1}$ for all $n \geq 1$. **The unilateral (unweighted) forward shift operator** $S \in \mathcal{B}(\mathcal{H})$ is the unilateral forward weighted shift with constant weight sequence $(w_n)_n$, where $w_n = 1$ for all $n \geq 1$. [Of course, such an operator may be chosen for any orthonormal basis $\{e_n\}_{n=1}^\infty$ for $\mathcal{H}$, but it is straightforward to show that all such operators are unitarily equivalent. For this reason, people usually refer to "the" unilateral forward shift, as opposed to "a" unilateral forward shift.]

(a) Prove that the unilateral (unweighted) forward shift S is essentially normal.
(b) Prove that there do not exist $N \in \mathcal{B}(\mathcal{H})$ normal and $K \in \mathcal{K}(\mathcal{H})$ such that $S = N + K$.
(c) Suppose that W is a unilateral forward *weighted* shift with weight sequence $(w_n)_{n=1}^\infty$. If W is compact, find $\sigma(W)$.
(d) State and prove a necessary and sufficient condition on the weight sequence $(w_n)_n$ of a unilateral forward shift W for W to be essentially normal.
(e) An operator $V \in \mathcal{B}(\mathcal{H})$ is said to be a **bilateral weighted shift** if there exists an orthonormal basis $(f_n)_{n\in\mathbb{Z}}$ for $\mathcal{H}$ and a weight sequence $(v_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ so that $Vf_n = v_n f_{n+1}$ for all $n \in \mathbb{Z}$. As with "the unilateral forward shift", we refer to V as "the bilateral shift" if all weights v_n are equal to 1. Note that in the case of bilateral shifts, we do not refer to "forward" nor "backward" shifts (why not?). A proof similar to the one appearing in Assignment One shows that any bilateral weighted shift is unitarily equivalent to a weighted shift with non-negative weights, and so we usually assume *a priori* that $v_n \geq 0$ for all $n \in \mathbb{Z}$.

Let V be a bilateral weighted shift with weight sequence $(v_n)_{n\in\mathbb{Z}}$, where $v_n \geq 0$ for all $n \in \mathbb{Z}$. Suppose furthermore that there exists $\delta > 0$ so that $\delta \leq v_n \leq 1$ for all $n \in \mathbb{Z}$. Show that

$$\sigma(V) \subseteq \{z \in \mathbb{C} : \delta \leq |z| \leq 1\}.$$

(f) Find an example of a bilateral weighted shift $U \in \mathcal{B}(\ell^2(\mathbb{Z}))$ and a rank-one operator $F \in \mathcal{B}(\ell^2(\mathbb{Z}))$ so that
(i) $\sigma(U) = \mathbb{T}$, and
(ii) $\sigma(U + F) = \overline{\mathbb{D}}$.

Question 9. Weyl's Theorem

Prove the following Theorem due to Weyl (1909):

Theorem. *Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space and let $A, B \in \mathcal{B}(\mathcal{H})$. If $A - B \in \mathcal{K}(\mathcal{H})$, and if $\lambda \in \sigma(A) \setminus \sigma_p(A)$, or if λ is an eigenvalue of infinite multiplicity, then $\lambda \in \sigma(B)$.*

Hint: We can assume without loss of generality that $\lambda = 0$ (why?). Then write $A = B + (A - B) = \ldots$.

Remark: Let $K := B - A \in \mathcal{K}(\mathcal{H})$. Then $B = A + K$ is called a *compact perturbation* of A. The underlying notion is that compact perturbations lie in an ideal of $\mathcal{B}(\mathcal{H})$, and hence are in some sense "small". As such, they shouldn't cause radical changes in the "essential" behaviour of the operator.

Question 10. The closure of the set of finite rank nilpotent operators

Let $\mathcal{F}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \operatorname{rank} T < \infty\}$, and

$$\mathrm{Nil}_0(\mathcal{H}) = \{T \in \mathcal{F}(\mathcal{H}) : T^k = 0 \text{ for some } k \geq 1\}.$$

Calculate the norm closure $\overline{\mathrm{Nil}_0(\mathcal{H})}$ of $\mathrm{Nil}_0(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$.

Question 11. Isometries on Hilbert space

First let us establish some notation. Given a family $\{\mathcal{H}_\beta\}_{\beta\in\Gamma}$ of complex Hilbert spaces, we define the Hilbert space $\mathcal{H} = \oplus_{\beta\in\Gamma}\mathcal{H}_\beta$ to be the Hilbert space $\mathcal{H} = \{(x_\beta)_\beta : \|(x_\beta)_\beta\| := (\sum_\beta \|x_\beta\|^2)^{1/2} < \infty\}$. Given operators $T_\beta \in \mathcal{B}(\mathcal{H}_\beta)$ with $\sup_\beta \|T_\beta\| < \infty$, we can define a bounded linear operator $T = \oplus_\beta T_\beta$ on $\mathcal{H}$ via $T(x_\beta)_\beta = (T_\beta x_\beta)_\beta$. If the cardinality of Γ is α, we write $T^{(\alpha)}$ to denote the operator $T^{(\alpha)} = \oplus_{\beta\in\Gamma}T$ acting on $\mathcal{H}^{(\alpha)} = \oplus_{\beta\in\Gamma}\mathcal{H}$.

Finally, for two Hilbert spaces $\mathcal{M}$ and $\mathcal{N}$ and two operators $A \in \mathcal{B}(\mathcal{M})$ and $B \in \mathcal{B}(\mathcal{N})$, we write $A \simeq B$, and say that A is *unitarily equivalent* to B, if there exists a unitary operator $U : \mathcal{M} \to \mathcal{N}$ such that $B = UAU^*$.

Let $\mathcal{H}$ be an infinite dimensional, complex Hilbert space. Suppose that $W : \mathcal{H} \to \mathcal{H}$ is an isometry; that is, $\|Wx\| = \|x\|$ for all $x \in \mathcal{H}$. Prove that there is a unitary operator U and a cardinal α such that $W \simeq U \oplus S^{(\alpha)}$, where S denotes *the* unilateral forward shift (all of whose weights are 1). That is, S acts on an infinite-dimensional, complex, separable Hilbert space $\mathcal{H}_0$ with orthonormal basis $\{e_n\}_{n=1}^\infty$ and $Se_n = e_{n+1}$ for all $n \geq 1$.

Question 12. Spectrum for unital abelian Banach algebras

Let $\mathcal{A}$ be a unital abelian Banach algebra

(a) Show that $\sigma(a+b) \subseteq \sigma(a) + \sigma(b)$ and $\sigma(ab) \subseteq \sigma(a)\sigma(b)$ for all $a, b \in \mathcal{A}$. Show by example that there exist (non-unital) Banach algebras for which this fails.

(b) Suppose a_1 and a_2 lie in $\mathcal{A}$. Let $\mathcal{B}$ be the Banach subalgebra of $\mathcal{A}$ *generated by* a_1 and a_2 - that is, $\mathcal{B}$ is the smallest unital Banach subalgebra of $\mathcal{A}$ containing both a_1 and a_2. Show that $\Sigma_\mathcal{B}$ is homeomorphic to the compact set $\sigma(a_1, a_2) := \{(\tau(a_1), \tau(a_2)) : \tau \in \Sigma_\mathcal{B}\}$. (It should be clear from your proof that this result extends to Banach algebras generated by n elements for any $n > 1$.)

Question 13. Idempotents in Banach algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras.

(a) Show that if $\mathcal{A}$ is abelian and it contains an idempotent e other than 0 and 1, then $\Sigma_\mathcal{A}$ is disconnected.

(b) Show that if $\mathcal{A}$ is abelian and the idempotents in $\mathcal{A}$ have dense linear span, then $\Sigma_\mathcal{A}$ is totally disconnected.

(c) Show that the linear span $\mathcal{E}$ of the idempotents in $\mathcal{B}$ forms a *Lie ideal* of $\mathcal{B}$; that is, they form a vector subspace of $\mathcal{B}$ which satisfies $[b, m] := bm - mb \in \mathcal{E}$ whenever $b \in \mathcal{B}$ and $m \in \mathcal{E}$.

Question 14. Ideals of spaces of continuous functions

Let X be a locally compact, Hausdorff space. Show that every closed ideal of $\mathcal{C}_0(X)$ has the form

$$\mathfrak{K}_E = \{f \in \mathcal{C}_0(X) : f|_E = 0\}$$

for some closed subset E of X.

Question 15. The Gelfand map need not be surjective nor isometric

Let $\mathcal{A} = \{f \in \mathcal{C}([0,1]) : f' \in \mathcal{C}([0,1])\}$, and for $f \in \mathcal{A}$, define $\|f\| = \|f\|_\infty + \|f'\|_\infty$, where $\|\cdot\|_\infty$ denotes the usual supremum norm on $\mathcal{C}([0,1])$. Then $\mathcal{A}$ can be shown to be a commutative Banach algebra under the usual pointwise operations for multiplication and addition. (Although you don't have to hand in the proof of that statement, you should at least verify it for yourselves.)

Show that the Gelfand map is neither isometric nor surjective in this case.

Question 16. Sets with property S for abelian, unital Banach algebras

Let $\mathcal{A}$ be an abelian, unital Banach algebra. We shall say that a weak*-closed subset $F \subseteq \Sigma_{\mathcal{A}}$ has **property** S for A if for each $x \in A$,

$$\sup\{|\widehat{x}(\varphi)| : \varphi \in F\} = \mathrm{spr}(x).$$

(a) Show that $\mathcal{A}$ admits a set Δ which is minimal (with respect to inclusion) amongst all sets with property S for $\mathcal{A}$.

(b) Let $\mathcal{A} = \mathcal{A}(\mathbb{D})$ be the disk algebra. Find a minimal set Δ as above for $\mathcal{A}$.

Question 17. Unitary groups in C^*-algebras.

Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{U} = \{u \in \mathcal{A} : uu^* = u^*u = 1\}$ denote the *unitary group* of $\mathcal{A}$.

(a) Show that if $u \in \mathcal{U}$ and $\|u - 1\| < 2$, then there exists a self-adjoint element $h \in \mathcal{A}$ such that $u = \exp(ih)$.

(b) Define $\mathcal{U}_\circ = \{\exp(ih_1)\exp(ih_2)\ldots\exp(ih_n) : h_k = h_k^* \in \mathcal{A}, 1 \le k \le n\}$. Show that $\mathcal{U}_\circ$ is both open and closed in $\mathcal{U}$, and hence that $\mathcal{U}_\circ$ is the connected component of the identity in $\mathcal{U}$. (Of course, $\mathcal{U}$ is just a group, not an algebra!)

(c) If $\mathcal{A}$ is abelian, show that $\mathcal{U}_\circ = \{\exp(ih) : h = h^* \in \mathcal{A}\}$.

Question 18. Every idempotent in a C^*-algebra is similar to a projection.

Question 19. Let $\mathcal{H}$ be a complex Hilbert space and $A, B \in \mathcal{B}(\mathcal{H})$. Prove that if $\langle Ax, x\rangle = \langle Bx, x\rangle$ for all $x \in \mathcal{H}$, then $A = B$. (Hint: prove that the above condition implies that $\langle Ax, y\rangle = \langle Bx, y\rangle$ for all $x, y \in \mathcal{H}$.)

Question 20. The Jacobson radical

Let $\mathcal{A}$ be a unital C^*-algebra.

(a) Show that if $a \in \mathcal{A}$ and $\lambda \in \sigma(a^*a)$, then $a^*a - \lambda 1$ has neither a left nor a right inverse in $\mathcal{A}$.

(b) Show that the intersection of the maximal left ideals of $\mathcal{A}$ is $\{0\}$. That is, C^*-algebras are *semisimple*.

Question 21. Multiplier algebras

Let $\mathcal{A}$ be a C^*-algebra. A *double centraliser* for $\mathcal{A}$ is a pair (L, R) of bounded linear maps on $\mathcal{A}$ satisfying

$$L(ab) = L(a)b, \qquad R(ab) = aR(b), \qquad \text{and } R(a)b = aL(b).$$

For example, if $c \in \mathcal{A}$, set $L_c(b) = cb$ and $R_c(b) = bc$ for all $b \in \mathcal{A}$. Then (L_c, R_c) is a double centraliser for $\mathcal{A}$.

(a) Show that if (L, R) is a double centraliser for $\mathcal{A}$, then $\|L\| = \|R\|$.
Let $\mathcal{M}(\mathcal{A})$ denote the set of all double centralisers for $\mathcal{A}$.
For (L_1, R_1), $(L_2, R_2) \in \mathcal{M}(\mathcal{A})$ and $\alpha, \beta \in \mathbf{C}$, we define
 (i) $\alpha(L_1, R_1) + \beta(L_2, R_2) = (\alpha L_1 + \beta L_2, \alpha R_1 + \beta R_2)$;
 (ii) $(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1)$;
 (iii) $(L_1, R_1)^* = (R_1^*, L_1^*)$, where a map $B : \mathcal{A} \to \mathcal{A}$ induces the map $B^* : \mathcal{A} \to \mathcal{A}$ via $B^*(a) := (B(a^*))^*$.

(b) Show that $\mathcal{M}(\mathcal{A})$ is a unital C^*-algebra (called the *multiplier algebra* of $\mathcal{A}$) under the norm $\|(L, R)\| = \|L\|(= |R\|)$.

Culture: One way to add a unit to a non-unital C^*-algebra $\mathcal{A}$ is to embed the Banach algebra unitisation of $\mathcal{A}$, namely $\mathcal{A}_u := \{(a, \lambda) : a \in \mathcal{A}, \lambda \in \mathbf{C}\}$ with $(a, \lambda)(b, \beta) = (ab + \lambda b + \beta a, \lambda\beta)$ into $\mathcal{M}(\mathcal{A})$ via the map

$$\begin{array}{cccc} \Phi : & \mathcal{A}_u & \to & \mathcal{M}(\mathcal{A}) \\ & (a, \lambda) & \mapsto & (L_a, R_a) + (\lambda I, \lambda I). \end{array}$$

(Here, $I : \mathcal{A} \to \mathcal{A}$ is the identity map.) If, for $(a, \lambda) \in \mathcal{A}_u$, we define $(a, \lambda) = (a^*, \overline{\lambda})$, then this gives us an involution. The map Φ proves to be a *-isomorphism, so that we may impose a C^*-algebra norm on $\mathcal{A}_u$ via $\|(a, \lambda)\| := \|\Phi(a, \lambda)\|$.

Question 22. Lifting elements from quotient C^*-algebras

Let $\mathcal{A}$ denote a unital C*-algebra and $\mathcal{K}$ denote a closed, two-sided ideal of $\mathcal{A}$. Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{K}$ denote the canonical *-homomorphism.

(a) Suppose r is a *self-adjoint* element of $\mathcal{A}/\mathcal{K}$. Show that there exists $R = R^* \in \mathcal{A}$ such that $\pi(R) = r$.

(b) Suppose r is a *positive* element of $\mathcal{A}/\mathcal{K}$. Show that there exists R positive in $\mathcal{A}$ such that $\pi(R) = r$.

(c) Suppose $u \in \mathcal{A}/\mathcal{K}$ is *unitary* and that $\sigma(u) = \{\lambda \in \mathbb{T} : \mathrm{Re}(\lambda) \geq 0\}$. Show that there exists $U \in \mathcal{A}$ unitary such that $\pi(U) = u$.

(d) Does the result hold for all unitaries in $\mathcal{A}/\mathcal{K}$? For example, suppose that $u \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a unitary element. Does there exist a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $\pi(U) = u$, where $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the canonical quotient map?

Question 23. Representations of the Calkin algebra

Let $\mathcal{H}$ be an infinite-dimensional, complex, separable Hilbert space. The GNS construction may be applied to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Thus, we obtain an isometric *-monomorphism (i.e. an injective *-homomorphism which is then automatically isometric) $\rho : \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_0)$ for some Hilbert space $\mathcal{H}_0$. Prove that in this case, $\mathcal{H}_0$ can not be separable.

Hints:

- Show that there exist *uncountably* many infinite subsets $\{F_\alpha\}_{\alpha\in\Lambda}$ of the rational numbers $\mathbb{Q}$ with the property that $F_\alpha \cap F_\beta$ is a finite set for all $\alpha \neq \beta$.
- Let $\{e_q\}_{q\in\mathbb{Q}}$ be an orthonormal basis for $\mathcal{H}$. Consider the orthogonal projections P_α of $\mathcal{H}$ onto span $\{e_q : q \in F_\alpha\}$. What can you say about $\{\pi(P_\alpha) : \alpha \in \Lambda\}$?

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