CHAIN RULE for MAPPINGS

Let $G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x,y) \longmapsto (f(x,y), g(x,y)) = (u,v)$

and

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(u,v) \longmapsto (p(u,v), q(u,v)) = (s,t)$.

Then,

$$F \circ G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (s,t)$$

and, if the component functions of F & G are ALL differentiable, then

$$D(F \circ G)(x_0, y_0) = DF(u_0, v_0) \cdot DG(x_0, y_0)$$

with $(u_0, v_0) = G(x_0, y_0)$.

matrix multiplication

OR, equivalently,

$$D(F \circ G)(x_0, y_0) = DF\big(G(x_0, y_0)\big) \cdot DG(x_0, y_0),$$

which is like the Chain Rule formula for functions of one variable.

NOTE: The Chain Rule formula is useful in proofs or when computing messy examples.

Ex. 1) Let $G(x,y) = (xy + x^2,\ 2x - y^3) = (u, v)$

and $F(u,v) = (ve^u,\ u - v) = (s, t)$.

Find $D(F \circ G)(-1,1)$ using the Chain Rule.

HERE: $(x_0, y_0) = (-1, 1)$ so that $(u_0, v_0) = F(x_0, y_0) = (0, -3)$.

So, by the Chain Rule,

$$D(F \circ G)(-1,1) = DF(0,-3) \cdot DG(-1,1).$$

Now,

$$DF = \begin{array}{c c} & \begin{array}{cc} u & v \end{array} \\ \begin{array}{c} s \\ t \end{array} & \begin{pmatrix} ve^u & e^u \\ 1 & -1 \end{pmatrix} \end{array} \Rightarrow DF(0,-3) = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$DG = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} u \\ v \end{array} & \begin{pmatrix} y + 2x & x \\ 2 & -3y^2 \end{pmatrix} \end{array} \Rightarrow DG(-1,1) = \begin{pmatrix} -1 & -1 \\ 2 & -3 \end{pmatrix}$$

$$\Rightarrow D(F \circ G)(-1,1) = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ -3 & 2 \end{pmatrix}.$$

2) Let $G(x,y) = \left(\sin(xy) + 2y^3, (2x+1)e^{4x^2-y}\right) = (u,v)$

and

$$F(u,v) = \left(4(u+v)\ln v,\ 2uv\right) = (s,t)$$

Find $D(F \circ G)(0,-1)$.

We see that the expressions of F & G are complicated enough that it's much easier to use the Chain Rule to compute $D(F \circ G)(0,-1)$ rather than find the derivative matrix of the composite map $F \circ G$ directly.

HERE: $(x_0, y_0) = (0,-1)$ so that $(u_0, v_0) = G(x_0, y_0) = (-2, e)$

Then,

$$D(F \circ G)(0,-1) = DF(-2,e) \cdot DG(0,-1).$$

Now,

$$DF = \begin{array}{c|cc} & u & v \\ \hline s & 4\ln v & 4\ln v + \frac{4(u+v)}{v} \\ t & 2v & 2u \end{array}$$

$$\Rightarrow DF(-2,e) = \begin{pmatrix} 4\ln e & 4\ln e + \frac{4(-2+e)}{e} \\ 2e & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 8 - 8e^{-1} \\ 2e & -4 \end{pmatrix}$$

ALSO,

$$
DG = \begin{array}{c c}
 & \begin{array}{cc} \quad x \quad & \quad y \quad \end{array} \\
\begin{array}{c} u \\ v \end{array} & \begin{pmatrix} y\cos(xy) & x\cos(xy) + 6y^2 \\ 2e^{4x^2-y} + (2x+1)8xe^{4x^2-y} & -(2x+1)e^{4x^2-y} \end{pmatrix}
\end{array}
$$

$$
\Rightarrow DG(0,-1) = \begin{pmatrix} -1 \cdot \cos 0 & 0 + 6 \\ 2e + 0 & -e \end{pmatrix}
$$

$$
= \begin{pmatrix} -1 & 6 \\ 2e & -e \end{pmatrix}
$$

$$
\Rightarrow D(F \circ G)(0,-1) = \begin{pmatrix} 4 & 8-8e^{-1} \\ 2e & -4 \end{pmatrix} \begin{pmatrix} -1 & 6 \\ 2e & -e \end{pmatrix}
$$

$$
= \begin{pmatrix} 16e - 20 & 32 - 8e \\ -10e & 16e \end{pmatrix}
$$